CESÀRO ORLICZ SEQUENCE SPACES AND THEIR KÖTHE-TOEPLITZ DUALS

KULDIP RAJ, RENU ANAND AND SURUCHI PANDOH

Abstract. The present paper focus on introducing certain classes of Cesàro Orlicz sequences over \( n \)-normed spaces. We study some topological and algebraic properties of these spaces. Further, we examine relevant relations among the classes of these sequences. We show that these spaces are made \( n \)-BK-spaces under certain conditions. Finally, we compute the Köthe-Toeplitz duals of these spaces.

1. Introduction and Preliminaries

Let \( w, \ell_\infty, \ell_p, \ell_1, c \) and \( c_0 \) represent the spaces of all, bounded, \( p \)-absolutely summable, absolutely summable, convergent and null sequences \( x = (x_k) \) with complex terms, respectively. The zero element of a normed linear space is denoted by \( \theta \).

The space of all complex sequences \( \ell_p(0 < p < \infty) \) such that \( \sum_k |x_k|^p < \infty \), known as the space of \( p \)-absolutely summable sequences. The space \( \ell_p \) for \( p \geq 1 \) is complete under the norm defined by \( \|x\| = (\sum_k |x_k|^p)^{1/p} \) and for

\[ 0 < p < 1, \ell_p \] is a complete \( p \)-normed space, \( p \)-normed by \( \|x\| = \sum_{k=1}^{\infty} |x_k|^p \).

A BK-space \( (X, \|\|) \) is a Banach space of complex sequences \( x = (x_k) \), in which the co-ordinate maps are continuous, i.e., \( |x^a_n - x_k| \to 0 \), whenever \( \|x^a_n - x\| \to 0 \) as \( n \to \infty \), where \( x_n = (x^n_k) \) for all \( n \in \mathbb{N} \) (see [33]).

Let \( (X, \|\|) \) be a normed linear space and \( \lambda \) is a scalar-valued sequence space, then the vector-valued sequence space or \( X \)-valued sequence space \( \lambda(X) \) is defined by

\[ \lambda(X) = \{(x_k) : x_k \in X \text{ for all } k \in \mathbb{N} \text{ and } \|x\| \in \lambda\} \]

Clearly, \( \lambda(X) \) is a linear space under coordinatewise addition and scalar multiplication over the field of scalars of \( X \). Similarly, if \( X \) is a Banach space, then \( \ell_p(1 \leq p < \infty) \) is a Banach space with the norm given by

Mathematics Subject Classification. Primary 40A05, 46A20; Secondary 46D05, 46A45, 46E30.

Key words and phrases. Orlicz function, Musielak-Orlicz function, \( n \)-normed spaces, difference sequence spaces, Köthe-Toeplitz dual.
\[ \|x\| = \left( \sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}}. \]

Cesàro sequence spaces \( \text{Ces}_p \), \( 1 \leq p < \infty \), were introduced for the first time in 1968 in connection with the problem of finding their duals, which was posed by the Dutch Mathematical Society [1]. Shiue [27] and Leibowitz [14] studied the basic properties of these spaces. In 1974, Jagers [11] found the dual space of \( \text{Ces}_p \) [15].

The Cesàro sequence spaces is defined by

\[
\text{Ces}_p = \left\{ x = (x_k) : \|x\|_p = \left( \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}} < \infty, \ 1 \leq p < \infty \right\}
\]

and

\[
\text{Ces}_\infty = \left\{ x = (x_k) : \|x\|_\infty = \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k| < \infty \right\}.
\]

It was observed that \( \ell_p \subset \text{Ces}_p (1 < p < \infty) \) is strict, although it does not hold for \( p = 1 \). Nag and Lee [22] defined and studied the Cesàro sequence space \( X_p \) of non-absolute type as follows:

\[
X_p = \left\{ x = (x_k) : \|x\|_p = \left( \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}} < \infty, \ 1 \leq p < \infty \right\}
\]

and

\[
X_\infty = \left\{ x = (x_k) : \|x\|_\infty = \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k| < \infty \right\}.
\]

The inclusion \( \text{Ces}_p \subset X_p \), \( 1 \leq p < \infty \) is strict. Orhan [23] defined and studied the Cesàro difference sequence spaces \( X_p(\Delta) \) and \( X_\infty(\Delta) \) by replacing \( x = (x_k) \) with \( \Delta x = (\Delta x_k) = (x_k - x_{k+1}), k = 1, 2, ... \) and proved that for \( 1 \leq p < \infty \), the inclusions \( X_p \subset X_p(\Delta) \) and \( X_\infty \subset X_\infty(\Delta) \) are strict.

In fact, Orhan [23] used \( C_p \) instead of \( X_p(\Delta) \) and \( C_\infty \) instead of \( X_\infty(\Delta) \). Further, Orhan [23] also defined and studied the following sequence spaces

\[
O_p(\Delta) = \left\{ x = (x_k) : \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} |\Delta x_k| \right)^p < \infty, \ 1 \leq p < \infty \right\}
\]

and

\[
O_\infty(\Delta) = \left\{ x = (x_k) : \sup_{n \geq 1} \frac{1}{n} \sum_{k=1}^{n} |\Delta x_k| < \infty \right\}.
\]

He established that for \( 1 \leq p < \infty \), the inclusions \( O_p(\Delta) \subset X_p(\Delta) \) and \( \text{Ces}_p \subset O_p(\Delta) \) are strict.

Mursaleen et al. [19] studied the Cesàro difference sequence spaces which
were defined as
\[
X_p(\Delta^2) = \left\{ x = (x_k) : \sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \Delta^2 x_k \right|^p < \infty, \ 1 \leq p < \infty \right\}
\]
and
\[
X_\infty(\Delta^2) = \left\{ x = (x_k) : \sup_{n \geq 1} \left| \frac{1}{n} \sum_{k=1}^{n} \Delta^2 x_k \right| < \infty \right\},
\]
where \( \Delta^2 x_k = \Delta x_k - \Delta x_{k+1} \).

For uniformity of the literature, henceforth, we shall write \( C_p \) instead of \( X_p \) and \( C_\infty \) instead of \( X_\infty \).

Let \( E \) and \( F \) be two sequence spaces. Then the \( F \) dual of \( E \) is defined as 
\[
E^F = \{(x_k) \in w : (x_k y_k) \in F \text{ for all } (y_k) \in E \}.
\]
For \( F = \ell_1 \), the dual is termed as \( \alpha \)-dual (Köthe-Toeplitz dual) of \( E \) and denoted by \( E^\alpha \). If \( X \subset Y \), then \( Y^\alpha \subset X^\alpha \).

For more details about Cesàro-type summable spaces and Köthe-Toeplitz dual one can refer to ([3], [20], [21], [22], [28], [29], [31], [32]).

The concept of 2-normed spaces was initially developed by Gähler [6] in the mid of 1960’s, while that of \( n \)-normed spaces one can see in Misiak [18].

