TRAVELING CURVED FRONTS OF A MEAN CURVATURE FLOW WITH CONSTANT DRIVING FORCE

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Abstract. In this paper, traveling curved fronts by a mean curvature flow with constant driving force are studied. In the two-dimensional Euclidean space, the classification of all traveling fronts is completely carried out. It is proved that if the interface is a traveling front, then there are the following three possibilities: a line, a stationary circle, and a family of some traveling curved fronts. The explicit forms of all traveling curved fronts of this family are also obtained. It is proved that all traveling fronts can be represented by the graph except for stationary circles. Moreover we classify of all traveling fronts in the half plane with a prescribed contact angle on the boundary. For higher dimensional Euclidean spaces, the existence of rotationally symmetric traveling curved fronts is obtained.

1. Introduction

The theory of interfacial phenomena is one of the most prosperous fields in applied mathematics. Actually, if at least two different chemical or physical states coexist, interfaces appear as the phase separation boundary between them, and the analysis of their motions is quite important in such fields.

We begin with the definition of interfaces. A pair $(\Gamma, \boldsymbol{\nu})$ is called an *interface* provided that there exists a family of connected open sets $D(t)$ such that $\Gamma(t)$ is a smooth connected boundary of $D(t)$ and $\boldsymbol{\nu}$ is a outer normal vector on $\Gamma(t)$ pointing from $D(t)$ to $D(t)^{\mathrm{c}}$. Then $\Gamma(t)$ has no self-intersection points. We consider an interface $(\Gamma(t), \boldsymbol{\nu})$ which satisfies

$$V = H + k \tag{1.1}$$

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where V is a normal velocity of $\Gamma(t)$ from $D(t)$ to $D(t)^{\mathrm{c}}$, H is the mean curvature, that is, the sum of the principal curvatures, and $k \neq 0$ is a given constant. In this paper

$$k \neq 0$$

is always assumed. This equation comes from the several fields. This equation represents motions of interfaces in the Allen-Cahn equations as in [6] and [11] and in Belousov-Zhabotinsky reaction [8], and it also describes the motion of the filamentary vortex of the Ginzburg-Landau equation confined in a plane as in [5] .

We are interested in traveling curved fronts which have a characteristic shape, say, a parabola or a 'V-shape'. The equation (1.1) is one of the simplest equations which admit such traveling curved fronts. We study traveling curved fronts of 'V-shape' in $\mathbf{R}^2$, and show that (1.1) has a rotationally symmetric traveling curved front like a parabola in $\mathbf{R}^n$ with $(n \geq 3)$ in §3.

If the interface $(\Gamma(t), \boldsymbol{\nu})$ in $\mathbf{R}^2$ satisfies (1.1) and

$$\Gamma(t) = \Gamma(0) + \boldsymbol{v}t,$$

for $t > 0$ where some constant vector $\boldsymbol{v} \in \mathbf{R}^n$ $(n \geq 2)$, then this interface $\Gamma(t)$ is called a *traveling front* of $V = H + k$ with velocity $\boldsymbol{v}$, and $|\boldsymbol{v}|$ is called the speed of this traveling front.

In what follows we deal with two-dimensional cases. Consider the interface

$$\Gamma(t) = \{(x(t,\theta), y(t,\theta)) \mid \theta \in \Theta\}, \quad \boldsymbol{\nu} = {}^t\left(-\frac{y_\theta}{\sqrt{x_\theta + y_\theta}}, \frac{x_\theta}{\sqrt{x_\theta + y_\theta}}\right),$$

where Θ is an interval. If

$$\left(\frac{d}{dt}\begin{pmatrix} x(t,\theta) \\ y(t,\theta) \end{pmatrix}, \boldsymbol{\nu}\right) = H + k, \tag{1.2}$$

then $(\Gamma(t), \boldsymbol{\nu})$ is a solution to (1.1). If $\Gamma(t)$ is parametrized, we take the normal vector $\boldsymbol{\nu}$ as in the above. So, we can omit $\boldsymbol{\nu}$ in the case.

We remark that after the suitable rotation, $\boldsymbol{v}$ can be transformed into ${}^t(0, |\boldsymbol{v}|)$. Note that the velocity $\boldsymbol{v}$ and the speed $|\boldsymbol{v}|$ of traveling fronts are not uniquely defined. For example, consider the line $\Gamma(t) = \{y = x\tan\gamma\}$ with $\boldsymbol{\nu} = {}^t(-k\sin\gamma, k\cos\gamma)$ for some γ. Then $\Gamma(t) = \Gamma(0) + \boldsymbol{\nu}t$, which implies that $\boldsymbol{v} = \boldsymbol{\nu}$ and that its speed is k. However, the following equality also holds:

$$\Gamma(t) = \Gamma(0) + {}^t(0, k\sec\gamma)t,$$

which implies that $\boldsymbol{v} = {}^t(0, k\sec\gamma)$ and that its speed is $k\sec\gamma (\geq k)$ for any $\gamma \in (-\pi/2, \pi/2)$.

If $\Gamma(t)$ is represented by the graph $y = u(x,t)$ and

$$\boldsymbol{\nu} = {}^t\left(-\frac{u_x}{\sqrt{1+u_x^2}}, \frac{1}{\sqrt{1+u_x^2}}\right),$$

then the equation (1.1) is reduced to the following Cauchy problem

$$u_t = \frac{u_{xx}}{1+u_x^2} + k\sqrt{1+u_x^2} \qquad x \in \mathbf{R},\ t > 0, \tag{1.3}$$

with initial condition

$$u(x,0) = u_0(x) \qquad x \in \mathbf{R}.$$

If the solution $u(x,t)$ is a traveling front with $u(x,t) = \varphi(x) + ct$, then the equation is transformed into

$$c = \frac{\varphi_{xx}}{1+\varphi_x^2} + k\sqrt{1+\varphi_x^2}. \tag{1.4}$$

There exists a unique solution $\varphi(x)$ of (1.4) with $\varphi(0) = 0$ and $\varphi_x(0) = 0$, which is denoted by $\varphi(x;c)$ as in §2. In this paper we call $\varphi(x;c)$ a *traveling curved front* of V-shape. See also [2] and [3].

