Domino tilings of Aztec rectangles with connected holes

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joint work with Fumihiko Nakano, Taizo Sadahiro and Hiroyuki Tagawa.

Abstract

It is famous that the number of domino tilings of an Aztec diamond is 2 to $n+1$ choose 2. We study the number of domino tilings of an Aztec rectangle with even number of connected holes in a line and we obtain a formula which express the number of such domino tilings by a product of a similar power of 2, linear factors and a polynomial of the coordinates of the holes in a line. We will find a formula which expresses this polynomial as a determinant of terminating Gauss hypergeometric series and show that this polynomial possesses interesting properties. First we use the Lindström-Gessel-Viennot theorem to enumerate the domino tilings of an Aztec rectangle with connected holes and obtain a determinant whose entries are generalized large Schröder numbers.

Abstract

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Then we consider a more general determinant whose entries are the moments of the Laurent biorthogonal polynomials, which enable us to apply the Desnanot-Jacobi adjoint matrix theorem. This general determinant reduces to the case $q = t = 1$ in Kamioka's result if we have no hole, i.e., the Aztec diamond case. Then the evaluation of the determinant reduces to a quadratic relation of the above polynomials. This project is still a work in progress and we believe that we are very close to the complete proof. This is a joint work with Fumihiko Nakano, Taizo Sadahiro and Hiroyuki Tagawa.

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1. Large Schröder numbers
2. Laurent biorthogonal polynomials
3. Aztec diamond theorem
4. Aztec Rectangle with connected holes in line
5. How to count the tilings?
6. Introducing a new parameter c
7. Strategy for proof
8. Future problems

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2. **[EKLP]** N. Elkies, G. Kuperberg, M. Larsen, and J. Propp, “Alternating-sign matrices and domino tilings (Part I, II)”, *J. Algebraic Combinatorics* **1** (1992), 111–132, 219–234.
3. **[GV]** I. M. Gessel and X. G. Viennot, Determinants, Paths, and Plane Partitions, 1989 preprint.
4. **[Ka]** S. Kamioka, “Laurent biorthogonal polynomials, q-Narayana polynomials and domino tilings of the Aztec diamonds”, *J. Combin. Theory Ser. A* **123** (2014), 14–29.

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Large Schröder numbers

The large Schröder number $S(n)$

Definition

A *Schröder path* is a path in the plane, starting from the origin $(0,0)$ and ending at $(2n,0)$ never going below the x-axis, using the steps

$$U = (1,1) \quad \text{up}, \qquad D = (1,-1) \quad \text{down}, \qquad L = (2,0) \quad \text{level}.$$

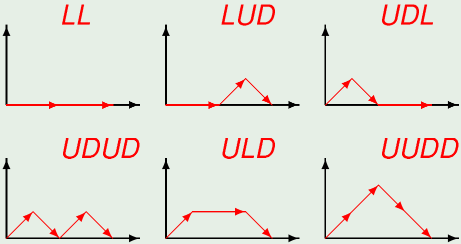
The *large Schröder number*, denoted by $S(n)$, is the number of such paths.

UDULUDD $\qquad S(4) = 90$

Example of the large Schröder numbers

Example

If $n = 2$, then the Schröder paths are the followings. hence we obtain $S(2) = 6$.

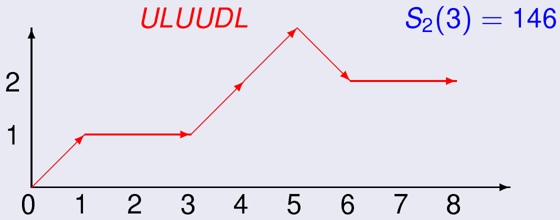


Similarly one easily gets

$$S(0) = 1, \quad S(1) = 2, \quad S(2) = 6, \quad S(3) = 22$$
$$S(4) = 90, \quad S(5) = 394, \quad S(6) = 1806, \ldots$$

Definition

More generally, we consider a path starting from the origin $(0, 0)$ and ending at $(2n+m, m)$ never going below the x-axis, using the steps U, D and L.

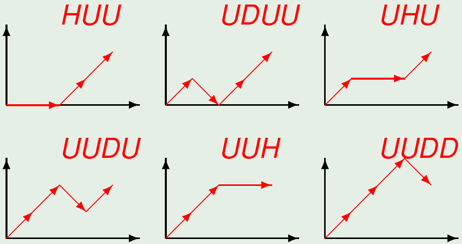


Let $S_m(n)$ denote the number of such paths. Especially, $S(n) = S_0(n)$.

Example of $S_m(n)$

Example

If $m = 2$ and $n = 1$, then such paths are the followings. hence we obtain $S_2(1) = 6$.



Similarly one easily gets

$$S_2(0) = 1, \qquad S_2(1) = 6, \qquad S_2(2) = 30$$
$$S_2(3) = 146, \qquad S_2(4) = 714, \ldots$$

Gauss hypergeometric series

The *Gauss hypergeometric function* is defined by the power series

Definition

$$ {}_2F_1\left({a, b; \atop c;} x\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n! (c)_n} x^n, $$

where $(a)_n = a(a+1)\cdots(a+n-1)$ is called the *rising factorial* or *Pochhammer symbol*. If a or b is a negative integer, then ${}_2F_1(a, b; c; x)$ is called *terminating*.

Proposition

If $m, n \geq 0$ are integers, then $S_m(n)$ has that following expression by the Gauss hypergeometric series:

$$ S_m(n) = \begin{cases} 1 & \text{if } n = 0, \\ 2\binom{n+m}{m} {}_2F_1\left({-n+1, m+n+2; \atop m+2;} -1\right) & \text{if } n \geq 1. \end{cases} $$

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$$ {}_2F_1\left(\begin{matrix} a, b; \\ c; \end{matrix} x\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} x^n, $$

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Proof

Proof. The key idea is

$$S_m(n) = S_{m+1}(n-1) + S_m(n-1) + S_{m-1}(n) \text{ if } m > 0.$$

$$\begin{tikzcd}
 & S_{m+1}(n-1) \arrow[dr] & \\
S_m(n-1) \arrow[rr] & & S_m(n) \\
 & S_{m-1}(n) \arrow[ur] &
\end{tikzcd}$$

Corollary

If $n \geq 0$ are integers, then

$$S(n) = \begin{cases} 1 & \text{if } n = 0, \\ 2\,{}_2F_1\left(\begin{matrix} -n+1, 2; \\ 2; \end{matrix}\; -1\right) & \text{if } n \geq 1. \end{cases}$$

Laurent biorthogonal polynomials

The Laurent biorthogonal polynomials $P_n(z)$

Definition (The Laurent biorthogonal polynomials)

The (monic) *Laurent biorthogonal polynomials* (LBPs) $P_n(z)$, $n \in \mathbb{N}$, are the polynomials determined from the three term relation

$$P_{n+1}(z) = (z-1)P_n(z) - z\,P_{n-1}(z) \qquad (n \geq 1)$$

with $P_{-1}(z) = 0$ and $P_0(z) = 1$. (Kamioka defined the LBPs with parameters $\{b_n\}$ and $\{c_n\}$, but here we restrict our attention to the case where we need.)

Example

The first few terms are as follows:

$$P_1(z) = z - 1$$
$$P_2(z) = z^2 - 3z + 1$$
$$P_3(z) = z^3 - 5z^2 + 5z - 1$$

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Linear functional $\mathscr{F}$ and Moments $f(n)$

Theorem (Favard type theorem for LBPs [Ka])

There exists a linear functional $\mathscr{F}$ defined over Laurent polynomials in z with respect to which the LBPs $P_n(z)$ satisfy the orthogonality

$$\mathscr{F}[P_n(z)z^{-k}] = h_n\delta_{n,k}, \quad 0 \le k \le n$$

with some constants $h_n \neq 0$, where $\delta_{n,k}$ denotes the Kronecker delta. The linear functional $\mathscr{F}$ is unique up to a constant factor. Hence we assume $\mathscr{F}[1] = 1$ hereafter.