Since then, many others have studied this concept and obtained various results, see Gunawan ([7], [8]) and Gunawan and Mashadi [9]. Let \( n \in \mathbb{N} \) and \( X \) be a linear space over the field of real numbers \( \mathbb{R} \) of dimension \( d \), where \( d \geq n \geq 2 \). A real valued function \( ||\cdot, \cdot, \cdot|| \) on \( X^n \) satisfying the following four conditions:

1. \( ||(x_1, x_2, \cdots, x_n)|| = 0 \) if and only if \( x_1, x_2, \cdots, x_n \) are linearly dependent in \( X \),
2. \( ||(x_1, x_2, \cdots, x_n)|| \) is invariant under permutation,
3. \( ||(\alpha x_1, x_2, \cdots, x_n)|| = |\alpha| ||(x_1, x_2, \cdots, x_n)|| \) for any \( \alpha \in \mathbb{R} \), and
4. \( ||(x + x', x_2, \cdots, x_n)|| \leq ||(x, x_2, \cdots, x_n)|| + ||(x', x_2, \cdots, x_n)|| \)

is called an \( n \)-norm on \( X \) and the pair \((X, ||\cdot, \cdot, \cdot||)\) is called an \( n \)-normed space over the field \( \mathbb{R} \).

As an example, we may take \( X = \mathbb{R}^n \) being equipped with the \( n \)-norm 
\[
||(x_1, x_2, \cdots, x_n)||_E = \text{the volume of the } n \text{-dimensional parallelepiped spanned by the vectors } x_1, x_2, \cdots, x_n \text{ which may be given explicitly by the formula}
\]
\[
||(x_1, x_2, \cdots, x_n)||_E = |\det(x_{ij})|,
\]
where \( x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n \) for each \( i = 1, 2, \cdots, n \).

Let \((X, ||\cdot, \cdot, \cdot||)\) be an \( n \)-normed space of dimension \( d \geq n \geq 2 \) and \( \{a_1, a_2, \cdots, a_n\} \) be linearly independent set in \( X \). Then the following function \( ||\cdot, \cdot, \cdot||_\infty \) on \( X^{n-1} \) as defined by
\[
||(x_1, x_2, \cdots, x_{n-1})||_\infty = \max\{|||(x_1, x_2, \cdots, x_{n-1}, a_i)|| : i = 1, 2, \cdots, n\}
\]
is called an \((n - 1)\)-norm on \(X\) with respect to \(\{a_1, a_2, \ldots, a_n\}\).

A sequence \(\{x_k\}\) in an \(n\)-normed space \((X, ||\cdot||, \ldots, ||\cdot||)\) is said to converge to some \(L \in X\) if

\[
\lim_{k \to \infty} ||(x_k - L, z_1, \ldots, z_{n-1})|| = 0 \text{ for every } z_1, \ldots, z_{n-1} \in X.
\]

A sequence \(\{x_k\}\) in an \(n\)-normed space \((X, ||\cdot||, \ldots, ||\cdot||)\) is said to be Cauchy if

\[
\lim_{k,p \to \infty} ||(x_k - x_p, z_1, \ldots, z_{n-1})|| = 0 \text{ for every } z_1, \ldots, z_{n-1} \in X.
\]

If every Cauchy sequence in \(X\) converges to some \(L \in X\), then \(X\) is said to be complete with respect to the \(n\)-norm. A complete \(n\)-normed space is said to be \(n\)-Banach space. For more details about sequence spaces and \(n\)-normed spaces (see [2], [24], [25], [26]) and references therein.

An Orlicz function \(M : [0, \infty) \to [0, \infty)\) is a continuous, non-decreasing and convex such that \(M(0) = 0, M(x) > 0\) for \(x > 0\) and \(M(x) \to \infty\) as \(x \to \infty\). If convexity of Orlicz function is replaced by \(M(x + y) \leq M(x) + M(y)\), then this function is called modulus function. Lindenstrauss and Tzafriri [13] used the idea of Orlicz function to define the following sequence space,

\[
\ell_M = \left\{ x = (x_k) \in w : \sum_{k=1}^\infty M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}
\]

is known as an Orlicz sequence space. The space \(\ell_M\) is a Banach space with the norm

\[
||x|| = \inf \left\{ \rho > 0 : \sum_{k=1}^\infty M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.
\]

A sequence \(\mathcal{M} = (M_k)\) of Orlicz functions is said to be Musielak-Orlicz function (see [16], [17]). A Musielak-Orlicz function \(\mathcal{M} = (M_k)\) is said to satisfy \(\Delta_2\)-condition if there exist constants \(a, K > 0\) and a sequence \(c = (c_k)_{k=1}^\infty \in l_1^+\) (the positive cone of \(l_1^+\)) such that the inequality

\[
M_k(2u) \leq KM_k(u) + c_k
\]

holds for all \(k \in \mathbb{N}\) and \(u \in \mathbb{R}^+\), whenever \(M_k(u) \leq a\).

The notion of difference sequence spaces was introduced by Kizmaz [12], who studied the difference sequence spaces \(\ell_\infty(\Delta), c(\Delta)\) and \(c_0(\Delta)\). The notion was further generalized by Et and Çolak [4] by introducing the spaces \(\ell_\infty(\Delta^m), c(\Delta^m)\) and \(c_0(\Delta^m)\). Let \(n, m\) be non-negative integers, then for \(Z = c, c_0\) and \(\ell_\infty\) we have sequence spaces

\[
Z(\Delta^m_n) = \{ x = (x_k) \in w : (\Delta^m_n x_k) \in Z \},
\]
where $\Delta^m_n x = (\Delta^m_n x_k) = (\Delta^{m-1}_n x_k - \Delta^{m-1}_n x_{k+1})$ and $\Delta^0_n = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta^m_n x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+nv}.$$ 

Taking $n = 1$, we get the spaces $\ell_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$ studied by Et and Çolak [4]. Taking $n = m = 1$, we get the spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kizmaz [12].

Let $(X, \|\cdot\|)$ be an $n$-normed real linear space, $w(n - X)$ denotes $X$-valued sequence space. Let $\mathcal{M} = (M_i)$ be a sequence of Orlicz functions and $u = (u_k)$ be a sequence of positive real numbers. Then we define the following sequence spaces for every nonzero $z_1, \ldots, z_n \in X$:

$$C_p(\mathcal{M}, u, \Delta^m_n, \|\cdot\|) = \left\{ (x_k) \in w(n - X) : \sum_{i=1}^{\infty} M_i \left( \left\| \frac{1}{i} \sum_{k=1}^{i} \frac{u_k \Delta^m_n x_k}{\rho}, z_1, \ldots, z_{n-1} \right\| \right)^p < \infty, \right.$$ 

for some $\rho > 0$,

$$C_\infty(\mathcal{M}, u, \Delta^m_n, \|\cdot\|) = \left\{ (x_k) \in w(n - X) : \sup_i M_i \left( \left\| \frac{1}{i} \sum_{k=1}^{i} \frac{u_k \Delta^m_n x_k}{\rho}, z_1, \ldots, z_{n-1} \right\| \right) < \infty, \right.$$ 

for some $\rho > 0$,

$$\ell_p(\mathcal{M}, u, \Delta^m_n, \|\cdot\|) = \left\{ (x_k) \in w(n - X) : \sum_{k=1}^{\infty} M_k \left( \left\| \frac{u_k \Delta^m_n x_k}{\rho}, z_1, \ldots, z_{n-1} \right\| \right)^p < \infty, \right.$$ 

for some $\rho > 0$,

$$O_p(\mathcal{M}, u, \Delta^m_n, \|\cdot\|) = \left\{ (x_k) \in w(n - X) : \sum_{i=1}^{\infty} M_i \left( \left\| \frac{1}{i} \sum_{k=1}^{i} \frac{u_k \Delta^m_n x_k}{\rho}, z_1, \ldots, z_{n-1} \right\| \right)^p < \infty, \right.$$ 

for some $\rho > 0$. 

