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AN $(N-1)$-DIMENSIONAL CONVEX COMPACT SET GIVES AN N-DIMENSIONAL TRAVELING FRONT IN THE ALLEN–CAHN EQUATION*

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Abstract. This paper studies traveling fronts to the Allen–Cahn equation in $\mathbb{R}^N$ for $N \geq 3$. Let $(N-2)$-dimensional smooth surfaces be the boundaries of compact sets in $\mathbb{R}^{N-1}$ and assume that all principal curvatures are positive everywhere. We define an equivalence relation between them and prove that there exists a traveling front associated with a given surface and that it is asymptotically stable for given initial perturbation. The associated traveling fronts coincide up to phase transition if and only if the given surfaces satisfy the equivalence relation.

Key words. traveling front, Allen–Cahn equation, nonsymmetric

AMS subject classifications. 35C07, 35B20, 35K57

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1. Introduction.

In this paper we study the Allen–Cahn equation

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u + f(u), \qquad && \boldsymbol{x} \in \mathbb{R}^N, t > 0, \\
u(\boldsymbol{x}, 0) &= u_0, && \boldsymbol{x} \in \mathbb{R}^N.
\end{aligned}
\tag{1.1}
$$

Here $\Delta = \sum_{j=1}^{N} D_{jj}$ with $D_j = \partial/\partial x_j$ and $D_{jj} = (\partial/\partial x_j)^2$ for $1 \leq j \leq N$. Now $N \geq 3$ is a given integer, and u_0 is a given bounded and uniformly continuous function from $\mathbb{R}^N$ to $\mathbb{R}$.

The assumption on f is as follows.

(A1) $f \in C^1[-1,1]$ satisfies $f(1) = 0$, $f(-1) = 0$, $f'(1) < 0$, $f'(-1) < 0$ and

$$\int_{-1}^{1} f(s)ds > 0.$$

(A2) There exists $a_* \in (-1,1)$ such that

$$
\begin{aligned}
f(s) &< 0 \qquad \text{for all } s \in (-1, -a_*), \\
f(s) &> 0 \qquad \text{for all } s \in (-a_*, 1).
\end{aligned}
$$

The profile equation of a one-dimensional traveling front with speed k is given by

$$
\begin{gathered}
-\Phi''(x) - k\Phi'(x) - f(\Phi(x)) = 0, \qquad -\infty < x < \infty, \\
\Phi(-\infty) = 1, \ \Phi(\infty) = -1.
\end{gathered}
\tag{1.2}
$$

It is known that (1.2) has a solution Φ under (A1) and (A2), and it is unique up to translation. See [1, 2, 9, 10, 4, 3], for instance. Now (A1) gives $k > 0$. In

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particular, one has $k = \sqrt{2}a_*$ and $\Phi(x) = -\tanh(x/\sqrt{2})$ when $0 < a_* < 1$ and $f(u) = -(u+1)(u+a_*)(u-1)$.

The Allen–Cahn equation by a moving coordinate system with speed c toward the x_N-direction is given by

$$(1.3)\qquad \begin{aligned} (D_t - \Delta - cD_N)\, w - f(w) &= 0, && \boldsymbol{x} \in \mathbb{R}^N, t > 0,\\ w(\boldsymbol{x}, 0) &= u_0(\boldsymbol{x}), && \boldsymbol{x} \in \mathbb{R}^N. \end{aligned}$$

In this paper we assume $c > k$. We denote the solution of (1.3) by $w(\boldsymbol{x}, t; u_0)$. The profile equation of a traveling front in $\mathbb{R}^N$ is given by

$$(1.4)\qquad (-\Delta - cD_N)\, v - f(v) = 0, \qquad \boldsymbol{x} \in \mathbb{R}^N.$$

Here we put $\boldsymbol{x}' = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$ and $\boldsymbol{x} = (\boldsymbol{x}', x_N)$.

For the Allen–Cahn equation, multidimensional traveling fronts have been studied by many mathematicians. Two-dimensional V-form fronts are studied by Ninomiya and myself [13, 14], Hamel, Monneau, and Roquejoffre [6, 7], Haragus and Scheel [8], and so on. Cylindrically symmetric traveling fronts in $\mathbb{R}^N$ are studied by [6, 7]. Traveling fronts of pyramidal shapes and convex polyhedral shapes are studied by [15, 16, 11, 17]. See [12] for a related work. Let a compact set in $\mathbb{R}^2$ be given, and assume its smooth boundary is a curve that has a positive curvature everywhere. A traveling front associated with such a curve is studied for the Allen–Cahn equation in $\mathbb{R}^3$ by [17]. Let a surface be the boundary of a convex compact set in $\mathbb{R}^{N-1}$ and assume that all principal curvatures are positive everywhere. The purpose of this paper is to show that there exists a traveling front in the Allen–Cahn equation in $\mathbb{R}^N$ associated with such a surface by using a clear and concise argument. Since the Allen–Cahn equation is one of the simplest reaction-diffusion equations, the argument in this paper might be useful for studies on other reaction-diffusion equations or reaction-diffusion systems that admit comparison principles.

As is seen in section 4, there exists a cylindrically symmetric traveling front solution U that satisfies

$$(1.5)\qquad \begin{aligned} \left(-D_{rr} - \frac{N-2}{r} D_r - D_{zz} - cD_z\right) U - f(U(r,z)) &= 0 \quad \text{for } r > 0,\ z \in \mathbb{R},\\ U_r(0,z) &= 0 \quad \text{for } z \in \mathbb{R},\\ -U_z(r,z) &> 0 \quad \text{for } r \geq 0,\ z \in \mathbb{R},\\ U(0,0) &= 0. \end{aligned}$$

Here $D_r U = \partial U/\partial r$, $D_{rr} U = \partial^2 U/\partial r^2$, $D_z U = \partial U/\partial z$, and $D_{zz} U = \partial^2 U/\partial z^2$. See Lemmas 4.1 and 4.3 for detailed properties of U.