Definition (moments)

We write the *moments* of the linear functional $\mathscr{F}$,

$$f(n) = \mathscr{F}[z^n], \qquad n \in \mathbb{Z}.$$

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Example

For example, we have

$$P_1(z) = z - 1 \Rightarrow f(1) = f(0) = 1,$$
$$P_2(z) = z^2 - 3z + 1 \Rightarrow f(2) = 3f(1) - f(0) = 2,$$
$$P_3(z) = z^3 - 5z^2 + 5z - 1 \Rightarrow f(3) = 5f(2) - 5f(1) + f(0) = 6.$$

Similarly we obtain

$$P_2(z) = z^2 - 3z + 1 \Rightarrow f(-1) = -\{3f(0) - f(1)\} = 2,$$
$$P_3(z) = z^3 - 5z^2 + 5z - 1$$
$$\Rightarrow f(-2) = -\{-5f(2) + 5f(1) - f(0)\} = 6.$$

The Laurent biorthogonal polynomials $P_n(z)$

Definition

For $m \in \mathbb{N}$ and $n \in \mathbb{Z}$ we define $f_m(n)$ by

$$f_m(n) = \mathscr{F}\left[P_m(z)z^n\right].$$

Example

Actually, we can compute $f_m(n)$ by

$$f_m(n) = \sum_{k=0}^{m} [z^k]P_m(z) \cdot f(k+n)$$

where $[z^k]P_m(z)$ stands for the coefficient of z^k in $P_m(z)$.

$$f_1(-3) = -16, \qquad f_1(-2) = -4, \qquad f_1(-1) = -1,$$
$$f_1(0) = 0, \qquad f_1(1) = 1, \qquad f_1(2) = 4, \qquad f_1(4) = 16, \dots$$

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The moments and the large Schröder numbers

Proposition

For $m \in \mathbb{N}$ and $n \in \mathbb{Z}$ we obtain

$$f_m(n) = \begin{cases} S_m(n-1) & (n \geq 1), \\ (-1)^m S_m(-m-n) & (n \leq 0). \end{cases}$$

Aztec diamond Theorem

Aztec diamond AD_n

Definition

For $n \in \mathbb{N}$, *the Aztec diamond* AD_n of order n is the union of all unit squares which lie inside the closed region $|x| + |y| \le n + 1$.

Example

AD_5

Aztec diamond AD_n

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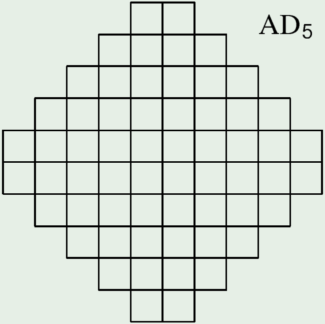

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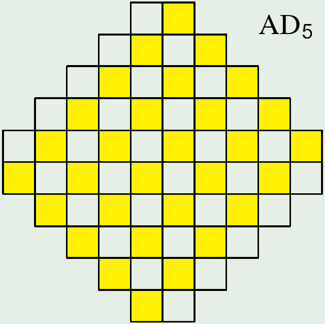

Domino

Definition

A domino denotes a one-by-two or two-by-one rectangle.

Example

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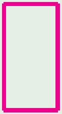
Example

1×2



2×1

Domino tiling

Definition

A *domino tiling*, or simply a *tiling*, of AD_n is a collection of non-overlapping dominoes which exactly covers AD_n.

Example

A tiling of AD_5

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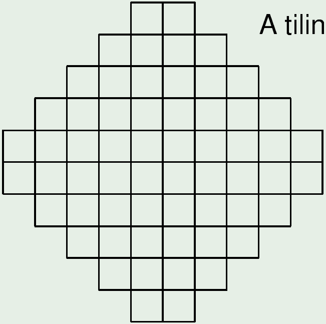

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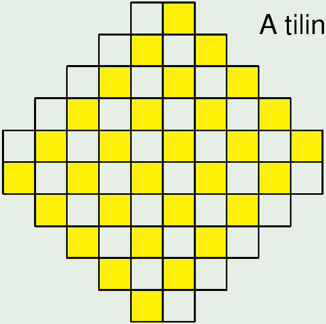

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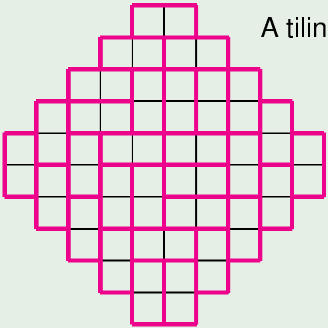

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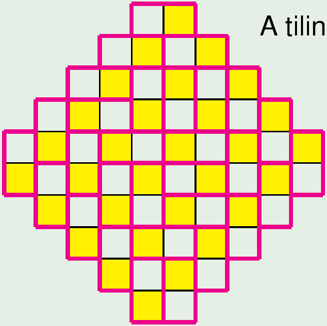

A tiling of AD_5

Theorem (Aztec diamond theorem)

For $n \in \mathbb{N}$, Let T_n denote the set of all tilings of AD_n. Then

$$\sharp T_n = 2^{\frac{n(n+1)}{2}}$$

Many proofs are known, e.g., Elkies-Kuperberg-Larsen-Propp (1992), Ciucu (1996), Brualdi-Kirkland (2003), Kuo (2004), Eu-Fu (2005), and Kamioka (2014).

Aztec Rectangle with connected holes in line

Aztec rectangle $\mathrm{AR}_{a,b}$

Definition

For $a, b \in \mathbb{N}$, *the Aztec rectangle* $\mathrm{AR}_{a,b}$ is the union of all unit squares which lie inside the closed region

$$b - 2a - 1 \leq x + y \leq b + 1, \; -b - 1 \leq y - x \leq b + 1.$$

(Hereafter we assume $a \leq b$.) Let $S_{x,y}$ denote the square with the vertex (x, y), $(x + 1, y)$, $(x + 1, y + 1)$ and $(x, y + 1)$. We call $S_{x,y}$ *white* (resp. *black*) if $x + y + b$ is even (resp. odd).

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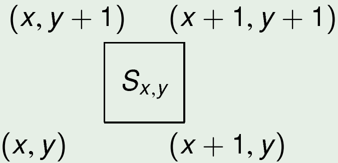
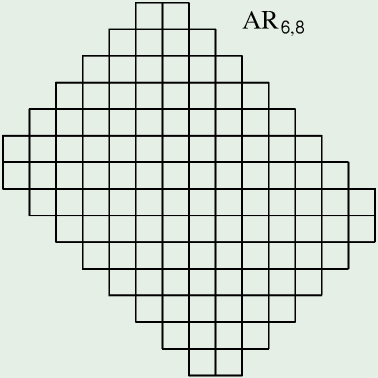


Example of Aztec rectangle

Example

The Aztec rectangle $\mathrm{AR}_{a,b}$ has $(a+1)b$ black squares and $a(b+1)$ white squares, so that there are $b-a$ more black squares than white ones. Meanwhile, each domino occupies 1 black square and 1 white square.