and
\[ O_\infty(M, u, \Delta^n_m, ||\cdot||, \ldots) = \{ (x_k) \in w(n - X) : \sup_i M_i\left( \frac{1}{i} \sum_{k=1}^i \left( u_k \Delta^n_m x_k \rho, z_1, \ldots, z_{n-1} \right) \right) < \infty, \]
for some \( \rho > 0 \}.

Lemma 1.1. [30] (a) Let \( 1 \leq p < \infty \). Then

(i) The space \( C_p \) is a Banach space, normed by
\[ \|x\| = \left( \sum_{i=1}^\infty \frac{1}{i} \sum_{k=1}^i |x_k|^p \right)^{\frac{1}{p}}. \]

(ii) The space \( O_p \) is a Banach space, normed by
\[ \|x\| = \left( \sum_{i=1}^\infty \frac{1}{i} \sum_{k=1}^i |x_k|^p \right)^{\frac{1}{p}}. \]

(iii) The space \( \ell_p \) is a Banach space, normed by
\[ \|x\| = \left( \sum_{k=1}^\infty |x_k|^p \right)^{\frac{1}{p}}. \]

(b) (i) The space \( C_\infty \) is a Banach space, normed by
\[ \|x\| = \sup_i \frac{1}{i} \sum_{k=1}^i |x_k|. \]

(ii) The space \( O_\infty \) is a Banach space, normed by
\[ \|x\| = \sup_i \frac{1}{i} \sum_{k=1}^i |x_k|. \]

Definition 1. An \( n \)-BK-space \( (X, ||\cdot||, \ldots) \) is an \( n \)-Banach space of real sequences \( x = (x_k) \) in which the co-ordinate maps are continuous.

Let us consider a few special cases of the above sequence spaces:

(i) If \( M_i(x) = x \) for all \( i \in N \), then we have
\[ C_p(M, u, \Delta^n_m, ||\cdot||, \ldots) = C_p(u, \Delta^n_m, ||\cdot||, \ldots), \]
\[ C_\infty(M, u, \Delta^n_m, ||\cdot||, \ldots) = C_\infty(u, \Delta^n_m, ||\cdot||, \ldots), \]
\[ O_p(M, u, \Delta^n_m, ||\cdot||, \ldots) = O_p(u, \Delta^n_m, ||\cdot||, \ldots) \]
\[ = O_\infty(u, \Delta^n_m, ||\cdot||, \ldots). \]

(ii) If \( u = (u_k) = 1 \) for all \( k \in N \), then we have
Let convex so by using inequality (1.1), we have and 

\[ C_p(\mathcal{M}, u, \Delta_n^m, \|\cdot\|) = C_p(\mathcal{M}, u, \Delta_n^m, \|\cdot\|, 1) \]

and 

\[ C_d(\mathcal{M}, u, \Delta_n^m, \|\cdot\|) = C_d(\mathcal{M}, u, \Delta_n^m, \|\cdot\|, 1) \]

also 

\[ O_p(\mathcal{M}, u, \Delta_n^m, \|\cdot\|) = O_p(\mathcal{M}, u, \Delta_n^m, \|\cdot\|, 1) \]

and 

\[ O_d(\mathcal{M}, u, \Delta_n^m, \|\cdot\|) = O_d(\mathcal{M}, u, \Delta_n^m, \|\cdot\|, 1) \] 

The following inequality will be used throughout the paper. Let \( p = (p_k) \) be a sequence of positive real numbers with \( 0 \leq p_k \leq \sup p_k = H \) and let 

\[ K = \max \{1, 2^{H-1}\} \] 

Then for the factorable sequences \((a_k)\) and \((b_k)\) in the complex plane, we have

\[ (1.1) \quad |a_k + b_k|^{p_k} \leq K(|a_k|^{p_k} + |b_k|^{p_k}). \]

Also \(|a_k|^{p_k} \leq \max \{1, |a|^H\} \) for all \( a \in \mathbb{C} \).

The main purpose of this paper is to introduce and study certain classes of multiplier sequences of Cesàro-type defined by a sequence of Orlicz functions over \( n \)-normed space. We make an effort to study completeness and some interesting inclusion relations between these spaces. Finally, we compute the Köthe-Toeplitz duals of these spaces.

2. Main Results

**Theorem 2.1.** Let \( \mathcal{M} = (M_i) \) be a sequence of Orlicz functions and \( u = (u_k) \) be a sequence of positive real numbers. Then the classes of sequences

\[ C_p(\mathcal{M}, u, \Delta_n^m, \|\cdot\|) \]

and 

\[ C_d(\mathcal{M}, u, \Delta_n^m, \|\cdot\|) \]

for \( 1 \leq p < \infty \) are linear spaces over the real field \( \mathbb{R} \).

**Proof.** We shall prove the result for the space \( O_p(\mathcal{M}, u, \Delta_n^m, \|\cdot\|) \) and for the other spaces, it will follow on applying similar arguments. Suppose \( x = (x_k), y = (y_k) \in O_p(\mathcal{M}, u, \Delta_n^m, \|\cdot\|) \) and \( \alpha, \beta \in \mathbb{R} \). Then there exist positive numbers \( \rho_1, \rho_2 \) such that

\[ \sum_{i=1}^{\infty} M_i \left( \frac{1}{i} \sum_{k=1}^{i} \left\| \frac{u_k \Delta_n^m x_k}{\rho_1}, z_1, \ldots, z_{n-1} \right\| \right)^p < \infty, \text{ for some } \rho_1 > 0 \]

and

\[ \sum_{i=1}^{\infty} M_i \left( \frac{1}{i} \sum_{k=1}^{i} \left\| \frac{u_k \Delta_n^m y_k}{\rho_2}, z_1, \ldots, z_{n-1} \right\| \right)^p < \infty, \text{ for some } \rho_2 > 0. \]