For any positive-valued function $g \in C^2(S^{N-2})$, let

$$D_g = \left\{ r\boldsymbol{\xi} \,|\, 0 \leq r < g(\boldsymbol{\xi}), \boldsymbol{\xi} \in S^{N-2} \right\}$$

and let $C_g = \partial D_g = \{ g(\boldsymbol{\xi})\boldsymbol{\xi} \,|\, \boldsymbol{\xi} \in S^{N-2} \}$. Now we choose the signs of principal curvatures of C_g such that the principal curvatures of the boundary of S^{N-2} are $+1$ in this paper. Then, if all principal curvatures of C_g are positive at every point of C_g, $\overline{D_g}$ is a strictly convex compact set in $\mathbb{R}^{N-1}$. Let $\mathcal{G}$ be given by

$$\begin{aligned} &\left\{ g \in C^2(S^{N-2}) \,| \right.\\ &\qquad \left. g > 0, \text{ all principal curvatures of } C_g \text{ are positive at every point of } C_g \right\}. \end{aligned}$$

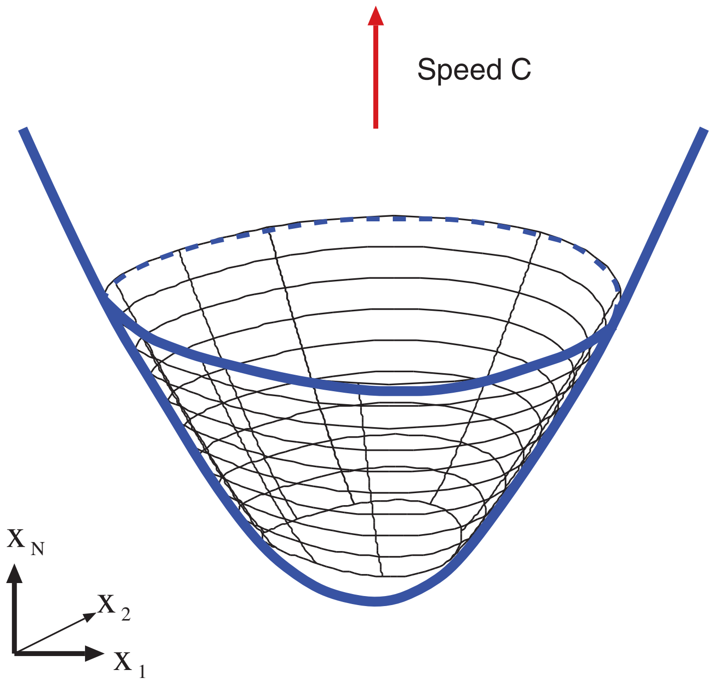

Fig. 1. *The graph of a level set of* $\widetilde{U}$.

For any $g \in \mathcal{G}$ and $a \geq 0$, we define $g_1 = \tau_a g$ by

$$C_{g_1} = \left\{ \boldsymbol{z} \in \mathbb{R}^{N-1} \backslash D_g \mid \operatorname{dist}(\boldsymbol{z}, C_g) = a \right\}.$$

Then τ_a becomes a mapping in $\mathcal{G}$ by Lemma 5.1 in section 5. We define an equivalence relation $g_1 \sim g_2$ if and only if one has either $g_1 = \tau_a g_2$ or $g_2 = \tau_a g_1$ for some $a \geq 0$. Roughly speaking, we define $g_1 \sim g_2$ if and only if one can expand D_{g_1} with a constant width and the expanded one equals D_{g_2} or one can expand D_{g_2} with a constant width and the expanded one equals D_{g_1}. See section 5 for the details.

The following is the main assertion in this paper.

Theorem 1.1. *Assume $c > k$. For any given $g \in \mathcal{G}$, there exists a unique solution $\widetilde{U}$ to*

$$\left(-\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} - c \frac{\partial}{\partial x_N} \right) \widetilde{U} - f(\widetilde{U}) = 0 \quad \text{in } \mathbb{R}^N, \tag{1.6}$$

$$\lim_{s \to \infty} \sup_{|\boldsymbol{x}| \geq s} \left| \widetilde{U}(\boldsymbol{x}) - \min_{\boldsymbol{\xi} \in S^{N-2}} U(|\boldsymbol{x}' - g(\boldsymbol{\xi})\boldsymbol{\xi}|, x_N) \right| = 0. \tag{1.7}$$

Here $\boldsymbol{x} = (\boldsymbol{x}', x_N)$. Let $\widetilde{U}_j$ be the solution to (1.6)–(1.7) *associated with $g_j \in \mathcal{G}$ for $j = 1, 2$, respectively. Then one has*

$$\widetilde{U}_2(x_1, \ldots, x_{N-1}, x_N) = \widetilde{U}_1(x_1, \ldots, x_{N-1}, x_N - \zeta) \tag{1.8}$$

for some $\zeta \in \mathbb{R}$ if and only if $g_1 \sim g_2$.

Thus each element of a quotient set $\mathcal{G}/\sim$ gives an N-dimensional traveling front $\widetilde{U}$ in the Allen–Cahn equation. Figure 1 shows the graph of a level set $\{\boldsymbol{x} \in \mathbb{R}^N \mid \widetilde{U}(\boldsymbol{x}) =$

$-a_*\}$. This paper is organized as follows. We state preliminaries in section 2 and give a uniform estimate on pyramidal traveling fronts in section 3 with respect to the number of lateral faces. Using this estimate, we show that pyramidal traveling fronts converge to a cylindrically symmetric traveling front U as the number of lateral faces goes to infinity, and we state properties of U in section 4. In section 5, we define an equivalence relation in $\mathcal{G}$. In section 6, we give a proof of Theorem 1.1. We construct a supersolution and a subsolution by using U, prove the existence of a cylindrically nonsymmetric traveling front $\widetilde{U}$ between them, and show the stability of $\widetilde{U}$.

