Advances in Differential Equations Volume 8, Number 1, January 2003, Pages 29–54

A UNIFORM CONVERGENCE THEOREM FOR SINGULAR LIMIT EIGENVALUE PROBLEMS

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Abstract. For reaction-diffusion equations, equilibrium solutions or traveling wave solutions with thin transition layers are constructed by singular perturbation methods. It is usually difficult to study their stability. This is because the linearized eigenvalue problem has a critical eigenvalue in a small neighborhood of zero, and its location is difficult to determine. The SLEP method is known as one of the most powerful tools to study this critical eigenvalue. To apply this method rigorously, a uniform convergence theorem for the inverse of a differential operator, for instance the inverse Allen-Cahn operator, in some function space plays a crucial role. However, there has been a significant difficulty in the cases of unbounded intervals including those of traveling waves, and no rigorous result was available previously. This paper presents a uniform convergence theorem in a general framework. Our new uniform convergence theorem makes the SLEP method applicable to various kinds of problems including stability of traveling waves.

1. Introduction

In a coupled system of reaction-diffusion equations of bistable type, solutions often have thin transition layers. We study them in one-dimensional intervals. Equilibrium solutions or traveling wave solutions with such layers are constructed by singular perturbation methods and are called singularly perturbed solutions. Studying their stability is usually quite difficult. This is because the linearized eigenvalue problem has a critical eigenvalue in a small neighborhood of zero in the complex plane. Let $\varepsilon > 0$ be a parameter associated with the thickness of transition layers. In some cases this critical eigenvalue approaches zero as $\varepsilon \to 0$. The location of the critical eigenvalue is usually very difficult to determine. The so-called SLEP (singular limit eigenvalue problem) method is known as one of the most powerful tools to

Accepted for publication: May 2002.
AMS Subject Classifications: 35K57, 35B35, 35B25.

study such a critical eigenvalue. This method was introduced by Nishiura and Fujii [9] to prove the stability of singularly perturbed equilibrium solutions in a finite interval for activator-inhibitor systems in chemistry or mathematical biology.

The SLEP method is very powerful because it determines the location of the critical eigenvalue completely, and it has been modified and used in various forms by many people. Nishiura, Mimura, Ikeda and Fujii [10] study traveling waves of activator-inhibitor systems by using this method, Tsujikawa [12] applies this method to a chemotaxis model with a growth effect. Kan-on and Mimura [6] and Ikeda [4, 5] apply this method to three component reaction-diffusion systems. Kan-on and Mimura [6] study a model of two competing preys with a common predator, and Ikeda [4, 5] studies that of three competing species.

To apply the SLEP method, an uniform convergence theorem for the inverse of a differential operator, for instance the inverse Allen-Cahn operator, in some function space plays a crucial role. However, in the case of unbounded intervals, no rigorous uniform convergence theorem was available previously, which has posed a difficulty in applying the SLEP method to solutions on unbounded intervals. In the following we briefly explain where the problem lies for two-component systems.

A typical example of a two-component system is as follows:

$$\varepsilon\tau U_t = \varepsilon^2 U_{xx} + U - U^3 - V, \quad V_t = V_{xx} + U - \beta_1 V + \beta_2, \quad x \in \boldsymbol{R},\ t > 0. \tag{1.1}$$

Here, $U(x,t)$, $V(x,t)$ represent the density of an activator and an inhibitor, respectively, and β_1, β_2 are positive constants with $3 + 3\sqrt{3}\beta_2 < 2\beta_1$. These nonlinear terms are those of the bistable FitzHugh-Nagumo equations. To analyze the stability of a singularly perturbed solution for a two-component system, one studies the linearized eigenvalue problem, which, in a typical situation, can be stated as follows: find $\lambda \in \boldsymbol{C}_+ = \{\lambda \in \boldsymbol{C} : \operatorname{Re}\lambda \geq 0\}$ and $(\widehat{u}(x,\varepsilon), \widehat{v}(x,\varepsilon)) \in L^2(I) \times L^2(I)$ that satisfy

$$L^\varepsilon \widehat{u} + p(x,\varepsilon)\widehat{v} = -\varepsilon\tau\lambda\widehat{u} \tag{1.2}$$

$$-g(x,\varepsilon)\widehat{u} + (-D_{xx} + \sigma(x,\varepsilon)D_x + q(x,\varepsilon))\widehat{v} = -\lambda\widehat{v} \tag{1.3}$$

in I, where $I = (-1,1)$ or $I = \boldsymbol{R}$. Here, L^ε is the so-called Allen-Cahn operator

$$L^\varepsilon = -\varepsilon^2 D_{xx} + \varepsilon s(x,\varepsilon)D_x + a(x,\varepsilon),$$

and $\varepsilon > 0$ is a small number. If $I = (-1,1)$, then the Neumann boundary condition $\widehat{u}_x = 0$, $\widehat{v}_x = 0$ are imposed at $x = \partial I$. See §2 for the assumptions on the coefficients. This operator L^ε has a spectral gap near $\lambda = 0$.

See Nishiura and Fujii [9] and Hale and Sakamoto [3]. See also Carr and Pego [2], de Mottoni and Schatzman [8] and Chen [1]. To explain the situation precisely, let us consider the case where the solution of the original reaction-diffusion system has a single layer at $x = 0$ for simplicity. Then $-L^\varepsilon$ has a positive eigenvalue ζ^ε near the origin, which goes to zero as $\varepsilon \to 0$, while the rest of the spectra lie in $\{\lambda \in \boldsymbol{C} : \operatorname{Re}\lambda < 0\}$ and they are separated away from the imaginary axis uniformly in $\varepsilon > 0$. Let P^ε and Q^ε be the projections in $L^2(I)$ associated with ζ^ε and $\sigma(L^\varepsilon)\backslash\{\zeta^\varepsilon\}$, respectively.

The SLEP method goes as follows. For simplicity, we put $p(x,\varepsilon) \equiv 1$. By using (1.2), one solves $Q^\varepsilon\widehat{u}$ in terms of $P^\varepsilon\widehat{u}$ and $\widehat{v}$, and by using (1.3) one solves $Q^\varepsilon\widehat{u}$ in terms of $P^\varepsilon\widehat{u}$ and $\widehat{v}$ again. Then the compatibility condition gives a scalar equation on eigenvalues called the SLEP equation, and it gives the location of the critical eigenvalue. In carrying out the argument, one should study the convergence of $(L^\varepsilon)^{-1}Q^\varepsilon\widehat{v}(x,\varepsilon)$ as ε goes to zero as is shown in §2. If necessary, one can assume $\|\widehat{v}(x,\varepsilon)\|_{H^1(I)} = 1$ without loss of generality. Since $\widehat{v}(x,\varepsilon)$ depends on ε, one needs the uniform convergence of $(L^\varepsilon)^{-1}Q^\varepsilon$ in some function space. Let $a^* = a^*(x)$ be as in §2. Nishiura and Fujii [9], Nishiura, Mimura, Ikeda and Fujii [10] and Tsujikawa [12] showed the strong convergence of $(L^\varepsilon)^{-1}Q^\varepsilon$, that is, the convergence

$$\lim_{\varepsilon\to 0}(L^\varepsilon)^{-1}Q^\varepsilon h = (a^*)^{-1}h \qquad \text{in } L^2(I) \tag{1.4}$$

for fixed $h \in L^2(I)\cap L^\infty(I)$. If the given interval I is bounded, the imbedding of $H^1(I)$ into $L^2(I)\cap L^\infty(I)$ is compact, and then (1.4) implies

$$\lim_{\varepsilon\to 0}(L^\varepsilon)^{-1}Q^\varepsilon = (a^*)^{-1} \qquad \text{in } \mathcal{L}(H^1(I), L^2(I)).$$

However, if I is unbounded, the imbedding is no longer compact, and the strong convergence theorem of $(L^\varepsilon)^{-1}Q^\varepsilon$ is insufficient, which cannot be recovered if one uses weighted normed spaces as explained in §2. A new uniform convergence theorem is needed for the application of the SLEP method if I is unbounded.

This paper shows that $(L^\varepsilon)^{-1}Q^\varepsilon$ converges to $(a^*)^{-1}$ in $\mathcal{L}(L^2(I), (H^\theta(I))')$ with any $\theta \in (0, \frac{1}{2})$. Here, $\big(H^\theta(I)\big)'$ is the dual space of $H^\theta(I)$. For the definition of the interpolation space see Lions and Magenes [7] for instance. This uniform convergence theorem enables the SLEP method to be applied to the stability analysis of singularly perturbed solutions for $I = \boldsymbol{R}$ including traveling wave solutions as in §2.

A similar uniform convergence theorem is presented for a matrix of differential operators $\boldsymbol{L}^\varepsilon$ in §4. This theorem can also apply both in the case I is a finite interval and in the case I is unbounded including $I = \boldsymbol{R}$. It allows the SLEP method to apply to the stability analysis of singularly perturbed traveling waves for three component reaction-diffusion systems. We discuss it in a general framework in §5.

2. PRELIMINARIES

In this section we explain how the SLEP method applies to singular limit linearized eigenvalue problems. First we study differential operators appearing in [9], [10] and [12]. More generally, we consider

$$L^\varepsilon = -\varepsilon^2 D_{xx} + \varepsilon s(x,\varepsilon) D_x + a(x,\varepsilon) \qquad \text{in } I. \tag{2.1}$$

Here, I is $\boldsymbol{R} = (-\infty, \infty)$ or a finite interval, say, $(-1,1)$. We assume $\varepsilon > 0$ is small, say, $\varepsilon \in (0, \varepsilon_0)$. A given function $a(x,\varepsilon)$ is smooth in x. The symbols D_x, D_{xx} stand for d/dx, $(d/dx)^2$, respectively. Define $y = x/\varepsilon$ and $\widetilde{a}(y,\varepsilon) = a(\varepsilon y, \varepsilon)$ for y in $\widetilde{I} = \{y \in \boldsymbol{R} : \varepsilon y \in I\}$. Define $\widetilde{s}(y,\varepsilon) = s(\varepsilon y, \varepsilon)$ and set

$$T^\varepsilon = -D_{yy} + \widetilde{s}(y,\varepsilon) D_y + \widetilde{a}(y,\varepsilon) \qquad \text{in } \widetilde{I}$$

for all $\varepsilon \in (0, \varepsilon_0)$. Put $\widetilde{a}(y,0) = \widetilde{a}^*(y)$, where $\widetilde{a}^*(y)$ is as in (A2), and put $\widetilde{s}(y,0) \equiv \widetilde{s}^*$, where $\widetilde{s}^*$ is a constant as in (A3). Then T^ε is defined up to $\varepsilon = 0$. Denote the spectrum sets of $-L^\varepsilon$ and $-T^\varepsilon$ by $\sigma(-L^\varepsilon)$ and $\sigma(-T^\varepsilon)$, respectively. They agree with each other for $\varepsilon \in (0, \varepsilon_0)$. Let $B(0;r) = \{\mu \in \boldsymbol{C} : |\mu| < r\}$ and define a counterclockwise circle $\Gamma(r) = \{\mu \in \boldsymbol{C} : |\mu| = r\}$ for $r > 0$. The assumptions on L^ε in §2 and §3 are as follows.

- (A1) $\lim_{\varepsilon \to 0} \|a(x,\varepsilon) - a^*(x)\|_{L^\infty(I \setminus J(\varepsilon))} = 0$ is valid, where $J(\varepsilon) = (-r(\varepsilon), r(\varepsilon))$ and $r(\varepsilon)$ is a positive number with $\lim_{\varepsilon \to 0} r(\varepsilon) = 0$. $\sup_{x \in I \setminus J(\varepsilon)} |a_x(x,\varepsilon)|$ is bounded uniformly in $\varepsilon \in (0, \varepsilon_0)$. Here, $a^*(x)$ is a bounded function that is continuous except $x = 0$ with $a^*(x) > k > 0$ for all $x \in \boldsymbol{R}$.
- (A2) $\sup_{y \in \widetilde{I}} |D_y \widetilde{a}(y,\varepsilon)|$ is bounded uniformly in $\varepsilon \in (0, \varepsilon_0)$. $\lim_{\varepsilon \to 0} \widetilde{a}(y,\varepsilon) = \widetilde{a}^*(y)$ holds uniformly on any fixed compact subset of $\boldsymbol{R}$. Here, $\widetilde{a}^*(y)$ is a smooth function. There exists a constant $m > 0$ that is independent of ε so that $\widetilde{a}(y,\varepsilon) > k$ holds for all $\varepsilon \in (0, \varepsilon_0)$, $|y| > m$.
- (A3) $s(x,\varepsilon)$ satisfies one of the following conditions.
  - (a1) $I = \boldsymbol{R}$ and $s(x,\varepsilon) = c^\varepsilon$, where c^ε is a constant independent of x and satisfies $\lim_{\varepsilon \to 0} c^\varepsilon = c^* \in (-\infty, \infty)$. Define $\widetilde{s}^* \overset{\text{def}}{=} c^*$.
  - (a2) $I = (-1,1)$ and $L^\varepsilon u$ is equipped with the Neumann boundary condition $u_x(\pm 1) = 0$ for given $u = u(x)$. In this case $s(x,\varepsilon) \equiv 0$ is assumed, and define $\widetilde{s}^* \overset{\text{def}}{=} 0$.