Deckelnick et al in [5] considered this problem (1.3) in the half plane $\{(x,y) \in \mathbf{R}^2 \mid x > 0\}$. They have proved the existence of a solution of (1.3) with $u(x,0) = u_0(x)$ which satisfies

$$u_x(x,0) = \frac{\sqrt{c^2-k^2}}{k} \quad \text{for all } x > R; \quad u_x(x,0) > -\frac{\sqrt{c^2-k^2}}{k} \quad \text{for all } x > 0$$

with some $R > 0$ and all solutions converge to some traveling front if $k > 0$. It is preferable if their condition for initial data can be relaxed and generalized.

Our aim in this paper is to prove the unique existence of the solution to (1.3) globally in time for general initial data, and to classify all traveling fronts in $\mathbf{R}^2$ or in the half plane $\{(x,y) \in \mathbf{R}^2 \mid x > 0\}$.

The following assertion concretely characterizes the graph of $y = \varphi(x;c)$ for $c > k$.

Proposition 1.1. *For each $k > 0$ and $c \in (k,\infty)$, $\varphi(x;c)$ is represented by a parameter θ as follows:*

$$x(\theta;c) := \frac{\theta}{c} + \frac{k}{c\sqrt{c^2-k^2}} \log \left| \frac{1+\sqrt{\frac{c+k}{c-k}}\tan\frac{\theta}{2}}{1-\sqrt{\frac{c+k}{c-k}}\tan\frac{\theta}{2}} \right|, \tag{1.5}$$

$$y(\theta;c) := -\frac{1}{c}\log\left(\frac{c\cos\theta - k}{c-k}\right), \tag{1.6}$$

for $\theta \in (-\theta_0, \theta_0)$ where $\theta_0 := \arctan(\sqrt{c^2-k^2}/k) \in (0,\pi/2)$ (see Fig. 1.1).

Note that $y = \varphi(x;c)$ has asymptotic lines as $x \to \pm\infty$.

For a curve shortening equation $V = H$, there exists the following traveling curved front, for $c > 0$,

$$y = -\frac{1}{c}\log(\cos(cx)) + ct \qquad (|x| < \frac{\pi}{2c}). \tag{1.7}$$

This traveling curved front is called the *Grim Reaper* by M. Grayson (cf. [1]). We remark that if $k = 0$, then $\theta = cx$ and $\varphi(x;c) + ct$ are equivalent to (1.7).

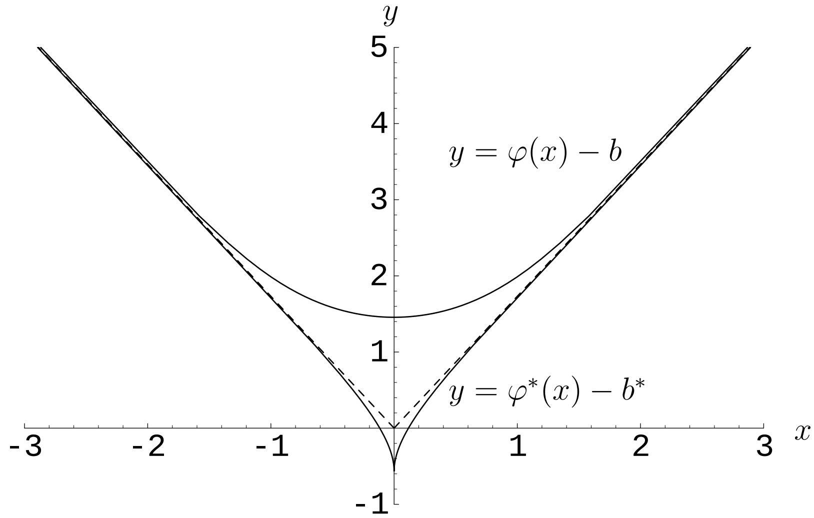

Figure 1.1: The graph of $\varphi - b$ and $\varphi^* - b^*$ in the case $c = 2,\ k = 1$ where the constants b and b^* are defined in (2.11) and (2.12).

For simplicity, we define

$$\mathrm{TF}_\pm(t;c) := \{(x(\theta;|c|), \pm y(\theta;|c|) \pm |c|t) \mid -\theta_0 < \theta < \theta_0\}.$$

We have the following two theorems.

Theorem 1.2. *Let $(\Gamma(t), \boldsymbol{\nu})$ be any smooth traveling front with velocity ${}^t(0,c)$ to (1.1) in $\mathbf{R}^2$. If $c \geq k > 0$ (resp. $c \leq k < 0$), $\Gamma(t)$ should be one of the following two types:*

(i) *a line with angle θ_0 from the horizontal,*

(ii) *a traveling front $TF_+(t;c) + (x_0, y_0)$ (resp. $TF_-(t;c) + (x_0, y_0)$) for some x_0, y_0.*

For the other cases, that is,

$$\begin{aligned} c < k \quad &\text{and} \quad k > 0, \\ c > k \quad &\text{and} \quad k < 0, \end{aligned}$$

no smooth traveling fronts exist except for the case where the interface is a stationary circle with radius $1/|k|$ and $c = 0$.