2. Preliminaries. We extend f as a function of class $C^1(\mathbb{R})$ with $f'(s) < 0$ for $|s| > 1$. Setting

$$\beta = \frac{1}{2}\min\{-f'(-1), -f'(1)\} > 0,$$

we choose $\delta_* \in (0, 1/4)$ with

$$-f'(s) > \beta \qquad \text{if } |s+1| \le 2\delta_* \text{ or } |s-1| \le 2\delta_*.$$

Let

$$M = \max_{|s|\le 1+\delta_*} |f'(s)| > 0,$$
$$m_* = \frac{\sqrt{c^2 - k^2}}{k},$$

and define $\theta_* \in (0, \pi/2)$ by

$$\tan\theta_* = m_*.$$

Let $n \ge 2$ be a given integer and let$\{\boldsymbol{a}_j\}_{j=1}^n$ be a set of unit vectors in $\mathbb{R}^{N-1}$ with $\boldsymbol{a}_i \ne \boldsymbol{a}_j$ for $i \ne j$. Then $\boldsymbol{a}_j = (a_j^1, \dots, a_j^{N-1})$ satisfies

$$|\boldsymbol{a}_j|^2 = \sum_{i=1}^{N-1} (a_j^i)^2 = 1 \qquad \text{for all } 1 \le j \le n.$$

Here we put $\boldsymbol{x}' = (x_1, \dots, x_{N-1}) \in \mathbb{R}^{N-1}$ and $\boldsymbol{x} = (\boldsymbol{x}', x_N) = (x_1, \dots, x_N) \in \mathbb{R}^N$ with $|\boldsymbol{x}'| = \sqrt{\sum_{i=1}^{N-1} x_i^2}$ and $|\boldsymbol{x}| = \sqrt{\sum_{i=1}^{N} x_i^2}$, respectively. For $\boldsymbol{x}' \in \mathbb{R}^{N-1}$, we set

$$h_j(\boldsymbol{x}') = m_*(\boldsymbol{a}_j, \boldsymbol{x}'), \tag{2.1}$$
$$h(\boldsymbol{x}') = \max_{1\le j\le n} h_j(\boldsymbol{x}') = m_* \max_{1\le j\le n} (\boldsymbol{a}_j, \boldsymbol{x}'). \tag{2.2}$$

Here $(\boldsymbol{a}_j, \boldsymbol{x}')$ denotes the inner product of vectors $\boldsymbol{a}_j$ and $\boldsymbol{x}'$. In this paper we call $\{(\boldsymbol{x}', x_N) \in \mathbb{R}^N \,|\, x_N \ge h(\boldsymbol{x}')\}$ a pyramid. Setting

$$\Omega_j = \left\{\boldsymbol{x}' \in \mathbb{R}^{N-1} \,|\, h(\boldsymbol{x}') = h_j(\boldsymbol{x}')\right\}$$

for $j = 1, \dots, n$, we have

$$\mathbb{R}^{N-1} = \cup_{j=1}^n \Omega_j.$$

We denote the boundary of Ω_j by $\partial\Omega_j$. Now we put

$$S_j = \left\{\boldsymbol{x} \in \mathbb{R}^N \,|\, x_N = h_j(\boldsymbol{x}') \quad \text{for } \boldsymbol{x}' \in \Omega_j\right\}$$

for each j, and call $\cup_j^n S_j \subset \mathbb{R}^N$ the lateral faces of a pyramid. We put

$$\Gamma_j = \left\{\boldsymbol{x} \in \mathbb{R}^N \,|\, x_N = h_j(\boldsymbol{x}') \quad \text{for } \boldsymbol{x}' \in \partial\Omega_j\right\}$$

for $j = 1, \ldots, n$. Then $\cup_{j=1}^n \Gamma_j$ represents the set of all edges of a pyramid. For $\gamma > 0$, let

$$D(\gamma) = \left\{\boldsymbol{x} \,|\, \operatorname{dist}\left(\boldsymbol{x}, \cup_{j=1}^n \Gamma_j\right) > \gamma\right\}.$$

Now we define $\underline{v}(\boldsymbol{x})$ by

$$\underline{v}(\boldsymbol{x}) = \Phi\left(\frac{k}{c}(x_N - h(\boldsymbol{x}'))\right) = \max_{1\le j\le n} \Phi\left(\frac{k}{c}(x_N - h_j(\boldsymbol{x}'))\right).$$

Pyramidal traveling fronts are stated as follows. For the proof see [13] for $N = 2$, and see [15, 11] for $N \ge 3$.

THEOREM 2.1 (see [13, 15, 11]). *Let h be given in* (2.2). *Let V be defined by*

$$V(\boldsymbol{x}) = \lim_{t\to\infty} w(\boldsymbol{x}, t; \underline{v}) \qquad \textit{for all } \boldsymbol{x} \in \mathbb{R}^N.$$

Then V satisfies

$$(-\Delta - cD_N)V - f(V) = 0, \qquad \boldsymbol{x} \in \mathbb{R}^N \tag{2.3}$$

with

$$\lim_{\gamma\to\infty} \sup_{\boldsymbol{x}\in D(\gamma)} |V(\boldsymbol{x}) - \underline{v}(\boldsymbol{x})| = 0,$$

$$-1 < \underline{v}(\boldsymbol{x}) < V(\boldsymbol{x}) < 1 \qquad \textit{for all } \boldsymbol{x} \in \mathbb{R}^N.$$

Here we state lemmas that we will use later.

LEMMA 2.2. *Let h be given in* (2.2) *and let V be as in Theorem* 2.1. *For any given $\boldsymbol{t} = (\boldsymbol{t}', t_N) \in \mathbb{R}^N$ with $t_N > 0$ and $m_*|\boldsymbol{t}'| \le t_N$, one has*

$$-\frac{\partial V}{\partial \boldsymbol{t}} > 0 \qquad \textit{in } \mathbb{R}^N. \tag{2.4}$$

Moreover, one has

$$-D_N V \ge \frac{k}{c}|\nabla V| \qquad \textit{in } \mathbb{R}^N.$$

Proof. For any $\varepsilon > 0$, we have

$$\underline{v}(\boldsymbol{x} + \varepsilon\boldsymbol{t}) \le \underline{v}(\boldsymbol{x}) \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^N.$$

Then, from the definition of V, we get

$$V(\boldsymbol{x} + \varepsilon\boldsymbol{t}) \le V(\boldsymbol{x}) \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^N.$$