(a3) $I = \boldsymbol{R}$. $|s(x,\varepsilon)|$ and $|s_x(x,\varepsilon)|$ are bounded uniformly in $\varepsilon \in (0,\varepsilon_0)$ and $x \in I$. $\lim_{\varepsilon\to 0} s(x,\varepsilon) = s^*(x)$ holds uniformly in $x \in \boldsymbol{R}$. $s^*(x)$ is a bounded continuous function on $\boldsymbol{R}$. $\lim_{\varepsilon\to 0} \widetilde{s}(y,\varepsilon) = s^*(0)$ uniformly on any compact subset of $\boldsymbol{R}$. In this case define $\widetilde{s}^* \stackrel{\text{def}}{=} s^*(0)$.

(A4) $\sigma(-T^\varepsilon) = \sigma_1 \cup \sigma_2$ for all $\varepsilon \in [0,\varepsilon_0)$. Here, $\sigma_1 = \{\zeta^\varepsilon\}$ and $\sigma_1 \subset B(0;r_0)$ holds with $r_0 < k$. Also $\sigma_2 \subset B(0;r_1)^{\mathrm{c}} \cap \{\mu \in \boldsymbol{C} : \frac{\pi}{2} + \theta_0 < |\arg\mu| \le \pi\}$ holds. Here, θ_0, r_0, r_1 ($r_0 < r_1$) are positive constants independent of ε and λ. The eigenspace associated with ζ^ε is one-dimensional. For $\varepsilon > 0$, let $\phi(x,\varepsilon)$ be the eigenfunction of $-L^\varepsilon$ associated with ζ^ε, and let $\phi(x,\varepsilon)$ be normalized in $L^2(I)$. Then

$$\lim_{\varepsilon\to 0} \|\phi(x,\varepsilon)\|_{L^1(I)} = \lim_{\varepsilon\to 0} \int_I |\phi(x,\varepsilon)|\,dx = 0$$

holds true.

(A5) For all $\varepsilon \in (0,\varepsilon_0)$, $|a(x,\varepsilon)| < M$ and $\varepsilon|a_x(x,\varepsilon)| < M$ hold for $x \in I$. Here, $M > 0$ is a constant independent of ε.

Usually $r(\varepsilon) = O(\varepsilon|\log\varepsilon|)$ as $\varepsilon \to 0$, and so $m\varepsilon \le r(\varepsilon)$ holds for small $\varepsilon > 0$. Without loss of generality, we assume $m\varepsilon \le r(\varepsilon)$ for all $\varepsilon \in (0,\varepsilon_0)$. The underground function spaces in §2 and §3 are as follows. Let $Y = L^2(I)$, $X = H^1(I)$ and let the dual space of X be denoted by X'. If $I = \boldsymbol{R}$, $X' = H^{-1}(\boldsymbol{R})$ is valid. We denote the inner product of $Y = L^2(I)$ by $(\,\cdot\,,\,\cdot\,)$ in §2 and §3. Let P^ε and Q^ε be the projections associated with σ_1 and σ_2, respectively. Define $Y_1 = P^\varepsilon Y$ and $Y_2 = Q^\varepsilon Y$. Then L^ε is completely reduced by (Y_1, Y_2). Fix positive constants r_2, r_3 independently of ε, λ so that $r_0 < r_3 < r_2 < k < r_1$ is valid. It holds that

$$P^\varepsilon \stackrel{\text{def}}{=} \frac{1}{2\pi i}\int_{\Gamma(r_2)} (\mu + L^\varepsilon)^{-1}\,d\mu, \qquad Q^\varepsilon \stackrel{\text{def}}{=} I - P^\varepsilon.$$

Define $\widetilde{Y} = L^2(\widetilde{I})$ and $\widetilde{X} = H^1(\widetilde{I})$. Let $\widetilde{X}'$ be the dual space of $\widetilde{X}$. The following projections

$$\widetilde{P}^\varepsilon \stackrel{\text{def}}{=} \frac{1}{2\pi i}\int_{\Gamma(r_2)} (\mu + T^\varepsilon)^{-1}\,d\mu, \quad \widetilde{Q}^\varepsilon \stackrel{\text{def}}{=} I - \widetilde{P}^\varepsilon$$

give a direct sum decomposition $\widetilde{Y} = \widetilde{Y}_1 \oplus \widetilde{Y}_2$ with $\widetilde{Y}_1 = \widetilde{P}^\varepsilon\widetilde{Y}$ and $\widetilde{Y}_2 = \widetilde{Q}^\varepsilon\widetilde{Y}$. We denote $\{\lambda \in \boldsymbol{C} : \operatorname{Re}\lambda \ge 0\}$ by $\boldsymbol{C}_+$.

Remark 1. The assumption (A4) implies that ζ^ε should be a real number. If one assumes in addition to (A4) that ζ^ε is a pole of order 1 for the resolvent

$(\mu + L^\varepsilon)^{-1}$ for $\mu \in \boldsymbol{C}$, then $P^\varepsilon = (\phi, \psi)^{-1}(\,\cdot\,, \psi)\phi$ is valid. Here, $\phi = \phi(x, \varepsilon)$ is as in (A4), and ψ is a non-zero function in the null space $\mathcal{N}(\zeta^\varepsilon + (L^\varepsilon)^*)$, where $(L^\varepsilon)^*$ is the formal adjoint operator given by

$$(L^\varepsilon)^* = -\varepsilon^2 D_{xx} - \varepsilon s D_x - \varepsilon s_x + a(x, \varepsilon).$$

This fact follows from $\mathcal{R}(\zeta^\varepsilon + L^\varepsilon) = (\mathcal{N}(\zeta^\varepsilon + (L^\varepsilon)^*))^\perp$. See for [11] for instance.

We consider (1.2)–(1.3) in §1. If $I = (-1, 1)$, then the Neumann boundary condition $\widehat{u}_x = 0$, $\widehat{v}_x = 0$ are imposed at $x = \partial I$. We assume either $\tau > 0$ is a given constant or $\tau = \varepsilon^{-1}$. Functions $p(x, \varepsilon)$, $g(x, \varepsilon)$, $q(x, \varepsilon)$, $\sigma(x, \varepsilon)$ and $\sigma_x(x, \varepsilon)$ are uniformly bounded in $x \in I$ and $\varepsilon \in (0, \varepsilon_0)$ with

$$\lim_{\varepsilon \to 0} \|p(x, \varepsilon) - p^*(x)\|_{L^2(I)} = 0, \quad \lim_{\varepsilon \to 0} \|q(x, \varepsilon) - q^*(x)\|_{L^2(I)} = 0, \tag{2.2}$$

$$\lim_{\varepsilon \to 0} \|g(x, \varepsilon) - g^*(x)\|_{L^2(I)} = 0, \quad \lim_{\varepsilon \to 0} \|g(x, \varepsilon) - g^*(x)\|_{H^1(I \setminus (-\frac{1}{2}, \frac{1}{2}))} = 0, \tag{2.3}$$

$$\lim_{\varepsilon \to 0} \|\sigma(x, \varepsilon) - \sigma^*(x)\|_{L^\infty(I \setminus J(\varepsilon))} = 0, \quad \lim_{\varepsilon \to 0} \|\sigma_x(x, \varepsilon) - \sigma_x^*(x)\|_{L^\infty(I \setminus J(\varepsilon))} = 0. \tag{2.4}$$

Functions $p^*(x)$, $q^*(x)$, $g^*(x)$, $\sigma^*(x)$ and $\sigma_x^*(x)$ are continuous except at $x = 0$, and belong to $H^1(I \cap (0, +\infty))$ and $H^1(I \cap (-\infty, 0))$. Assume that

$$\inf_{x \in I} \left(q^*(x) + a^*(x)^{-1} p^*(x) g^*(x) - \frac{1}{2} \sigma_x^*(x) \right) > 0. \tag{2.5}$$

Under the assumptions stated above, one can obtain a scalar equation for $\lambda \in \boldsymbol{C}_+$ by the SLEP method. The procedure is as follows. First

$$(L^\varepsilon + \varepsilon\tau\lambda)\widehat{u} = -p(x, \varepsilon)\widehat{v}. \tag{2.6}$$

Then it is necessary that

$$\widehat{u} = -(L^\varepsilon + \varepsilon\tau\lambda)^{-1} Q^\varepsilon(p(x, \varepsilon)\widehat{v}) + k_0 \phi(x, \varepsilon) \tag{2.7}$$

holds with some $k_0 \in \boldsymbol{C}$. Substituting this into (1.3), one has

$$A^{\varepsilon,\lambda}\widehat{v} \stackrel{\text{def}}{=} (-D_{xx} + \sigma(x, \varepsilon) D_x + q + \lambda)\widehat{v} + g(L^\varepsilon + \varepsilon\tau\lambda)^{-1} Q^\varepsilon(p\widehat{v}) = k_0 g \phi(x, \varepsilon). \tag{2.8}$$

Define a bilinear form

$$B(z^1, z^2) = (z_x^1, z_x^2) + \left(\sigma(x, \varepsilon) z_x^1 + (q + \lambda) z^1, z^2\right) + ((L^\varepsilon + \varepsilon\tau\lambda)^{-1} Q^\varepsilon(p z^1), g z^2). \tag{2.9}$$

for z^1, $z^2 \in X = H^1(I)$. Now one proves that this bilinear form is coercive and applies the Lax-Milgram theorem. For this purpose one should study

$g(x,\varepsilon)(L^\varepsilon+\varepsilon\tau\lambda)^{-1}Q^\varepsilon(p(x,\varepsilon)\,\cdot\,)$. The following is a strong convergence theorem due to Nishiura and Fujii [9] and Nishiura, Mimura, Ikeda and Fujii [10].

Theorem 1 ([9], [10]). *For each fixed $h(x) \in L^2(I) \cap L^\infty(I)$,*

$$\lim_{\varepsilon\to 0}(L^\varepsilon+\varepsilon\tau\lambda)^{-1}Q^\varepsilon(p(x,\varepsilon)h(x)) = (a^*)^{-1}p^*h(x) \qquad \text{in } L^2(I).$$

If I is bounded, the imbedding $H^1(I) \subset L^2(I) \cap L^\infty(I)$ is compact, and Theorem 1 gives a uniform convergence of $g(x,\varepsilon)(L^\varepsilon+\varepsilon\tau\lambda)^{-1}Q^\varepsilon(p(x,\varepsilon)\,\cdot\,)$ in $\mathcal{L}(H^1(I), L^2(I))$. However, if I is unbounded, say $I = \boldsymbol{R}$, then it is no longer compact. If one considers $B(\,\cdot\,,\,\cdot\,)$ on a space with an exponentially weighted norm:

$$H^1_\rho(\boldsymbol{R}) = \left\{ u \in H^1(\boldsymbol{R}) : \|u\|^2_{H^1_\rho(\boldsymbol{R})} = \int_{\boldsymbol{R}} \exp(\rho|x|)(|u|^2+|u_x|^2)\,dx < \infty \right\}$$

with $\rho > 0$, then the imbedding $H^1_\rho(\boldsymbol{R}) \subset L^2(\boldsymbol{R}) \cap L^\infty(\boldsymbol{R})$ is compact, and $g(x,\varepsilon)(L^\varepsilon+\varepsilon\tau\lambda)^{-1}Q^\varepsilon(p(x,\varepsilon)\,\cdot\,)$ converges to $g^*(a^*)^{-1}p^*$ in $\mathcal{L}(H^1_\rho(\boldsymbol{R}), L^2(\boldsymbol{R}))$ by virtue of Theorem 1. However, $B(\,\cdot\,,\,\cdot\,)$ is not coercive on $H^1_\rho(\boldsymbol{R})$ unfortunately. Therefore, $B(\,\cdot\,,\,\cdot\,)$ should be a bilinear form on $H^1(I)$, and the strong convergence theorem is insufficient.