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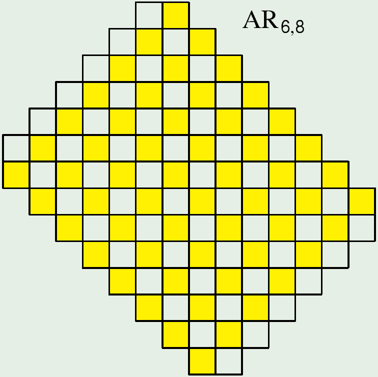
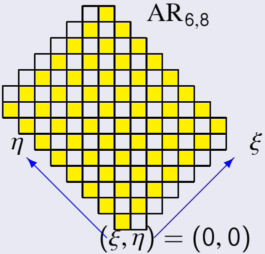


$\mathrm{AR}_{6,8}$

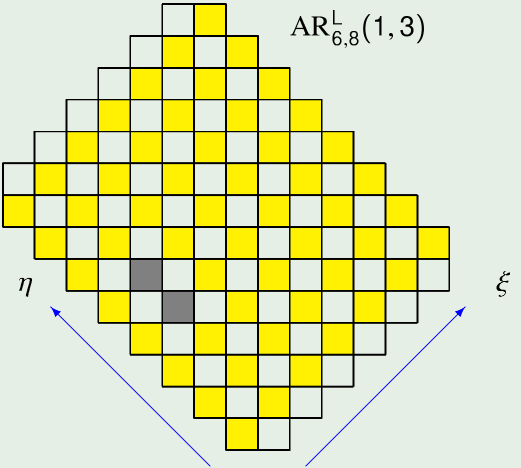
Definition ($\mathrm{AR}^{\mathrm{L}}_{a,b}(\xi,\eta)$ and $\mathrm{AR}^{\mathrm{S}}_{a,b}(\xi,\eta)$)

We remove $r = b - a$ connected black squares in a line parallel to the *long* or *short* side of $\mathrm{AR}_{a,b}$, which we call holes. We use the coordinates (ξ,η) of the lowest hole of the series of holes in line, where ξ increases along the short side and η increases along the long side. Here we assume the coordinates of the lowest black square is $(0,0)$. If the holes are parallel to the long (resp. short) side, we write the remaining area as $\mathrm{AR}^{L}_{a,b}(\xi,\eta)$ (resp. $\mathrm{AR}^{S}_{a,b}(\xi,\eta)$).



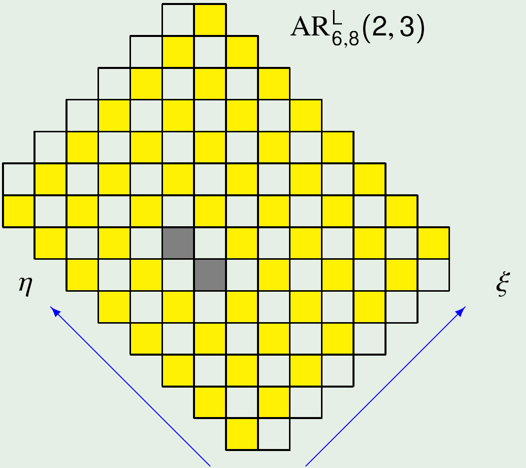
Example of $\mathrm{AR}^{\mathsf{L}}_{a,b}(\xi,\eta)$

Example



Example of $\mathrm{AR}^{\mathsf{L}}_{a,b}(\xi, \eta)$

Example



Example of $\mathrm{AR}^{\text{L}}_{a,b}(\xi,\eta)$

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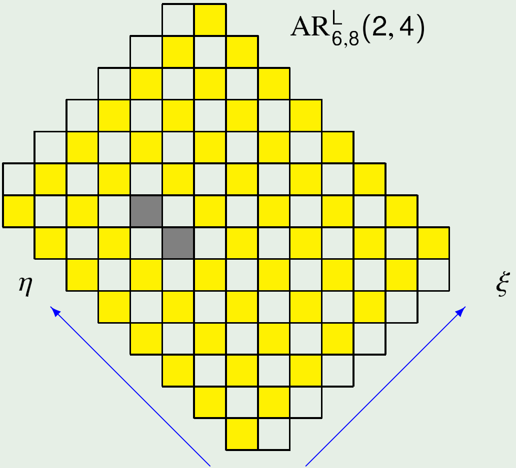
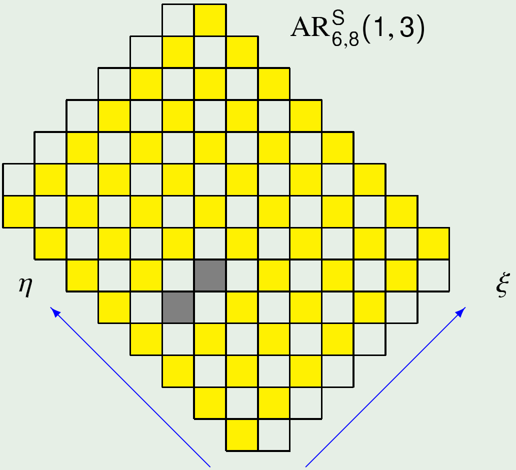


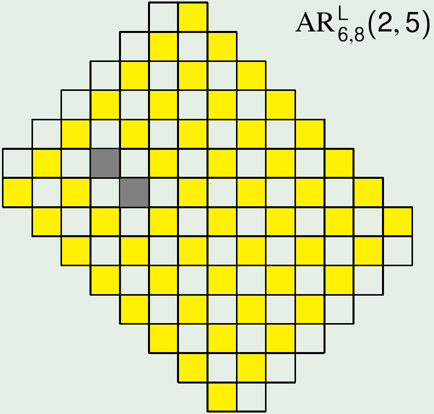
Example of $\mathrm{AR}^{\mathrm{S}}_{a,b}(\xi,\eta)$

Example



Example of domino tiling of Aztec tectangle

An example of domino tiling of $\mathrm{AR}^{\llcorner}_{6,8}(2,5)$



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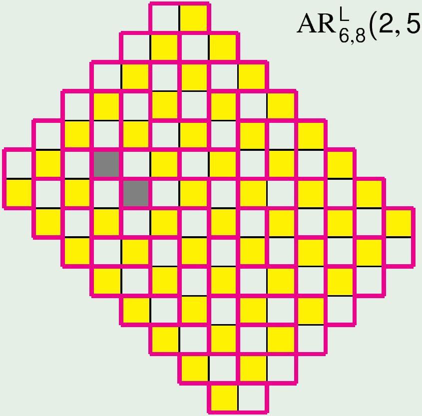

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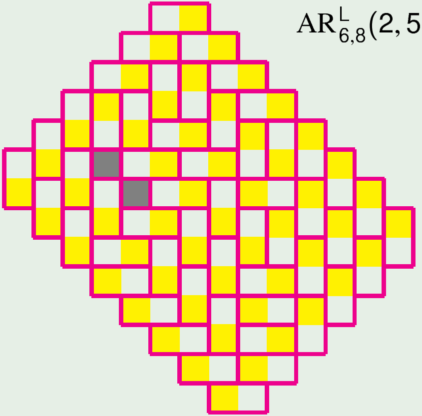

Our Qestion

Problem

What is the number of domino tilings of $\mathrm{AR}^{\mathrm{L}}_{a,b}(\xi,\eta)$ and $\mathrm{AR}^{\mathrm{S}}_{a,b}(\xi,\eta)$?
Is it nice or ugly?

Definition

Let $T^{L}_{a,b}(\xi,\eta)$ (resp. $T^{S}_{a,b}(\xi,\eta)$) denote the set of domino tilings of $\mathrm{AR}^{\mathrm{L}}_{a,b}(\xi,\eta)$ (resp. $\mathrm{AR}^{\mathrm{S}}_{a,b}(\xi,\eta)$), where (ξ,η) ranges

$$0 \le \xi \le a \text{ and } 0 \le \eta \le a$$
$$(\text{resp. } 0 \le \xi \le a - r + 1 \text{ and } 0 \le \eta \le a + r - 1).$$

Here $r = b - a$.