By combining with the maximum principle, this gives

$$\frac{\partial V}{\partial \boldsymbol{t}} < 0 \qquad \text{in } \mathbb{R}^N.$$

The latter inequality follows from

$$\left(-\frac{\nabla V}{|\nabla V|}, \boldsymbol{e}_N \right) \geq \cos\theta_* = \frac{k}{c},$$

where $\boldsymbol{e}_N = {}^t(0, \ldots, 0, 1) \in \mathbb{R}^N$. This completes the proof. $\square$

LEMMA 2.3. *Let h be given in* (2.2) *and let V be as in Theorem* 2.1. *Then one can choose a constant $m_0 > 0$ that is independent of h and has*

$$\sup_{\boldsymbol{x} \in \mathbb{R}^N} |\nabla V(\boldsymbol{x})| \leq m_0. \tag{2.5}$$

Proof. Let $p \in (1, \infty)$ and γ_0 satisfy

$$0 < \gamma_0 < 1 - \frac{N}{p}.$$

For any $\boldsymbol{x}_0 \in \mathbb{R}^N$, we have

$$\|V\|_{W^{2,p}(B(\boldsymbol{x}_0;1))} \leq k_1 \left(\|V\|_{L^p(B(\boldsymbol{x}_0;1))} + \|f(V)\|_{L^p(B(\boldsymbol{x}_0;1))} \right)$$

by applying the Schauder interior estimate to (2.3). Here a constant $k_1 > 0$ depends on c and f and is independent of h. Using the Sobolev imbedding $W^{2,p}(B(\boldsymbol{x}_0; 2)) \subset C^{1,\gamma_0}(\overline{B(\boldsymbol{x}_0; 1)})$, we obtain (2.5). This completes the proof. $\square$

LEMMA 2.4. *Let h be given by* (2.2), *let V be as in Theorem* 2.1, *and let $1 \leq j \leq N-1$. Assume*

$$h(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_{N-1}) = h(x_1, \ldots, x_{j-1}, |x_j|, x_{j+1}, \ldots, x_{N-1})$$

for $(x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$. Then one has

$$V(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_N) = V(x_1, \ldots, x_{j-1}, |x_j|, x_{j+1}, \ldots, x_N)$$

for $(x_1, \ldots, x_N) \in \mathbb{R}^N$, and

$$D_j V(x_1, x_2, \ldots, x_N) \geq 0 \ \text{ for } x_j > 0,\ (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N) \in \mathbb{R}^{N-1},$$
$$D_j V(x_1, \ldots, x_{j-1}, 0, x_{j+1}, x_N) = 0 \ \text{ for } (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N) \in \mathbb{R}^{N-1}.$$

Proof. Without loss of generality, we can assume $j = 1$. The former statement follows from the definition of V in Theorem 2.1 and

$$\underline{v}(x_1, x_2, \ldots, x_{N-1}) = \underline{v}(|x_1|, x_2, \ldots, x_{N-1})$$

for $(x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$.

For the latter statement, we have

$$D_1 h(x_1, x_2, \ldots, x_{N-1}) \geq 0 \text{ for } x_1 > 0, (x_2, \ldots, x_{N-1}) \in \mathbb{R}^{N-2},$$
$$D_1 h(0, x_2, \ldots, x_{N-1}) = 0 \text{ for } (x_2, \ldots, x_{N-1}) \in \mathbb{R}^{N-2}.$$

Then we get

$$\begin{aligned} D_1\underline{v}(x_1, x_2, \ldots, x_{N-1}) \geq 0 \text{ for } x_1 > 0, (x_2, \ldots, x_{N-1}) \in \mathbb{R}^{N-2}, \\ D_1\underline{v}(0, x_2, \ldots, x_{N-1}) = 0 \text{ for } (x_2, \ldots, x_{N-1}) \in \mathbb{R}^{N-2}. \end{aligned}$$

Now $w_1(\boldsymbol{x}, t) = D_1 w(\boldsymbol{x}, t; \underline{v})$ satisfies

$$\begin{aligned} (D_t - \Delta - cD_N - f'(w(\boldsymbol{x}, t; \underline{v})))\, w_1 = 0 &\ \text{ for } x_1 > 0, (x_2, \ldots, x_N) \in \mathbb{R}^{N-1}, t > 0, \\ D_1 w_1(0, x_2, \ldots, x_N) = 0 &\ \text{ for } (x_2, \ldots, x_N) \in \mathbb{R}^{N-1}, \\ w_1(\boldsymbol{x}, 0) \geq 0 &\ \text{ for } x_1 > 0, (x_2, \ldots, x_N) \in \mathbb{R}^{N-1}. \end{aligned}$$

Then we get

$$D_1 w(\boldsymbol{x}, t; \underline{v}) \geq 0 \qquad \text{for } x_1 > 0,\ (x_2, \ldots, x_N) \in \mathbb{R}^{N-1},\ t > 0.$$

Sending $t \to \infty$, we obtain

$$\begin{aligned} D_1 V(\boldsymbol{x}) \geq 0 &\qquad \text{for } x_1 > 0,\ (x_2, \ldots, x_N) \in \mathbb{R}^{N-1}, \\ D_1 V(0, x_2, \ldots, x_N) = 0 &\qquad \text{for } (x_2, \ldots, x_N) \in \mathbb{R}^{N-1}. \end{aligned}$$

This completes the proof. $\square$

Here we write the Harnack inequality. For the proof, see [5, Corollary 9.25], for example.

