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CESÀRO ORLICZ SEQUENCE SPACES AND THEIR KÖTHE-TOEPLITZ DUALS

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Abstract. The present paper focus on introducing certain classes of Cesàro Orlicz sequences over n-normed spaces. We study some topological and algebraic properties of these spaces. Further, we examine relevant relations among the classes of these sequences. We show that these spaces are made n-BK-spaces under certain conditions. Finally, we compute the Köthe-Toeplitz duals of these spaces.

1. Introduction and Preliminaries

Let w, ℓ_∞, ℓ_p, ℓ_1, c and c_0 represent the spaces of all, bounded, p-absolutely summable, absolutely summable, convergent and null sequences $x = (x_k)$ with complex terms, respectively. The zero element of a normed linear space is denoted by θ.
The space of all complex sequences $\ell_p(0 < p < \infty)$ such that $\sum_k |x_k|^p < \infty$, known as the space of p-absolutely summable sequences. The space ℓ_p for $p \geq 1$ is complete under the norm defined by $\|x\| = (\sum_k |x_k|^p)^{\frac{1}{p}}$ and for $0 < p < 1$, ℓ_p is a complete p-normed space, p-normed by $\|x\| = \sum_{k=1}^{\infty} |x_k|^p$.
A BK-space $(X, \|.\|)$ is a Banach space of complex sequences $x = (x_k)$, in which the co-ordinate maps are continuous, i.e., $|x_k^n - x_k| \to 0$, whenever $\|x^n - x\| \to 0$ as $n \to \infty$, where $x^n = (x_k^n)$ for all $n \in \mathbb{N}$ (see [33]).
Let $(X, \|.\|)$ be a normed linear space and λ is a scalar-valued sequence space, then the vector-valued sequence space or X-valued sequence space $\lambda(X)$ is defined by

$$\lambda(X) = \{(x_k) : x_k \in X \text{ for all } k \in \mathbb{N} \text{ and } \|x\| \in \lambda\}.$$

Clearly, $\lambda(X)$ is a linear space under coordinatewise addition and scalar multiplication over the field of scalars of X. Similarly, if X is a Banach space, then $\ell_p(1 \leq p < \infty)$ is a Banach space with the norm given by

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$\|x\| = (\sum_{k=1}^{\infty} \|x_k\|^p)^{\frac{1}{p}}$.

Cesàro sequence spaces Ces_p, $1 \leq p < \infty$, were introduced for the first time in 1968 in connection with the problem of finding their duals, which was posed by the Dutch Mathematical Society [1]. Shiue [27] and Leibowitz [14] studied the basic properties of these spaces. In 1974, Jagers [11] found the dual space of Ces_p [15].

The Cesàro sequence spaces is defined by

$$Ces_p = \left\{ x = (x_k) : \|x\|_p = \Big(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k|^p \Big)^{\frac{1}{p}} < \infty,\ 1 \leq p < \infty \right\}$$

and

$$Ces_\infty = \left\{ x = (x_k) : \|x\|_\infty = \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k| < \infty \right\}.$$

It was observed that $\ell_p \subset Ces_p (1 < p < \infty)$ is strict, although it does not hold for $p = 1$. Nag and Lee [22] defined and studied the Cesàro sequence space X_p of non-absolute type as follows:

$$X_p = \left\{ x = (x_k) : \|x\|_p = \Big(\sum_{n=1}^{\infty} \Big| \frac{1}{n} \sum_{k=1}^{n} x_k \Big|^p \Big)^{\frac{1}{p}} < \infty,\ 1 \leq p < \infty \right\}$$

and

$$X_\infty = \left\{ x = (x_k) : \|x\|_\infty = \sup_n \Big| \frac{1}{n} \sum_{k=1}^{n} x_k \Big| < \infty \right\}.$$

The inclusion $Ces_p \subset X_p,\ 1 \leq p < \infty$ is strict. Orhan [23] defined and studied the Cesàro difference sequence spaces $X_p(\Delta)$ and $X_\infty(\Delta)$ by replacing $x = (x_k)$ with $\Delta x = (\Delta x_k) = (x_k - x_{k+1}), k = 1, 2, ...$ and proved that for $1 \leq p < \infty$, the inclusions $X_p \subset X_p(\Delta)$ and $X_\infty \subset X_\infty(\Delta)$ are strict. In fact, Orhan [23] used C_p instead of $X_p(\Delta)$ and C_∞ instead of $X_\infty(\Delta)$. Further, Orhan [23] also defined and studied the following sequence spaces

$$O_p(\Delta) = \left\{ x = (x_k) : \sum_{n=1}^{\infty} \Big(\frac{1}{n} \sum_{k=1}^{n} |\Delta x_k| \Big)^p < \infty,\ 1 \leq p < \infty \right\}$$

and

$$O_\infty(\Delta) = \left\{ x = (x_k) : \sup_{n \geq 1} \frac{1}{n} \sum_{k=1}^{n} |\Delta x_k| < \infty \right\}.$$

He established that for $1 \leq p < \infty$, the inclusions $O_p(\Delta) \subset X_p(\Delta)$ and $Ces_p \subset O_p(\Delta)$ are strict.