Answer

Let $r = b - a \geq 0$.
If r is even,

$$\sharp T^{\mathrm{L}}_{a,b}(\xi,\eta) = 2^{\frac{a(a+1)}{2}+r\eta} \cdot \prod_{k=0}^{r-1} \frac{k!}{(k+\eta)!} \cdot f^{(r)}_{\eta}\left(a, \xi - \frac{a}{2}\right)$$

$$= 2^{\frac{a(a+1)}{2}+r\xi} \cdot \prod_{k=0}^{r-1} \frac{k!(k+a-\xi)!}{(k+\eta)!(k+a-\eta)!} \cdot f^{(r)}_{\xi}\left(a, \eta - \frac{a}{2}\right)$$

where $f^{(r)}_{n}(a,x)$ is a polynomial of degree rn with respect to x such that $f^{(r)}_{n}(a,-x) = f^{(r)}_{n}(a,x)$.
If r is odd, $\sharp T^{\mathrm{L}}_{a,b}(\xi,\eta)$ is ugly!

Our Guess

Answer

Let $r = b - a \geq 0$.
If r is even, then $\sharp T^{S}_{a,b}(\xi, \eta)$ equals

$$2^{\frac{a(a+1)}{2}+r\eta-\frac{r(r-1)}{2}} \prod_{k=0}^{r-1} \frac{k!}{(k+\eta-r+1)!} \cdot g^{(r)}_{\eta-r+1}\left(a, \xi - \frac{a-r+1}{2}\right)$$

$$= 2^{\frac{a(a+1)}{2}+r\xi+\frac{r(r-1)}{2}} \cdot \prod_{k=0}^{r-1} \frac{k!(k+a-r+1-\xi)!}{(k+\eta-r+1)!(k+a-\eta)!}$$

$$\times g^{(r)}_{\xi}\left(a, \eta - \frac{a+r-1}{2}\right)$$

for $r - 1 \leq \eta \leq a$, otherwise $\sharp T^{S}_{a,b}(\xi, \eta)$ equals 0. Here $g^{(r)}_{n}(a, x)$ is also a polynomial of degree rn with respect to x such that
$g^{(r)}_{n}(a, -x) = g^{(r)}_{n}(a, x)$.
If r is odd, $\sharp T^{S}_{a,b}(\xi, \eta)$ is ugly again!

Example of $f_n^{(r)}(a,x)$

Example

$$f_0^{(2)}(a,x) = 1$$

$$f_1^{(2)}(a,x) = x^2 + \frac{a-1}{4}$$

$$f_2^{(2)}(a,x) = x^4 - x^2 + \frac{3}{16}(a-1)^2$$

$$f_3^{(2)}(a,x) = x^6 - \frac{3\,a+5}{4}\,x^4 + \frac{9\,a^2 - 24\,a + 31}{16}\,x^2 + \frac{9}{64}\,(a-3)(a-1)^2$$

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$$g_3^{(2)}(a,x) = x^6 - \frac{3a+8}{4}x^4 + \frac{1}{16}(9a^2 - 6a + 16)x^2 + \frac{9}{64}a^2(a-2)$$

Definition

Let $r, n \geq 0$. Define $f_n^{(r)}(a,x)$ by

$$f_n^{(r)}(a,x) = 2^{-rn} \prod_{k=0}^{r-1} \frac{(n+k)!(-a-k)_n}{k!} \cdot \det \left(\frac{{}_2F_1\left({-n-j+i, -\frac{a}{2}-x; \atop -a-j+1;} \, 2 \right)}{(n+j-i)!} \right)_{1 \leq i,j \leq r}.$$

Here we use the convention that $\frac{1}{m!} = 0$ if $m < 0$. Note that this is a determinant of size r (the number of holes).

Define $g_n^{(r)}(a,x)$ by

$$g_n^{(r)}(a,x) = f_n^{(r)}(a-r+1, x).$$

Example of $f_n^{(r)}(a,x)$

Example

If $r = 1$ then

$$f_n^{(1)}(a,x) = \frac{(-a)_n}{2^n} \cdot {}_2F_1\left(\begin{matrix} -n, -\frac{a}{2} - x; \\ -a; \end{matrix}\; 2\right).$$

skip If $r = 2$ then

$$f_n^{(2)}(a,x) = \frac{n!(n+1)!(-a)_n(-a-1)_n}{2^{2n}} \times \begin{vmatrix} \frac{1}{n!} \cdot {}_2F_1\left(\begin{smallmatrix} -n,-\frac{a}{2}-x; \\ -a; \end{smallmatrix}\; 2\right) & \frac{1}{(n+1)!} \cdot {}_2F_1\left(\begin{smallmatrix} -n-1,-\frac{a}{2}-x; \\ -a-1; \end{smallmatrix}\; 2\right) \\ \frac{1}{(n-1)!} \cdot {}_2F_1\left(\begin{smallmatrix} -n+1,-\frac{a}{2}-x; \\ -a; \end{smallmatrix}\; 2\right) & \frac{1}{n!} \cdot {}_2F_1\left(\begin{smallmatrix} -n,-\frac{a}{2}-x; \\ -a-1; \end{smallmatrix}\; 2\right). \end{vmatrix}$$

How to count the tilings?

Lindström-Gessel-Viennot Lemma

Definition

Let G be a locally finite directed acyclic graph. Consider base vertices $\boldsymbol{u} = (u_1, \ldots, u_n)$ and destination vertices $\boldsymbol{v} = (v_1, \ldots, v_n)$, and also assign a weight ω_e to each directed edge e. For each directed path P between two vertices, let $\omega(P)$ be the product of the weights of the edges of the path. For any two vertices u and v, write $h(u, v)$ for the sum $h(u, v) = \sum_{P:u \to v} \omega(P)$ over all paths from u to v. If one assigns the weight 1 to each edge, then $h(u, v)$ counts the number of paths from u to v.

Theorem (Lindström-Gessel-Viennot Lemma)

$$\sum_{\sigma \in S_n} \sum_{(P_1, \ldots, P_n):\ \boldsymbol{u} \to \boldsymbol{v}^{\sigma}} \operatorname{sgn}(\sigma(P)) \prod_{i=1}^{n} \omega(P_i) = \det\left(h(u_i, v_j)\right)_{1 \leq i,j, \leq n}$$

where the sum is over all n-tuples $(P_1, \ldots, P_n)$ of non-intersecting

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$$\sum_{\sigma \in S_n} \sum_{(P_1, \ldots, P_n) : \, \boldsymbol{u} \to \boldsymbol{v}^{\sigma}} \operatorname{sgn}(\sigma(P)) \prod_{i=1}^{n} \omega(P_i) = \det \left(h(u_i, v_j) \right)_{1 \le i, j, \le n}$$

where the sum is over all n-tuples $(P_1, \ldots, P_n)$ of non-intersecting paths with P taking u to v

Definition

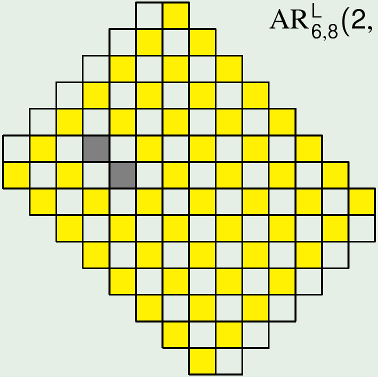
We consider the directed graph $G = (V, E)$ where V is the set of (x, y) such that $x + y$ is even, and the edge set E is composed of U, D and L. We use the coordinates $\langle x, y\rangle = (2x + y, y)$.