Let \( \rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2) \). Since \( \mathcal{M} = (M_i) \) is a non-decreasing and convex so by using inequality (1.1), we have
\[
\sum_{i=1}^{\infty} M_i \left( \frac{1}{i} \sum_{k=1}^{i} \left\| \frac{\alpha u_k \Delta_n^m x_k + \beta u_k \Delta_n^m y_k}{\rho_3}, z_1, ..., z_{n-1} \right\| \right)^p
\]
\[
\leq \sum_{i=1}^{\infty} M_i \left( \frac{1}{i} \left[ \sum_{k=1}^{i} \left\| \frac{u_k \Delta_n^m x_k}{\rho_3}, z_1, ..., z_{n-1} \right\| + \left\| \sum_{k=1}^{i} \frac{u_k \Delta_n^m y_k}{\rho_3}, z_1, ..., z_{n-1} \right\| \right)^p
\]
\[
\leq K \sum_{i=1}^{\infty} M_i \left( \frac{1}{i} \sum_{k=1}^{i} \left\| \frac{u_k \Delta_n^m x_k}{\rho_1}, z_1, ..., z_{n-1} \right\| \right)^p
\]
\[
+ K \sum_{i=1}^{\infty} M_i \left( \frac{1}{i} \sum_{k=1}^{i} \left\| \frac{u_k \Delta_n^m y_k}{\rho_2}, z_1, ..., z_{n-1} \right\| \right)^p
\]
\[
< \infty.
\]
Thus, \( \alpha x + \beta y \in O_p(\mathcal{M}, u, \Delta_n^m, \|\cdot\|, \|\cdot\|) \). This proves that \( O_p(\mathcal{M}, u, \Delta_n^m, \|\cdot\|, \|\cdot\|) \) is a linear space.

**Theorem 2.2.** Let \( \mathcal{M} = (M_i) \) be a sequence of Orlicz functions and \( u = (u_k) \) be a sequence of positive real numbers. Let \( 1 \leq p < \infty \) and the base space \( X \) is a \( n \)-Banach space. Then

(i) The space \( C_p(\mathcal{M}, u, \Delta_n^m, \|\cdot\|, \|\cdot\|) \) is an \( n \)-Banach space, \( n \)-normed by

\[
\left\| x^1, x^2, ..., x^n \right\|_{C_p(\mathcal{M}, u, \Delta_n^m)} = 0 \text{ if } x^1, x^2, ..., x^n \text{ are linearly dependent and}
\]
\[
= \sum_{k=1}^{m} \left\| x_k, z_1, ..., z_{n-1} \right\| + \left( \sum_{i=1}^{\infty} M_i \left( \frac{1}{i} \sum_{k=1}^{i} \left\| \frac{u_k \Delta_n^m x_k}{\rho}, z_1, ..., z_{n-1} \right\| \right)^p \right)^{\frac{1}{p}}
\]
for every \( z_1, ..., z_{n-1} \in X \) if \( x^1, x^2, ..., x^n \) are linearly independent.

(ii) The space \( O_p(\mathcal{M}, u, \Delta_n^m, \|\cdot\|, \|\cdot\|) \) is an \( n \)-Banach space, \( n \)-normed by

\[
\left\| x^1, x^2, ..., x^n \right\|_{O_p(\mathcal{M}, u, \Delta_n^m)} = 0 \text{ if } x^1, x^2, ..., x^n \text{ are linearly dependent and}
\]
\[
= \sum_{k=1}^{m} \left\| x_k, z_1, ..., z_{n-1} \right\| + \left( \sum_{i=1}^{\infty} M_i \left( \frac{1}{i} \sum_{k=1}^{i} \left\| \frac{u_k \Delta_n^m x_k}{\rho}, z_1, ..., z_{n-1} \right\| \right)^p \right)^{\frac{1}{p}}
\]
for every \( z_1, ..., z_{n-1} \in X \) if \( x^1, x^2, ..., x^n \) are linearly independent.

(iii) The space \( \ell_p(\mathcal{M}, u, \Delta_n^m, \|\cdot\|, \|\cdot\|) \) is an \( n \)-Banach space, \( n \)-normed by

\[
\left\| x^1, x^2, ..., x^n \right\|_{\ell_p(\mathcal{M}, u, \Delta_n^m)} = 0 \text{ if } x^1, x^2, ..., x^n \text{ are linearly dependent and}
\]
\[
= \sum_{k=1}^{m} \left\| x_k, z_1, ..., z_{n-1} \right\| + \left( \sum_{k=1}^{\infty} M_k \left\| \frac{u_k \Delta_n^m x_k}{\rho}, z_1, ..., z_{n-1} \right\| \right)^{\frac{1}{p}}
\]
for every \( z_1, ..., z_{n-1} \in X \) if \( x^1, x^2, ..., x^n \) are linearly independent.

(b) (i) The space \( C_{\infty}(\mathcal{M}, u, \Delta_n^m, \|\cdot\|, \|\cdot\|) \) is an \( n \)-Banach space, \( n \)-normed by
\[ \|x^1, x^2, \ldots, x^n\|_{C_\infty(M, u, \Delta_n^m)} = 0 \text{ if } x^1, x^2, \ldots, x^n \text{ are linearly dependent and} \]
\[ = \sum_{k=1}^{m} \|x_k, z_1, \ldots, z_{n-1}\| + \sup M_i \left| \frac{1}{i} \sum_{k=1}^{i} \frac{\Delta_n^m x_k}{\rho}, z_1, \ldots, z_{n-1} \right|, \]
for every \(z_1, \ldots, z_{n-1} \in X\) if \(x^1, x^2, \ldots, x^n\) are linearly independent.

(ii) The space \(O_\infty(M, u, \Delta_n^m, ||\cdot||)\) is an \(n\)-Banach space, \(n\)-normed by
\[ \|x^1, x^2, \ldots, x^n\|_{O_\infty(M, u, \Delta_n^m)} = 0 \text{ if } x^1, x^2, \ldots, x^n \text{ are linearly dependent and} \]
\[ = \sum_{k=1}^{m} \|x_k, z_1, \ldots, z_{n-1}\| + \sup M_i \left| \frac{1}{i} \sum_{k=1}^{i} \frac{\Delta_n^m x_k}{\rho}, z_1, \ldots, z_{n-1} \right|, \]
for every \(z_1, \ldots, z_{n-1} \in X\) if \(x^1, x^2, \ldots, x^n\) are linearly independent.

\textbf{Proof.} It is easy to show that the spaces \(C_p(M, u, \Delta_n^m, ||\cdot||), O_p(M, u, \Delta_n^m, ||\cdot||), \ell_p(M, u, \Delta_n^m, ||\cdot||), C_\infty(M, u, \Delta_n^m, ||\cdot||)\) and \(O_\infty(M, u, \Delta_n^m, ||\cdot||)\) are \(n\)-normed spaces under the \(n\)-norm as defined above.

Now, we prove the completeness for the space \(C_\infty(M, u, \Delta_n^m, ||\cdot||)\) only. The other parts can be proved in a similar way.