Mursaleen et al. [19] studied the Cesàro difference sequence spaces which were defined as

$$X_p(\Delta^2) = \left\{ x = (x_k) : \sum_{n=1}^{\infty} \Big| \frac{1}{n} \sum_{k=1}^{n} \Delta^2 x_k \Big|^p < \infty,\ 1 \le p < \infty \right\}$$

and

$$X_\infty(\Delta^2) = \left\{ x = (x_k) : \sup_{n \ge 1} \Big| \frac{1}{n} \sum_{k=1}^{n} \Delta^2 x_k \Big| < \infty \right\},$$

where $\Delta^2 x_k = \Delta x_k - \Delta x_{k+1}$.
For uniformity of the literature, henceforth, we shall write C_p instead of X_p and C_∞ instead of X_∞.
Let E and F be two sequence spaces. Then the F dual of E is defined as $E^F = \{(x_k) \in w : (x_k y_k) \in F \text{ for all } (y_k) \in E\}$.
For $F = \ell_1$, the dual is termed as α-dual (Köthe-Toeplitz dual) of E and denoted by E^α. If $X \subset Y$, then $Y^\alpha \subset X^\alpha$.
For more details about Cesàro-type summable spaces and Köthe-Toeplitz dual one can refer to ([3], [20], [21], [22], [28], [29], [31], [32]).
The concept of 2-normed spaces was initially developed by Gähler [6] in the mid of 1960's, while that of n-normed spaces one can see in Misiak [18]. Since then, many others have studied this concept and obtained various results, see Gunawan ([7], [8]) and Gunawan and Mashadi [9]. Let $n \in \mathbb{N}$ and X be a linear space over the field of real numbers $\mathbb{R}$ of dimension d, where $d \ge n \ge 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the following four conditions:

(1) $||(x_1, x_2, \cdots, x_n)|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X,
(2) $||(x_1, x_2, \cdots, x_n)||$ is invariant under permutation,
(3) $||(\alpha x_1, x_2, \cdots, x_n)|| = |\alpha|\ \ ||(x_1, x_2, \cdots, x_n)||$ for any $\alpha \in \mathbb{R}$, and
(4) $||(x + x', x_2, \cdots, x_n)|| \le ||(x, x_2, \cdots, x_n)|| + ||(x', x_2, \cdots, x_n)||$

is called an n-norm on X and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called an *n-normed space* over the field $\mathbb{R}$.
As an example, we may take $X = \mathbb{R}^n$ being equipped with the n-norm $||(x_1, x_2, \cdots, x_n)||_E =$ the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||(x_1, x_2, \cdots, x_n)||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$.
Let $(X, ||\cdot, \cdots, \cdot||)$ be an n-normed space of dimension $d \ge n \ge 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $||\cdot, \cdots, \cdot||_\infty$ on X^{n-1} as defined by

$$||(x_1, x_2, \cdots, x_{n-1})||_\infty = \max\{||(x_1, x_2, \cdots, x_{n-1}, a_i)|| : i = 1, 2, \cdots, n\}$$

is called an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.
A sequence (x_k) in an n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to *converge* to some $L \in X$ if

$$\lim_{k\to\infty} ||(x_k - L, z_1, \cdots, z_{n-1})|| = 0 \ \text{ for every } \ z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in an n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to be *Cauchy* if

$$\lim_{k,p\to\infty} ||(x_k - x_p, z_1, \cdots, z_{n-1})|| = 0 \ \text{ for every } \ z_1, \cdots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be *complete* with respect to the n-norm. A complete n-normed space is said to be *n-Banach space*. For more details about sequence spaces and n-normed spaces (see [2], [24], [25], [26]) and references therein.

An Orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$. If convexity of Orlicz function is replaced by $M(x+y) \leq M(x) + M(y)$, then this function is called modulus function. Lindenstrauss and Tzafriri [13] used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \Big\{x = (x_k) \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\Big\}$$

is known as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is said to be Musielak-Orlicz function (see [16], [17]). A Musielak-Orlicz function $\mathcal{M} = (M_k)$ is said to satisfy Δ_2-*condition* if there exist constants a, $K > 0$ and a sequence $c = (c_k)_{k=1}^{\infty} \in l_+^1$ (the positive cone of l^1) such that the inequality

$$M_k(2u) \leq K M_k(u) + c_k$$

holds for all $k \in \mathbb{N}$ and $u \in \mathbb{R}^+$, whenever $M_k(u) \leq a$.

The notion of difference sequence spaces was introduced by Kizmaz [12], who studied the difference sequence spaces $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [4] by introducing the spaces $\ell_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$. Let n, m be non-negative integers, then for $Z = c$, c_0 and ℓ_∞ we have sequence spaces

$$Z(\Delta_n^m) = \{x = (x_k) \in w : (\Delta_n^m x_k) \in Z\},$$

where $\Delta_n^m x = (\Delta_n^m x_k) = (\Delta_n^{m-1} x_k - \Delta_n^{m-1} x_{k+1})$ and $\Delta_n^0 = x_k$ for all $k \in \mathbb{N}$ which is equivalent to the following binomial representation

$$\Delta_n^m x_k = \sum_{v=0}^{m} (-1)^v \begin{pmatrix} m \\ v \end{pmatrix} x_{k+nv}.$$

Taking $n = 1$, we get the spaces $\ell_\infty(\Delta^m), c(\Delta^m)$ and $c_0(\Delta^m)$ studied by Et and Çolak [4]. Taking $n = m = 1$, we get the spaces $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kizmaz [12].