We can extend this result to traveling fronts of the same interface equation with prescribed contact angle α in the half plane $\{x > 0\}$ (see Fig. 1.2). Namely,

$$\varphi_{xx} = f(\varphi_x), \tag{1.8}$$

$$\varphi(0) = 0, \varphi_x(0) = \beta, \tag{1.9}$$

where

$$\beta = \tan\left(\frac{\pi}{2} - \alpha\right). \tag{1.10}$$

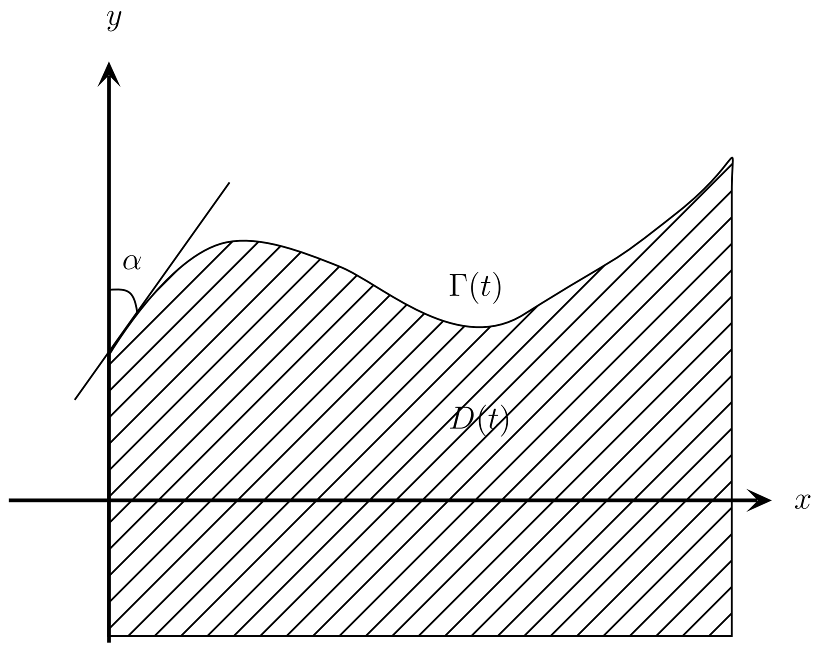

Figure 1.2: The definition of α.

By studying the system (1.8)–(1.9), we have the following theorem.

Theorem 1.3. *For any given* $\alpha \in [0, \pi)$, *assume that* $\Gamma(t)$ *has the constant contact angle* α *as the boundary condition at the* y*-axis. Consider* $V = H + k$ *in the half plane* $\{(x, y) \in \mathbf{R}^2 \mid x > 0\}$. *Then there exists a unique traveling front with the velocity* ${}^t(0, c)$ *with* $c \neq 0$ *of* $V = H + k$, *if and only if*

$$k > 0 \quad \text{and} \quad c \geq c(\alpha),$$

or

$$k < 0 \quad \text{and} \quad c \leq -c(\alpha),$$

where

$$c(\alpha) := \begin{cases} |k| & \text{if} \quad \alpha \leq \dfrac{\pi}{2}, \\ |k| \sec\left(\alpha - \dfrac{\pi}{2}\right) & \text{if} \quad \alpha > \dfrac{\pi}{2}. \end{cases}$$

Especially the solution to (1.4) corresponding to the traveling front tangential to the boundary is denoted by $\varphi^*(x; c)$. We take $\varphi^*(0; c) = 0$. $\mathrm{TF}^*(t; c)$ is defined as follows:

$$\mathrm{TF}^*_{\pm}(t; c) := \left\{ (x^*(\theta; |c|), \pm y^*(\theta; |c|) \pm |c|t) \;\middle|\; \theta_0 < \theta < \frac{\pi}{2} \right\}$$

(see Fig. 1.1).

The existence and the uniqueness of the solution to the Cauchy problem (1.3) are not trivial. The following restriction of the growth order of the initial data leads us to guarantee the unique existence of the solution and the comparison principle. See also [4] for existence without the restriction of growth orders.

Define

$$\begin{aligned} BC^1 &:= \{w(x) \in C^1(\mathbf{R}) \mid \sup_x |w(x)| < \infty,\ \sup_x |w_x(x)| < \infty\}, \\ \|w\|_{BC^1} &:= \sup_x |w(x)| + \sup_x |w_x(x)| < \infty. \end{aligned}$$

Theorem 1.4. *Assume $c > k > 0$. If $u_0(x) - \varphi(x;c)$ belongs to BC^1, then there exists a unique classical solution $u(x,t)$ of (1.3) satisfying $u(x,t) - \varphi(x;c) - ct \in BC^1$ for every $t \in (0,+\infty)$. This solution satisfies $\sup_{t\geq 0} \|u(x,t) - \varphi(x;c) - ct\|_{BC^1} < +\infty$.*

Let $u_j(x,t)$ be the solution of (1.3) with an initial value $u_j(x,0) - \varphi(x;c) \in BC^1$ for $j = 1,2$. If $u_1(x,0) \leq u_2(x,0)$ for $x \in \mathbf{R}$ holds true, then $u_1(x,t) \leq u_2(x,t)$ holds true for all $t > 0,\ x \in \mathbf{R}$.

In the forthcoming paper [9] it is shown that the traveling curved front in Proposition 1.1 is asymptotically stable and moreover it is asymptotically stable globally in space.

The higher dimensional case is discussed in §3. The existence of traveling fronts like parabola are shown.

2. Exact Representation of Traveling Fronts

It is easily seen that if the interface $(\Gamma(t), \boldsymbol{\nu})$ satisfies $V = H + k$, then $(\Gamma(t), -\boldsymbol{\nu})$ satisfies $V = H - k$. Therefore we mainly consider the case

$$k > 0$$

in this section. Since we can treat the case $k < 0$ similarly to the case $k > 0$, we will sometimes omit the case.