Let \((x^s)_{s=1}^\infty\) be a Cauchy sequence in \(C_\infty(M, u, \Delta_n^m, ||\cdot||)\), where \(x^s = (x^s_1, x^s_2, \ldots) \in C_\infty(M, u, \Delta_n^m, ||\cdot||)\) for each \(s \in N\). Let \(\epsilon > 0\) be given. Then there exists a positive integer \(n_0\) such that
\[ \|x^s - x^t, w^2, \ldots, w^n\|_{C_\infty(M, u, \Delta_n^m)} < \epsilon \]
for all \(s, t \geq n_0\) and for every \(w^2, \ldots, w^n \in C_\infty(M, u, \Delta_n^m, ||\cdot||)\), we have
\[ \sum_{k=1}^{m} \|x^s_k - x^t_k, z_1, \ldots, z_{n-1}\| + \sup M_i \left| \frac{1}{i} \sum_{k=1}^{i} \frac{\Delta_n^m (x^s_k - x^t_k)}{\rho}, z_1, \ldots, z_{n-1} \right| < \epsilon \]
for all \(s, t \geq n_0\) and for every \(z_1, \ldots, z_{n-1} \in X\). This implies
\[ \sum_{k=1}^{m} \|x^s_k - x^t_k, z_1, \ldots, z_{n-1}\| < \epsilon \]
and \( \sup M_i \left| \frac{1}{i} \sum_{k=1}^{i} \frac{\Delta_n^m (x^s_k - x^t_k)}{\rho}, z_1, \ldots, z_{n-1} \right| < \epsilon \)
for all \(s, t \geq n_0\) and for every \(z_1, \ldots, z_{n-1} \in X\). Hence, \(\|x^s_k - x^t_k, z_1, \ldots, z_{n-1}\| < \epsilon\) for all \(k = 1, 2, \ldots, m\) and for every \(z_1, \ldots, z_{n-1} \in X\).

Therefore, \((x^s_k)\) is a Cauchy sequence for all \(k = 1, 2, \ldots, m\) in \(X\), an \(n\)-Banach space.
Hence, \((x^s_k)\) converges in \(X\) for all \(k = 1, 2, \ldots, m\). Let \(\lim_{s \to \infty} x^s_k = x_k\) for all \(k = 1, 2, \ldots, m\). Next, we have

\[
\sup_i M_i \left\| \frac{1}{t} \sum_{k=1}^i u_k \Delta^m_n(x^s_k - x^t_k) - \frac{1}{\rho} \right\|, z_1, \ldots, z_{n-1} < \epsilon,
\]

for all \(s, t \geq n_0\) and for every \(z_1, \ldots, z_{n-1} \in X\). This implies for every \(z_1, \ldots, z_{n-1} \in X\)

\[
M_i \left\| \frac{1}{t} \sum_{k=1}^i u_k \Delta^m_n(x^s_k - x^t_k), z_1, \ldots, z_{n-1} \right\| < \epsilon,
\]

for all \(s, t \geq n_0\) and \(i \in N\).

Thus, \((\Delta^m_n x^s_k)\) is a Cauchy sequence in \(C_\infty(M, u, \|\cdot\|, \|\cdot\|)\) which is complete. Hence, \((\Delta^m_n x^s_k)\) converges for each \(k \in N\).

Let \(\lim_{s \to \infty} \Delta^m_n x^s_k = y_k\) for each \(k \in N\). Let \(k = 1\), we have

\[
(2.1) \quad \lim_{s \to \infty} \Delta^m_n x^s_1 = \lim_{s \to \infty} \sum_{v=0}^m (-1)^v \binom{m}{v} x_1 + n v = y_1,
\]

we have

\[
(2.2) \quad \lim_{s \to \infty} x^s_k = x_k, \text{ for } k = 1 + nv, \text{ for } v = 1, 2, \ldots, m - 1.
\]

Thus, from equation (2.1) and (2.2), we have \(\lim_{s \to \infty} x^s_k = x_k\) exists. Let \(\lim_{s \to \infty} x^s_k = x_k\) for each \(k \in N\). Proceeding in this way inductively \(\lim_{s \to \infty} x^s_k = x_k\) exists for each \(k \in N\).

Now, for every \(z_1, \ldots, z_{n-1} \in X\)

\[
\lim_{s \to \infty} \sum_{k=1}^m \|x^s_k - x^t_k, z_1, \ldots, z_{n-1}\| = \sum_{k=1}^m \|x^s_k - x_k, z_1, \ldots, z_{n-1}\| < \epsilon,
\]

for all \(s \geq n_0\). Again, using the continuity of \(n\)-norm, we find that for every \(z_1, \ldots, z_{n-1} \in X\)

\[
M_i \left\| \frac{1}{i} \sum_{k=1}^i u_k \Delta^m_n x^s_k - \frac{1}{\rho} \right\|, z_1, \ldots, z_{n-1} < \epsilon,
\]

for all \(s \geq n_0\) and \(i \in N\). Hence, for every \(z_1, \ldots, z_{n-1} \in X\)

\[
\sup_i M_i \left\| \frac{1}{i} \sum_{k=1}^i u_k \Delta^m_n x^s_k - u_k \Delta^m_n x^t_k, z_1, \ldots, z_{n-1} \right\| < \epsilon \text{ for all } s \geq n_0.
\]

Thus, for every \(w^2, \ldots, w^m \in C_\infty(M, \Delta^m_n, \|\cdot\|, \|\cdot\|)\)

\[
\|x^s - x, w^2, \ldots, w^m\|_{C_\infty(M, u, \Delta^m_n)} < 2\epsilon \text{ for all } s \geq n_0.
\]
Hence, \((x^s - x) \in C_{\infty}(M, u, \Delta_n^m, \|\cdot\|, \ldots, \|\cdot\|)\). Since \(C_{\infty}(M, u, \Delta_n^m, \|\cdot\|, \ldots, \|\cdot\|)\) is a linear space, so we have for all \(s \geq n_0\), \(x = x^s - (x^s - x) \in C_{\infty}(M, u, \Delta_n^m, \|\cdot\|, \ldots, \|\cdot\|)\). Hence, \(C_{\infty}(M, u, \Delta_n^m, \|\cdot\|, \ldots, \|\cdot\|)\) is complete and as such is an \(n\)-Banach space.  

**Corollary 2.3.** The spaces \(C_p(M, u, \Delta_n^m, \|\cdot\|, \ldots, \|\cdot\|)\), \(C_{\infty}(M, u, \Delta_n^m, \|\cdot\|, \ldots, \|\cdot\|)\), \(\ell_p(M, u, \Delta_n^m, \|\cdot\|, \ldots, \|\cdot\|)\), \(O_p(M, u, \Delta_n^m, \|\cdot\|, \ldots, \|\cdot\|)\) and \(O_{\infty}(M, u, \Delta_n^m, \|\cdot\|, \ldots, \|\cdot\|)\) for \(1 \leq p < \infty\) are \(n\)-BK-spaces if the base space \(X\) is an \(n\)-Banach space.