Let $(X, \|\cdot, \cdots, \cdot\|)$ be an n-normed real linear space, $w(n-X)$ denotes X-valued sequence space. Let $\mathcal{M} = (M_i)$ be a sequence of Orlicz functions and $u = (u_k)$ be a sequence of positive real numbers. Then we define the following sequence spaces for every nonzero $z_1, ..., z_n \in X$;

$$C_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|) =$$

$$\Big\{(x_k) \in w(n-X) : \sum_{i=1}^{\infty} M_i \Big(\Big\| \frac{1}{i} \sum_{k=1}^{i} \frac{u_k \Delta_n^m x_k}{\rho}, z_1, ..., z_{n-1} \Big\|\Big)^p < \infty, \text{ for some } \rho > 0\Big\},$$

$$C_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|) =$$

$$\Big\{(x_k) \in w(n-X) : \sup_i M_i \Big(\Big\| \frac{1}{i} \sum_{k=1}^{i} \frac{u_k \Delta_n^m x_k}{\rho}, z_1, ..., z_{n-1} \Big\|\Big) < \infty, \text{ for some } \rho > 0\Big\},$$

$$\ell_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|) =$$

$$\Big\{(x_k) \in w(n-X) : \sum_{k=1}^{\infty} M_k \Big(\Big\| \frac{u_k \Delta_n^m x_k}{\rho}, z_1, ..., z_{n-1} \Big\|\Big)^p < \infty, \text{ for some } \rho > 0\Big\},$$

$$O_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|) =$$

$$\Big\{(x_k) \in w(n-X) : \sum_{i=1}^{\infty} M_i \Big(\frac{1}{i} \sum_{k=1}^{i} \Big\| \frac{u_k \Delta_n^m x_k}{\rho}, z_1, ..., z_{n-1} \Big\|\Big)^p < \infty, \text{ for some } \rho > 0\Big\}$$

and
$O_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|) =$

$$\Big\{(x_k) \in w(n-X) : \sup_i M_i\Big(\frac{1}{i}\sum_{k=1}^{i}\Big\|\frac{u_k\Delta_n^m x_k}{\rho}, z_1, ..., z_{n-1}\Big\|\Big) < \infty, \text{ for some } \rho > 0\Big\}.$$

Lemma 1.1. [30] *(a) Let $1 \leq p < \infty$. Then*
(i) The space C_p is a Banach space, normed by

$$\|x\| = \Big(\sum_{i=1}^{\infty}\Big|\frac{1}{i}\sum_{k=1}^{i} x_k\Big|^p\Big)^{\frac{1}{p}}.$$

(ii) The space O_p is a Banach space, normed by

$$\|x\| = \Big(\sum_{i=1}^{\infty}\frac{1}{i}\sum_{k=1}^{i}|x_k|^p\Big)^{\frac{1}{p}}.$$

(iii) The space ℓ_p is a Banach space, normed by

$$\|x\| = \Big(\sum_{k=1}^{\infty}|x_k|^p\Big)^{\frac{1}{p}}.$$

(b) (i) The space C_∞ is a Banach space, normed by

$$\|x\| = \sup_i\Big|\frac{1}{i}\sum_{k=1}^{i} x_k\Big|.$$

(ii) The space O_∞ is a Banach space, normed by

$$\|x\| = \sup_i \frac{1}{i}\sum_{k=1}^{i}|x_k|.$$

Definition 1. An n-BK-space $(X, \|., ..., .\|)$ is an n-Banach space of real sequences $x = (x_k)$ in which the co-ordinate maps are continuous.

Let us consider a few special cases of the above sequence spaces:

(i) If $M_i(x) = x$ for all $i \in N$, then we have
$C_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|) = C_p(u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$, $C_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ $= C_\infty(u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$, $\ell_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|) = \ell_p(u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$, $O_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|) = O_p(u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ and $O_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ $= O_\infty(u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$.

(ii) If $u = (u_k) = 1$ for all $k \in N$, then we have

$C_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|) = C_p(\mathcal{M}, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$, $C_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ $= C_\infty(\mathcal{M}, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$, $\ell_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|) = \ell_p(\mathcal{M}, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$, $O_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|) = O_p(\mathcal{M}, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ and $O_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ $= O_\infty(\mathcal{M}, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$.

The following inequality will be used throughout the paper. Let $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k \leq \sup_k p_k = H$ and let $K = \max\left\{1, 2^{H-1}\right\}$. Then for the factorable sequences (a_k) and (b_k) in the complex plane, we have

$$|a_k + b_k|^{p_k} \leq K(|a_k|^{p_k} + |b_k|^{p_k}). \tag{1.1}$$

Also $|a_k|^{p_k} \leq \max\left\{1, |a|^H\right\}$ for all $a \in \mathbb{C}$.

The main purpose of this paper is to introduce and study certain classes of multiplier sequences of Cesàro-type defined by a sequence of Orlicz functions over n-normed space. We make an effort to study completeness and some interesting inclusion relations between these spaces. Finally, we compute the Köthe-Toeplitz duals of these spaces.

2. Main Results

Theorem 2.1. *Let $\mathcal{M} = (M_i)$ be a sequence of Orlicz functions and $u = (u_k)$ be a sequence of positive real numbers. Then the classes of sequences $C_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$, $C_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$, $\ell_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$, $O_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ and $O_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ for $1 \leq p < \infty$ are linear spaces over the real field $\mathbb{R}$.*