Given a, b, ξ and η, define the vertices u_i and v_j $(1 \le i, j \le b)$ as follows:

$$u_i = \langle b - i, 0\rangle \quad (1 \le i \le b),$$

$$v_j = \begin{cases} \langle j + b - 1, 0\rangle & (1 \le j \le a) \\ \langle a + b - \eta - j, j - a - 1 + \xi + \eta\rangle & (a < j \le b) \end{cases}.$$

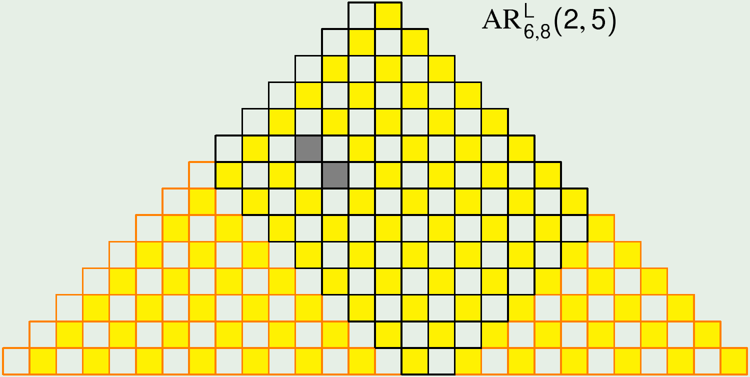
Example



$\mathrm{AR}^{\mathsf{L}}_{6,8}(2,5)$

Example

Example



$\mathrm{AR}^{\mathsf{L}}_{6,8}(2,5)$

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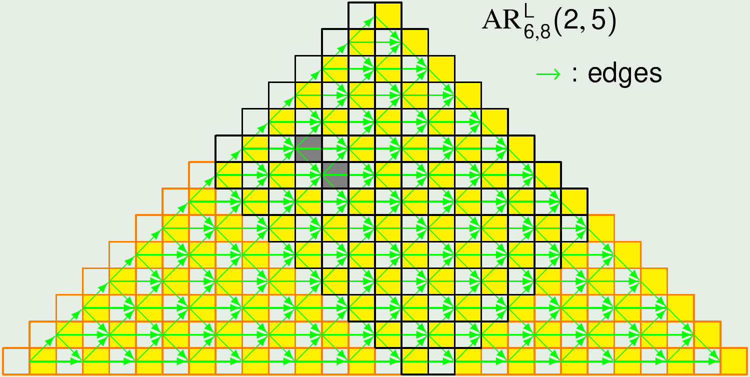


$\mathrm{AR}^{L}_{6,8}(2,5)$

$\longrightarrow$: edges

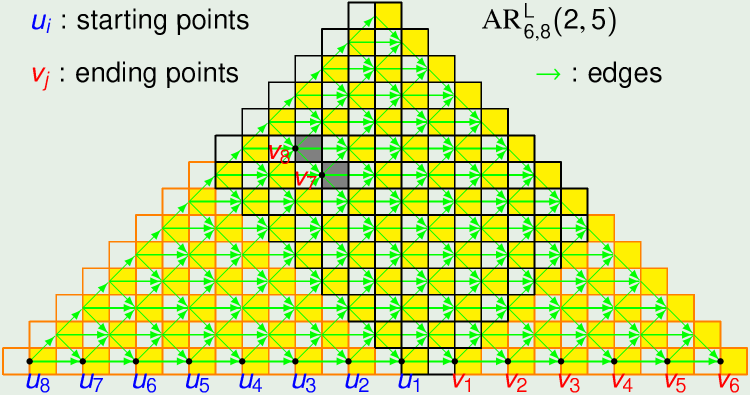
Example

u_i : starting points

v_j : ending points

$\mathrm{AR}^{\mathrm{L}}_{6,8}(2,5)$

$\rightarrow$: edges



Bijection

Theorem

There is a bijection between domino tilings and non-intersecting lattice paths.

Domino and path

There are four kinds of dominos:

1×2 2×1

Bijection

Theorem

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Bijection

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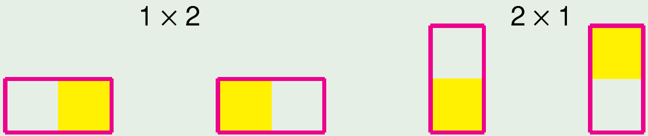

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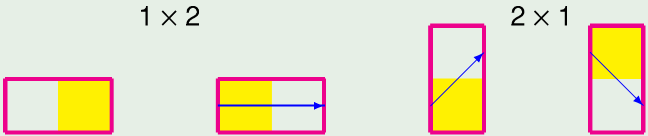
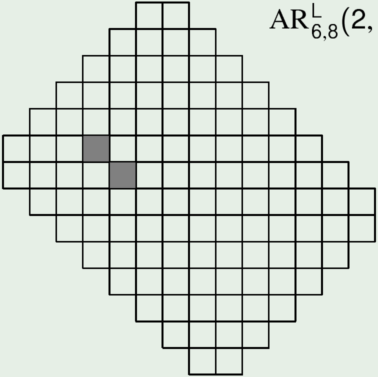
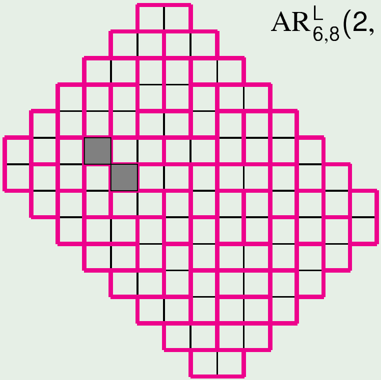


Bijection



$\mathrm{AR}^{\mathsf{L}}_{6,8}(2,5)$

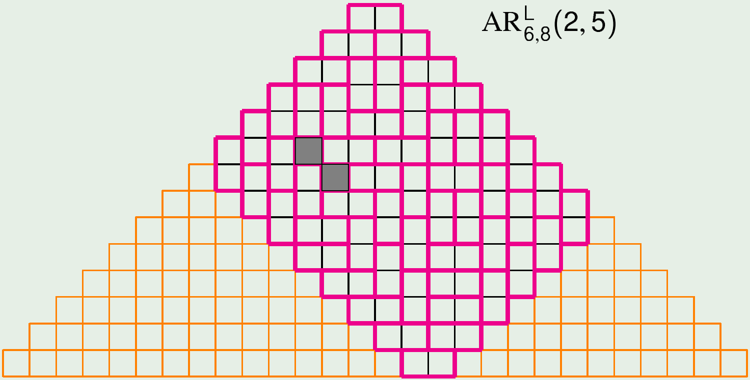
Bijection



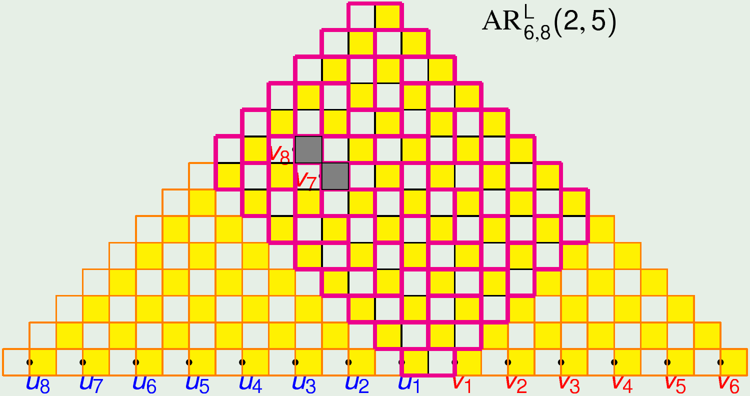
$\mathrm{AR}^{\mathsf{L}}_{6,8}(2,5)$

From domino tiling to path

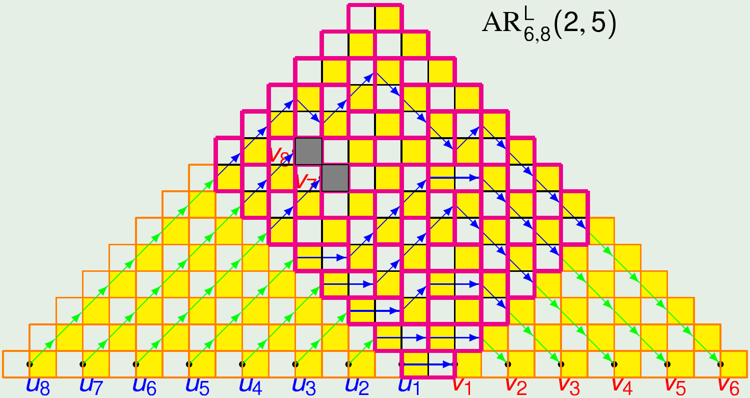
Bijection



Bijection


$\mathrm{AR}^{\mathrm{L}}_{6,8}(2,5)$
v_8
v_7
u_8 u_7 u_6 u_5 u_4 u_3 u_2 u_1 v_1 v_2 v_3 v_4 v_5 v_6