Before the proof of Proposition 1.1, we prepare some notations. Set

$$v = \varphi_x, \quad f(v) = c(1+v^2) - k(1+v^2)^{\frac{3}{2}}.$$

By the definition of f, (1.4) is reduced to

$$\varphi_{xx} = f(\varphi_x), \tag{2.1}$$

which implies

$$v_x = f(v). \tag{2.2}$$

By the shape of the graph f in the case $c > k > 0$ (see Case (ii) in Fig. 2.1), there exist two equilibria and there exist three orbits of (2.2), that is,

$$v \equiv \pm \frac{\sqrt{c^2 - k^2}}{k},$$

which correspond to two lines

$$y = \pm \frac{\sqrt{c^2 - k^2}}{k} x,$$

and orbits in three intervals:

$$I_1 := \Big(-\frac{\sqrt{c^2 - k^2}}{k}, \frac{\sqrt{c^2 - k^2}}{k} \Big), \quad I_2 := \Big(\frac{\sqrt{c^2 - k^2}}{k}, \infty \Big), \quad I_3 := \Big(-\infty, -\frac{\sqrt{c^2 - k^2}}{k} \Big).$$

Lemma 2.1. *The followings assertions* (i), (ii), (iii) *and* (iv) *hold true.*

(i) $\displaystyle \int \frac{dv}{f(v)} = \frac{\theta}{c} + \frac{k}{c\sqrt{c^2 - k^2}} \log \left| \frac{1 + \sqrt{\dfrac{c+k}{c-k}} \tan \dfrac{\theta}{2}}{1 - \sqrt{\dfrac{c+k}{c-k}} \tan \dfrac{\theta}{2}} \right|,$

(ii) $\displaystyle \int \frac{v}{f(v)} dv = -\frac{1}{c} \log |c \cos \theta - k|,$

(iii) $\displaystyle \int \log \left(1 + a \tan \frac{\arctan v}{2} \right) dv$
$\displaystyle \quad = \left(v + \frac{2a}{a^2 - 1} \right) \log \left(1 + a \tan \frac{\theta}{2} \right) + \frac{a}{a+1} \log \left(1 - \tan \frac{\theta}{2} \right) - \frac{a}{a-1} \log \left(1 + \tan \frac{\theta}{2} \right),$

(iv) $\displaystyle \int \arctan v \; dv = v \arctan v - \frac{1}{2} \log(1 + v^2),$

where $\theta = \arctan v$ *and* $a \in \mathbf{R}$.

Proof. We only show (ii). Setting $v = \tan \theta$, we have

$$\int \frac{v}{f(v)} dv = \int \frac{\sin \theta}{c \cos \theta - k} d\theta = -\frac{1}{c} \log |c \cos \theta - k|.$$

It is tedious but not difficult to show the other equalities. So, we omit the proof. $\square$

Proof of Proposition 1.1. The equation (1.5) follows from (i) of the above lemma. Setting

$$\Phi(\xi) := \int_0^{\xi} \frac{dv}{f(v)},$$

we have

$$\Phi(\varphi_x(x)) = x.$$

It follows from the positivity of $f(v)$ for $v \in I_1$ that there exists the inverse function Φ^{-1} from $\mathbf{R}$ to I_1 of Φ such that $\varphi_x(x) = \Phi^{-1}(x)$. Here $\Phi^{-1}(0) = \beta$ and $\Phi^{-1}(\pm\infty) = \pm\sqrt{c^2 - k^2}/k$. Then we can use $v = \varphi_x$, (or θ) to parametrize x and y. The above fact and (2.1) yield

$$\frac{d\varphi}{dv} = \frac{v}{f(v)}. \tag{2.3}$$

By (2.3) and Lemma 2.1 (ii),

$$\varphi = -\frac{1}{c} \log \left(\frac{c \cos \theta - k}{c - k} \right),$$

which implies (1.6) (see Fig. 1.1). $\square$

To prove Theorem 1.2 and Theorem 1.3, we prepare several propositions and lemmas.

Proposition 2.2. *For $c > k > 0$, there exists a unique traveling front $y = \varphi^*(x; c) + ct$ tangential to the boundary in the half plane except for translations $\{x > 0\}$. More exactly, the graph $y = \varphi^*(x; c)$ is parametrized in terms of $\theta \in (\theta_0, \pi/2)$ as*

$$x^*(\theta) := \frac{2\theta - \pi}{2c} + \frac{k}{c\sqrt{c^2 - k^2}} \log \frac{\left(\sqrt{\frac{c+k}{c-k}} \tan \frac{\theta}{2} + 1\right)\left(\sqrt{\frac{c+k}{c-k}} - 1\right)}{\left(\sqrt{\frac{c+k}{c-k}} \tan \frac{\theta}{2} - 1\right)\left(\sqrt{\frac{c+k}{c-k}} + 1\right)}, \tag{2.4}$$

$$y^*(\theta) := -\frac{1}{c} \log \frac{k - c \cos \theta}{k} \tag{2.5}$$

(*see Fig. 1.1*).

Proof. Solve (2.2) in I_2. Set

$$\Phi^*(\xi) := \int_{\infty}^{\xi} \frac{dv}{f(v)}$$

(cf. Case (ii) in Fig. 2.1). We obtain (2.4) from Lemma 2.1 (i) easily. Similarly Φ^* has an inverse function and (2.3) holds for φ^*. Integrating (2.3), we get (2.5) in I_2. $\square$

We will classify all the solutions of (2.1) for any c and k.