LEMMA 2.5. *Assume that* $v \in W^{2,N}_{\mathrm{loc}}(\mathbb{R}^N)$ *satisfies*

$$\begin{aligned} (-\Delta - cD_N + h_0(\boldsymbol{x}))\, v = 0 &\qquad \textit{in } B(\boldsymbol{x}_0; R), \\ v \geq 0 &\qquad \textit{in } B(\boldsymbol{x}_0; R), \end{aligned}$$

where $\boldsymbol{x}_0 \in \mathbb{R}^N$ *and* $R > 0$. *Here* $h_0 \in L^\infty(B(\boldsymbol{x}_0; R))$ *satisfies*

$$\max\left\{ |h_0(\boldsymbol{x})| \;\middle|\; \boldsymbol{x} \in \overline{B(\boldsymbol{x}_0; R)} \right\} \leq M.$$

Then, for all $\boldsymbol{x}_0 \in \mathbb{R}^N$ *and all* $R > 0$, *one has*

$$\max\left\{ v(\boldsymbol{x}) \,|\, \boldsymbol{x} \in \overline{B(\boldsymbol{x}_0; R)} \right\} \leq K_R \min\left\{ v(\boldsymbol{x}) \,|\, \boldsymbol{x} \in \overline{B(\boldsymbol{x}_0; R)} \right\},$$

where a constant K_R *depends only on* (R, M, c, N) *and is independent of* $\boldsymbol{x}_0$.

3. A uniform estimate on pyramidal traveling fronts. In this section we give an estimate of the widths of transition layers of pyramidal traveling fronts with n lateral faces uniformly in n. This uniform estimate enables us to take the limit of $n \to \infty$ in section 4.

LEMMA 3.1. *Let* h *and* V *be as in* (2.2) *and Theorem* 2.1, *respectively. Assume*

$$h(x_1, \ldots, x_{N-1}) = h(|x_1|, \ldots, |x_{N-1}|) \qquad \text{for } (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}.$$

Then, for any $\delta \in (0, \delta_*)$ *there exists* $\varepsilon_0 > 0$ *such that one has*

$$\begin{aligned} \inf\left\{ |\nabla V(\boldsymbol{x})| \;\middle|\; |f(V(\boldsymbol{x}))| > \delta \right\} &\geq \varepsilon_0 > 0, \\ \inf\left\{ -D_N V(\boldsymbol{x}) \;\middle|\; |f(V(\boldsymbol{x}))| > \delta \right\} &\geq \frac{k\varepsilon_0}{c} > 0, \end{aligned}$$

where ε_0 *depends on* δ *and is independent of* h.

Proof. The second inequality follows from the first inequality and Lemma 2.2. It suffices to prove the first inequality. Assume the contrary. Then there exist $\boldsymbol{x}_j \in \mathbb{R}^N$, h_j, and V_j for each $j \in \mathbb{N}$ such that we have

$$(-\Delta - cD_N)V_j - f(V_j) = 0 \qquad \text{in } \mathbb{R}^N, \\ |f(V_j(\boldsymbol{x}_j))| > \delta, \quad \lim_{j\to\infty} |\nabla V_j(\boldsymbol{x}_j)| = 0. \tag{3.1}$$

Lemmas 2.2 and 2.4 give $-D_N V > 0$ and $D_i V \geq 0$ for each $1 \leq i \leq N-1$. Applying Lemma 2.5 to $D_i V$ for each $1 \leq i \leq N$, we have

$$\lim_{j\to\infty} \max\left\{ |\nabla V_j(\boldsymbol{x})| \;\middle|\; \boldsymbol{x} \in \overline{B(\boldsymbol{x}_j; 1)} \right\} = 0.$$

Sending $j \to \infty$ in (3.1) and using $\lim_{j\to\infty} |\Delta V_j(\boldsymbol{x}_j)| = 0$, we get $\lim_{j\to\infty} f(V_j(\boldsymbol{x}_j)) = 0$, which contradicts $|f(V_j(\boldsymbol{x}_j))| > \delta$. This completes the proof. $\square$

Now define

$$F(u) = \int_{-1}^{u} f(s)\,ds,$$

and define $u_* \in (-a_*, 1)$ by $F(u_*) = 0$. We choose $u_1 \in (-1, u_*)$ arbitrarily and set κ by

$$2\kappa = -F(u_1) > 0.$$

We define $u_\kappa \in (-1, u_1)$ by $-F(u_\kappa) = \kappa$ and have

$$-1 < u_\kappa < u_1 < u_* < 1.$$

PROPOSITION 3.2. *Let V satisfy* (1.4), (2.4), *and $V(\boldsymbol{x}', \pm\infty) = \mp 1$ for all $\boldsymbol{x}' \in \mathbb{R}^{N-1}$. Let $u_1 \in (-1, u_*)$ be chosen arbitrarily. Then one has*

$$\inf\left\{ |D_N V(\boldsymbol{x})|^2 \;\middle|\; V(\boldsymbol{x}) = u_1 \right\} \geq \frac{k^3 \ell^{N-1}(-F(u_1))}{2c^3 K_\ell^2} > 0,$$

where a positive constant ℓ is given by (3.7) *and K_ℓ is a positive constant in Lemma* 2.5. *Both ℓ and K_ℓ are independent of h.*

Proof. Multiplying (2.3) by $D_N V$ and using

$$(D_N V)\Delta V = \operatorname{div}\left((D_N V)\nabla V\right) - \frac{1}{2} D_N\left(|\nabla V|^2\right),$$

we find

$$-\operatorname{div}\left((D_N V)\nabla V\right) + \frac{1}{2} D_N\left(|\nabla V|^2\right) - c\,(D_N V)^2 - f(V) D_N V = 0. \tag{3.2}$$