Bijection


$\mathrm{AR}^{\mathrm{L}}_{6,8}(2,5)$
v_8
v_7
u_8 u_7 u_6 u_5 u_4 u_3 u_2 u_1 v_1 v_2 v_3 v_4 v_5 v_6

The result obtained by applying the LGV Lemma

Theorem

Assume $r = b - a$ is even integer. Then

$$\sharp T^{\mathrm{L}}_{a,b}(\xi,\eta) = \det(\widetilde{h}^{\mathrm{L}}_{i,j}(a,b,\xi,\eta))_{1\le i,j\le b}$$

where

$$\widetilde{h}^{\mathrm{L}}_{i,j}(a,b,\xi,\eta) = \begin{cases} S(i+j-1) & \text{for } 1 \le j \le a, \\ S_{j-a-1+\xi+\eta}(a-\eta+i-j) & \text{for } a+1 \le j \le b \end{cases}$$

$$\sharp T^{\mathrm{S}}_{a,b}(\xi,\eta) = \det(\widetilde{h}^{\mathrm{S}}_{i,j}(a,b,\xi,\eta))_{1\le i,j\le b}$$

where

$$\widetilde{h}^{\mathrm{S}}_{i,j}(a,b,\xi,\eta) = \begin{cases} S(i+j-1) & \text{for } 1 \le j \le a, \\ S_{j-a-1+\xi+\eta}(i-\eta-1) & \text{for } a+1 \le j \le b \end{cases}$$

Note that these are determinants of size b (long side).

Example

If $a = 4$, $b = 7$ ($r = b - a = 3$), $\xi = 2$ and $\eta = 1$, then $(\widetilde{h}^{\text{L}}_{i,j}(a, b, \xi, \eta))_{1 \le i,j \le b}$ is

$$\left[\begin{array}{cccc|ccc} 2 & 6 & 22 & 90 & 0 & 0 & 0 \\ 6 & 22 & 90 & 394 & 1 & 0 & 0 \\ 22 & 90 & 394 & 1806 & 8 & 1 & 0 \\ 90 & 394 & 1806 & 8558 & 48 & 10 & 1 \\ 394 & 1806 & 8558 & 41586 & 264 & 70 & 12 \\ 1806 & 8558 & 41586 & 206098 & 1408 & 430 & 96 \\ 8558 & 41586 & 206098 & 1037718 & 7432 & 2490 & 652 \end{array}\right]$$

Example

If $a = 4$, $b = 8$ ($r = b - a = 4$), $\xi = 2$ and $\eta = 1$, then $(\widetilde{h}^{S}_{i,j}(a, b, \xi, \eta))_{1 \le i,j \le b}$ is

$$\left[\begin{array}{cccc|ccc}
2 & 6 & 22 & 90 & 0 & 0 & 0 \\
6 & 22 & 90 & 394 & 1 & 1 & 1 \\
22 & 90 & 394 & 1806 & 8 & 10 & 12 \\
90 & 394 & 1806 & 8558 & 48 & 70 & 96 \\
394 & 1806 & 8558 & 41586 & 264 & 430 & 652 \\
1806 & 8558 & 41586 & 206098 & 1408 & 2490 & 4080 \\
8558 & 41586 & 206098 & 1037718 & 7432 & 14002 & 24396
\end{array}\right]$$

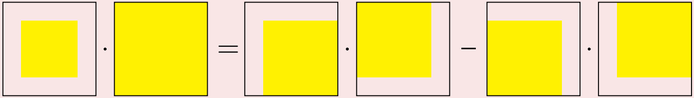
Introducing a new parameter c.

The Desnanot-Jacobi adjoint matrix theorem

The Desnanot-Jacobi adjoint matrix theorem

Let M be an n by n matrix. M_j^i is matrix M with row i and column j removed.

$$\det M_{1,n}^{1,n} \cdot \det M = \det M_1^1 \cdot \det M_n^n - \det M_n^1 \cdot \det M_1^n.$$



This formula is also called the *Luis Carroll condensation formula* or *Sylvester's determinant identity*.

Why the Laurent biorthogonal polynomials?

Fact

The condensation formula DO NOT work with $\det(\widetilde{h}^{L}_{i,j}(a,b,\xi,\eta))_{1\le i,j\le b}$ nor $\det(\widetilde{h}^{S}_{i,j}(a,b,\xi,\eta))_{1\le i,j\le b}$.

But, the large Schröder numbers are the moments of the Laurent biorthogonal polynomials. Hence we extend $S_m(n)$ for $n<0$ as

$$f_m(n) = S_m(n-1)$$

always holds.

Definition

Let m be a nonnegative integer.

$$S_m(n) = \begin{cases} 0 & \text{if } -m \le< n < 0, \\ (-1)^m S_m(-n-m-1) & \text{if } N < -m. \end{cases}$$

Introducing a new parameter c

Definition

Let $a, b \in \mathbb{N}$ ($a \leq b$) and $c \in \mathbb{Z}$. Let's consider the $b \times b$ matrices $H^{\mathrm{L}}(a,b,c,\xi,\eta) = \left(h^{\mathrm{L}}_{i,j}(a,b,c,\xi,\eta)\right)_{1\leq i,j\leq b}$ and $H^{\mathrm{S}}(a,b,c,\xi,\eta) = \left(h^{\mathrm{S}}_{i,j}(a,b,c,\xi,\eta)\right)_{1\leq i,j\leq b}$ defined by

$$h^{\mathrm{L}}_{i,j}(a,b,c,\xi,\eta) = \begin{cases} S(i+j+c-1) & \text{for } 1 \leq j \leq a, \\ S_{j-a-1+\xi+\eta}(a-\eta+i-j+c) & \text{for } a+1 \leq j \leq b, \end{cases}$$

and

$$h^{\mathrm{S}}_{i,j}(a,b,c,\xi,\eta) = \begin{cases} S(i+j+c-1) & \text{for } 1 \leq j \leq a, \\ S_{j-a-1+\xi+\eta}(i+c-\eta-1) & \text{for } a+1 \leq j \leq b. \end{cases}$$

Note that $h^{\mathrm{L}}_{i,j}(a,b,0,\xi,\eta) = \widetilde{h}^{\mathrm{L}}_{i,j}(a,b,\xi,\eta)$ and $h^{\mathrm{S}}_{i,j}(a,b,0,\xi,\eta) = \widetilde{h}^{\mathrm{S}}_{i,j}(a,b,\xi,\eta)$. We introduced a new parameter c to apply D-J adjoint matrix theorem, but c has no combinatorial meaning!. (It comes from the moments.)