To prove that $B(\,\cdot\,,\,\cdot\,)$ is coercive even when I is unbounded, the uniform convergence theorem as in Theorem 2 in §3 is useful as follows. As is shown in Remark 2,

$$\lim_{\varepsilon\to 0}\|g(x,\varepsilon)z(x) - g^*(x)z(x)\|_{H^\theta(I)} = 0 \tag{2.10}$$

holds true for $z(x) \in X$ with some $\theta \in (0, \frac{1}{2})$ and the convergence is uniform on $\{z \in X : \|z\|_X = 1\}$. Here, $X = H^1(I)$. Theorem 2 in §3 implies $(L^\varepsilon + \varepsilon\tau\lambda)^{-1}Q^\varepsilon(p(x,\varepsilon)\cdot)$ converges to $(a^*)^{-1}p^*$ in $\mathcal{L}(Y, (H^\theta(I))')$. Then $((L^\varepsilon + \varepsilon\tau\lambda)^{-1}Q^\varepsilon(p(x,\varepsilon)z), g(x,\varepsilon)z)$ converges to $\int_I (a^*)^{-1}p^*zg^*\overline{z}\,dx$ uniformly on $\|z\|_X = 1$. For all $z \in X$ with $\|z\|_X = 1$, one has

$$\operatorname{Re} B(z,z) = \|z_x\|^2_Y + \left((-\tfrac{1}{2}\sigma_x(x,\varepsilon) + q + \operatorname{Re}\lambda)|z|^2, 1\right) + \left(g(L^\varepsilon+\varepsilon\tau\lambda)^{-1}Q^\varepsilon(pz), z\right).$$

The right-hand side equals

$$\|z_x\|^2_Y + \int_I \left(-\tfrac{1}{2}\sigma^*_x + q^* + \operatorname{Re}\lambda + (a^*)^{-1}p^*g^*\right)|z|^2\,dx + \widehat{r}(\varepsilon)\|z\|^2_Y$$

for all $z \in X$, where $\widehat{r}(\varepsilon)$ satisfies $\lim_{\varepsilon\to 0}\widehat{r}(\varepsilon) = 0$. From the assumption (2.5), $B(\,\cdot\,,\,\cdot\,)$ is proved to be coercive. Thus, the Lax-Milgram theorem is applicable and it gives a bounded linear inverse operator $K^{\varepsilon,\lambda} \overset{\text{def}}{=} (A^{\varepsilon,\lambda})^{-1}$

from X' to X. $K^{\varepsilon,\lambda}$ is bounded in $\mathcal{L}(X',X)$ uniformly in ε and λ . Thus, (2.8) yields

$$\widehat{v} = k_0 K^{\varepsilon,\lambda}(g\phi). \tag{2.11}$$

From (2.7), $P^\varepsilon \widehat{u} = k_0\phi(x,\varepsilon)$ follows. Acting P^ε on (2.6), one has

$$(L^\varepsilon + \varepsilon\tau\lambda)P^\varepsilon \widehat{u} = -P^\varepsilon(p(x,\varepsilon)\widehat{v}).$$

Substituting (2.11) and $P^\varepsilon\widehat{u} = k_0\phi$ into this equation and using $k_0 \neq 0$, one obtains

$$\Lambda(\lambda)\phi \overset{\text{def}}{=} (-\varepsilon^{-1}\zeta^\varepsilon + \tau\lambda)\phi + \varepsilon^{-1}P^\varepsilon(p(x,\varepsilon)K^{\varepsilon,\lambda}(g(x,\varepsilon)\phi)) = 0. \tag{2.12}$$

Conversely, we show (2.12), (2.11) and (2.7) give (1.2) and (1.3). It suffices to prove the solvability condition for (2.6). For this purpose it suffices to show when $L^\varepsilon + \varepsilon\tau\lambda$ has zero-eigenvalue. That is, $\zeta^\varepsilon + \varepsilon\tau\lambda = 0$. Then (2.12) and (2.11) give $P^\varepsilon(p\widehat{v}) = 0$. Because $(L^\varepsilon + \varepsilon\tau\lambda)|_{Y_2}$ is invertible, (2.6) satisfies the solvability condition. Thus, one obtains a scalar equation (2.12) from a singular limit eigenvalue problem as follows.

Proposition 1. *For sufficiently small $\varepsilon > 0$, $\lambda \in \boldsymbol{C}_+$ is an eigenvalue of* (1.2)-(1.3) *if and only if λ satisfies* (2.12).

Equation (2.12) is called the SLEP equation. Theorem 2 also implies that $K^{\varepsilon,\lambda}$ converges to $K^{0,\lambda}$ in $\mathcal{L}(X',X)$ uniformly in $\lambda \in \boldsymbol{C}_+$. Here, $K^{0,\lambda} = (A^{0,\lambda})^{-1}$ and

$$A^{0,\lambda} = -D_{xx} + \sigma^*(x)D_x + q^*(x) + a^*(x)^{-1}p^*(x)g^*(x) + \lambda.$$

Under the assumption of Remark 1, define

$$\widehat{\zeta}(\varepsilon) = \varepsilon^{-1}\zeta^\varepsilon, \quad \beta(\varepsilon) = (\phi(x,\varepsilon), \psi(x,\varepsilon))^{-1},$$
$$h_1(x,\varepsilon) = \varepsilon^{-\frac{1}{2}}p(x,\varepsilon)\psi(x,\varepsilon), \quad h_2(x,\varepsilon) = \varepsilon^{-\frac{1}{2}}g(x,\varepsilon)\phi(x,\varepsilon).$$

Then

$$\Lambda(\lambda) = -\widehat{\zeta}(\varepsilon) + \tau\lambda + \beta(\varepsilon)(K^{\varepsilon,\lambda}h_2(x,\varepsilon), h_1(x,\varepsilon)).$$

In the cases as in [9], [10] and [12],

$$\widehat{\zeta}(0) \overset{\text{def}}{=} \lim_{\varepsilon\to 0}\widehat{\zeta}(\varepsilon) \ \text{ and } \beta(0) \overset{\text{def}}{=} \lim_{\varepsilon\to 0}\beta(\varepsilon)$$

exist, and $h_j(x,\varepsilon)$ converges to $k_j^*\delta(x)$ in X' as $\varepsilon \to 0$. Here, $\delta(x)$ is the Dirac function concentrated on $x = 0$, and k_j^* is a constant $(j = 1,2)$. Then sending $\varepsilon \to 0$ in (2.12), one has

$$-\widehat{\zeta}(0) + \tau\lambda + \beta(0)k_1^*k_2^*z^*(0) = 0, \tag{2.13}$$

where $z^*(x) = K^{0,\lambda}\delta(x)$ satisfies

$$\begin{aligned} \left(-D_{xx} + \sigma^* D_x + q^* + (a^*)^{-1} p^* g^* + \lambda\right) z^*(x) &= 0 \quad \text{in } I\backslash\{0\}, \\ z^*_x(+0) - z^*_x(-0) &= -1. \end{aligned}$$

Equation (2.13) is called the SLEP equation for $\varepsilon = 0$. This equation characterizes the critical eigenvalue λ, and is useful to prove stability and bifurcation of equilibrium solutions and traveling wave solutions. Using Theorem 2 in this paper, one can obtain the SLEP equation (2.13) rigorously even when $I = \boldsymbol{R}$.

Remark 2. The proof of (2.10) is as follows. First note that $g^*(x)$ belongs to $H^\theta(I)$ if $0 < \theta < \frac{1}{2}$. See [7, Theorem 11.4] for instance. Put $\widehat{g}(x,\varepsilon) = g(x,\varepsilon) - g^*(x)$. From (2.3), $\|\widehat{g}(x,\varepsilon)z(x)\|_{H^1(I\backslash(-\frac{1}{2},\frac{1}{2}))}$ converges to 0 as $\varepsilon \to 0$ uniformly in z with $\|z\|_X = 1$. Because

$$\|\widehat{g}(x,\varepsilon)z\|_{H^\theta(I)} = \|\widehat{g}(x,\varepsilon)z\|_{H^\theta(I\backslash(-\frac{1}{2},\frac{1}{2}))} + \|\widehat{g}(x,\varepsilon)z\|_{H^\theta(-\frac{1}{2},\frac{1}{2})}$$

is valid, it suffices to prove that $\|\widehat{g}(x,\varepsilon)z(x)\|_{H^1(-\frac{1}{2},\frac{1}{2})}$ converges to 0 as $\varepsilon \to 0$ uniformly in z with $\|z\|_X = 1$. From $g^*(x) \in H^\theta(-\frac{1}{2},\frac{1}{2})$ and $\lim_{\varepsilon\to 0}\|g(x,\varepsilon) - g^*\|_Y = 0$, $\{\|g(x,\varepsilon)\|_{H^\theta(-\frac{1}{2},\frac{1}{2})}\}_{0<\varepsilon<\varepsilon_0}$ turns out bounded. Using the intrinsic norm of $H^\theta(-\frac{1}{2},\frac{1}{2})$, one has

$$\begin{aligned} &\|g(x,\varepsilon)z\|^2_{H^\theta(-\frac{1}{2},\frac{1}{2})} \\ &= \|g(x,\varepsilon)z\|^2_{L^2(-\frac{1}{2},\frac{1}{2})} + \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{|g(x_1,\varepsilon)z(x_1) - g(x_2,\varepsilon)z(x_2)|^2}{|x_1 - x_2|^{1+2\theta}}\, dx_1 dx_2. \end{aligned}$$

From $\|z\|_X = 1$ and the boundedness of $g(x,\varepsilon)$ stated above, the right-hand side is bounded uniformly in $\varepsilon \in (0,\varepsilon_0)$ and $\|z\|_X = 1$. Take θ' with $0 < \theta' < \theta$. From the compact imbedding $H^\theta(-\frac{1}{2},\frac{1}{2}) \subset H^{\theta'}(-\frac{1}{2},\frac{1}{2})$, $g(x,\varepsilon)z$ converges in $H^{\theta'}(-\frac{1}{2},\frac{1}{2})$ as $\varepsilon \to 0$. The convergence limit is $g^*(x)z(x)$. This convergence is uniform in z with $\|z\|_X = 1$. Denoting θ' simply by θ, one obtains (2.10).

3. A uniform convergence theorem for differential operators

In this section, we present Theorem 2 which already appeared in §2. First we show that $\|(\lambda + L^\varepsilon)^{-1}Q^\varepsilon\|_{\mathcal{L}(Y)}$ is uniformly bounded in ε and $\lambda \in \boldsymbol{C}_+ \stackrel{\text{def}}{=} \{\lambda : \operatorname{Re}\lambda \geq 0\}$. If L^ε is self-adjoint, this fact follows immediately from (A4). However, if L^ε is not self-adjoint, it needs a proof. The following is a brief proof of this boundedness. We put

$$T^* \stackrel{\text{def}}{=} -D_{yy} + \widetilde{s}^* D_y + \widetilde{a}^*(y).$$

Lemma 1. *There exists a constant $M_1 > 0$ such that*

$$\left\|(\lambda + L^\varepsilon)^{-1} Q^\varepsilon\right\|_{\mathcal{L}(Y)} < M_1(1+|\lambda|)^{-1}$$

holds true for sufficiently small $\varepsilon > 0$ and all $\lambda \in \boldsymbol{C}_+$.

Proof. It suffices to prove

$$\left\|(\lambda + T^\varepsilon)^{-1} \widetilde{Q}^\varepsilon\right\|_{\mathcal{L}(\widetilde{Y})} < M_1(1+|\lambda|)^{-1}$$

for all small $\varepsilon > 0$ and all $\lambda \in \boldsymbol{C}_+$. Assume $\widetilde{u}(y) \in \widetilde{Y}_2$ and $\widetilde{f}(y) \in \widetilde{Y}_2$ satisfy

$$(\lambda + T^\varepsilon)\widetilde{u} = \widetilde{f} \qquad \text{in } \widetilde{I}.$$

First we prove the lemma when $|\lambda|$ is large enough. In this case the Lax-Milgram theorem implies that $(T^* + \lambda)^{-1}$ is well defined and satisfies

$$\left\|(T^* + \lambda)^{-1}\right\|_{\mathcal{L}(\widetilde{X}', \widetilde{X})} \le (\text{const.})|\lambda|^{-1}. \tag{3.1}$$

From the definitions of T^ε and T^*,

$$T^\varepsilon + \lambda = (T^* + \lambda)\left(I - (T^* + \lambda)^{-1}\left((\widetilde{s}(y,\varepsilon) - \widetilde{s}^*)D_y - \widetilde{a}(y,\varepsilon) + \widetilde{a}^*\right)\right).$$

If $|\lambda|$ is large enough,

$$\left\|(T^* + \lambda)^{-1}\left((\widetilde{s}(y,\varepsilon) - \widetilde{s}^*)D_y - \widetilde{a}(y,\varepsilon) + \widetilde{a}^*(y)\right)\right\|_{\mathcal{L}(\widetilde{X})} < \frac{1}{2}$$

and thus

$$\|(T^\varepsilon + \lambda)^{-1}\|_{\mathcal{L}(\widetilde{X}', \widetilde{X})} < 2\|(T^* + \lambda)^{-1}\|_{\mathcal{L}(\widetilde{X}', \widetilde{X})}.$$

Combining this inequality and (3.1), one proves the lemma if $|\lambda|$ is large enough.