Let $\boldsymbol{x}'_0 \in \mathbb{R}^{N-1}$ be arbitrarily chosen. Defining

$$\Omega = \left\{ (\boldsymbol{x}', x_N) \in \mathbb{R}^N \;\middle|\; |\boldsymbol{x}' - \boldsymbol{x}'_0| < \ell, \quad u_\kappa < V(\boldsymbol{x}', x_N) < u_1 \right\},$$
$$\Gamma_1 = \left\{ (\boldsymbol{x}', x_N) \in \mathbb{R}^N \;\middle|\; |\boldsymbol{x}' - \boldsymbol{x}'_0| \leq \ell, \quad V(\boldsymbol{x}', x_N) = u_1 \right\},$$
$$\Gamma_\kappa = \left\{ (\boldsymbol{x}', x_N) \in \mathbb{R}^N \;\middle|\; |\boldsymbol{x}' - \boldsymbol{x}'_0| \leq \ell, \quad V(\boldsymbol{x}', x_N) = u_\kappa \right\},$$
$$\Gamma_{\mathrm{f}} = \left\{ (\boldsymbol{x}', x_N) \in \mathbb{R}^N \;\middle|\; |\boldsymbol{x}' - \boldsymbol{x}'_0| = \ell, \quad u_\kappa \leq V(\boldsymbol{x}', x_N) \leq u_1 \right\},$$

we have $\partial\Omega = \Gamma_1 \cup \Gamma_\kappa \cup \Gamma_{\mathrm{f}}$. Let $\boldsymbol{\nu} = {}^t(\nu_1, \ldots, \nu_N)$ denote the unit outward normal vector on $\partial\Omega$. Putting

$$F_1(u) = \int_{u_1}^{u} f(s)\,ds$$

and integrating both sides of (3.2) over Ω, we obtain

$$\int_{\partial\Omega} \left(-(D_N V)(\nabla V, \boldsymbol{\nu}) + \frac{1}{2}|\nabla V|^2 \nu_N - F_1(V)\nu_N \right) ds = c \int_{\Omega} (D_N V)^2 \, d\boldsymbol{x}. \tag{3.3}$$

Using

$$\boldsymbol{\nu} = \frac{\nabla V}{|\nabla V|} \qquad \text{on } \Gamma_1,$$

we have

$$-(D_N V)(\nabla V, \boldsymbol{\nu}) + \frac{1}{2}|\nabla V|^2 \nu_N = -\frac{1}{2}|\nabla V| D_N V \qquad \text{on } \Gamma_1.$$

Combining this equality and $F_1(u_1) = 0$, we obtain

$$\begin{aligned} \int_{\Gamma_1} \left(-(D_N V)(\nabla V, \boldsymbol{\nu}) + \frac{1}{2}|\nabla V|^2 \nu_N - F_1(V)\nu_N \right) ds & \\ = \int_{\Gamma_1} \left(-\frac{1}{2}|\nabla V| D_N V \right) ds > 0. & \end{aligned} \tag{3.4}$$

Using

$$\boldsymbol{\nu} = -\frac{\nabla V}{|\nabla V|} \qquad \text{on } \Gamma_\kappa,$$

we get

$$-(D_N V)(\nabla V, \boldsymbol{\nu}) + \frac{1}{2}|\nabla V|^2 \nu_N = \frac{1}{2}|\nabla V| D_N V < 0 \qquad \text{on } \Gamma_\kappa.$$

Lemma 2.2 gives

$$\min_{\Gamma_\kappa} \nu_N \geq \frac{k}{c}.$$

Using $F_1(u_\kappa) = \kappa$, we have

$$\int_{\Gamma_\kappa} F_1(V)\nu_N \, ds = \kappa \int_{\Gamma_\kappa} \nu_N \, ds \geq \frac{k\kappa}{c} |\Gamma_\kappa|.$$

Here $|\Gamma_\kappa|$ is the measure of Γ_κ. Recall that the measure of

$$B^{(N-1)}(0; \ell) = \left\{ \boldsymbol{x}' \in \mathbb{R}^{N-1} \;\middle|\; |\boldsymbol{x}'| < \ell \right\}$$

is given by $\mathcal{V}_{N-1}\ell^{N-1}$, where $\mathcal{V}_{N-1}$ is the volume of the unit ball in $\mathbb{R}^{N-1}$. Then we get

$$|\Gamma_\kappa| \geq \mathcal{V}_{N-1}\ell^{N-1}$$

and

$$\int_{\Gamma_\kappa} F_1(V)\nu_N \, ds \geq \frac{k\kappa}{c}\mathcal{V}_{N-1}\ell^{N-1}.$$

Now we obtain

$$\int_{\Gamma_\kappa} \left(-(D_N V)(\nabla V, \boldsymbol{\nu}) + \frac{1}{2}|\nabla V|^2 \nu_N - F_1(V)\nu_N \right) ds \leq -\frac{k\kappa}{c}\mathcal{V}_{N-1}\ell^{N-1}. \tag{3.5}$$

Using $\nu_N = 0$ on Γ_{f}, we get

$$\int_{\Gamma_{\mathrm{f}}} \left(-(D_N V)(\nabla V, \boldsymbol{\nu}) + \frac{1}{2}|\nabla V|^2 \nu_N - F_1(V)\nu_N \right) ds = \int_{\Gamma_{\mathrm{f}}} \left(-(D_N V)(\nabla V, \boldsymbol{\nu}) \right) ds.$$

Applying Lemma 2.3, we find

$$\left| \int_{\Gamma_{\mathrm{f}}} \left((-D_N V)(\nabla V, \boldsymbol{\nu}) \right) ds \right| \leq m_0 \int_{\Gamma_{\mathrm{f}}} (-D_N V) \, ds.$$

Now we continue the calculation as

$$\int_{\Gamma_{\mathrm{f}}} (-D_N V) \, ds = \int_{\partial B^{(N-1)}(0;\ell)} (u_1 - u_\kappa) \, d\boldsymbol{x}',$$

where

$$\partial B^{(N-1)}(0;\ell) = \left\{ \boldsymbol{x}' \in \mathbb{R}^{N-1} \;\middle|\; |\boldsymbol{x}'| = \ell \right\}.$$

Recall that the measure of $\partial B^{(N-1)}(0;\ell)$ is given by $\mathcal{A}_{N-2}\ell^{N-2}$, where $\mathcal{A}_{N-2}$ is the surface area of the unit ball in $\mathbb{R}^{N-1}$. Then we have

$$\left| \int_{\partial B^{(N-1)}(0;\ell)} (u_1 - u_\kappa) \, d\boldsymbol{x}' \right| \leq 2\mathcal{A}_{N-2}\ell^{N-2},$$

where we used $0 < u_1 - u_\kappa < 2$. Thus we obtain

$$\left| \int_{\Gamma_{\mathrm{f}}} \left((-D_N V)(\nabla V, \boldsymbol{\nu}) + \frac{1}{2}|\nabla V|^2 \nu_N - F_1(V)\nu_N \right) ds \right| \leq 2m_0\mathcal{A}_{N-2}\ell^{N-2}. \tag{3.6}$$