**Theorem 2.4.** Let \(M = (M_i)\) be a sequence of Orlicz functions and \(u = (u_k)\) be a sequence of positive real numbers. Then \(Z(M, u, \Delta_n^{m-1}, \|\cdot\|, \ldots, \|\cdot\|) \subset Z(M, u, \Delta_n^m, \|\cdot\|, \ldots, \|\cdot\|)\) (in general \(Z(M, u, \Delta^i_n, \|\cdot\|, \ldots, \|\cdot\|) \subset Z(M, u, \Delta_n^m, \|\cdot\|, \ldots, \|\cdot\|)\) for \(i = 1, 2, \ldots, m-1\)) for \(Z = C_p, O_p, \ell_p, C_{\infty}\) and \(O_{\infty}\).

**Proof.** We shall prove the result for the space \(Z = C_p\) only and others can be proved in the similar way. Let \(x = (x_k) \in C_p(M, u, \Delta_n^{m-1}, \|\cdot\|, \ldots, \|\cdot\|)\), \(1 \leq p < \infty\). Then for every nonzero \(z_1, \ldots, z_{n-1} \in X\),

\[
\sum_{i=1}^{\infty} M_i\left(\left\|\frac{1}{i} \sum_{k=1}^{i} \frac{u_k \Delta_n^{m-1} x_k}{\rho} , z_1, \ldots, z_{n-1}\right\|\right)^p < \infty.
\]

Now, we have for every nonzero \(z_1, \ldots, z_{n-1} \in X\)

\[
M_i\left(\left\|\frac{1}{i} \sum_{k=1}^{i} \frac{u_k \Delta_n^{m} x_k}{\rho} , z_1, \ldots, z_{n-1}\right\|\right)
\]

\[
\leq M_i\left(\left\|\frac{1}{i} \sum_{k=1}^{i} \frac{u_k \Delta_n^{m-1} x_k}{\rho} , z_1, \ldots, z_{n-1}\right\|\right)
\]

\[
+ \ M_i\left(\left\|\frac{1}{i} \sum_{k=1}^{i} \frac{u_k \Delta_n^{m} x_{k+1}}{\rho} , z_1, \ldots, z_{n-1}\right\|\right).
\]

It is known that for \(1 \leq p < \infty\), \(|a + b|^p \leq 2^p(|a|^p + |b|^p)\). Hence, for \(1 \leq p < \infty\),

\[
M_i\left(\left\|\frac{1}{i} \sum_{k=1}^{i} \frac{u_k \Delta_n^{m} x_k}{\rho} , z_1, \ldots, z_{n-1}\right\|\right)^p
\]

\[
\leq 2^p \left\{ M_i\left(\left\|\frac{1}{i} \sum_{k=1}^{i} \frac{u_k \Delta_n^{m-1} x_k}{\rho} , z_1, \ldots, z_{n-1}\right\|\right)^p
\right. 
\]

\[
+ \ M_i\left(\left\|\frac{1}{i} \sum_{k=1}^{i} \frac{u_k \Delta_n^{m} x_{k+1}}{\rho} , z_1, \ldots, z_{n-1}\right\|\right)^p \right\}.
\]
Then for each positive integer \( r \), we get
\[
\sum_{i=1}^{r} M_i \left( \frac{\| \frac{1}{i} \sum_{k=1}^{i} \frac{u_k \Delta_n^m x_k}{\rho}, z_1, \ldots, z_{n-1} \|}{\rho} \right)^p \\
\leq 2^p \left\{ \sum_{i=1}^{r} M_i \left( \frac{\| \frac{1}{i} \sum_{k=1}^{i} \frac{u_k \Delta_n^{m-1} x_k}{\rho}, z_1, \ldots, z_{n-1} \|}{\rho} \right)^p \\
+ \sum_{i=1}^{r} M_i \left( \frac{\| \frac{1}{i} \sum_{k=1}^{i} \frac{u_k \Delta_n^{m-1} x_{k+1}}{\rho}, z_1, \ldots, z_{n-1} \|}{\rho} \right)^p \right\}.
\]
Taking \( r \to \infty \) and using equation (2.3), we get
\[
\sum_{i=1}^{\infty} M_i \left( \frac{\| \frac{1}{i} \sum_{k=1}^{i} \frac{u_k \Delta_n^m x_k}{\rho}, z_1, \ldots, z_{n-1} \|}{\rho} \right)^p < \infty.
\]
Thus, \( C_p(\mathcal{M}, u, \Delta_n^{m-1}, \| \cdot, \cdot, \cdot, \cdot \|) \subseteq C_p(\mathcal{M}, u, \Delta_n^m, \| \cdot, \cdot, \cdot, \cdot \|) \) for \( 1 \leq p < \infty \). The inclusion is strict and it follows from the following example.

**Example 2.5.** Let \( X = \mathbb{R}^3 \) be a real linear space. Define \( \| \cdot, \cdot, \cdot \| : X \times X \to \mathbb{R} \) by \( \| x, y \| = \max \{ |x_1 y_2 - x_2 y_1|, |x_2 y_3 - x_3 y_2|, |x_3 y_1 - x_1 y_3| \} \), where \( x = (x_1, x_2, x_3) \), \( y = (y_1, y_2, y_3) \) \( \in \mathbb{R}^3 \). Then \( (X, \| \cdot, \cdot, \cdot \|) \) is a 2-normed linear space. Let \( (u_k) = 1 \), \( (M_i) = I \), the identity map, for all \( i \in N \), \( m = 2 \) and \( n = 1 \). Consider the sequence \( x = (x_k) = (k, k, k) \) for all \( k \in N \). Then \( \Delta^2 x_k = (0, 0, 0) \) for all \( k \in N \). Hence, \( (x_k) \in C_p(\mathcal{M}, u, \Delta^2, \| \cdot, \cdot, \cdot \|) \), we have \( \Delta(x_k) = (-1, -1, -1) \) for all \( k \in N \). Hence, \( (x_k) \notin C_p(\mathcal{M}, u, \Delta, \| \cdot, \cdot, \cdot \|) \). The inclusion is strict.

**Theorem 2.6.** Let \( \mathcal{M} = (M_i) \) be a sequence of Orlicz functions and \( u = (u_k) \) be a sequence of positive real numbers. Then
(a) \( O_p(\mathcal{M}, u, \Delta_n^m, \| \cdot, \cdot, \cdot, \cdot \|) \subseteq C_p(\mathcal{M}, u, \Delta_n^m, \| \cdot, \cdot, \cdot, \cdot \|) \) and the inclusions are strict.
(b) \( O_p(\mathcal{M}, u, \Delta_n^m, \| \cdot, \cdot, \cdot, \cdot \|) \subseteq C_\infty(\mathcal{M}, u, \Delta_n^m, \| \cdot, \cdot, \cdot, \cdot \|) \) and the inclusions are strict.