Next we prove the lemma when λ belongs to a bounded closed subset in $(\boldsymbol{C}_+ \cap B(0; r_3)^{\mathrm{c}}) \cup \mathit{\Gamma}(r_2)$ for all sufficiently small $\varepsilon > 0$. We denote this subset by B_0, which is independent of ε. Here, r_2, r_3 are as in §2. Assume the contrary. Then there exists $(\widetilde{u}^\varepsilon, \widetilde{f}^\varepsilon)$ that satisfies

$$(\lambda^\varepsilon + T^\varepsilon)\widetilde{u}^\varepsilon = \widetilde{f}^\varepsilon \qquad \text{in } \widetilde{I} \tag{3.2}$$

with $\lambda^\varepsilon \in B_0$, $\|\widetilde{u}^\varepsilon\|_{\widetilde{Y}} = 1$ and $\lim_{\varepsilon \to 0} \|\widetilde{f}^\varepsilon\|_{\widetilde{Y}} = 0$. Multiplying (3.2) by the complex conjugate of $\widetilde{u}^\varepsilon$ and integrating the real parts over $\widetilde{I}$, one has

$$\begin{aligned}\|D_y \widetilde{u}^\varepsilon\|^2_{\widetilde{Y}} - \frac{\varepsilon}{2}\int_{\widetilde{I}} s_x(\varepsilon x, \varepsilon)|\widetilde{u}^\varepsilon(y)|^2\,dy + \int_{\widetilde{I}} (\widetilde{a}(y,\varepsilon) + \operatorname{Re}\lambda)\,|\widetilde{u}^\varepsilon(y)|^2\,dy \\ = \operatorname{Re}(\widetilde{f}^\varepsilon, \widetilde{u}^\varepsilon) \le \|\widetilde{f}^\varepsilon\|_{\widetilde{Y}}\|\widetilde{u}^\varepsilon\|_{\widetilde{Y}}.\end{aligned}$$

Thus, $\|D_y\widetilde{u}^\varepsilon\|_{\widetilde{Y}}$ is bounded uniformly in ε. From this boundedness and (3.2), $\|D_{yy}\widetilde{u}^\varepsilon\|_{\widetilde{Y}}$ turns out bounded uniformly in ε. By the Sobolev imbedding theorem, $\widetilde{u}^\varepsilon(y)$ converges in C^1 on any compact subset of $\boldsymbol{R}$ as $\varepsilon \to 0$. Let $\widetilde{u}^*(y)$ be the convergence limit. Then $\widetilde{u}^*(y)$ belongs to $L^2(\boldsymbol{R})$. If $\widetilde{u}^*(y) \not\equiv 0$, then one obtains a contradiction to the assumption (A4) by sending $\varepsilon \to 0$ in (3.2). Indeed, by taking a subsequence one can assume that λ^ε converges to $\lambda^0 \in B_0$. Multiplying (3.2) by a test function $z(x) \in \widetilde{X}$ and integrating over $\widetilde{I}$, one has

$$\int_{\widetilde{I}} D_y\widetilde{u}^\varepsilon D_y\bar{z}\,dy + \int_{\widetilde{I}} (\lambda^\varepsilon\widetilde{u}^\varepsilon + \widetilde{s}(y,\varepsilon)D_y\widetilde{u}^\varepsilon + \widetilde{a}(y,\varepsilon)\widetilde{u}^\varepsilon)\,\bar{z}\,dy = \int_{\widetilde{I}} \widetilde{f}^\varepsilon(y)\bar{z}(y)\,dy.$$

Sending $\varepsilon \to 0$, one sees that $\widetilde{u}^*(y)$ is a weak solution of

$$T^*\widetilde{u}^*(y) \stackrel{\text{def}}{=} \left(-D_{yy} + \widetilde{s}^*D_y + \widetilde{a}^*(y) + \lambda^0\right)\widetilde{u}^*(y) = 0.$$

Because $\widetilde{a}^*(y)$ is a smooth function, $\widetilde{u}^*(y)$ becomes a strong solution of $T^*\widetilde{u}^*(y) = 0$, which contradicts (A4). If $\widetilde{u}^* \equiv 0$, then one has

$$\lim_{\varepsilon\to 0}\left(\|D_y\widetilde{u}^\varepsilon\|^2_{\widetilde{Y}} + \int_{-\infty}^{\infty}(\widetilde{a}(y,\varepsilon) + \operatorname{Re}\lambda^\varepsilon)|\widetilde{u}^\varepsilon(y)|^2\,dy\right) = 0$$

and also $\lim_{\varepsilon\to 0}\widetilde{u}^\varepsilon(y) = 0$ in $C^2[-m,m]$. Thus,

$$\lim_{\varepsilon\to 0}\int_{\widetilde{I}\setminus[-m,m]}|\widetilde{u}^\varepsilon(y)|^2\,dy = 0$$

follows from this equality and (A2). This contradicts $\|\widetilde{u}^\varepsilon\|_{\widetilde{Y}} = 1$. Thus, $(\lambda + T^\varepsilon)^{-1}$ is bounded in $\mathcal{L}(\widetilde{Y})$ uniformly in $\varepsilon \in [0,\varepsilon_0)$ and $\lambda \in (\boldsymbol{C}_+ \cap B(0;r_3)^{\mathrm{c}}) \cup \Gamma(r_2)$.

Third we prove the lemma when $\lambda \in \boldsymbol{C}_+ \cap B(0;r_3)$. By using

$$\frac{1}{2\pi i}\int_{\Gamma(r_2)}(\mu-\lambda)^{-1}\,d\mu = I,$$

the projection $\widetilde{Q}^\varepsilon$ is given by

$$\widetilde{Q}^\varepsilon = \frac{1}{2\pi i}\int_{\Gamma(r_2)}(\mu-\lambda)^{-1}\,d\mu - \frac{1}{2\pi i}\int_{\Gamma(r_2)}(\mu+T^\varepsilon)^{-1}\,d\mu.$$

From

$$(\mu-\lambda)^{-1} - (\mu+T^\varepsilon)^{-1} = (\mu-\lambda)^{-1}(\lambda+T^\varepsilon)(\mu+T^\varepsilon)^{-1},$$

one has

$$Q^\varepsilon = \frac{1}{2\pi i}\int_{\Gamma(r_2)}(\mu-\lambda)^{-1}(\lambda+T^\varepsilon)(\mu+T^\varepsilon)^{-1}\,d\mu,$$

and thus

$$(\lambda + T^\varepsilon)^{-1} Q^\varepsilon = \frac{1}{2\pi i} \int_{\Gamma(r_2)} (\mu - \lambda)^{-1} (\mu + T^\varepsilon)^{-1}\, d\mu.$$

Because $(\mu + T^\varepsilon)^{-1}$ is bounded in $\mathcal{L}(\widetilde{Y})$ uniformly in $\varepsilon \in [0, \varepsilon_0)$ and $\mu \in \Gamma(r_2)$, the right-hand side is bounded uniformly in sufficiently small $\varepsilon > 0$ and all $\lambda \in \boldsymbol{C}_+ \cap B(0; r_3)$. This completes the proof. □

We begin with the following assertion.

Proposition 2. *Assume (A1)–(A5). Then*

$$(1 + |\lambda|) \left\| (\lambda + L^\varepsilon)^{-1} Q^\varepsilon - (\lambda + a^*(x))^{-1} \right\|_{\mathcal{L}(L^2(I), (H^1(I))')}$$

converges to zero as $\varepsilon \to 0$ uniformly in $\lambda \in \boldsymbol{C}_+$.

The main theorem is as follows.

Theorem 2. *Let $\theta \in (0, 1]$ be arbitrarily fixed. Assume (A1)–(A5). Then for sufficiently small $\varepsilon > 0$ and all $\lambda \in \boldsymbol{C}_+$,*

$$(\lambda + L^\varepsilon)^{-1} Q^\varepsilon = (\lambda + a^*(x))^{-1} + R_1^\varepsilon(\lambda) \qquad \text{in } \mathcal{L}(L^2(I), \left(H^\theta(I)\right)')$$

holds true, where $R_1^\varepsilon(\lambda)$ is a bounded linear operator with

$$\|R_1^\varepsilon(\lambda)\|_{\mathcal{L}(L^2(I), (H^\theta(I))')} \leq (1 + |\lambda|)^{-1}\, \widehat{r}_1(\varepsilon).$$

Here, $\widehat{r}_1(\varepsilon)$ is a positive number that is independent of λ and satisfies

$$\lim_{\varepsilon \to 0} \widehat{r}_1(\varepsilon) = 0.$$

First we prove Proposition 2, and then we will prove Theorem 2.

Proof of Proposition 2. Let $h \in Y$ be arbitrarily fixed. Set $f = Q^\varepsilon h$. Then $h = f + P^\varepsilon h$.

$$\left((\lambda + L^\varepsilon)^{-1} Q^\varepsilon - \frac{1}{\lambda + a^*(x)} \right) h = \left((\lambda + L^\varepsilon)^{-1} - \frac{1}{\lambda + a^*(x)} \right) f - \frac{P^\varepsilon h}{\lambda + a^*(x)}.$$

Note that $P^\varepsilon h = c_1 \phi(x, \varepsilon)$ is valid with $|c_1| \leq \|h\|_Y$. For any $z \in X = H^1(I)$,

$$\left| \int_I \frac{P^\varepsilon h}{\lambda + a^*(x)} z(x)\, dx \right| \leq (\text{const.}) \frac{\|h\|_Y}{|\lambda| + 1} \int_I |\phi(x) z(x)|\, dx.$$

The letter (const.) means a constant that is independent of ε and λ. Because $\int_I |\phi(x) z(x)|\, dx = (\text{const.}) \|z\|_X \int_I |\phi(x, \varepsilon)|\, dx$ and $\lim_{\varepsilon \to 0} \int_I |\phi(x, \varepsilon)|\, dx = 0$, it suffices to estimate the term $\left((\lambda + L^\varepsilon)^{-1} - (\lambda + a^*)^{-1} \right) f$. Thus, it suffices to prove the proposition for $h = f \in Y_2$. Without loss of generality, we assume f is real-valued.

Assume $f \in Y_2$ and $u \in Y_2$ satisfy $(\lambda + L^\varepsilon)u = f$ with $f \not\equiv 0$. From Lemma 1, $\|u\|_Y \le M_1(1+|\lambda|)^{-1}\|f\|_Y$. Define $\alpha(x,\varepsilon)$ so that one has

(1) $\alpha(x,\varepsilon) = a(x,\varepsilon)$ for $|x| \ge m\varepsilon$.
(2) $\alpha(x,\varepsilon)$ belongs to $C^\infty[-m\varepsilon, m\varepsilon]$ and satisfies

$$\begin{aligned} |\alpha(x,\varepsilon)| &\le M_2 < \infty && \text{for all } x \in I,\\ \varepsilon|\alpha_x(x,\varepsilon)| &\le M_2 < \infty && \text{for all } x \in [-m\varepsilon, m\varepsilon]. \end{aligned}$$

(3) $\alpha(x,\varepsilon) \ge k > 0$ for all $x \in I$.

Here, $M_2 > 0$ is a constant. From the definition of $\alpha(x,\varepsilon)$ and the assumptions on $a(x,\varepsilon)$, $\sup\limits_{x\in J(\varepsilon)} \varepsilon|\alpha_x(x,\varepsilon)|$ is bounded uniformly in ε. $\sup\limits_{x\in I\setminus J(\varepsilon)} |\alpha_x(x,\varepsilon)|$ is bounded uniformly in ε. As ε goes to zero,

$$(1+|\lambda|)^2 \big\| \,(\alpha(x,\varepsilon)+\lambda)^{-1} - (a^*(x)+\lambda)^{-1}\, \big\|_{\mathcal{L}(Y)}$$

converges to 0 uniformly in $\lambda \in \boldsymbol{C}_+$. Therefore, it suffices to show that

$$(1+|\lambda|)\big\|(L^\varepsilon+\lambda)^{-1}Q^\varepsilon - (\alpha(x,\varepsilon)+\lambda)^{-1}\,\big\|_{\mathcal{L}(Y,X')}$$

converges to 0 uniformly in $\lambda \in \boldsymbol{C}_+$. Define $w = (\alpha(x,\varepsilon)+\lambda)^{-1}f$. Then

$$u - w = -(\alpha(x,\varepsilon)+\lambda)^{-1}(L^\varepsilon - \alpha(x,\varepsilon))u.$$

For every $z(x) \in X$,

$$-{}_X\langle z, u-w\rangle_{X'} = \int_I (\alpha(x,\varepsilon)+\bar\lambda)^{-1} z(x)\,(L^\varepsilon - \alpha(x,\varepsilon))\,u\,dx = K_1 + K_2 + K_3, \tag{3.3}$$

where

$$\begin{aligned} K_1 &= \int_I (\alpha(x,\varepsilon)+\bar\lambda)^{-1} z(-\varepsilon^2 u_{xx})\,dx,\\ K_2 &= \int_I (\alpha(x,\varepsilon)+\bar\lambda)^{-1} z(-\varepsilon s(x,\varepsilon)u_x)\,dx,\\ K_3 &= \int_I (\alpha(x,\varepsilon)+\bar\lambda)^{-1} z(a(x,\varepsilon)-\alpha(x,\varepsilon))u\,dx. \end{aligned}$$

By applying the Schwarz inequality to K_3,

$$|K_3| \le \Big(\int_I |\alpha(x,\varepsilon)+\bar\lambda|^{-2}(a(x,\varepsilon)-\alpha(x,\varepsilon))^2|z|^2\,dx\Big)^{\frac12}\|u\|_Y$$

follows. From the Sobolev imbedding $H^1(I) \subset C^{\frac{1}{2}}(\bar{I})$, $\sup_{x\in I}|z(x)| \le$ (const.)$\|z\|_X$ holds. Then one has

$$\begin{aligned}
&\Big(\int_I |\alpha(x,\varepsilon)+\bar{\lambda}|^{-2}(a(x,\varepsilon)-\alpha(x,\varepsilon))^2|z|^2\,dx\Big)^{\frac{1}{2}}\\
&\le \Big(\int_I |\alpha(x,\varepsilon)+\bar{\lambda}|^{-2}(a(x,\varepsilon)-\alpha(x,\varepsilon))^2\,dx\Big)^{\frac{1}{2}}\sup_{x\in I}|z(x)|\\
&\le (\text{const.})(1+|\lambda|)^{-1}\varepsilon^{\frac{1}{2}}\|z\|_X.
\end{aligned}$$

Combining this estimate and $\|u\|_Y \le (\text{const.})(1+|\lambda|)^{-1}\|f\|_Y$, one obtains

$$|K_3| \le (\text{const.})(1+|\lambda|)^{-2}\varepsilon^{\frac{1}{2}}\|z\|_X\|f\|_Y. \tag{3.4}$$

Lemma 2. *There exists a constant M_3 such that*

$$\varepsilon\|u_x\|_Y \le M_3(1+|\lambda|)^{-\frac{1}{2}}\|f\|_Y$$

holds true for sufficiently small $\varepsilon > 0$ and all $\lambda \in \boldsymbol{C}_+$.