Lemma 2.3. *There exists a unique solution of (2.1) with $\varphi(0) = 0$ and $\varphi_x(0) = \beta$, which is denoted by $\psi(x; c, k, \beta)$. Moreover the following properties hold true:*

(i) **Case**: $c < k$ *and* $k > 0$.
For any $\beta \in \mathbf{R}$, there exists a positive constant x_1 satisfying

$$\begin{cases} |\psi_x(x; c, k, \beta)| \to \infty \quad \text{as } x \to x_1 - 0, \\ \limsup_{x \to x_1 - 0} |\psi(x; c, k, \beta)| < \infty. \end{cases} \tag{2.6}$$

(ii) **Case**: $c \geq k > 0$.
For $\beta < -\sqrt{c^2 - k^2}/k$, there exists a positive constant x_1 satisfying (2.6).
For $\beta \geq -\sqrt{c^2 - k^2}/k$, $\psi(x; c, k, \beta)$ exists globally in $\mathbf{R}$.

(iii) **Case**: $c \leq k < 0$
For $\beta > \sqrt{c^2 - k^2}/|k|$, there exists a positive constant x_1 satisfying (2.6).
For $\beta \leq \sqrt{c^2 - k^2}/|k|$, $\psi(x; c, k, \beta)$ exists globally in $\mathbf{R}$.

(iv) **Case**: $c > k$ *and* $k < 0$
For any $\beta \in \mathbf{R}$*, there exists a positive constant* x_1 *satisfying* (2.6).

Proof. The existence and uniqueness of $\psi(x; c, k, \beta)$ are easily shown. In the case (i), $f(v) > 0$ for any $v \in \mathbf{R}$ (see Case (i) in Fig. 2.1). If $c > 0$, then

$$\psi_{xx} \leq -(k-c)(1+\psi_x^2)^{3/2}$$

and if $c < 0$, then

$$\psi_{xx} \leq -k(1+\psi_x^2)^{3/2}.$$

In both cases, we can show

$$\begin{cases} \lim_{x \to x_1 - 0} \sqrt{x_1 - x}|\psi_x(x; c, k, \beta)| > 0, \\ \limsup_{x \to x_1 - 0} |\psi(x; c, k, \beta)| < \infty, \end{cases}$$

where

$$x_1 := -\int_{-\infty}^{\beta} \frac{dv}{f(v)} > 0. \tag{2.7}$$

In the case $c > k > 0$, there exist two equilibria and three orbits of (2.2), that is, two equilibria corresponding to two lines

$$\psi(x; c, k, \beta) = \pm \frac{\sqrt{c^2 - k^2}}{k} x + \beta$$

and orbits in three intervals I_i (i=1,2,3). By the positivity of f in I_1, ψ_x converges to $\sqrt{c^2-k^2}/k$ as $x \to \infty$ if $\beta \in I_1$. So, $\psi(x; c, k, \beta)$ exists globally in $x \geq 0$. By the uniqueness of solutions of (2.1),

$$\psi(x; c, k, \beta) = \varphi(x + \gamma; c) - \varphi(\gamma)$$

where

$$\gamma = \Phi^{-1}(\beta).$$

For the case $\beta \in I_2$, ψ_x also converges to $\sqrt{c^2-k^2}/k$ as $x \to \infty$ by the negativity of f. So, $\psi(x; c, k, \beta)$ also exists globally in $x \geq 0$. We also obtain

$$\psi(x; c, k, \beta) = \varphi^*(x + \gamma; c) - \varphi^*(\gamma)$$

where

$$\gamma = \Phi^{*-1}(\beta).$$

For the case $\beta \in I_3$,

$$\psi(x; c, k, \beta) = \varphi^*(-x + \gamma; c) - \varphi^*(\gamma)$$

where

$$\gamma = \Phi^{*-1}(-\beta).$$

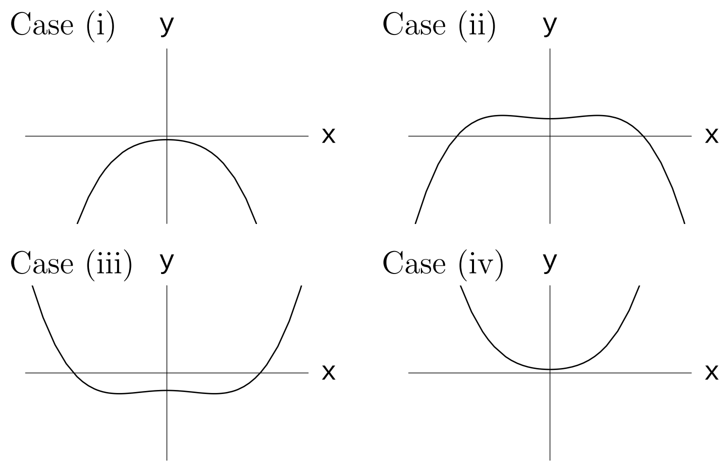

Figure 2.1: The graph of $y = f(x)$. Case (i): $c < k$, $k > 0$, Case (ii): $c > k > 0$, Case (iii): $c \leq k < 0$, and Case (iv): $c > k$ and $k < 0$.

This implies that ψ exists only on $[0, \gamma)$ and (2.6) holds.

If $c = k$, then there exist one equilibria 0 and two orbits in I_2 and I_3. If $\beta = 0$, then $\psi \equiv 0$. We can prove that (2.6) holds for $\beta < 0$ and the solution for $\beta > 0$ exists globally. The cases (iii) and (iv) can be similarly shown. □

Lemma 2.4. *Any smooth traveling front in* $\mathbf{R}^2$ *can be represented by the graph from the line which is vertical to the velocity of the traveling front, if the speed of the traveling front is not equal to zero. If the speed of traveling front is zero in* $\mathbf{R}^2$*, then this front is a stationary circle of radius* $1/|k|$.