Proof. We shall prove the result for the space $O_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ and for the other spaces, it will follow on applying similar arguments.
Suppose $x = (x_k)$, $y = (y_k) \in O_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ and $\alpha, \beta \in \mathbb{R}$. Then there exist positive numbers ρ_1, ρ_2 such that

$$\sum_{i=1}^{\infty} M_i\Big(\frac{1}{i}\sum_{k=1}^{i}\Big\|\frac{u_k\Delta_n^m x_k}{\rho_1}, z_1, ..., z_{n-1}\Big\|\Big)^p < \infty, \text{ for some } \rho_1 > 0$$

and

$$\sum_{i=1}^{\infty} M_i\Big(\frac{1}{i}\sum_{k=1}^{i}\Big\|\frac{u_k\Delta_n^m y_k}{\rho_2}, z_1, ..., z_{n-1}\Big\|\Big)^p < \infty, \text{ for some } \rho_2 > 0.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M} = (M_i)$ is a non-decreasing and convex so by using inequality (1.1), we have

$$\sum_{i=1}^{\infty} M_i\Big(\frac{1}{i}\sum_{k=1}^{i}\Big\|\frac{\alpha u_k\Delta_n^m x_k+\beta u_k\Delta_n^m y_k}{\rho_3}, z_1, ..., z_{n-1}\Big\|\Big)^p$$
$$\leq \sum_{i=1}^{\infty} M_i\Big(\frac{1}{i}|\alpha|\sum_{k=1}^{i}\Big\|\frac{u_k\Delta_n^m x_k}{\rho_3}, z_1, ..., z_{n-1}\Big\| + |\beta|\sum_{k=1}^{i}\Big\|\frac{u_k\Delta_n^m y_k}{\rho_3}, z_1, ..., z_{n-1}\Big\|\Big)^p$$
$$\leq K\sum_{i=1}^{\infty} M_i\Big(\frac{1}{i}\sum_{k=1}^{i}\Big\|\frac{u_k\Delta_n^m x_k}{\rho_1}, z_1, ..., z_{n-1}\Big\|\Big)^p$$
$$+ K\sum_{i=1}^{\infty} M_i\Big(\frac{1}{i}\sum_{k=1}^{i}\Big\|\frac{u_k\Delta_n^m y_k}{\rho_2}, z_1, ..., z_{n-1}\Big\|\Big)^p$$
$$< \infty.$$

Thus, $\alpha x+\beta y \in O_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$. This proves that $O_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ is a linear space. $\square$

Theorem 2.2. *Let $\mathcal{M} = (M_i)$ be a sequence of Orlicz functions and $u = (u_k)$ be a sequence of positive real numbers. Let $1 \leq p < \infty$ and the base space X is an n-Banach space. Then*
(i) The space $C_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ is an n-Banach space, n-normed by

$\|x^1, x^2, \cdots, x^n\|_{C_p(\mathcal{M},u,\Delta_n^m)} = 0$ *if $x^1, x^2, \cdots, x^n$ are linearly dependent and*
$$= \sum_{k=1}^{m}\|x_k, z_1, \cdots, z_{n-1}\| + \Big(\sum_{i=1}^{\infty} M_i\Big\|\frac{1}{i}\sum_{k=1}^{i}\frac{u_k\Delta_n^m x_k}{\rho}, z_1, ..., z_{n-1}\Big\|^p\Big)^{\frac{1}{p}}$$
for every $z_1, ..., z_{n-1} \in X$ if $x^1, x^2, \cdots, x^n$ are linearly independent.
(ii) The space $O_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ is an n-Banach space, n-normed by

$\|x^1, x^2, \cdots, x^n\|_{O_p(\mathcal{M},u,\Delta_n^m)} = 0$ *if $x^1, x^2, \cdots, x^n$ are linearly dependent and*
$$= \sum_{k=1}^{m}\|x_k, z_1, \cdots, z_{n-1}\| + \Big(\sum_{i=1}^{\infty} M_i\frac{1}{i}\sum_{k=1}^{i}\Big\|\frac{u_k\Delta_n^m x_k}{\rho}, z_1, ..., z_{n-1}\Big\|^p\Big)^{\frac{1}{p}}$$
for every $z_1, ..., z_{n-1} \in X$ if $x^1, x^2, \cdots, x^n$ are linearly independent.
(iii) The space $\ell_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ is an n-Banach space, n-normed by

$\|x^1, x^2, \cdots, x^n\|_{\ell_p(\mathcal{M},u,\Delta_n^m)} = 0$ *if $x^1, x^2, \cdots, x^n$ are linearly dependent and*
$$= \sum_{k=1}^{m}\|x_k, z_1, \cdots, z_{n-1}\| + \Big(\sum_{k=1}^{\infty} M_k\Big\|\frac{u_k\Delta_n^m x_k}{\rho}, z_1, ..., z_{n-1}\Big\|^p\Big)^{\frac{1}{p}}$$
for every $z_1, ..., z_{n-1} \in X$ if $x^1, x^2, \cdots, x^n$ are linearly independent.
(b) (i) The space $C_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ is an n-Banach space, n-normed by

$\|x^1, x^2, \cdots, x^n\|_{C_\infty(\mathcal{M},u,\Delta_n^m)} = 0$ *if* $x^1, x^2, \cdots, x^n$ *are linearly dependent and*
$= \sum_{k=1}^{m} \|x_k, z_1, \cdots, z_{n-1}\| + \sup_i M_i \Big\| \frac{1}{i} \sum_{k=1}^{i} \frac{u_k \Delta_n^m x_k}{\rho}, z_1, ..., z_{n-1} \Big\|,$
for every $z_1, ..., z_{n-1} \in X$ *if* $x^1, x^2, \cdots, x^n$ *are linearly independent.*
(ii) The space $O_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ *is an n-Banach space, n-normed by*

$\|x^1, x^2, \cdots, x^n\|_{O_\infty(\mathcal{M},u,\Delta_n^m)} = 0$ *if* $x^1, x^2, \cdots, x^n$ *are linearly dependent and*
$= \sum_{k=1}^{m} \|x_k, z_1, \cdots, z_{n-1}\| + \sup_i M_i \frac{1}{i} \sum_{k=1}^{i} \Big\| \frac{u_k \Delta_n^m x_k}{\rho}, z_1, ..., z_{n-1} \Big\|,$
for every $z_1, ..., z_{n-1} \in X$ *if* $x^1, x^2, \cdots, x^n$ *are linearly independent.*