**Proof.** The proof is trivial, so we omitted. \( \square \)

**Remark.** \( \ell_p(\mathcal{M}, u, \Delta_n^m, \| \cdot, \cdot, \cdot, \cdot \|) \subseteq O_p(\mathcal{M}, u, \Delta_n^m, \| \cdot, \cdot, \cdot, \cdot \|). \)

**Example 2.7.** Let \( p = 1 \) and 2-norm \( \| \cdot, \cdot \| \) on \( X = \mathbb{R}^3 \) in Example (2.5).
Let \( m = 2 \), \( n = 1 \), \( (u_k) = 1 \) and \( (M_i) = I \). Consider the sequence \( \{x_k\} = \{(1,1,1), (0,0,0), (0,0,0), (0,0,0), \ldots\} \). Then \( \Delta^2 x_k = (1,1,1) \) for \( k = 1 \) and \( \Delta^2 x_k = (0,0,0) \) for all \( k > 1 \). Then \( (x_k) \in \ell(\mathcal{M}, u, \Delta^2, \| \cdot, \cdot, \cdot \|) \) but \( (x_k) \notin O(\mathcal{M}, u, \Delta^2, \| \cdot, \cdot, \cdot \|) \).
Let \( F \) be an \( n \)-functional with domain \( A_1 \times \ldots \times A_n \). \( F \) is called a linear \( n \)-functional whenever for all \( 1 \leq a_1, \ldots, a_n \in A_1 \) and all \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \), we have
\[
F(\alpha_1 a_1, \ldots, \alpha_n a_n) = \alpha_1 F(a_1, \ldots, a_n) = \ldots = \alpha_n F(a_1, \ldots, a_n).
\]

Let \( F \) be an \( n \)-functional with domain \( D(F) \). \( F \) is called bounded if there is a real constant \( K \geq 0 \) such that \( |F(a_1, \ldots, a_n)| \leq K \|a_1, \ldots, a_n\| \) for all \( (a_1, \ldots, a_n) \in D(F) \). If \( F \) is bounded, we define the norm
\[
\|F\| = \text{glb}\{K : |F(a_1, \ldots, a_n)| \leq K \|a_1, \ldots, a_n\| \text{ for all } (a_1, \ldots, a_n) \in D(F)\}.
\]

If \( F \) is not bounded, we define \( \|F\| = +\infty \).

**Theorem 2.8.** If \( 1 \leq p < q \), then
(i) \( C_p(\mathcal{M}, u, \Delta_n^m, \|\cdot\|, \ldots, \|\cdot\|) \subseteq C_q(\mathcal{M}, u, \Delta_n^m, \|\cdot\|, \ldots, \|\cdot\|) \);
(ii) \( \ell_p(\mathcal{M}, u, \Delta_n^m, \|\cdot\|, \ldots, \|\cdot\|) \subseteq \ell_q(\mathcal{M}, u, \Delta_n^m, \|\cdot\|, \ldots, \|\cdot\|) \);
(iii) \( O_p(\mathcal{M}, u, \Delta_n^m, \|\cdot\|, \ldots, \|\cdot\|) \subseteq O_q(\mathcal{M}, u, \Delta_n^m, \|\cdot\|, \ldots, \|\cdot\|) \).

**Proof.** We prove the result for the space \( O_p(\mathcal{M}, u, \Delta_n^m, \|\cdot\|, \ldots, \|\cdot\|) \) only and for the other cases it can be proved in a similar way. Let \( x \in O_p(\mathcal{M}, u, \Delta_n^m, \|\cdot\|, \ldots, \|\cdot\|) \). Then there exists \( \rho > 0 \) such that
\[
\sum_{i=1}^{\infty} M_i \left( \frac{1}{i} \sum_{k=1}^{i} \left\| \frac{u_k \Delta_n^m x_k}{\rho}, z_1, \ldots, z_{n-1} \right\| \right)^p < \infty.
\]
This implies
\[
M_i \left( \frac{1}{i} \sum_{k=1}^{i} \left\| \frac{u_k \Delta_n^m x_k}{\rho}, z_1, \ldots, z_{n-1} \right\| \right)^p < 1
\]
for sufficiently large values of \( i \). Since (\( M_i \)) is non-decreasing, we get
\[
\sum_{i=1}^{\infty} M_i \left( \frac{1}{i} \sum_{k=1}^{i} \left\| \frac{u_k \Delta_n^m x_k}{\rho}, z_1, \ldots, z_{n-1} \right\| \right)^q
\leq \sum_{i=1}^{\infty} M_i \left( \frac{1}{i} \sum_{k=1}^{i} \left\| \frac{u_k \Delta_n^m x_k}{\rho}, z_1, \ldots, z_{n-1} \right\| \right)^p
\leq \infty.
\]
Thus, \( x \in O_q(\mathcal{M}, u, \Delta_n^m, \|\cdot\|, \ldots, \|\cdot\|) \). This completes the proof. \( \square \)

3. Köthe-Toeplitz duals

In order to compute Köthe-Toeplitz dual, we first define the following:
An \( n \)-functional is a real-valued mapping with domain \( A_1 \times \ldots \times A_n \), where \( A_1, \ldots, A_n \) are linear manifolds of a linear \( n \)-normed space.

Let \( F \) be an \( n \)-functional with domain \( A_1 \times \ldots \times A_n \). \( F \) is called an \( n \)-functional whenever for all \( a_1, a_2, \ldots, a_n \in A_1 \) and all \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \), we have
(i) \( F(a_1, a_2, \ldots, a_n) = \alpha_1 F(a_1, \ldots, a_n) = \ldots = \alpha_n F(a_1, \ldots, a_n) \)
(ii) \( F(a_1 a_1, \ldots, a_n a_n, \alpha_1 a_1, \ldots, \alpha_n a_n) = \alpha_1 F(a_1, \ldots, a_n) \)

Let \( F \) be an \( n \)-functional with domain \( D(F) \). \( F \) is called bounded if there is a real constant \( K \geq 0 \) such that \( |F(a_1, \ldots, a_n)| \leq K \|a_1, \ldots, a_n\| \) for all \( (a_1, \ldots, a_n) \in D(F) \). If \( F \) is bounded, we define the norm
\[
\|F\| = \text{glb}\{K : |F(a_1, \ldots, a_n)| \leq K \|a_1, \ldots, a_n\| \text{ for all } (a_1, \ldots, a_n) \in D(F)\}.
\]

If \( F \) is not bounded, we define \( \|F\| = +\infty \).
Proposition 3.1. \[10\] A linear $n$-functional $F$ is continuous if and only if it is bounded.

Proposition 3.2. \[10\] Let $B^*$ be the set of bounded linear $n$-functionals with domain $B_1 \times \ldots \times B_n$. Then $B^*$ is an $n$-Banach space up to linear dependence.

For any $n(> 1)$-normed space $E$, we denote by $E^*$ the continuous dual of $E$.