Proof. We give a proof only for $k > 0$, since a similar argument holds true for $k < 0$. Let $\boldsymbol{v}$ be a velocity of the traveling front. We can assume that $\boldsymbol{v}$ is parallel to y-axis. We will show that $\Gamma(t)$ is represented by the graph $y = u(x, t)$. Otherwise, there exist a points $(x_1, y_1) \in \Gamma(0)$ such that $\boldsymbol{\nu}$ at (x_1, y_1) is vertical to y-axis. So, V at (x_1, y_1) should vanish, that is $H + k = 0$. It means $D(t) \cap \{(x, y) \mid x < x_1,\ |y - y_1| \leq \epsilon\} = \emptyset$ for small ϵ. We only consider the case

$$\boldsymbol{\nu}|_{(x,y)=(x_1,y_1)} = {}^t(-1, 0). \tag{2.8}$$

The opposite case can be treated similarly. The interface $(\Gamma(0), \boldsymbol{\nu})$ consists of the two parts $(\Gamma_1(0), \boldsymbol{\nu}_1)$ and $(\Gamma_2(0), \boldsymbol{\nu}_2)$ represented by the graph of φ_i such that $\varphi_1(x) > \varphi_2(x)$ for $x (> x_1)$ close to x_1 and $\lim_{x \to x_1 + 0} \varphi_{1x} = \infty$, $\lim_{x \to x_1 + 0} \varphi_{2x} = -\infty$. By (2.8), φ_1 satisfies (2.1), and φ_2 satisfies

$$\varphi_{xx} = c(1 + \varphi_x^2) + k(1 + \varphi_x^2)^{3/2}.$$

Here we can assume $c \geq 0$. By the uniqueness of solutions of ordinary differential equations,

$$\varphi_1(x) \ := \ \psi(x - x_1; c, k, \infty) + y_1,$$

$$\varphi_2(x) \quad := \quad \psi(x - x_1; c, -k, -\infty) + y_1.$$

If $c \geq k > 0$, then φ_1 exists in (x_1, ∞) and converges to the line $y = \sqrt{c^2 - k^2}(x - x_1)/k + y_1 + \gamma_1$ as $x \to \infty$ for some γ_1 by the proof of Lemma 2.3 (ii). On the other hand, there exists $x_2 (> x_1)$ such that $\lim_{x \to x_2 - 0} \varphi_{2x} = \infty$ by Lemma 2.3 (iv). Then by the argument that the interface is vending at x_2, the interface must be extended to the left-hand side again. Namely, there exists $\Gamma_3(0)$ represented by the graph of φ_3 such that $\varphi_3(x) > \varphi_2(x)$ for $x (< x_2)$ close to x_2 and $\lim_{x \to x_2 - 0} \varphi_{3x} = -\infty$. Since φ_3 satisfies (2.1), $\varphi_3(x)$ converges to the line $y = -\sqrt{c^2 - k^2}(x - x_2)/k + \varphi(x_2) + \gamma_2$ as $x \to \infty$ for some γ_2 by the proof of Lemma 2.3 (ii). This implies that the interface $\Gamma(0)$ has a self-intersection point, which contradicts the definition of the interface.

Consider the case where $0 \leq c < k$ holds true. By Lemma 2.3 (i), (iv), there exist x_2, x_3 such that $\lim_{x \to x_2 - 0} \varphi_{1x} = -\infty$ and $\lim_{x \to x_3 - 0} \varphi_{2x} = \infty$ where

$$x_2 \quad := \quad x_1 + \int_{-\infty}^{\infty} \frac{dv}{k(1+v^2)^{3/2} - c(1+v^2)}, \tag{2.9}$$

$$x_3 \quad := \quad x_1 + \int_{-\infty}^{\infty} \frac{dv}{k(1+v^2)^{3/2} + c(1+v^2)}. \tag{2.10}$$

It is easily seen that $x_2 \neq x_3$ if $c > 0$. So, by an argument similar to the above, the interface has a self-intersection point.

In the case $c = 0$, then the above $x_2 = x_3$ given in (2.9) and (2.10). This is the case where $\Gamma(t)$ is a stationary circle of radius $1/|k|$. □

Proof of Theorem 1.2. First we consider the case $c > k > 0$. Using Lemma 2.4, we can assume that the traveling front $\Gamma(t)$ is represented by the graph $y = u(x) + ct$. Then u satisfies (2.1). Lemma 2.3 immediately implies Theorem 1.2. □

Proof of Theorem 1.3. In this proof we cannot assume $k > 0$ since the region $D(t)$ is fixed. By Lemma 2.3 and the proof of Lemma 2.4, we can conclude as follows.

(i) Case: $c < k$ and $k > 0$.
A similar argument as in the proof of Lemma 2.4 implies that the derivative of $\Gamma(0)$ becomes $-\infty$ at $x = x_4$, where

$$x_4 := \int_{-\infty}^{\beta} \frac{dv}{k(1+v^2)^{3/2} - c(1+v^2)}.$$

After turning back, $\Gamma(0)$ has the derivative $-\beta$ at $x = x_4 - x_5$, where

$$x_5 := \int_{-\beta}^{\infty} \frac{dv}{k(1+v^2)^{3/2} + c(1+v^2)}.$$

By $c \neq 0$, $x_4 - x_5 \neq 0$ follows. This implies that $\Gamma(0)$ does not satisfy the boundary condition at the y-axis, or has a self-intersection point.

(ii) Case: $c \geq k > 0$.
There exists a traveling front if and only if $\beta \geq -\sqrt{c^2 - k^2}/k$.

(iii) Case: $c \le k < 0$.
There exists a traveling front if and only if $\beta \le \sqrt{c^2-k^2}/|k|$.

(iv) Case: $k < c$ and $k < 0$.
A similar argument as in (i) leads to non-existence.

The condition $\beta \ge -\sqrt{c^2-k^2}/k$ is equivalent to $c \ge c(\alpha)$ and $\beta \le \sqrt{c^2-k^2}/|k|$ is equivalent to $c \le -c(\alpha)$. This completes the proof. ∎

We have the following result for the shape of the traveling curved fronts.