Proof. Multiplying $(L^\varepsilon+\lambda)u = f$ by $\bar{u}$, one gets

$$\begin{aligned}
&\varepsilon^2\|u_x\|_Y^2 + \frac{\varepsilon}{2}\int_I s_x(x,\varepsilon)|u|^2\,dx + \int_I (a(x,\varepsilon)+\operatorname{Re}\lambda)\,|u|^2\,dx\\
&\quad = \operatorname{Re}(f,u)_Y \le \|f\|_Y\|u\|_Y.
\end{aligned}$$

Combining this calculation and $\|u\|_Y \le M_1(1+|\lambda|)^{-1}\|f\|_Y$, one completes the proof of the lemma. □

Now we estimate K_2 and K_1 as follows by using this lemma. Integration by parts gives

$$\begin{aligned}
K_2 &= \varepsilon\int_I s(x,\varepsilon)(\alpha(x,\varepsilon)+\bar{\lambda})^{-1}z_x u\,dx + \varepsilon\int_{J(\varepsilon)}(s(x,\varepsilon)(\alpha(x,\varepsilon)+\bar{\lambda})^{-1})_x zu\,dx\\
&\quad + \varepsilon\int_{I\setminus J(\varepsilon)}\big(s(x,\varepsilon)(\alpha(x,\varepsilon)+\bar{\lambda})^{-1}\big)_x zu\,dx.
\end{aligned} \tag{3.5}$$

Now,

$$\Big|\int_I s(x,\varepsilon)(\alpha(x,\varepsilon)+\bar{\lambda})^{-1}z_x u\,dx\Big| \le (\text{const.})(1+|\lambda|)^{-1}\|z_x\|_Y\|u\|_Y$$

and thus,

$$\varepsilon\Big|\int_I s(x,\varepsilon)(\alpha(x,\varepsilon)+\bar{\lambda})^{-1}z_x u\,dx\Big| \le (\text{const.})\varepsilon(1+|\lambda|)^{-2}\|z\|_X\|f\|_Y$$

follows. For the second term of (3.5),

$$\varepsilon \int_{J(\varepsilon)} \left|(s(x,\varepsilon)(\alpha(x,\varepsilon)+\bar{\lambda})^{-1})_x z u\right| \, dx \leq (\text{const.}) r(\varepsilon)(1+|\lambda|)^{-1} \|z\|_X \|u\|_Y$$
$$\leq (\text{const.}) r(\varepsilon)(1+|\lambda|)^{-2} \|z\|_X \|f\|_Y.$$

For the third term of (3.5),

$$\varepsilon \int_{I \setminus J(\varepsilon)} \left|(s(x,\varepsilon)(\alpha(x,\varepsilon)+\bar{\lambda})^{-1})_x z u\right| \, dx$$
$$\leq (\text{const.})\, \varepsilon \int_{I \setminus J(\varepsilon)} (1+|\lambda|)^{-1} |z||u| \, dx \leq (\text{const.})\, \varepsilon (1+|\lambda|)^{-2} \|z\|_Y \|f\|_Y.$$

Combining the estimates stated above, one has

$$|K_2| \leq (\text{const.})(1+|\lambda|)^{-2} \max\{r(\varepsilon), \varepsilon\} \|z\|_X \|f\|_Y.$$

Finally, we estimate K_1 as follows:

$$\begin{aligned} K_1 &= \int_I \varepsilon \left((\alpha(x,\varepsilon)+\bar{\lambda})^{-1} z\right)_x \varepsilon u_x \, dx \\ &= \int_I \varepsilon (\alpha(x,\varepsilon)+\bar{\lambda})^{-1} z_x \varepsilon u_x \, dx + \int_I \varepsilon \left((\alpha(x,\varepsilon)+\bar{\lambda})^{-1}\right)_x z \varepsilon u_x \, dx. \end{aligned} \tag{3.6}$$

For the first term of (3.6), one has

$$\int_I \varepsilon \left|(\alpha(x,\varepsilon)+\bar{\lambda})^{-1}\right| |z_x| \varepsilon |u_x| \, dx \leq (\text{const.})(1+|\lambda|)^{-1} \|z_x\|_Y \|\varepsilon u_x\|_Y$$
$$\leq (\text{const.})(1+|\lambda|)^{-1} \|z_x\|_Y \|f\|_Y$$

by virtue of Lemma 2. For the second term of (3.6),

$$\int_I \varepsilon \left((\alpha(x,\varepsilon)+\bar{\lambda})^{-1}\right)_x z \varepsilon u_x \, dx \leq \left\|\varepsilon \left((\alpha(x,\varepsilon)+\bar{\lambda})^{-1}\right)_x z\right\|_Y \|\varepsilon u_x\|_Y. \tag{3.7}$$

Here,

$$\left\|\varepsilon \left((\alpha(x,\varepsilon)+\bar{\lambda})^{-1}\right)_x z\right\|_Y^2 = \left(\int_{J(\varepsilon)} + \int_{I \setminus J(\varepsilon)}\right) \varepsilon^2 \left|\left((\alpha(x,\varepsilon)+\bar{\lambda})^{-1}\right)_x\right|^2 |z|^2 \, dx$$
$$\leq (\text{const.}) r(\varepsilon)(1+|\lambda|)^{-2} \|z\|_X^2 + \varepsilon^2 (1+|\lambda|)^{-2} \|z\|_Y^2.$$

Thus, the second term of (3.6) is no more than

$$(\text{const.}) \max\{r(\varepsilon), \varepsilon\} (1+|\lambda|)^{-1} \|z\|_Y \|f\|_Y.$$

Combining the estimates on K_1, one has

$$|K_1| \leq (\text{const.})(1+|\lambda|)^{-1} \max\{r(\varepsilon), \varepsilon\} \|z\|_X \|f\|_Y.$$

Finally, combining the estimates on K_1, K_2 and K_3, one obtains

$$|K_1| + |K_2| + |K_3| \leq (\text{const.})(1+|\lambda|)^{-1} \max\{r(\varepsilon), \varepsilon, \varepsilon^{\frac{1}{2}}\} \|z\|_X \|f\|_Y.$$

From this estimate and (3.3),

$$\|u - w\|_{X'} \leq (\text{const.})(1+|\lambda|)^{-1} \max\{r(\varepsilon), \varepsilon, \varepsilon^{\frac{1}{2}}\} \|f\|_Y$$

follows. This completes the proof of Proposition 2. □

Proof of Theorem 2. From $H^\theta(I) = \left[H^1(I), L^2(I)\right]_{1-\theta}$ and the duality theorem, $\left(H^\theta(I)\right)' = \left[L^2(I), (H^1(I))'\right]_\theta$ follows. Thus, one has

$$\|T\|_{\mathcal{L}(L^2(I),(H^\theta(I))')} \leq c \|T\|^{1-\theta}_{\mathcal{L}(L^2(I))} \|T\|^\theta_{\mathcal{L}(L^2(I),(H^1(I))')} \tag{3.8}$$

with a constant $c > 0$. Then Theorem 2 follows from Lemma 1 and Proposition 2. □

Let us go back to the arguments in §2. The SLEP method and Theorem 2 give (2.12) in §2. In the following we briefly explain how to study (2.12) for the stability of traveling waves on $I = \boldsymbol{R}$. By the definition of $\Lambda(\,\cdot\,)$, $\lim_{\lambda\to+\infty} \Lambda(\lambda) = -\infty$ follows. $K^{\varepsilon,\lambda}$ is analytic in $\lambda \in \boldsymbol{C}_+$ with

$$\frac{\partial}{\partial\lambda} K^{\varepsilon,\lambda} = -K^{\varepsilon,\lambda} \left(I - \varepsilon\tau g(L^\varepsilon + \varepsilon\tau\lambda)^{-2} Q^\varepsilon(p\,\cdot\,)\right) K^{\varepsilon,\lambda} \qquad \text{in } \mathcal{L}(X', X).$$

Thus, $\Lambda(\lambda)$ is also analytic in $\lambda \in \boldsymbol{C}_+$. Particularly,

$$\Lambda_\lambda(0)\phi = \tau\phi - \varepsilon^{-1} P^\varepsilon \left(pK^{\varepsilon,0}\left(I - \varepsilon\tau g(L^\varepsilon)^{-2} Q^\varepsilon(p\,\cdot\,)\right), K^{\varepsilon,0}(g\phi)\right)$$

holds true.

In particular, we consider the case of $I = \boldsymbol{R}$ under the assumption as in Remark 1. Then

$$\Lambda_\lambda(0) = \tau - \beta(\varepsilon)\left(K^{\varepsilon,0}(I - \varepsilon\tau g(L^\varepsilon)^{-2}(p\,\cdot,\,))K^{\varepsilon,0}h_2, h_1\right). \tag{3.9}$$

If (1.2)–(1.3) is the linearized eigenvalue problem to a traveling wave, (1.2)–(1.3) has zero eigenvalue, that is, $\Lambda(0) = 0$. In this case $\Lambda_\lambda(0) \leq 0$ is always a necessary condition for stability of this traveling wave. Indeed, there exists at least one positive eigenvalue to (1.2)–(1.3) if $\Lambda_\lambda(0) > 0$ by virtue of $\lim_{\lambda\to\infty} \Lambda(\lambda) = -\infty$. If $\widehat{\zeta}(0) \stackrel{\text{def}}{=} \lim_{\varepsilon\to0} \widehat{\zeta}(\varepsilon)$ and $\beta(0) \stackrel{\text{def}}{=} \lim_{\varepsilon\to0} \beta(\varepsilon)$ exist, and $h_j(x,\varepsilon)$ converges to $k_j^*\delta(x)$ in X' as $\varepsilon \to 0$ as in §2, then the right-hand side of (3.9) converges to $\tau - \beta(0)k_1^*k_2^*{}_X\langle(K^{0,0})^2\delta, \delta\rangle_{X'} = \tau - \beta(0)k_1^*k_2^*\|K^{0,0}\delta\|_Y^2$ as $\varepsilon \to 0$. Thus, the SLEP method and Theorem 2 give a general necessary condition for stability of a traveling wave.

4. A uniform convergence theorem for systems of differential operators

The SLEP method is useful to study the stability of equilibrium solutions or traveling wave solutions for three component reaction-diffusion systems. In this section we present a uniform convergence theorem for this purpose. Theorem 3 in this section enables the SLEP method to apply to the cases of $I = \boldsymbol{R}$ including the stability analysis of traveling wave solutions for three component systems.

A typical example of a three-component system is as follows:

$$\begin{aligned} (U_1)_t &= \varepsilon^2 D_{xx} U_1 + U_1(1 - U_1 - a_3 U_2 - V) \\ (U_2)_t &= \varepsilon^2 D_{xx} U_2 + U_2(a_1 - a_2 U_1 - U_2 - kV) \\ V_t &= \tfrac{1}{\sigma_0} D_{xx} V + a_5 V(-a_4 + U_1 + a_6 k U_2) \end{aligned} \qquad \text{for } x \in I,\ t > 0. \tag{4.1}$$

Here, $U_j(x,t)$ $(j = 1,2)$ are two competing preys, and $V(x,t)$ is a common predator. All coefficients are positive constants. Kan-on and Mimura [6] study (4.1) for $I = (-1,1)$ with the Neumann boundary condition $(U_1)_x = (U_2)_x = V_x = 0$ for $x = \pm 1$, $t > 0$, and proved the stability of equilibrium solution with an internal layer by the SLEP method under some condition. This condition is satisfied, for instance, if one assumes $a_5 = a_6 = 1$, $a_1 a_3 < 1 < a_2 a_3$, $a_2 a_4 < a_1$, and that k, σ_0 are small enough. We assume $\varepsilon > 0$ is small enough compared with the other constants.