Example

If $a = 4$, $b = 7$ ($r = b - a = 3$), $c = -3$, $\xi = 2$ and $\eta = 1$, then $(\widetilde{h}^{\llcorner}_{i,j}(a, b, \xi, \eta))_{1 \le i,j \le b}$ is

$$\left[\begin{array}{cccc|ccc} 2 & 1 & 1 & 2 & -1 & 1 & -1 \\ 1 & 1 & 2 & 6 & 0 & 0 & 0 \\ 1 & 2 & 6 & 22 & 0 & 0 & 0 \\ 2 & 6 & 22 & 90 & 0 & 0 & 0 \\ 6 & 22 & 90 & 394 & 1 & 0 & 0 \\ 22 & 90 & 394 & 1806 & 8 & 1 & 0 \\ 90 & 394 & 1806 & 8558 & 48 & 10 & 1 \end{array}\right]$$

Example

If $a = 4$, $b = 8$ ($r = b - a = 4$), $c = -3$, $\xi = 2$ and $\eta = 1$, then $(\widetilde{h}^{S}_{i,j}(a, b, \xi, \eta))_{1 \le i,j \le b}$ is

$$\left[\begin{array}{cccc|ccc} 2 & 1 & 1 & 2 & -1 & 0 & 0 \\ 1 & 1 & 2 & 6 & 0 & 0 & 0 \\ 1 & 2 & 6 & 22 & 0 & 0 & 0 \\ 2 & 6 & 22 & 90 & 0 & 0 & 0 \\ 6 & 22 & 90 & 394 & 1 & 1 & 1 \\ 22 & 90 & 394 & 1806 & 8 & 10 & 12 \\ 90 & 394 & 1806 & 8558 & 48 & 70 & 96 \end{array}\right]$$

Main Result for $F^{\mathrm{L}}(a, b, c, \xi, \eta)$

Main Result (Not yet completely proved)

Let a, b, c be integers such that $0 \leq a \leq b$ and $-a \leq c \leq 0$. We put $b - a = r$.

(i) If $-c \leq \xi \leq a$ and $0 \leq \eta \leq a + c$, then $F^{\mathrm{L}}(a, b, c, \xi, \eta)$ is equal to

$$(-1)^{r(\xi+a)} \cdot 2^{\frac{(a+c)(a+c+1)}{2}+r\eta} \cdot \prod_{k=0}^{r-1} \frac{k!}{(k+\eta)!} \cdot f_{\eta}^{(r)}\left(a+c, \xi - \frac{a-c}{2}\right)$$

$$= (-1)^{r(\eta+a+c)} \cdot 2^{\frac{(a+c)(a+c+1)}{2}+r(\xi+c)} \cdot \prod_{k=0}^{r-1} \frac{k!(k+a-\xi)!}{(k+\eta)!(k+a+c-\eta)!}$$

$$\times f_{\xi+c}^{(r)}\left(a+c, \eta - \frac{a+c}{2}\right),$$

Main Result (Not yet completely proved)

(ii) If $-c \le \xi \le a-r+1$, then $F^{S}(a,b,c,\xi,\eta)$ is equal to

$$(-1)^{r(\xi+a)}\cdot 2^{\frac{(a+c)(a+c+1)}{2}+r\eta-\frac{r(r-1)}{2}}\cdot\prod_{k=0}^{r-1}\frac{k!}{(k+\eta-r+1)!}$$

$$\times\, g^{(r)}_{\eta-r+1}\left(a+c,\xi-\frac{a-c-r+1}{2}\right)$$

$$=(-1)^{r(\eta+a+c)}\cdot 2^{\frac{(a+c)(a+c+1)}{2}+r(\xi+c)+\frac{r(r-1)}{2}}$$

$$\times\prod_{k=0}^{r-1}\frac{k!(k+a-r+1-\xi)!}{(k+\eta-r+1)!(k+a+c-\eta)!}\cdot g^{(r)}_{\xi+c}\left(a+c,\eta-\frac{a+c+r-1}{2}\right)$$

for $r-1\le\eta\le a+c$, and 0 otherwise.

Strategy of our proof

The effect of the new parameter c

Theorem

By applying the D-J adjoint matrix theorem, we obtain the following quadratic formula, i.e., the both of $F^{\mathrm{L}}(a,b,c,\xi,\eta)$ and $F^{\mathrm{S}}(a,b,c,\xi,\eta)$ satisfy

$$\begin{aligned}
&F(a,b-1,c,\xi,\eta)F(a+1,b+1,c-2,\xi+1,\eta-1) \\
&= F(a+1,b,c-2,\xi+1,\eta-1)F(a,b,c,\xi,\eta) \\
&\qquad -F(a+1,b,c-1,\xi+1,\eta-1)F(a,b,c-1,\xi,\eta).
\end{aligned}$$

Definition

Let a, b, c be integers such that $0 \le a \le b$ and $-a \le c \le 0$. We put $b - a = r$.

Let $G_1^L(a,b,c,\xi,\eta)$ be

$$(-1)^{r(\xi+a)} \cdot 2^{\frac{(a+c)(a+c+1)}{2}+r\eta} \cdot \prod_{k=0}^{r-1} \frac{k!}{(k+\eta)!} \cdot f_\eta^{(r)}\left(a+c, \xi - \frac{a-c}{2}\right),$$

and let $G_2^L(a,b,c,\xi,\eta)$ be

$$(-1)^{r(\eta+a+c)} \cdot 2^{\frac{(a+c)(a+c+1)}{2}+r(\xi+c)} \cdot \prod_{k=0}^{r-1} \frac{k!(k+a-\xi)!}{(k+\eta)!(k+a+c-\eta)!} \times f_{\xi+c}^{(r)}\left(a+c, \eta - \frac{a+c}{2}\right),$$

Definition

Let $G_1^S(a,b,c,\xi,\eta)$ be

$$(-1)^{r(\xi+a)}\cdot 2^{\frac{(a+c)(a+c+1)}{2}+r\eta-\frac{r(r-1)}{2}}\cdot\prod_{k=0}^{r-1}\frac{k!}{(k+\eta-r+1)!}\times g^{(r)}_{\eta-r+1}\left(a+c,\xi-\frac{a-c-r+1}{2}\right),$$

and let $G_2^S(a,b,c,\xi,\eta)$ be

$$(-1)^{r(\eta+a+c)}\cdot 2^{\frac{(a+c)(a+c+1)}{2}+r(\xi+c)+\frac{r(r-1)}{2}}\times\prod_{k=0}^{r-1}\frac{k!(k+a-r+1-\xi)!}{(k+\eta-r+1)!(k+a+c-\eta)!}\cdot g^{(r)}_{\xi+c}\left(a+c,\eta-\frac{a+c+r-1}{2}\right).$$

Definition of $f_n^{(r)}(a,x)$ and $g_n^{(r)}(a,x)$

Recall the definition of $f_n^{(r)}(a,x)$ and $g_n^{(r)}(a,x)$:

Definition

Let $r, n \geq 0$. Define $f_n^{(r)}(a,x)$ by

$$f_n^{(r)}(a,x)$$

$$= 2^{-rn} \prod_{k=0}^{r-1} \frac{(n+k)!(-a-k)_n}{k!} \cdot \det\left(\frac{{}_2F_1\left(\begin{matrix} -n-j+i, -\frac{a}{2}-x; \\ -a-j+1; \end{matrix} 2\right)}{(n+j-i)!} \right)_{1 \leq i,j \leq r}.$$

Here we use the convention that $\frac{1}{m!} = 0$ if $m < 0$.