Definition 2. Let $E$ be an $n$-normed linear space, normed by $\|\cdot\|_E$. Then we define the Köthe-Toeplitz dual of the sequence space $Z(E)$ whose base space is $E$ as

$$Z(E) = \{ (y_k) : y_k \in E^*, k \in \mathbb{N} \text{ and } (\|x_k, w_2, \ldots, w_n\|_E \|y_k, v_2, \ldots, v_n\|_{E^*}^p) \leq 1 \text{ for every } v_2, \ldots, v_n \in E^*, w_2, \ldots, w_n \in E, (x_k) \in Z(E) \}.$$  

It is easy to check that $\phi \in X^\alpha$. If $X \subseteq Y$, then $Y^\alpha \subseteq X^\alpha$. Let us consider $SC_p(M, \Delta^m_n, \|\cdot\|_E, \|\cdot\|, \|\cdot\|_E)$ as

$$SC_p(M, \Delta^m_n, \|\cdot\|, \|\cdot\|_E) = \{ x = (x_k) : x \in C_p(M, \Delta^m_n, \|\cdot\|, \|\cdot\|_E), x_1 = \ldots = x_m = 0 \}.$$  

Then $SC_p(M, \Delta^m_n, \|\cdot\|, \|\cdot\|_E)$ is a subspace of $C_p(M, \Delta^m_n, \|\cdot\|, \|\cdot\|_E)$ for $1 < p < \infty$. We can have similar subspaces for other spaces as well.

Lemma 3.3. \[5\] $X \subseteq SC_{\infty}(\Delta^m)$ implies $\sup_{k} k^{-m}|x_k| < \infty$.

Lemma 3.4. $x \in SC_{\infty}(M, \Delta^m_n, \|\cdot\|_E)$ implies $\sup_{k} k^{-m}\|x_k, w_2, \ldots, w_n\|_E < \infty$ for every $w_2, \ldots, w_n \in E$.

Proof. The proof follows using similar techniques as applied in the proof of Lemma 3.3. Consider a set

$$U = \{ a = (a_k) : \sum_{k=1}^{\infty} k^m \|a_k, z_2, \ldots, z_n\|_{X^*} < \infty, \text{ for every } z_2, \ldots, z_n \in X^* \}.$$  

Theorem 3.5. Let $M = (M_i)$ be a sequence of Orlicz functions and $u = (u_k)$ be a sequence of positive real numbers. Then the Köthe-Toeplitz duals of the space $SC_p(M, \Delta^m_n, \|\cdot\|, \|\cdot\|_E)$ is $U$, that is, $[SC_{\infty}(M, \Delta^m_n, \|\cdot\|, \|\cdot\|_E)]^* = U$.

Proof. If $a \in U$, then

$$\sum_{k=1}^{\infty} \|a_k, z_2, \ldots, z_n\|_{X^*} \|x_k, w_2, \ldots, w_n\|_X = \sum_{k=1}^{\infty} k^m \|a_k, z_2, \ldots, z_n\|_{X^*} (k^{-m}\|x_k, w_2, \ldots, w_n\|_X)$$
for each $x \in SC_{\infty}(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdot, \cdot\|)$ by Lemma 3.4. Hence, $x \in [SC_{\infty}(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdot, \cdot\|)]^\alpha$.

Next, let $a \in [SC_{\infty}(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdot, \cdot\|)]^\alpha$. Then
\[
\sum_{k=1}^\infty \|a_k, z_2, \ldots, z_n\|_X \|x_k, w_2, \ldots, w_n\|_X < \infty,
\]
for each $x \in SC_{\infty}(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdot, \cdot\|)$.

Define the sequence $x = (x_k)$ by
\[
x_k = \begin{cases} 
0, & k \leq m, \\
k^m, & k > m
\end{cases}
\]
and choose $w_2, \ldots, w_n \in X$ such that
\[
\|k^m, w_2, \ldots, w_n\|_X = k^m \|1, w_2, \ldots, w_n\|_X = \begin{cases} 
0, & k \leq m, \\
k^m, & k > m.
\end{cases}
\]
Thus, we have $z_2, \ldots, z_n \in X^*$
\[
\sum_{k=1}^\infty k^m \|a_k, z_2, \ldots, z_n\|_X = \sum_{k=1}^\infty k^m \|1, w_2, \ldots, w_n\|_X = \sum_{k=1}^{\infty} \|k^m, w_2, \ldots, w_n\|_X \|a_k, z_2, \ldots, z_n\|_X^* \\
+ \sum_{k=1}^\infty \|k^m, w_2, \ldots, w_n\|_X \|a_k, z_2, \ldots, z_n\|_X^* \\
< \infty.
\]
This implies $a \in \mathcal{U}$.

**Theorem 3.6.** Let $\mathcal{M} = (M_i)$ be a sequence of Orlicz functions and $u = (u_k)$ be a sequence of positive real numbers. Then
\[
[SC_{\infty}(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdot, \cdot\|)]^\alpha = [C_{\infty}(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdot, \cdot\|)]^\alpha.
\]

**Proof.** Since $SC_{\infty}(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdot, \cdot\|) \subset C_{\infty}(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdot, \cdot\|)$, we have $[C_{\infty}(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdot, \cdot\|)]^\alpha \subset [SC_{\infty}(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdot, \cdot\|)]^\alpha$.

Let $a \in [SC_{\infty}(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdot, \cdot\|)]^\alpha$ and $x \in C_{\infty}(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdot, \cdot\|)$. Consider the sequence $x = (x_k)$ defined by
\[
x_k = \begin{cases} 
x_k, & k \leq m, \\
x'_k, & k > m,
\end{cases}
\]
where \( x' = (x'_k) \in SC_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot\|, \|\cdot\|) \). Then write
\[
\sum_{k=1}^{\infty} \|a_k, z_2, \ldots, z_n\|_X \|x_k, w_2, \ldots, w_n\|_X
\]
\[
= \sum_{k=1}^{m} \|a_k, z_2, \ldots, z_n\|_X \|x_k, w_2, \ldots, w_n\|_X
\]
\[
+ \sum_{k=1}^{\infty} \|a_k, z_2, \ldots, z_n\|_X \|x'_k, w_2, \ldots, w_n\|_X
\]
< \infty.
This implies \( a \in [C_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot\|, \|\cdot\|)]^\alpha \). \( \square \)

**Theorem 3.7.** Let \( \mathcal{M} = (M_i) \) be a sequence of Orlicz functions and \( u = (u_k) \) be a sequence of positive real numbers. Then
\[
[O_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot\|, \|\cdot\|)]^\alpha = [C_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot\|, \|\cdot\|)]^\alpha.
\]

**Proof.** The proof is easy, so omitted. \( \square \)

**Acknowledgement:** We would like to express our sincere thanks to the reviewer for the kind remarks which improved the presentation of the paper.

**References**


KULDIP RAJ
School of Mathematics
Shri Mata Vaishno Devi University
KATRA-182320, J&K, INDIA

e-mail address: kuldipraj68@gmail.com
Renu Anand
School of Mathematics
Shri Mata Vaishno Devi University
Katra-182320, J&K, India
e-mail address: remuanand71@gmail.com

Suruchi Pandoh
School of Mathematics
Shri Mata Vaishno Devi University
Katra-182320, J&K, India
e-mail address: suruchi.pandoh87@gmail.com

(Received July 12, 2017)
(Accepted February 19, 2018)