The following is a general singular limit eigenvalue problem to three-component systems as in [6] and [4, 5]. Find $\lambda \in \boldsymbol{C}_+$,

$$\widehat{\boldsymbol{u}}(x,\varepsilon) = {}^t(\widehat{u}_1(x,\varepsilon), \widehat{u}_2(x,\varepsilon)) \in Y$$

and $\widehat{v}(x,\varepsilon) \in L^2(I)$ that satisfy

$$\begin{aligned} (-\varepsilon^2 D_{xx} + \varepsilon s_1(x,\varepsilon) D_x + a_1(x,\varepsilon))\widehat{u}_1 + b_1(x,\varepsilon)\widehat{u}_2 + p_1(x,\varepsilon)\widehat{v} &= -\varepsilon\tau\lambda\widehat{u}_1, \\ b_2(x,\varepsilon)\widehat{u}_1 + (-\varepsilon^2 D_{xx} + \varepsilon s_2(x,\varepsilon) D_x + a_2(x,\varepsilon))\widehat{u}_2 + p_2(x,\varepsilon)\widehat{v} &= -\varepsilon\tau\lambda\widehat{u}_2, \\ -g_1(x,\varepsilon)\widehat{u}_1 - g_2(x,\varepsilon)\widehat{u}_2 + (-D_{xx} + \sigma(x,\varepsilon) D_x + q(x,\varepsilon))\widehat{v} &= -\lambda\widehat{v}. \end{aligned} \tag{4.2}$$

If $I = (-1,1)$, then the Neumann boundary conditions $\widehat{\boldsymbol{u}}_x = 0$, $\widehat{v}_x = 0$ are imposed at $x = \partial I$. We assume either $\tau > 0$ is a constant or $\tau = \varepsilon^{-1}$. Here, $\sigma(x,\varepsilon)$ and $q(x,\varepsilon)$ are as in §2. $p_j(x,\varepsilon)$ and $g_j(x,\varepsilon)$ satisfy the assumptions on $p(x,\varepsilon)$ and $g(x,\varepsilon)$ in §2 by replacing p^* by p_j^* and g^* by g_j^* for $j = 1,2$. Both $p_j^*(x)$ and $g_j^*(x)$ are bounded continuous functions except $x = 0$, and belong to $H^1(I \cap (0,\infty))$ and $H^1(I \cap (-\infty,0))$ for $j = 1,2$.

First we study a system of differential operators appearing in [6] and [4, 5]. More generally we consider

$$\boldsymbol{L}^{\varepsilon} = \begin{pmatrix} -\varepsilon^2 D_{xx} + \varepsilon s_1(x,\varepsilon) D_x + a_1(x,\varepsilon) & b_1(x,\varepsilon) \\ b_2(x,\varepsilon) & -\varepsilon^2 D_{xx} + \varepsilon s_2(x,\varepsilon) D_x + a_2(x,\varepsilon) \end{pmatrix} \tag{4.3}$$

for $x \in I$, where $I = (-1, 1)$ or $\boldsymbol{R}$. Here, ε is small enough, say, $\varepsilon \in (0, \varepsilon_0)$. The assumptions are as follows. Put $y = x/\varepsilon$ and

$$\widetilde{a}_j(y,\varepsilon) = a_j(\varepsilon y, \varepsilon), \quad \widetilde{b}_j(y,\varepsilon) = b_j(\varepsilon y, \varepsilon), \quad \widetilde{s}_j(y,\varepsilon) = s_j(\varepsilon y, \varepsilon)$$

for $y \in \widetilde{I} = \{y \in \boldsymbol{R} \,|\, \varepsilon y \in I\}$ and for $j = 1, 2$. Define

$$A(x,\varepsilon) = \begin{pmatrix} a_1(x,\varepsilon) & b_1(x,\varepsilon) \\ b_2(x,\varepsilon) & a_2(x,\varepsilon) \end{pmatrix}, \quad \widetilde{A}(y,\varepsilon) = \begin{pmatrix} \widetilde{a}_1(y,\varepsilon) & \widetilde{b}_1(y,\varepsilon) \\ \widetilde{b}_2(y,\varepsilon) & \widetilde{a}_2(y,\varepsilon) \end{pmatrix},$$

for $x \in I$ and $y \in \widetilde{I}$. We assume that $A(x,\varepsilon)$ is smooth in $x \in I$. Define

$$\boldsymbol{T}^{\varepsilon} = \begin{pmatrix} -D_{yy} + \widetilde{s}_1(y,\varepsilon) D_y + \widetilde{a}_1(y,\varepsilon) & \widetilde{b}_1(y,\varepsilon) \\ \widetilde{b}_2(y,\varepsilon) & -D_{yy} + \widetilde{s}_2(y,\varepsilon) D_y + \widetilde{a}_2(y,\varepsilon) \end{pmatrix},$$

and

$$\boldsymbol{T}_0^{\varepsilon} = \begin{pmatrix} -D_{yy} + \widetilde{a}_1(y,\varepsilon) & \widetilde{b}_1(y,\varepsilon) \\ \widetilde{b}_2(y,\varepsilon) & -D_{yy} + \widetilde{a}_2(y,\varepsilon) \end{pmatrix}.$$

We define $T^* = T^{\varepsilon}|_{\varepsilon=0}$ and $T_0^* = T_0^{\varepsilon}|_{\varepsilon=0}$ as

$$\boldsymbol{T}^{*} = \begin{pmatrix} -D_{yy} + \widetilde{s}_1^* D_y + \widetilde{a}_1^*(y) & \widetilde{b}_1^*(y) \\ \widetilde{b}_2^*(y) & -D_{yy} + \widetilde{s}_2^* D_y + \widetilde{a}_2^*(y) \end{pmatrix},$$

$$\boldsymbol{T}_0^{*} = \begin{pmatrix} -D_{yy} + \widetilde{a}_1^*(y) & \widetilde{b}_1^*(y) \\ \widetilde{b}_2^*(y) & -D_{yy} + \widetilde{a}_2^*(y) \end{pmatrix}.$$

Here, $\widetilde{a}_1^*(y)$, $\widetilde{a}_2^*(y)$, $\widetilde{b}_1^*(y)$ and $\widetilde{b}_2^*(y)$ are as in (B2) and $\widetilde{s}_1^*$, $\widetilde{s}_2^*$ are as in (B3).

In §4 and §5 let Y be $L^2(I) \times L^2(I)$. The following properties (B1)–(B6) hold true for $\boldsymbol{L}^{\varepsilon}$ appearing in [6] and [4, 5]. We assume (B1)–(B6) in §4 and §5.

(B1) For $j = 1, 2$, $\lim_{\varepsilon \to 0} \|a_j(x,\varepsilon) - a_j^*(x)\|_{L^\infty(I \setminus J(\varepsilon))}$ and $\lim_{\varepsilon \to 0} \|b_j(x,\varepsilon) - b_j^*(x)\|_{L^\infty(I \setminus J(\varepsilon))}$ are valid. $\sup_{x \in I \setminus J(\varepsilon)} |A_x(x,\varepsilon)|$ is uniformly bounded in $\varepsilon \in (0, \varepsilon_0)$. Here, $J(\varepsilon)$ is as in §2, and $a_j^*(x)$, $b_j^*(x)$ are bounded functions that are continuous except $x = 0$ with

$$\inf_{x \in I} a_j^*(x) > 0 \quad \text{for } j = 1, 2,$$

$$\inf_{x\in I,\lambda\in \boldsymbol{C}_+}\left|\det\begin{pmatrix}\lambda+a_1^*(x) & b_1^*(x)\\ b_2^*(x) & \lambda+a_2^*(x)\end{pmatrix}\right|>0.$$

(B2) $\sup_{y\in\widetilde{I}}|D_y\widetilde{a}_j(y,\varepsilon)|$ and $\sup_{y\in\widetilde{I}}|D_y\widetilde{b}_j(y,\varepsilon)|$ are bounded uniformly in $\varepsilon\in(0,\varepsilon_0)$ for $j=1,2$. $\lim_{\varepsilon\to 0}\widetilde{a}_j(y,\varepsilon)=\widetilde{a}_j^*(y)$ and $\lim_{\varepsilon\to 0}\widetilde{b}_j(y,\varepsilon)=\widetilde{b}_j^*(y)$ hold uniformly in y on any compact subset of $\boldsymbol{R}$. Here, $\widetilde{a}_j^*(y)$ and $\widetilde{b}_j^*(y)$ are smooth functions for $j=1,2$. There exists a constant $m>0$ that is independent of ε and λ such that one has

$$\inf_{\varepsilon\in(0,\varepsilon_0),|y|>m}\widetilde{a}_j(y,\varepsilon)>0\qquad\text{for } j=1,2,$$
$$\inf_{\lambda\in\boldsymbol{C}_+,\varepsilon\in(0,\varepsilon_0),|y|>m}\left|\det\left(\lambda+\widetilde{A}(y,\varepsilon)\right)\right|>0.$$

(B3) $s_j(x,\varepsilon)$ satisfies one of the following three conditions.

(b1) $I=\boldsymbol{R}$ and $s_1(x,\varepsilon)=s_2(x,\varepsilon)=c^\varepsilon$. Here, c^ε is a constant with $\lim_{\varepsilon\to 0}c^\varepsilon=c^*\in(-\infty,\infty)$. Define $\widetilde{s}_1^*=\widetilde{s}_2^*=c^*$.

(b2) $I=(-1,1)$ and $\boldsymbol{L}^\varepsilon\widehat{\boldsymbol{u}}$ is equipped with the Neumann boundary condition $\widehat{\boldsymbol{u}}_x(\pm1)=0$ for $\widehat{\boldsymbol{u}}={}^t(u_1,u_2)$. In this case $s_j(x,\varepsilon)\equiv 0$ is assumed for $j=1,2$. Define $\widetilde{s}_j^*=0$ for $j=1,2$.

(b3) $I=\boldsymbol{R}$. $s_j(x,\varepsilon)$ satisfies (a3) in (A3) by replacing $s^*(x)$ and $\widetilde{s}^*$ by $s_j^*(x)$ and $\widetilde{s}_j^*$. Define $\widetilde{s}_j^*=s_j^*(0)$ for $j=1,2$.

(B4) For all $\varepsilon\in[0,\varepsilon_0)$, the spectrum set $\sigma(-\boldsymbol{T}^\varepsilon)$ satisfies (A4). The eigenspace associated with ζ^ε is one-dimensional. Let $\boldsymbol{\phi}(x,\varepsilon)={}^t(\phi_1(x,\varepsilon),\phi_2(x,\varepsilon))$ be the associated eigenfunction of $-\boldsymbol{L}^\varepsilon$ with $\|\boldsymbol{\phi}(x,\varepsilon)\|_Y=1$. Then $\boldsymbol{\phi}(x,\varepsilon)$ satisfies $\lim\limits_{\varepsilon\to 0}\|\boldsymbol{\phi}(x,\varepsilon)\|_{L^1(I)\times L^1(I)}=0$.

(B5) $a_j(x,\varepsilon)$ and $b_j(x,\varepsilon)$ satisfy (A5).

(B6) Let $\widetilde{Y}_+=L^2(m,\infty)\times L^2(m,\infty)$ and $\widetilde{Y}_-=L^2(-\infty,-m)\times L^2(-\infty,-m)$. There exist positive constants k_*, θ_1^+, θ_1^-, θ_2^+, θ_2^- that are independent of ε such that

$$\left(\boldsymbol{T}_0^\varepsilon\begin{pmatrix}w_1\\w_2\end{pmatrix},\begin{pmatrix}\theta_1^+w_1\\\theta_2^+w_2\end{pmatrix}\right)_{\widetilde{Y}_+}\ge k_*\left\|\begin{pmatrix}w_1\\w_2\end{pmatrix}\right\|_{\widetilde{Y}_+}^2\ \text{for all } w_j\in C_0^\infty(m,\infty),\ j=1,2,$$
$$\left(\boldsymbol{T}_0^\varepsilon\begin{pmatrix}w_1\\w_2\end{pmatrix},\begin{pmatrix}\theta_1^-w_1\\\theta_2^-w_2\end{pmatrix}\right)_{\widetilde{Y}_-}\ge k_*\left\|\begin{pmatrix}w_1\\w_2\end{pmatrix}\right\|_{\widetilde{Y}_-}^2\ \text{for all } w_j\in C_0^\infty(-\infty,-m),\ j=1,2$$

hold true for all $\varepsilon\in[0,\varepsilon_0)$.

Without loss of generality, we assume $m\varepsilon\le r(\varepsilon)$ for all $\varepsilon\in(0,\varepsilon_0)$. We define

$$A^*(x)=\begin{pmatrix}a_1^*(x) & b_1^*(x)\\ b_2^*(x) & a_2^*(x)\end{pmatrix}.$$

In §4 and §5, the underground function space is $Y = L^2(I) \times L^2(I)$, and $X = H^1(I) \times H^1(I)$, and its dual space X'. The inner product of Y is denoted by $(\,\cdot\,,\,\cdot\,)$ in §4 and §5. We use the same notation as in §2 and §3 since no confusion will occur.