Proof. It is easy to show that the spaces $C_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$, $O_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$, $\ell_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$, $C_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ and $O_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ are n-normed spaces under the n-norm as defined above.
Now, we prove the completeness for the space $C_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ only. The other parts can be proved in a similar way.
Let $(x^s)_{s=1}^{\infty}$ be a Cauchy sequence in $C_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$, where $x^s = (x_i^s) = (x_1^s, x_2^s, ...) \in C_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ for each $s \in N$. Let $\epsilon > 0$ be given. Then there exists a positive integer n_0 such that

$$\|x^s - x^t, w^2, ..., w^n\|_{C_\infty(\mathcal{M},u,\Delta_n^m)} < \epsilon$$

for all $s, t \geq n_0$ and for every $w^2, ..., w^n \in C_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$, we have

$$\sum_{k=1}^{m} \|x_k^s - x_k^t, z_1, ..., z_{n-1}\| + \sup_i M_i \Big\| \frac{1}{i} \sum_{k=1}^{i} \frac{u_k \Delta_n^m (x_k^s - x_k^t)}{\rho}, z_1, ..., z_{n-1} \Big\| < \epsilon$$

for all $s, t \geq n_0$ and for every $z_1, ..., z_{n-1} \in X$. This implies

$$\sum_{k=1}^{m} \|x_k^s - x_k^t, z_1, ..., z_{n-1}\| < \epsilon$$

$$\text{and } \sup_i M_i \Big\| \frac{1}{i} \sum_{k=1}^{i} \frac{u_k \Delta_n^m (x_k^s - x_k^t)}{\rho}, z_1, ..., z_{n-1} \Big\| < \epsilon$$

for all $s, t \geq n_0$ and for every $z_1, ..., z_{n-1} \in X$. Hence, $\|x_k^s - x_k^t, z_1, ..., z_{n-1}\| < \epsilon$ for all $k = 1, 2, ..., m$ and for every $z_1, ..., z_{n-1} \in X$.
Therefore, (x_k^s) is a Cauchy sequence for all $k = 1, 2, , ..., m$ in X, an n-Banach space.

Hence, (x_k^s) converges in X for all $k = 1, 2, ..., m$. Let $\lim_{s\to\infty} x_k^s = x_k$ for all $k = 1, 2, , ..., m$. Next, we have

$$\sup_i M_i \Big\| \frac{1}{i} \sum_{k=1}^{i} \frac{u_k \Delta_n^m (x_k^s - x_k^t)}{\rho}, z_1, ..., z_{n-1} \Big\| < \epsilon,$$

for all $s, t \geq n_0$ and for every $z_1, ..., z_{n-1} \in X$. This implies for every $z_1, ..., z_{n-1} \in X$

$$M_i \Big\| \frac{1}{i} \sum_{k=1}^{i} \frac{u_k \Delta_n^m (x_k^s - x_k^t)}{\rho}, z_1, ..., z_{n-1} \Big\| < \epsilon,$$

for all $s, t \geq n_0$ and $i \in N$.
Thus, $(\Delta_n^m x_k^s)$ is a Cauchy sequence in $C_\infty(\mathcal{M}, u, \|\cdot, \cdots, \cdot\|)$ which is complete. Hence, $(\Delta_n^m x_k^s)$ converges for each $k \in N$.
Let $\lim_{s\to\infty} \Delta_n^m x_k^s = y_k$ for each $k \in N$. Let $k = 1$, we have

$$\lim_{s\to\infty} \Delta_n^m x_1^s = \lim_{s\to\infty} \sum_{v=0}^{m} (-1)^v \begin{pmatrix} m \\ v \end{pmatrix} x_{1+nv} = y_1, \tag{2.1}$$

we have

$$\lim_{s\to\infty} x_k^s = x_k, \text{ for } k = 1 + nv, \text{ for } v = 1, 2, ..., m-1. \tag{2.2}$$

Thus, from equation (2.1) and (2.2), we have $\lim_{s\to\infty} x_{1+m}^s$ exists. Let $\lim_{s\to\infty} x_{1+m}^s = x_{1+m}$. Proceeding in this way inductively $\lim_{s\to\infty} x_k^s = x_k$ exists for each $k \in N$.
Now, for every $z_1, ..., z_{n-1} \in X$

$$\lim_t \sum_{k=1}^{m} \|x_k^s - x_k^t, z_1, ..., z_{n-1}\| = \sum_{k=1}^{m} \|x_k^s - x_k, z_1, ..., z_{n-1}\| < \epsilon,$$

for all $s \geq n_0$. Again, using the continuity of n-norm, we find that for every $z_1, ..., z_{n-1} \in X$

$$M_i \Big\| \frac{1}{i} \sum_{k=1}^{i} \frac{u_k \Delta_n^m x_k^s}{\rho} - \lim_{t\to\infty} \frac{u_k \Delta_n^m x_k^t}{\rho}, z_1, ..., z_{n-1} \Big\| < \epsilon,$$

for all $s \geq n_0$ and $i \in N$. Hence, for every $z_1, ..., z_{n-1} \in X$

$$\sup_i M_i \Big\| \frac{1}{i} \sum_{k=1}^{i} \frac{u_k \Delta_n^m x_k^s - u_k \Delta_n^m x_k}{\rho}, z_1, ..., z_{n-1} \Big\| < \epsilon \text{ for all } s \geq n_0.$$