Proposition 2.5. *The following relations hold true:*

$$\begin{aligned}\varphi(x;c) &= \pm\frac{\sqrt{c^2-k^2}}{k}x + b + O(|x|e^{-c\sqrt{c^2-k^2}|x|/k}), \quad \text{as } x \to \pm\infty,\\ \varphi^*(x;c) &= \pm\frac{\sqrt{c^2-k^2}}{k}x + b^* + O(|x|e^{-c\sqrt{c^2-k^2}|x|/k}), \quad \text{as } x \to \pm\infty,\end{aligned}$$

respectively, where b and b^* are constants given by

$$b := -\frac{\sqrt{c^2-k^2}}{ck}\arctan\frac{\sqrt{c^2-k^2}}{k} - \frac{1}{c}\log\frac{2(c+k)}{c} < 0, \tag{2.11}$$

$$b^* := \frac{\sqrt{c^2-k^2}}{ck}\Big(\frac{\pi}{2} - \arctan\frac{\sqrt{c^2-k^2}}{k}\Big) + \frac{1}{c}\log\frac{c(c+\sqrt{c^2-k^2})}{2(c^2-k^2)} > 0. \tag{2.12}$$

Proof. It is easily seen that

$$1 - \sqrt{\frac{c+k}{c-k}}\tan\theta = O(e^{-c\sqrt{c^2-k^2}|x|/k}).$$

by (1.5). This implies

$$\varphi_x \mp \frac{\sqrt{c^2-k^2}}{k} = O(e^{-c\sqrt{c^2-k^2}|x|/k}) \tag{2.13}$$

as $x \to \pm\infty$. Recall $\varphi_x = \Phi^{-1}(x)$. We have

$$\begin{aligned}\varphi(x)-\varphi(0) &= \int_0^x \varphi_x(x)dx = \int_0^x \Phi^{-1}(x)dx = x\Phi^{-1}(x) - \int_0^x \frac{d}{dx}\Phi^{-1}(x)dx\\ &= x\varphi_x(x) - \int_0^{\varphi_x(x)} \Phi(\xi)d\xi\end{aligned}$$

integrating by parts and changing of variables. Then

$$\begin{aligned}\varphi(x) &= x\tan\theta - \frac{1}{c}\Bigg\{\theta\tan\theta - \frac{1}{2}\log(1+\tan^2\theta) + \log(1-\frac{c+k}{c-k}\tan^2\frac{\theta}{2})\\ &\quad + \frac{k}{\sqrt{c^2-k^2}}\tan\theta\log\frac{1+\sqrt{\frac{c+k}{c-k}}\tan\frac{\theta}{2}}{1-\sqrt{\frac{c+k}{c-k}}\tan\frac{\theta}{2}} - \log\Big(1-\tan^2\frac{\theta}{2}\Big)\Bigg\}\end{aligned} \tag{2.14}$$

follows from (iii) and (iv) in Lemma 2.1. We remark that the exact form of $y(\theta; c)$ in (1.6) can also be derived from the above equality.

By (2.13) and (2.14), we obtain the first statement of this proposition.

The remainder part can be shown almost similarly. Indeed, we have

$$\begin{aligned}\varphi^*(x) - \varphi^*(0) &= \int_0^x \varphi_x^*(x)dx \\ &= \int_0^x \Phi^{*-1}(x)dx \\ &= [x\varphi_x^*(x)]_0^x - \int_0^{\varphi_x^*(x)} \Phi^*(\xi)d\xi.\end{aligned}$$

Since

$$\lim_{\theta\to\pi/2} \tan\theta \log \frac{\left(\sqrt{\frac{c+k}{c-k}}\tan\frac{\theta}{2}+1\right)\left(\sqrt{\frac{c+k}{c-k}}-1\right)}{\left(\sqrt{\frac{c+k}{c-k}}\tan\frac{\theta}{2}-1\right)\left(\sqrt{\frac{c+k}{c-k}}+1\right)} = \frac{\sqrt{c^2-k^2}}{k},$$

we obtain

$$\begin{aligned} y^*(\theta) &= \Bigg[x\tan\theta - \frac{1}{c}\Bigg\{(\theta - \frac{\pi}{2})\tan\theta - \frac{1}{2}\log(1+\tan^2\theta) + \log(\frac{c+k}{c-k}\tan^2\frac{\theta}{2} - 1) \\ &\quad - \log(\tan^2\frac{\theta}{2} - 1) + \frac{k}{\sqrt{c^2-k^2}}\tan\theta\log\frac{\left(\sqrt{\frac{c+k}{c-k}}\tan\frac{\theta}{2}+1\right)\left(\sqrt{\frac{c+k}{c-k}}-1\right)}{\left(\sqrt{\frac{c+k}{c-k}}\tan\frac{\theta}{2}-1\right)\left(\sqrt{\frac{c+k}{c-k}}+1\right)}\Bigg\}\Bigg]_{\pi/2}^{\theta}\end{aligned}$$

using Lemma 2.1 (iii) and (iv). By the above equalities, we obtain the asymptotic behavior of φ^*. $\square$

Proof of Theorem 1.4. Set $w(x,t) = u(x,t) - \varphi(x;c) - ct$ and $v(x,t) = u_x(x,t)$. Then w and v satisfy

$$w_t - \frac{\partial}{\partial x}(\arctan(w_x + \varphi_x)) - k\sqrt{1+(w_x+\varphi_x)^2} + \frac{\varphi_{xx}}{1+\varphi_x^2} + k\sqrt{1+\varphi_x^2} = 0, \quad (2.15)$$

$$v_t - \frac{\partial}{\partial x}\left(\frac{v_x}{1+v^2}\right) - k\frac{vv_x}{\sqrt{1+v^2}} = 0. \quad (2.16)$$