Using (3.3), (3.4), (3.5), and (3.6), we obtain

$$\int_{\Gamma_1} \frac{1}{2}(-D_N V)|\nabla V| \, ds \geq \frac{k\kappa}{c}\mathcal{V}_{N-1}\ell^{N-1} - 2m_0\mathcal{A}_{N-2}\ell^{N-2}.$$

We define ℓ as

$$\ell = \frac{4m_0 c\mathcal{A}_{N-2}}{k\kappa\mathcal{V}_{N-1}} > 0. \tag{3.7}$$

Note that ℓ is independent of h. Then we find

$$\int_{\Gamma_1} \frac{1}{2}(-D_N V)|\nabla V| \, ds \geq \frac{k\kappa}{2c}\mathcal{V}_{N-1}\ell^{N-1}.$$

Applying Lemma 2.2, we get

$$\int_{\Gamma_1} |D_N V|^2 \, ds \geq \frac{\kappa k^2}{c^2} \mathcal{V}_{N-1} \ell^{N-1}.$$

Using Lemma 2.5, we find

$$K_\ell^2 |\Gamma_1| \min_{\Gamma_1} |D_N V|^2 \geq \frac{\kappa k^2}{c^2} \mathcal{V}_{N-1} \ell^{N-1}.$$

Let ψ_1 be defined by

$$\Gamma_1 = \left\{ (\boldsymbol{x}', \psi_1(\boldsymbol{x}')) \;\middle|\; |\boldsymbol{x}' - \boldsymbol{x}'_0| \leq \ell \right\}.$$

Now Lemma 2.2 gives

$$|\nabla \psi_1| \leq m_*.$$

Then we have

$$|\Gamma_1| = \int_{B^{(N-1)}(0;\ell)} \sqrt{1 + |\nabla \psi_1|^2} \, d\boldsymbol{x}' \leq \frac{c}{k} \mathcal{V}_{N-1}$$

and thus

$$\frac{c}{k} K_\ell^2 \min_{\Gamma_1} |D_N V|^2 \geq \frac{\kappa k^2}{c^2} \ell^{N-1}.$$

Recalling the definition of κ and the fact that $\boldsymbol{x}'_0 \in \mathbb{R}^{N-1}$ is arbitrary, we complete the proof of Proposition 3.2. $\square$

4. Cylindrically symmetric traveling fronts. Let $\mathbb{N}$ be the set of positive integers and and let $\bar{\mathbb{N}} = \mathbb{N} \cup \{0\}$. For $m \in \mathbb{N}$ with $m \geq 2$, we define J as

$$J = \left\{ j \in \bar{\mathbb{N}} \;\middle|\; 0 \leq j \leq 2^m - 1 \right\} \quad \text{if } N = 3,$$
$$J = \left\{ (j_1, \ldots, j_{N-2}) \in \bar{\mathbb{N}}^{N-2} \;\middle|\; 0 \leq j_i \leq 2^m \ (1 \leq i \leq N-3),\ 0 \leq j_{N-2} \leq 2^m - 1 \right\}$$

if $N \geq 4$. For each $\boldsymbol{j} = (j_1, \ldots, j_{N-2}) \in J$, we define

$$\boldsymbol{a_j} = \begin{pmatrix} \cos\left(\dfrac{2\pi j_1}{2^m}\right) \\ \sin\left(\dfrac{2\pi j_1}{2^m}\right) \end{pmatrix} \qquad \text{for } N = 3,$$

and

$$\boldsymbol{a_j} = \begin{pmatrix} \cos\left(\dfrac{\pi j_1}{2^m}\right) \\ \sin\left(\dfrac{\pi j_1}{2^m}\right) \cos\left(\dfrac{\pi j_2}{2^m}\right) \\ \sin\left(\dfrac{\pi j_1}{2^m}\right) \sin\left(\dfrac{\pi j_2}{2^m}\right) \\ \vdots \\ \sin\left(\dfrac{\pi j_1}{2^m}\right) \ldots \sin\left(\dfrac{\pi j_{N-3}}{2^m}\right) \cos\left(\dfrac{2\pi j_{N-2}}{2^m}\right) \\ \sin\left(\dfrac{\pi j_1}{2^m}\right) \ldots \sin\left(\dfrac{\pi j_{N-3}}{2^m}\right) \sin\left(\dfrac{2\pi j_{N-2}}{2^m}\right) \end{pmatrix} \qquad \text{for } N \geq 4.$$

Let $h^{(m)}$ be as in (2.2) associated with $\{\boldsymbol{a_j} \mid \boldsymbol{j} \in J\}$ and let $V^{(m)}$ be as in Theorem 2.1 for $h^{(m)}$. Since $h^{(m)}$ is symmetric with respect to a plane $(\boldsymbol{x'}, \boldsymbol{a_j}) = 0$, $V^{(m)}(\,\cdot\,, x_N)$ is symmetric with respect to the same plane for any fixed $x_N \in \mathbb{R}$ by the definition of $V^{(m)}$ in Theorem 2.1. We choose $\zeta_m \in \mathbb{R}$ by $V^{(m)}(0\ldots, 0, \zeta_m) = 0$ and define

$$U_\infty(\boldsymbol{x'}, x_N) = \lim_{m\to\infty} V^{(m)}(\boldsymbol{x'}, x_N + \zeta_m) \qquad \text{in } C^2_{\mathrm{loc}}(\mathbb{R}^N).$$

Since $V^{(m)}(\boldsymbol{x'}, x_N + \zeta_m)$ satisfies Lemma 3.1 and Proposition 3.2, $U_\infty(\boldsymbol{x'}, x_N)$ also satisfies Lemma 3.1 and Proposition 3.2. Now U_∞ is a function of $(|\boldsymbol{x'}|, x_N)$. We denote $|\boldsymbol{x'}|$ and x_N by r and z, respectively, and we write $U_\infty(\boldsymbol{x'}, x_N)$ by $U(r, z)$. Now we have (1.5) in section 1.