Thus, for every $w^2, ..., w^n \in C_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$

$$\|x^s - x, w^2, ..., w^n\|_{C_\infty(\mathcal{M}, u, \Delta_n^m)} < 2\epsilon \text{ for all } s \geq n_0.$$

Hence, $(x^s - x) \in C_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$. Since $C_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ is a linear space, so we have for all $s \geq n_0$, $x = x^s - (x^s - x) \in C_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$. Hence, $C_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ is complete and as such is an n-Banach space. □

Corollary 2.3. *The spaces* $C_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$, $C_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$, $\ell_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$, $O_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ *and* $O_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ *for* $1 \leq p < \infty$ *are n-BK-spaces if the base space X is an n-Banach space.*

Theorem 2.4. *Let* $\mathcal{M} = (M_i)$ *be a sequence of Orlicz functions and* $u = (u_k)$ *be a sequence of positive real numbers. Then* $Z(\mathcal{M}, u, \Delta_n^{m-1}, \|\cdot, \cdots, \cdot\|) \subset Z(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ *(in general* $Z(\mathcal{M}, u, \Delta_n^i, \|\cdot, \cdots, \cdot\|) \subset Z(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ *for* $i = 1, 2, ..., m-1$*) for* $Z = C_p, O_p, \ell_p, C_\infty$ *and* O_∞.

Proof. We shall prove the result for the space $Z = C_p$ only and others can be proved in the similar way.
Let $x = (x_k) \in C_p(\mathcal{M}, u, \Delta_n^{m-1}, \|\cdot, \cdots, \cdot\|)$, $1 \leq p < \infty$. Then for every nonzero $z_1, ..., z_{n-1} \in X$,

$$\sum_{i=1}^{\infty} M_i\Big(\Big\|\frac{1}{i}\sum_{k=1}^{i} \frac{u_k \Delta_n^{m-1} x_k}{\rho}, z_1, ..., z_{n-1}\Big\|\Big)^p < \infty. \tag{2.3}$$

Now, we have for every nonzero $z_1, ..., z_{n-1} \in X$

$$\begin{aligned} &M_i\Big(\Big\|\tfrac{1}{i}\sum_{k=1}^{i} \frac{u_k \Delta_n^m x_k}{\rho}, z_1, ..., z_{n-1}\Big\|\Big) \\ &\quad\leq M_i\Big(\Big\|\frac{1}{i}\sum_{k=1}^{i} \frac{u_k \Delta_n^{m-1} x_k}{\rho}, z_1, ..., z_{n-1}\Big\|\Big) \\ &\quad+ M_i\Big(\Big\|\frac{1}{i}\sum_{k=1}^{i} \frac{u_k \Delta_n^{m-1} x_{k+1}}{\rho}, z_1, ..., z_{n-1}\Big\|\Big). \end{aligned}$$

It is known that for $1 \leq p < \infty$, $|a+b|^p \leq 2^p(|a|^p + |b|^p)$. Hence, for $1 \leq p < \infty$, $M_i\Big(\Big\|\tfrac{1}{i}\sum_{k=1}^{i} \frac{u_k \Delta_n^m x_k}{\rho}, z_1, ..., z_{n-1}\Big\|\Big)^p$

$$\begin{aligned} &\leq 2^p\Big\{M_i\Big(\Big\|\frac{1}{i}\sum_{k=1}^{i} \frac{u_k \Delta_n^{m-1} x_k}{\rho}, z_1, ..., z_{n-1}\Big\|\Big)^p \\ &+ M_i\Big(\Big\|\frac{1}{i}\sum_{k=1}^{i} \frac{u_k \Delta_n^{m-1} x_{k+1}}{\rho}, z_1, ..., z_{n-1}\Big\|\Big)^p\Big\}. \end{aligned}$$

Then for each positive integer r, we get

$$\sum_{i=1}^{r} M_i\Big(\Big\|\frac{1}{i}\sum_{k=1}^{i}\frac{u_k\Delta_n^m x_k}{\rho}, z_1, ..., z_{n-1}\Big\|\Big)^p$$

$$\leq \; 2^p\Big\{\sum_{i=1}^{r} M_i\Big(\Big\|\frac{1}{i}\sum_{k=1}^{i}\frac{u_k\Delta_n^{m-1} x_k}{\rho}, z_1, ..., z_{n-1}\Big\|\Big)^p$$

$$+ \; \sum_{i=1}^{r} M_i\Big(\Big\|\frac{1}{i}\sum_{k=1}^{i}\frac{u_k\Delta_n^{m-1} x_{k+1}}{\rho}, z_1, ..., z_{n-1}\Big\|\Big)^p\Big\}.$$

Taking $r \to \infty$ and using equation (2.3), we get

$$\sum_{i=1}^{\infty} M_i\Big(\Big\|\frac{1}{i}\sum_{k=1}^{i}\frac{u_k\Delta_n^m x_k}{\rho}, z_1, ..., z_{n-1}\Big\|\Big)^p < \infty.$$

Thus, $C_p(\mathcal{M}, u, \Delta_n^{m-1}, \|\cdot, \cdots, \cdot\|) \subset C_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ for $1 \leq p < \infty$. The inclusion is strict and it follows from the following example. $\square$