It follows from [7, Theorem 8.1, Remark 8.1, p. 495] that there exists a unique classical solution of (2.16) for any positive time, if $\sup_x |v(x,0)| < M$ ($M > 0$). Take $M > \sqrt{c^2-k^2}/|k|$. By the Phragmèm-Lindelöf principle [10, Theorem 10, p. 183],

$$\sup_{x,t} |v(x,t)| \le \sup_x |v(x,0)| < M \quad (2.17)$$

follows. To apply the general theory for quasi-linear parabolic equations with principal parts in divergence form, we rewrite the equation as

$$w_t - \frac{\partial}{\partial x}a(x,w_x) - k\sqrt{1+(w_x+\varphi_x)^2} + \frac{\varphi_{xx}}{1+\varphi_x^2} + k\sqrt{1+\varphi_x^2} = 0, \quad (2.18)$$

where $a(x,p) = \arctan(\eta(p) + \varphi_x(x))$. Here $\eta(p)$ is a smooth and monotone increasing function of p with

$$\eta(p) = \begin{cases} -2M & \text{if } p \le -2M, \\ p & \text{if } |p| < M, \\ 2M & \text{if } p \ge 2M. \end{cases}$$

Then [7, Theorem 8.1, Remark 8.1, p. 495] is applicable to (2.15), which implies the existence and uniqueness of classical solutions of (2.15) up to any positive time.

The comparison principle also follows from the Phragmèm-Lindelöf principle [10, Theorem 10, p. 183].

Let $\underline{w}(x,t) \equiv \inf_x w(x,0)$ and $\overline{w}(x,t) \equiv \sup_x w(x,0)$. Then they are solutions of (2.15). By the comparison principle stated above, $|w(x,t)| \le \sup_x |w(x,0)|$ holds true for all $x \in \mathbf{R}$ and $t \ge 0$. Combining this inequality and (2.17), one has $\sup_{t\ge 0} \|w(x,t)\|_{BC^1} < +\infty$. This completes the proof. $\square$

3. Higher dimensional cases

In this section we consider a traveling curved front that is rotationally symmetric in $\mathbf{R}^{n+1}$ for $n \ge 2$. Put $x = {}^t(x_1, \ldots, x_{n+1})$. We seek for a moving surface $\Gamma(t)$ with a graph $x_{n+1} = u(r,t)$, where $r = (x_1^2 + \ldots + x_n^2)^{\frac{1}{2}}$. Then $V = H + k$ becomes

$$\frac{u_t}{\sqrt{1+u_r^2}} = \frac{(n-1)u_r}{r\sqrt{1+u_r^2}} + \frac{u_{rr}}{(1+u_r^2)^{\frac{3}{2}}} + k.$$

Put $u(r,t) = \phi(r) + ct$. Without loss of generality, we put $\phi(0) = 0$. Then $\phi(r)$ should satisfy

$$\begin{gathered} c = \frac{\phi_{rr}}{1+\phi_r^2} + \frac{(n-1)\phi_r}{r} + k\sqrt{1+\phi_r^2} \quad \text{for } r > 0, \\ \phi(0) = 0, \quad \phi_r(0) = 0. \end{gathered} \tag{3.1}$$

Defining $v(r) = \phi_r(r)$, we have

$$\begin{gathered} \frac{dv}{dr} = (1+v^2) f(r,v) \quad r > 0, \\ v(0) = 0, \end{gathered} \tag{3.2}$$

where

$$f(r,v) = c - k\sqrt{1+v^2} - \frac{n-1}{r} v.$$

Then the following assertion holds true. This assertion implies that $\phi(r)$ possesses no asymptotic lines as $r \to +\infty$.

Theorem 3.1. *For $n \ge 2$, there exists a unique monotone increasing solution $\phi(r)$ to (3.1). The following relations*

$$\lim_{r\to+0} \frac{\phi(r)}{r^2} = \frac{c-k}{2n}, \tag{3.3}$$

$$\lim_{r\to+\infty} \frac{\phi(r)}{r} = \frac{\sqrt{c^2-k^2}}{k} \tag{3.4}$$

hold true. Moreover,

$$\phi_r(r) = \frac{\sqrt{c^2-k^2}}{k} - \frac{(n-1)c}{k^2}\frac{1}{r} + \frac{1}{k\sqrt{c^2-k^2}}\left(\frac{(n-1)^2c^2}{k^2} - \frac{n^2-1}{2}\right)\frac{1}{r^2} + O\left(\frac{1}{r^3}\right) \quad (3.5)$$

is valid as $r \to +\infty$.

Proof. In the r-v plane, (3.2) can be rewritten as

$$\begin{aligned} \frac{dr}{d\xi} &= r \\ \frac{dv}{d\xi} &= (1+v^2)f(r,v) \end{aligned} \qquad \text{for } \xi \in (-\infty, \infty). \quad (3.6)$$

Note that $D := \{(r,v) \mid f(r,v) > 0, v > 0, r > 0\}$ is a positively invariant set by the flow (3.6). A positive-valued solution $v(r)$ $(r > 0)$ is an orbit which connects the unstable manifold of $(r,v) = (0,0)$ and a point $(r,v) = (+\infty, \sqrt{c^2-k^2}/k)$ in the r-v plane. This fact yields (3.3) and (3.4).

Define $s = 1/r$ and $V(s) = v(r)$. Then, in the s-V plane, we have

$$\begin{aligned} \frac{ds}{d\xi} &= -s^2 \\ \frac{dV}{d\xi} &= (1+V^2)\left(c - k\sqrt{1+V^2} - (n-1)sV\right) \end{aligned} \qquad \text{for } \xi \in (-\infty, \infty). \quad (3.7)$$

As $\xi \to +\infty$, the orbit (s,V) approaches $(0, \sqrt{c^2-k^2}/k)$ along the center manifold of this point. Then the Taylor expansion of $V(s)$ up to s^2 gives (3.5). $\square$

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