Let P^ε, Q^ε be the projections associated with σ_1, σ_2 respectively. Put $Y_1 = P^\varepsilon Y$ and $Y_2 = Q^\varepsilon Y$. Then $\boldsymbol{L}^\varepsilon$ is completely reduced by (Y_1, Y_2). Set

$$P^\varepsilon \overset{\text{def}}{=} \frac{1}{2\pi i}\int_{\Gamma(r_2)} (\mu + \boldsymbol{L}^\varepsilon)^{-1}\, d\mu, \quad Q^\varepsilon \overset{\text{def}}{=} I - P^\varepsilon.$$

Here, r_2 is as in §2. For $\widetilde{Y} = L^2(\widetilde{I}) \times L^2(\widetilde{I})$, the following projections

$$\widetilde{P}^\varepsilon \overset{\text{def}}{=} \frac{1}{2\pi i}\int_{\Gamma(r_2)} (\mu + \boldsymbol{L}^\varepsilon)^{-1}\, d\mu, \quad \widetilde{Q}^\varepsilon \overset{\text{def}}{=} I - \widetilde{P}^\varepsilon$$

give $\widetilde{Y} = \widetilde{Y}_1 \oplus \widetilde{Y}_2$ with $\widetilde{Y}_1 = \widetilde{P}^\varepsilon \widetilde{Y}$ and $\widetilde{Y}_2 = \widetilde{Q}^\varepsilon \widetilde{Y}$. Define $\widetilde{X} = H^1(\widetilde{I}) \times H^1(\widetilde{I})$ and $\widetilde{X}'$ be the dual space of $\widetilde{X}$.

Remark 3. The assumption (B4) implies that ζ^ε is a real number. If one assumes in addition to (B4) that ζ^ε is a pole of order 1 for $(\mu + \boldsymbol{L}^\varepsilon)^{-1}$, then $P^\varepsilon = (\boldsymbol{\phi}, \boldsymbol{\psi})^{-1}(\,\cdot\,, \boldsymbol{\psi})\boldsymbol{\phi}$ holds true. Here, $\boldsymbol{\psi}(x,\varepsilon) = {}^t(\psi_1(x,\varepsilon), \psi_2(x,\varepsilon))$ is a non-zero function in $\mathcal{N}(\zeta^\varepsilon + (\boldsymbol{L}^\varepsilon)^*)$. The formal adjoint operator of $\boldsymbol{L}^\varepsilon$ is given by

$$(\boldsymbol{L}^\varepsilon)^* = \begin{pmatrix} -\varepsilon^2 D_{xx} - \varepsilon s_1 D_x - \varepsilon(s_1)_x + a_1(x,\varepsilon) & b_2(x,\varepsilon) \\ b_1(x,\varepsilon) & -\varepsilon^2 D_{xx} - \varepsilon^2 s_2 D_x - \varepsilon(s_2)_x + a_2(x,\varepsilon) \end{pmatrix}.$$

Lemma 3. *There exists a constant M_4 such that*

$$\left\|(\lambda + \boldsymbol{L}^\varepsilon)^{-1} Q^\varepsilon\right\|_{\mathcal{L}(Y)} < M_4(1 + |\lambda|)^{-1}$$

holds true for sufficiently small $\varepsilon > 0$ and all $\lambda \in \boldsymbol{C}_+$.

Proof. The proof is almost parallel to the proof of Lemma 1, where the argument consists of three cases. We only show the second case, because the other two cases can be shown by similar arguments as in Lemma 1.

We prove the lemma when λ belongs to a compact subset in $(\boldsymbol{C}_+ \cap B(0; r_3)^{\mathrm{c}}) \cup \Gamma(r_2)$ for all sufficiently small $\varepsilon > 0$. We denote this subset by B_0, which is independent of ε. Here, r_2, r_3 are as in §2. Assume the contrary, then there exists $(\widetilde{\boldsymbol{u}}^\varepsilon, \widetilde{\boldsymbol{f}}^\varepsilon)$ that satisfy

$$(\lambda^\varepsilon + \boldsymbol{T}^\varepsilon)\widetilde{\boldsymbol{u}}^\varepsilon = \widetilde{\boldsymbol{f}}^\varepsilon \qquad \text{in } \widetilde{I} \tag{4.4}$$

with $\lambda^\varepsilon \in B_0$, $\|\widetilde{\boldsymbol{u}}^\varepsilon\|_{\widetilde{Y}} = 1$ and $\lim_{\varepsilon\to 0} \|\widetilde{\boldsymbol{f}}^\varepsilon\|_{\widetilde{Y}} = 0$. Put

$$\Theta^\pm \overset{\text{def}}{=} \begin{pmatrix} \theta_1^\pm & 0 \\ 0 & \theta_2^\pm \end{pmatrix},$$

respectively. Multiplying (4.4) by the complex conjugate of $\Theta^{\pm}\widetilde{\boldsymbol{u}}^{\varepsilon}$ and integrating the real parts over $\widetilde{I}$, one has

$$\begin{aligned}\mathrm{Re}\,\left(\boldsymbol{T}_0^{\varepsilon}\widetilde{\boldsymbol{u}}^{\varepsilon},\Theta^{\pm}\widetilde{\boldsymbol{u}}^{\varepsilon}\right)+(\mathrm{Re}\,\lambda^{\varepsilon})\left(\theta_1^{\pm}\|\widetilde{u}_1\|_{\widetilde{Y}}^2+\theta_2^{\pm}\|\widetilde{u}_2\|_{\widetilde{Y}}^2\right)\\ -\frac{1}{2}\varepsilon\int_{\widetilde{I}}\left(\theta_1^{\pm}(D_xs_1)(\varepsilon y,\varepsilon)|\widetilde{u}_1^{\varepsilon}|^2+\theta_2^{\pm}(D_xs_2)(\varepsilon y,\varepsilon)|\widetilde{u}_2^{\varepsilon}|^2\right)\,dy\\ \le\min\{\theta_1^{\pm},\theta_2^{\pm}\}\|\widetilde{\boldsymbol{f}}^{\varepsilon}\|_{\widetilde{Y}}\|\widetilde{\boldsymbol{u}}^{\varepsilon}\|_{\widetilde{Y}}.\end{aligned}$$

Thus, $\|(\widetilde{\boldsymbol{u}}^{\varepsilon})_y\|_{\widetilde{Y}}$ is bounded uniformly in ε. Then (4.4) implies $\|\,(\widetilde{\boldsymbol{u}}^{\varepsilon})_{yy}\,\|_{\widetilde{Y}}$ is also bounded uniformly in ε. From the Sobolev imbedding theorem, $\widetilde{\boldsymbol{u}}^{\varepsilon}$ converges to a limiting function, say $\widetilde{\boldsymbol{u}}^{*}(y)$, in C^1 on any compact subset of $\boldsymbol{R}$ as $\varepsilon\to 0$. Then $\widetilde{\boldsymbol{u}}^{*}(y)$ belongs to $L^2(\boldsymbol{R})$. From the above inequality, (B6) and $\|\widetilde{\boldsymbol{u}}^{\varepsilon}\|_{\widetilde{Y}}=1$, one obtains $\widetilde{\boldsymbol{u}}^{*}\not\equiv 0$. Sending $\varepsilon\to 0$ in (4.4), one has $(\lambda^0+\boldsymbol{T}^{*})\widetilde{\boldsymbol{u}}^{*}=\boldsymbol{0}$, which contradicts the assumption (B4). Indeed, by taking a subsequence we can assume that λ^{ε} converges to $\lambda^0\in B_0$ as $\varepsilon\to 0$. Multiplying (4.4) by a test function in $\widetilde{X}$, integrating over $\widetilde{I}$ and sending $\varepsilon\to 0$, one sees that $\widetilde{\boldsymbol{u}}^{*}(y)$ is a weak solution of $\left(\lambda^0+\boldsymbol{T}^{*}\right)\widetilde{\boldsymbol{u}}^{*}=0$. Because $\widetilde{a}_j^{*}(y)$ and $\widetilde{b}_j^{*}(y)$ are smooth functions for $j=1,2$, $\widetilde{\boldsymbol{u}}^{*}(y)$ becomes a strong solution of $\left(\lambda^0+\boldsymbol{T}^{*}\right)\widetilde{\boldsymbol{u}}^{*}=0$, which contradicts (B4). Thus, $(\lambda+\boldsymbol{T}^{\varepsilon})^{-1}$ is bounded in $\mathcal{L}(\widetilde{Y})$ uniformly in $\varepsilon\in[0,\varepsilon_0)$ and $\lambda\in(\boldsymbol{C}_+\cap B(0;r_3)^{\mathrm{c}})\cup\Gamma(r_2)$. □

Associated with Proposition 2, we have

Proposition 3. *Assume (B1)–(B6). Then*

$$\left\|(\lambda+\boldsymbol{L}^{\varepsilon})^{-1}Q^{\varepsilon}-(\lambda+A^{*})^{-1}\right\|_{\mathcal{L}(L^2(I)\times L^2(I),(H^1(I))'\times(H^1(I))')}$$

converges to zero as $\varepsilon\to 0$ uniformly in $\lambda\in\boldsymbol{C}_+$.

The main assertion is as follows.

Theorem 3. *Let $\theta\in(0,1]$ be arbitrarily fixed. Assume (B1)–(B6). Then for sufficiently small $\varepsilon>0$ and all $\lambda\in\boldsymbol{C}_+$,*

$$(\lambda+\boldsymbol{L}^{\varepsilon})^{-1}Q^{\varepsilon}=(\lambda+A^{*})^{-1}+R_2^{\varepsilon}(\lambda)\quad\text{in }\mathcal{L}(L^2(I)\times L^2(I),(H^{\theta}(I))'\times(H^{\theta}(I))')$$

holds true, where $R_2^{\varepsilon}(\lambda)$ is a bounded linear operator with

$$\|R_2^{\varepsilon}(\lambda)\|_{\mathcal{L}(L^2(I)\times L^2(I),(H^{\theta}(I))'\times(H^{\theta}(I))')}\le(1+|\lambda|)^{-1}\,\widehat{r}_2(\varepsilon)$$

and $\widehat{r}_2(\varepsilon)$ is a positive number with $\lim_{\varepsilon\to 0}\widehat{r}_2(\varepsilon)=0$. Here, $\widehat{r}_2(\varepsilon)$ is independent of λ.

Proof of Proposition 3. By a similar argument as in the proof of Proposition 2, it suffices to prove

$$\lim_{\varepsilon\to 0}\left\|(\lambda+\boldsymbol{L}^\varepsilon)^{-1}Q^\varepsilon-(\lambda+A^*)^{-1}\right\|_{\mathcal{L}(Y_2,(H^1(I))'\times(H^1(I))')}=0.$$

Assume $\boldsymbol{f}\in Y_2$ and $\boldsymbol{u}\in Y_2$ satisfy $(\lambda+\boldsymbol{L}^\varepsilon)\boldsymbol{u}=\boldsymbol{f}$, where $\boldsymbol{f}$ is a real vector-valued function that does not vanish identically. From Lemma 3, $\|\boldsymbol{u}\|_Y\le M_4(1+|\lambda|)^{-1}\|\boldsymbol{f}\|_Y$.

Let $\eta(x)\in C^\infty(\boldsymbol{R})$ be any cut-off function with

$$\begin{cases}\eta(x)\equiv 1 & \text{if } x\ge 1,\\ 0<\eta(x)<1 & \text{if } 0<x<1,\\ \eta(x)\equiv 0 & \text{if } x\le 0.\end{cases}$$

Define $B(x,\lambda,\varepsilon)$ as

$$B(x,\lambda,\varepsilon)=(\lambda+A(x,\varepsilon))^{-1}\qquad\text{if } x\notin(-m\varepsilon,m\varepsilon),$$

and

$$\begin{aligned}B(x,\lambda,\varepsilon)&=\eta\left(\frac{x}{m\varepsilon}\right)\left(-(\lambda+A(m\varepsilon,\varepsilon))^{-2}A_x(m\varepsilon,\varepsilon)(x-m\varepsilon)+(\lambda+A(m\varepsilon,\varepsilon))^{-1}\right)\\&+\eta\left(\frac{-x}{m\varepsilon}\right)\left(-(\lambda+A(-m\varepsilon,\varepsilon))^{-2}A_x(-m\varepsilon,\varepsilon)(x+m\varepsilon)+(\lambda+A(-m\varepsilon,\varepsilon))^{-1}\right),\end{aligned}$$

if $x\in(-m\varepsilon,m\varepsilon)$. Then, by using (B1) and $(-m\varepsilon,m\varepsilon)\subset J(\varepsilon)$, $B(x,\lambda,\varepsilon)$ satisfies

$$\begin{aligned}|B(x,\lambda,\varepsilon)|&\le M_5(1+|\lambda|)^{-1}&&\text{for all }\varepsilon\in(0,\varepsilon_0),\ \lambda\in\boldsymbol{C}_+,\ x\in I\\ \varepsilon|B_x(x,\lambda,\varepsilon)|&\le M_5(1+|\lambda|)^{-1}&&\text{for all }\varepsilon\in(0,\varepsilon_0),\ \lambda\in\boldsymbol{C}_+,\ x\in I\\ |B_x(x,\lambda,\varepsilon)|&\le M_5(1+|\lambda|)^{-1}&&\text{for all }\varepsilon\in(0,\varepsilon_0),\ \lambda\in\boldsymbol{C}_+,\ x\in I\backslash J(\varepsilon),\end{aligned}$$

with a constant $M_5>0$ that is independent of ε and λ. The adjoint matrix $B^*(x,\lambda,\varepsilon)\overset{\text{def}}{=}{}^t\overline{B(x,\lambda,\varepsilon)}$ also satisfies the above estimates. As ε goes to zero, $(1+|\lambda|)^2\left\|B(x,\lambda,\varepsilon)-(\lambda+A^*(x))^{-1}\right\|_{\mathcal{L}(Y)}$ converges to 0 uniformly in $\lambda\in\boldsymbol{C}_+$. Thus, it suffices to show that

$$(1+|\lambda|)\left\|(\lambda+\boldsymbol{L}^\varepsilon)^{-1}Q^\varepsilon-B(x,\lambda,\varepsilon)\right\|_{\mathcal{L}(Y_2,(H^1(I))'\times(H^1(I))')}$$

converges to 0 uniformly in $\lambda\in\boldsymbol{C}_+$. Define $\boldsymbol{w}=B(x,\lambda,\varepsilon)\boldsymbol{f}$ for $\boldsymbol{f}\in Y_2$. Then

$$-\boldsymbol{u}+\boldsymbol{w}=(-I+B(x,\lambda,\varepsilon)(\lambda+\boldsymbol{L}^\varepsilon))\,\boldsymbol{u}.\tag{4.5}$$