Example 2.5. *Let $X = \mathbb{R}^3$ be a real linear space. Define $\|., .\| : X \times X \to \mathbb{R}$ by $\|x, y\| = \max\{|x_1y_2 - x_2y_1|, |x_2y_3 - x_3y_2|, |x_3y_1 - x_1y_3|\}$, where $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3$. Then $(X, \|., .\|)$ is a 2-normed linear space. Let $(u_k) = 1$, $(M_i) = I$, the identity map, for all $i \in N$, $m = 2$ and $n = 1$. Consider the sequence $x = (x_k) = (k, k, k)$ for all $k \in N$. Then $\Delta^2 x_k = (0, 0, 0)$ for all $k \in N$. Hence, $(x_k) \in C_p(\mathcal{M}, u, \Delta^2, \|., .\|)$, we have $\Delta(x_k) = (-1, -1, -1)$ for all $k \in N$. Hence, $(x_k) \notin C_p(\mathcal{M}, u, \Delta, \|., .\|)$. The inclusion is strict.*

Theorem 2.6. *Let $\mathcal{M} = (M_i)$ be a sequence of Orlicz functions and $u = (u_k)$ be a sequence of positive real numbers. Then*
(a) $O_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|) \subset C_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|) \subset C_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ and the inclusions are strict.
(b) $O_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|) \subset O_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|) \subset C_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ and the inclusions are strict.

Proof. The proof is trivial, so we omitted. $\square$

Remark. $\ell_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|) \subsetneq O_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$.

Example 2.7. *Let $p = 1$ and 2-norm $\|., .\|$ on $X = \mathbb{R}^3$ in Example (2.5). Let $m = 2$, $n = 1$, $(u_k) = 1$ and $(M_i) = I$. Consider the sequence $\{x_k\} = \{(1, 1, 1), (0, 0, 0), (0, 0, 0), (0, 0, 0), ...\}$. Then $\Delta^2 x_k = (1, 1, 1)$ for $k = 1$ and $\Delta^2 x_k = (0, 0, 0)$ for all $k > 1$. Then $(x_k) \in \ell(\mathcal{M}, u, \Delta^2, \|., .\|)$ but $(x_k) \notin O(\mathcal{M}, u, \Delta^2, \|., .\|)$.*

Theorem 2.8. *If $1 \leq p < q$, then*
(i) $C_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|) \subset C_q(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$;
(ii) $\ell_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|) \subset \ell_q(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$;
(iii) $O_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|) \subset O_q(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$.

Proof. We prove the result for the space $O_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ only and for the other cases it can be proved in a similar way. Let $x \in O_p(\mathcal{M},\ u,\ \Delta_n^m, \|\cdot, \cdots, \cdot\|)$. Then there exists $\rho > 0$ such that

$$\sum_{i=1}^{\infty} M_i\Big(\frac{1}{i}\sum_{k=1}^{i}\Big\|\frac{u_k\Delta_n^m x_k}{\rho}, z_1, ..., z_{n-1}\Big\|\Big)^p < \infty.$$

This implies

$$M_i\Big(\frac{1}{i}\sum_{k=1}^{i}\Big\|\frac{u_k\Delta_n^m x_k}{\rho}, z_1, ..., z_{n-1}\Big\|\Big)^p < 1$$

for sufficiently large values of i. Since (M_i) is non-decreasing, we get

$$\begin{aligned}
&\sum_{i=1}^{\infty} M_i\Big(\frac{1}{i}\sum_{k=1}^{i}\Big\|\frac{u_k\Delta_n^m x_k}{\rho}, z_1, ..., z_{n-1}\Big\|\Big)^q \\
&\qquad \leq \sum_{i=1}^{\infty} M_i\Big(\frac{1}{i}\sum_{k=1}^{i}\Big\|\frac{u_k\Delta_n^m x_k}{\rho}, z_1, ..., z_{n-1}\Big\|\Big)^p \\
&\qquad < \infty.
\end{aligned}$$

Thus, $x \in O_q(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$. This completes the proof. □

3. Köthe-Toeplitz duals

In order to compute Köthe-Toeplitz dual, we first define the following:
An n-functional is a real-valued mapping with domain $A_1 \times ... \times A_n$, where $A_1, ..., A_n$ are linear manifolds of a linear n-normed space.
Let F be an n-functional with domain $A_1 \times ... \times A_n$. F is called a linear n-functional whenever for all ${}^1a_1, {}^1a_2, ..., {}^1a_n \in A_1$, ${}^2a_1, {}^2a_2, ..., {}^2a_n \in A_2$ and ${}^na_1, {}^na_2, ..., {}^na_n \in A_n$ and all $\alpha_1, ..., \alpha_n \in \mathbb{R}$, we have
(i) $F({}^1a_1, {}^1a_2, ..., {}^1a_n, {}^2a_1, {}^2a_2, ..., {}^2a_n, ..., {}^na_1, {}^na_2, ..., {}^na_n) = \sum_{1 \leq i_1, i_2, ..., i_n \leq n} F({}^1a_{i_1}, {}^2a_{i_2}, ..., {}^na_{i_n})$ and
(ii) $F(\alpha_1 a_1, ..., \alpha_n a_n) = \alpha_1, ..., \alpha_n F(a_1, ..., a_n)$.
Let F be an n-functional with domain $D(F)$. F is called bounded if there is a real constant $K \geq 0$ such that $|F(a_1, ..., a_n)| \leq K\|a_1, ..., a_n\|$ for all $(a_1, ..., a_n) \in D(F)$. If F is bounded, we define the norm

$$\|F\| = \text{glb}\{K : |F(a_1, ..., a_n)| \leq K\|a_1, ..., a_n\| \text{ for all } (a_1, ..., a_n) \in D(F)\}.$$

If F is not bounded, we define $\|F\| = +\infty$.

Proposition 3.1. [10] *A linear n-functional F is continuous if and only if it is bounded.*

Proposition 3.2. [10] *Let B^* be the set of bounded linear n-functionals with domain $B_1 \times ... \times B_n$. Then B^* is an n-Banach space upto linear dependence.*

For any $n(>1)$-normed space E, we denote by E^* the continuous dual of E.