Define $g_n^{(r)}(a,x)$ by

$$g_n^{(r)}(a,x) = f_n^{(r)}(a-r+1,x).$$

Applying the D-J adjoint matrix theorem, we obtain

Lemma

$$(r+n)\, f_n^{(r)}(a,x)\, f_n^{(r)}\left(a+1, x-\frac{1}{2}\right) - n\, f_{n-1}^{(r)}(a,x)\, f_{n+1}^{(r)}\left(a+1, x-\frac{1}{2}\right) = r\, f_n^{(r-1)}\left(a+1, x-\frac{1}{2}\right) f_n^{(r+1)}(a,x).$$

From this lemma we obtain

Proposition

$$f_n^{(r)}(a,x) = (-1)^{rn} f_r^{(n)}(a,x) \quad \text{and} \quad f_n^{(r)}(-a,x) = f_r^{(n)}(a,x).$$

Important quadratic equations for $f_n^{(r)}(a,x)$

Theorem

The $f_n^{(r)}(a,x)$ satisfies the following quadratic formulas:

$$f_{n-1}^{(r-1)}\left(a-1,x-\frac{1}{2}\right)f_n^{(r)}(a,x)-f_{n-1}^{(r-1)}(a,x)\,f_n^{(r)}\left(a-1,x-\frac{1}{2}\right)$$
$$=\frac{n+r-1}{2}\cdot f_n^{(r-1)}(a,x)\,f_{n-1}^{(r)}\left(a-1,x-\frac{1}{2}\right),$$

$$\left(x-\frac{a}{2}\right)f_n^{(r-1)}(a,x-1)\,f_{n-1}^{(r)}\left(a-1,x+\frac{1}{2}\right)$$
$$+\left(x+\frac{a}{2}+r-1\right)f_n^{(r-1)}(a,x)\,f_{n-1}^{(r)}\left(a-1,x-\frac{1}{2}\right)$$
$$=2\,f_{n-1}^{(r-1)}\left(a-1,x-\frac{1}{2}\right)f_n^{(r)}(a,x).$$

Important quadratic relations for $g_n^{(r)}(a,x)$

Theorem

The $g_n^{(r)}(a,x)$ satisfies the following quadratic formulas:

$$g_n^{(r-1)}\left(a-1,x-\frac{1}{2}\right)g_n^{(r)}\left(a,x+\frac{1}{2}\right)-g_n^{(r-1)}(a,x)\,g_n^{(r)}(a-1,x)$$
$$=\frac{n}{2}\cdot g_{n+1}^{(r-1)}(a,x)\,g_{n-1}^{(r)}(a-1,x),$$

$$\left(x-\frac{a-r+1}{2}\right)g_n^{(r-1)}\left(a,x-\frac{1}{2}\right)g_{n-1}^{(r)}\left(a-1,x+\frac{1}{2}\right)$$
$$+\left(x+\frac{a-r+1}{2}\right)g_n^{(r-1)}\left(a,x+\frac{1}{2}\right)g_{n-1}^{(r)}\left(a-1,x-\frac{1}{2}\right)$$
$$=2\,g_{n-1}^{(r-1)}(a-1,x)\,g_n^{(r)}(a,x).$$

The RHS satisfies the same quadratic equation!

Theorem

All of $G_1^{\mathrm{L}}(a, b, c, \xi, \eta)$, $G_2^{\mathrm{L}}(a, b, c, \xi, \eta)$, $G_1^{\mathrm{S}}(a, b, c, \xi, \eta)$, and $G_2^{\mathrm{S}}(a, b, c, \xi, \eta)$ satisfy

$$\begin{aligned} & G(a, b-1, c, \xi, \eta) G(a+1, b+1, c-2, \xi+1, \eta-1) \\ & = G(a+1, b, c-2, \xi+1, \eta-1) G(a, b, c, \xi, \eta) \\ & \qquad - G(a+1, b, c-1, \xi+1, \eta-1) G(a, b, c-1, \xi, \eta). \end{aligned}$$

A special value for $F^{\mathrm{L}}(a, b, c, \xi, \eta)$

Proposition

Let $a, r \geq 0$ be integers.
(i) If $-2\,a - r - 1 \leq c \leq r$ then

$$F^{\mathrm{L}}(a, a+r, c, \xi, 0) = (-1)^{r(\xi+a)}\, 2^{\frac{(a+c)(a+c+1)}{2}}.$$

(ii) If $c \leq 0$ then

$$F^{\mathrm{L}}(a, a+r, c, -c, \eta) = (-1)^{r(\eta+a+c)}\, 2^{\frac{(a+c)(a+c+1)}{2}} \prod_{k=0}^{r-1} \frac{k!(a+c)!}{(k+\eta)!(k+a+c-\eta)!}.$$

A special value for $F^{\mathrm{S}}(a, b, c, \xi, \eta)$

Proposition

(iii) If $-2a - r - 1 \le c \le r$ then

$$F^{\mathrm{S}}(a, a+r, c, \xi, r-1) = (-1)^{r(\xi+a)}\, 2^{\frac{(a+c)(a+c+1)}{2} + \frac{r(r-1)}{2}}.$$

(iv) If $c \le 0$ then

$$F^{\mathrm{S}}(a, a+r, c, -c, \eta) = (-1)^{r(\eta+a+c)}\, 2^{\frac{(a+c)(a+c+1)}{2} + \frac{r(r-1)}{2}} \times \prod_{k=0}^{r-1} \frac{k!(k+a+c-r+1)!}{(k+\eta-r+1)!(k+a+c-\eta)!}.$$

Miscellanies

Another expression for $f_n^{(r)}(a,x)$.

Proposition

If $r=2$, $f_n^{(2)}(a,x)$ is expressed by the following terminating hypergeometric series ${}_4F_3$:

$$
\begin{aligned}
& f_n^{(2)}(a,x) \\
&= (-1)^n \left(\frac{a}{2}+1-n+x\right)_n \left(\frac{a}{2}+1-n-x\right)_n \\
&\times {}_4F_3\left(\begin{matrix} \frac{a}{2}+1-n, \frac{a+1}{2}-n, a+2-n, -n; \\ \frac{a}{2}+1-n+x, \frac{a}{2}+1-n-x, a+1-2n; \end{matrix} \; 1\right).
\end{aligned}
$$

For $r \geq 3$ we CANNOT find an expression by one hypergeometric series.

A recursive equation for $f_n^{(r)}(a,x)$.

Proposition

If $r = 1$, then $f_n^{(1)}(a,x)$ satisfies the initial condition $f_0^{(1)}(a,x) = 1$ and the following 3 term recurrence:

$$f_{n+2}^{(1)}(a,x) - x f_{n+1}^{(1)}(a,x) + \frac{(n+1)(a-n)}{4} \cdot f_n^{(1)}(a,x) = 0$$

for $n \geq -1$.

A recursive equation for $f_n^{(r)}(a,x)$.

Proposition

If $r=2$, then $f_n^{(2)}(a,x)$ satisfies the initial condition $f_0^{(2)}(a,x)=1$ and the following 4 term recurrence:

$$\begin{aligned}
&f_{n+3}^{(2)}(a,x)-\left\{x^2-\frac{(n+1)a-(n-2)(n+2)}{4}\right\}f_{n+2}^{(2)}(a,x)\\
&+\frac{(n+2)(a-n)}{4}\cdot\left\{x^2-\frac{(n+4)a-(n+1)(n+5)}{4}\right\}f_{n+1}^{(2)}(a,x)\\
&-\frac{(n+1)(n+2)^2(a-n)^2(a-n+1)}{64}\cdot f_n^{(2)}(a,x)=0 \qquad (n\geq -2).
\end{aligned}$$

For $r\geq 3$ we CANNOT find any recursive equation for $f_n^{(r)}(a,x)$.

Future Problem

Weighted version?

For Aztec diamond one can define the *hight function* and *rank* of tiling. In the case of Aztec rectangle with holes, we don't know how to define the hight functions. For Aztec diamond one can consider the weighted enumeration with the number of vertical dominos and rank of tiling.

The end

Thank you for your attention!