Thus, for every $\boldsymbol{z}\in X$,

$${}_X\langle\boldsymbol{z},-\boldsymbol{u}+\boldsymbol{w}\rangle_{X'}=\int_I(\boldsymbol{z},(-I+B(x,\lambda,\varepsilon)(\lambda+\boldsymbol{L}^\varepsilon))\boldsymbol{u})\,dx.$$

Then

$$-_X\langle \boldsymbol{z}, -\boldsymbol{u}+\boldsymbol{w}\rangle_{X'} = L_1 + L_2 + L_3$$

follows, where

$$\begin{aligned} L_1 &= \int_I (\boldsymbol{z}, (-I + B(x,\lambda,\varepsilon)(\lambda + A(x,\varepsilon)))\,\boldsymbol{u})\;dx, \\ L_2 &= \varepsilon \int_I \left(B^*(x,\lambda,\varepsilon) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} s_1 D_x u_1 \\ s_2 D_x u_2 \end{pmatrix} \right) dx, \\ L_3 &= -\varepsilon^2 \int_I (B^*(x,\lambda,\varepsilon)\boldsymbol{z}, D_{xx}\boldsymbol{u})\;dx. \end{aligned}$$

The following assertion holds true.

Lemma 4. *There exists a positive constant M_6 with*

$$\varepsilon \|D_x \boldsymbol{u}\|_Y \le M_6 (1+|\lambda|)^{-\frac{1}{2}} \|\boldsymbol{f}\|_Y,$$

for sufficiently small $\varepsilon > 0$ and all $\lambda \in \boldsymbol{C}_+$.

Proof. We multiply $(\lambda + \boldsymbol{L}^\varepsilon)\boldsymbol{u} = \boldsymbol{f}$ by $\overline{\boldsymbol{u}}$, and take the real parts, and apply Lemma 3. Then by a similar argument as in the proof of Lemma 2 we complete the proof. □

By similar calculations as in the proof of Proposition 2, we finally obtain

$$\begin{aligned} |L_1| &\le (\text{const.})(1+|\lambda|)^{-2}\varepsilon^{\frac{1}{2}} \|\boldsymbol{f}\|_Y \|\boldsymbol{z}\|_X, \\ |L_2| &\le (\text{const.})(1+|\lambda|)^{-1} \max\{\varepsilon, r(\varepsilon)\} \|\boldsymbol{f}\|_Y \|\boldsymbol{z}\|_X, \\ |L_3| &\le (\text{const.})(1+|\lambda|)^{-\frac{3}{2}} \max\{\varepsilon, r(\varepsilon)\} \|\boldsymbol{f}\|_Y \|\boldsymbol{z}\|_X. \end{aligned}$$

The letter (const.) means a constant that is independent of ε and λ. These estimates on L_1, L_2 and L_3 complete the proof of Proposition 3. □

Proof of Theorem 3. For any $\lambda \in \boldsymbol{C}_+$ and $\boldsymbol{h} = {}^t(h_1, h_2) \in Y$ with $\|\boldsymbol{h}\|_Y = 1$, define

$$\boldsymbol{v} = (\lambda + \boldsymbol{L}^\varepsilon)^{-1}\boldsymbol{h} - (\lambda + A^*(x))^{-1}\boldsymbol{h}$$

with $\boldsymbol{v} = {}^t(v_1, v_2)$. Then Proposition 3 implies that

$$\|v_j\|_{(H^1(I))'} \le (1+|\lambda|)^{-1} r_0(\varepsilon) \quad \text{for } j = 1, 2.$$

Here, $r_0(\varepsilon)$ satisfies $\lim_{\varepsilon\to 0} r_0(\varepsilon) = 0$ and is independent of $\lambda \in \boldsymbol{C}_+$. Lemma 3 implies that $\|v_j\|_{L^2(I)} \le M_4(1+|\lambda|)^{-1}$. Then, using

$$\|v_j\|_{(H^\theta(I))'} \le c \|v_j\|^\theta_{(H^1(I))'} \|v_j\|^{1-\theta}_{L^2(I)},$$

one has $\|v_j\|_{(H^\theta(I))'} \le (1+|\lambda|)^{-1}\widehat{r}_2(\varepsilon)$. Here, $\widehat{r}_2(\varepsilon)$ satisfies $\lim_{\varepsilon\to 0}\widehat{r}_2(\varepsilon) = 0$ and is independent of $\lambda \in \boldsymbol{C}_+$. Thus $\|\boldsymbol{v}\|_{(H^\theta(I))'\times(H^\theta(I))'} \le (1+|\lambda|)^{-1}\widehat{r}_2(\varepsilon)$ holds true. This completes the proof of Theorem 3. $\square$

5. APPLICATION OF THEOREM 3 TO SINGULAR LIMIT EIGENVALUE PROBLEMS

This section is devoted to the application of Theorem 3 to a linearized eigenvalue problem (4.2).

We put $\boldsymbol{p}(x,\varepsilon) = {}^t(p_1(x,\varepsilon), p_2(x,\varepsilon))$, $\boldsymbol{g}(x,\varepsilon) = {}^t(g_1(x,\varepsilon), g_2(x,\varepsilon))$. We define $\boldsymbol{p}^*(x) = {}^t(p_1^*(x), p_2^*(x))$ and $\boldsymbol{g}^*(x) = {}^t(g_1^*(x), g_2^*(x))$. The operator $\boldsymbol{L}^\varepsilon$ given by (4.3) satisfies (B1)–(B6). Assume that

$$\inf_{x\in I}\left(q^*(x) + {}^t(\boldsymbol{g}^*(x))(A^*(x))^{-1}\boldsymbol{p}^*(x) - \frac{1}{2}\sigma_x^*(x)\right) > 0. \tag{5.1}$$

Then (4.2) becomes

$$(\boldsymbol{L}^\varepsilon + \varepsilon\tau\lambda)\widehat{\boldsymbol{u}} = -\boldsymbol{p}(x,\varepsilon)\widehat{v}, \tag{5.2}$$

$$-{}^t\boldsymbol{g}(x,\varepsilon)\widehat{\boldsymbol{u}} + (-D_{xx} + \sigma(x,\varepsilon)D_x + q(x,\varepsilon) + \lambda)\widehat{v} = 0. \tag{5.3}$$

Associated with

$$A^{\varepsilon,\lambda}\widehat{v} \overset{\text{def}}{=} (-D_{xx}+\sigma(x,\varepsilon)D_x+q(x,\varepsilon)+\lambda)\widehat{v}+{}^t\boldsymbol{g}(x,\varepsilon)(\boldsymbol{L}^\varepsilon+\varepsilon\tau\lambda)^{-1}Q^\varepsilon(\widehat{v}\boldsymbol{p}(x,\varepsilon)),$$

we define a bilinear form

$$B(z^1,z^2) = \left(z_x^1, z_x^2\right) + \left(\sigma(x,\varepsilon)z_x^1 + (q(x,\varepsilon)+\lambda)z^1, z^2\right) + \left((\boldsymbol{L}^\varepsilon + \varepsilon\tau\lambda)^{-1}Q^\varepsilon(\boldsymbol{p}(x,\varepsilon)z^1), z^2\boldsymbol{g}(x,\varepsilon)\right)$$

for z^1, $z^2 \in H^1(I)$. As is shown in §2, one has

$$\lim_{\varepsilon\to 0}\|z(x)\boldsymbol{g}(x,\varepsilon) - z(x)\boldsymbol{g}^*(x)\|_{H^\theta(I)\times H^\theta(I)}.$$

The convergence is uniform on $\{z \in H^1(I) : \|z\|_{H^1(I)} = 1\}$. Then Theorem 3 implies that this bilinear form is coercive and has a bounded linear operator $K^{\varepsilon,\lambda} \overset{\text{def}}{=} (A^{\varepsilon,\lambda})^{-1}$ in $\mathcal{L}((H^1(I))', H^1(I))$. This operator $K^{\varepsilon,\lambda}$ is bounded in $\mathcal{L}((H^1(I))', H^1(I))$ uniformly in $\varepsilon \in (0,\varepsilon_0)$ and $\lambda \in \boldsymbol{C}_+$. Theorem 3 also implies that

$$\lim_{\varepsilon\to 0}\|K^{\varepsilon,\lambda} - K^{0,\lambda}\|_{\mathcal{L}((H^1(I))',H^1(I))} = 0$$

uniformly in $\lambda \in \boldsymbol{C}_+$, where $K^{0,\lambda} = (A^{0,\lambda})^{-1}$ and $A^{0,\lambda}$ is given by

$$A^{0,\lambda} = -D_{xx} + \sigma^*(x)D_x + q^*(x) + {}^t(\boldsymbol{g}^*(x))A^*(x)^{-1}\boldsymbol{p}^*(x) + \lambda.$$

Associated with Proposition 1 of the scalar case, we have

Proposition 4. *For sufficiently small $\varepsilon > 0$, $\lambda \in \boldsymbol{C}_+$ is an eigenvalue of the linearized eigenvalue problem* (4.2) *if and only if λ satisfies*

$$\Lambda(\lambda)\boldsymbol{\phi} \stackrel{\text{def}}{=} (-\varepsilon^{-1}\zeta^\varepsilon + \tau\lambda)\boldsymbol{\phi}(x,\varepsilon) + \varepsilon^{-1}P^\varepsilon(\boldsymbol{p}(x,\varepsilon)K^{\varepsilon,\lambda}({}^t\boldsymbol{g}(x,\varepsilon)\boldsymbol{\phi}(x,\varepsilon))) = 0. \tag{5.4}$$

In this case $(\widehat{\boldsymbol{u}}, \widehat{v})$ is given by

$$\widehat{v} = K^{\varepsilon,\lambda}({}^t\boldsymbol{g}(x,\varepsilon)\boldsymbol{\phi}(x,\varepsilon)), \quad \widehat{\boldsymbol{u}} = \boldsymbol{\phi} - (\boldsymbol{L}^\varepsilon + \varepsilon\tau\lambda)^{-1}Q^\varepsilon(\boldsymbol{p}(x,\varepsilon)\widehat{v}).$$

We omit the proof, since the argument is the same as in Proposition 1.

Equation (5.4) is called the SLEP equation, which is a scalar equation for the critical eigenvalues. By the definition of $\Lambda(\,\cdot\,)$, $\lim_{\lambda\to+\infty}\Lambda(\lambda) = -\infty$ follows. $\Lambda(\lambda)$ is analytic in $\lambda \in \boldsymbol{C}_+$ with

$$\begin{aligned}\Lambda_\lambda(0)\boldsymbol{\phi} = \tau\boldsymbol{\phi} - \varepsilon^{-1}P^\varepsilon\Big(&\boldsymbol{p}(x,\varepsilon)K^{\varepsilon,0}\left(I - \varepsilon\tau\,{}^t\boldsymbol{g}(x,\varepsilon)(\boldsymbol{L}^\varepsilon)^{-2}Q^\varepsilon(\,\cdot\,\boldsymbol{p}(x,\varepsilon))\right) \\ &\times \left(K^{\varepsilon,0}({}^t\boldsymbol{g}(x,\varepsilon)\boldsymbol{\phi})\right)\Big).\end{aligned}$$

Then the procedure is the same as in the scalar case in §2 and §3. By a similar argument as in §3, one can study the stability of traveling waves for three component systems by using the SLEP method.

Acknowledgements. The author thanks Professor Seiji Ukai of Yokohama National University for stimulating discussions on Lemma 1. Special thanks go to Professor Yasumasa Nishiura of Hokkaido University for his comment on this paper.

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