Definition 2. Let E be an n-normed linear space, normed by $\|., ..., .\|_E$. Then we define the Köthe-Toeplitz dual of the sequence space $Z(E)$ whose base space is E as

$[Z(E)]^\alpha = \{(y_k) : y_k \in E^*, k \in N$ and $(\|x_k, w_2, ..., w_n\|_E \|y_k, v_2, ..., v_n\|_{E^*}) \in \ell_1$ for every

$v_2, ..., v_n \in E^*, w_2, ..., w_n \in E, (x_k) \in Z(E)\}$.

It is easy to check that $\phi \in X^\alpha$. If $X \subset Y$, then $Y^\alpha \subset X^\alpha$. Let us consider $SC_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|) =$

$$\{x = (x_k) : x \in C_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|), x_1 = ... = x_m = 0\}.$$

Then $SC_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ is a subspace of $C_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ for $1 \le p < \infty$. We can have similar subspaces for other spaces as well.

Lemma 3.3. [5] *$x \in SC_\infty(\Delta^m)$ implies $\sup_k k^{-m}|x_k| < \infty$.*

Lemma 3.4. *$x \in SC_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ implies $\sup_k k^{-m}\|x_k, w_2, ..., w_n\| < \infty$ for every $w_2, ..., w_n \in X$.*

Proof. The proof follows using similar techniques as applied in the proof of Lemma 3.3. Consider a set

$$\mathcal{U} = \Big\{a = (a_k) : \sum_{k=1}^{\infty} k^m \|a_k, z_2, ..., z_n\|_{X^*} < \infty, \text{ for every } z_2, ..., z_n \in X^*\Big\}.$$

□

Theorem 3.5. *Let $\mathcal{M} = (M_i)$ be a sequence of Orlicz functions and $u = (u_k)$ be a sequence of positive real numbers. Then the Köthe-Toeplitz duals of the space $SC_p(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ is $\mathcal{U}$, that is, $[SC_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)]^\alpha = \mathcal{U}$.*

Proof. If $a \in \mathcal{U}$, then

$$\sum_{k=1}^{\infty} \|a_k, z_2, ..., z_n\|_{X^*} \|x_k, w_2, ..., w_n\|_X$$

$$= \sum_{k=1}^{\infty} k^m \|a_k, z_2, ..., z_n\|_{X^*} (k^{-m}\|x_k, w_2, ..., w_n\|_X)$$

$$< \quad \infty,$$

for each $x \in SC_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$ by Lemma 3.4. Hence, $x \in [SC_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)]^\alpha$.
Next, let $a \in [SC_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)]^\alpha$. Then

$$\sum_{k=1}^{\infty} \|a_k, z_2, ..., z_n\|_{X^*} \|x_k, w_2, ..., w_n\|_X < \infty,$$

for each $x \in SC_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$.
Define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} 0, & k \leq m, \\ k^m, & k > m \end{cases}$$

and choose $w_2, ..., w_n \in X$ such that

$$\|k^m, w_2, ..., w_n\|_X = k^m \|1, w_2, ..., w_n\|_X = \begin{cases} 0, & k \leq m \\ k^m, & k > m. \end{cases}$$

Thus, we have $z_2, ..., z_n \in X^*$

$$\begin{aligned} \sum_{k=1}^{\infty} k^m \|a_k, z_2, ..., z_n\|_{X^*} &= \sum_{k=1}^{\infty} \|k^m, w_2, ..., w_n\|_X \|a_k, z_2, ..., z_n\|_{X^*} \\ &= \sum_{k=1}^{m} \|k^m, w_2, ..., w_n\|_X \|a_k, z_2, ..., z_n\|_{X^*} \\ &+ \sum_{k=1}^{\infty} \|k^m, w_2, ..., w_n\|_X \|a_k, z_2, ..., z_n\|_{X^*} \\ &< \infty. \end{aligned}$$

This implies $a \in \mathcal{U}$. □

Theorem 3.6. *Let $\mathcal{M} = (M_i)$ be a sequence of Orlicz functions and $u = (u_k)$ be a sequence of positive real numbers. Then*

$$[SC_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)]^\alpha = [C_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)]^\alpha.$$

Proof. Since $SC_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|) \subset C_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$, we have $[C_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)]^\alpha \subset [SC_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)]^\alpha$.
Let $a \in [SC_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)]^\alpha$ and $x \in C_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$. Consider the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} x_k, & k \leq m, \\ x'_k, & k > m, \end{cases}$$

where $x' = (x'_k) \in SC_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)$. Then write

$$\begin{aligned}
&\sum_{k=1}^{\infty} \|a_k, z_2, ..., z_n\|_{X^*} \|x_k, w_2, ..., w_n\|_X \\
&\qquad = \sum_{k=1}^{m} \|a_k, z_2, ..., z_n\|_{X^*} \|x_k, w_2, ..., w_n\|_X \\
&\qquad + \sum_{k=1}^{\infty} \|a_k, z_2, ..., z_n\|_{X^*} \|x_{k'}, w_2, ..., w_n\|_X \\
&\qquad < \infty.
\end{aligned}$$

This implies $a \in [C_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)]^\alpha$. □

Theorem 3.7. *Let $\mathcal{M} = (M_i)$ be a sequence of Orlicz functions and $u = (u_k)$ be a sequence of positive real numbers. Then*

$$[O_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)]^\alpha = [C_\infty(\mathcal{M}, u, \Delta_n^m, \|\cdot, \cdots, \cdot\|)]^\alpha.$$

Proof. The proof is easy, so omitted. □

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