The property of U is as follows.

Lemma 4.1. *For any given $\boldsymbol{s} = (s_1, s_2)$ with $s_2 > 0$ and $m_*|s_1| \le s_2$, one has*

$$-\frac{\partial U}{\partial \boldsymbol{s}} > 0 \quad \textit{for all } r \ge 0,\ z \in \mathbb{R}.$$

Moreover, one has

$$D_r U(r, z) \ge 0 \qquad \textit{for all } r \ge 0,\ z \in \mathbb{R}.$$

For any $\delta \in (0, \delta_*)$, there exists $\varepsilon_0 > 0$ such that one has

$$\inf\left\{-D_z U(r, z) \;\middle|\; |f(U(r, z))| > \delta\right\} \ge \frac{k\varepsilon_0}{c} > 0.$$

For any $u_1 \in (-1, u_*)$, one has

$$\inf\left\{|D_z U(r, z)|^2 \;\middle|\; U(r, z) = u_1\right\} \ge \frac{k^3\ell^{N-1}(-F(u_1))}{2c^3K_\ell^2} > 0,$$

where ℓ and K_ℓ are as in Proposition 3.2.

Proof. The inequalities in this lemma follow from the definition of U, Lemmas 2.2, 2.4, and 3.1, and Proposition 3.2. ∎

Defining $\phi(r)$ by $U(r, \phi(r)) = 0$, we obtain

$$0 \le \phi'(r) \le m_* \qquad \text{for all } r \ge 0.$$

Then we have

$$\begin{aligned} &\lim\nolimits_{R\to\infty} \sup\left\{|U(r, z) + 1| \,\middle|\, z - \phi(r) \ge R\right\} = 0, \\ &\lim\nolimits_{R\to\infty} \sup\left\{|U(r, z) - 1| \,\middle|\, z - \phi(r) \le -R\right\} = 0, \end{aligned} \tag{4.1}$$

and thus

$$\lim_{R\to\infty} \sup\,\{|D_z U(r, z)| \mid |z - \phi(r)| \ge R\} = 0 \tag{4.2}$$

by applying the Schauder estimate to (2.3).

First we show the following lemma.

Lemma 4.2. *One has*

$$\lim_{r\to\infty} \phi'(r) = m_*.$$

Proof. It suffices to prove $\liminf_{r\to\infty} \phi'(r) = m_*$. Assume the contrary. Then there exists $(s_i)_{i\in\mathbb{N}}$ with

$$0 < s_1 < s_2 < \cdots < s_i < \cdots \to +\infty,$$
$$\sup_i \phi'(s_i) < m_*.$$

Assume, in addition, that there exists $\{\beta_i\}_{i\in\mathbb{N}}$ with $\lim_i \beta_i = +\infty$ and $0 < 2\beta_i < s_i$ for all $i \in \mathbb{N}$ such that we have

$$0 \le \max_{[s_i, s_i+2\beta_i]} \phi' < m' < m_* \qquad \text{for all } i \in \mathbb{N}$$

or

$$0 \le \max_{[s_i-2\beta_i, s_i]} \phi' < m' < m_* \qquad \text{for all } i \in \mathbb{N}$$

for $m' \in (0, m_*)$. Then we set

$$\overline{U}(r,z) = \lim_{i\to\infty} U(r + s_i + \beta_i, z + \phi(s_i + \beta_i)) \qquad \text{for } (r,z) \in \mathbb{R}^2$$

or

$$\overline{U}(r,z) = \lim_{i\to\infty} U(r + s_i - \beta_i, z + \phi(s_i - \beta_i)) \qquad \text{for } (r,z) \in \mathbb{R}^2,$$

respectively. Then we have

$$\begin{aligned} (-D_{rr} - D_{zz} - cD_z)\,\overline{U}(r,z) &= f(\overline{U}(r,z)) \quad && \text{for } (r,z) \in \mathbb{R}^2, \\ \overline{U}(0,0) &= 0, && \\ D_N\overline{U}(r,z) &< 0 && \text{for } (r,z) \in \mathbb{R}^2. \end{aligned}$$

Defining $\bar{\phi}(r)$ by $\overline{U}(r, \bar{\phi}(r)) = 0$, we have $\bar{\phi}(0) = 0$ and, with some $m' \in (0, m_*)$,

$$0 \le \bar{\phi}'(r) \le m' < m_* \qquad \text{for all } r \in \mathbb{R}.$$

Let v_* be the two-dimensional front V-form associated with $x_N = m'|x|$ in Theorem 2.1 for $N = 2$. Then we have

$$\overline{U}(r,z) \le v_*(r, z - \lambda) + \delta_*,$$

by taking $\lambda > 0$ large enough. If $\sigma > 0$ is large enough, $v_*(r, z - \sigma\delta_*(1 - e^{-\beta t})) + \delta_* e^{-\beta t}$ is a supersolution. Taking the sides of

$$\overline{U}(r,z) \le\, v_*(r, z - \lambda - \sigma\delta_*(1 - e^{-\beta t})) + \delta_* e^{-\beta t}\big|_{t=0}$$

as initial values of

$$w_t = (D_{rr} + D_{zz})\, w + f(w) \qquad (r,z) \in \mathbb{R}^2,\ t > 0,$$

we obtain

$$\overline{U}(r, z - ct) \le v_*(r, z - \lambda - c't - \sigma\delta_*(1 - e^{-\beta t})) + \delta_* e^{-\beta t} \tag{4.3}$$

for $(r,z) \in \mathbb{R}^2$ and $t \ge 0$, where

$$c' = k\sqrt{1 + (m')^2} < c.$$

Sending $t \to +\infty$, we have a contradiction from (4.3) and $\overline{U}(0,0) = 0$.

Thus we can choose $b_* > 0$ that is independent of i such that we have

$$\lim_{i\to\infty} \phi'(\xi_i) = m_*$$

for some $\xi_i \in [s_i - b_*, s_i + b_*]$. Then we have

$$\lim_{i\to\infty} \left(-\frac{\partial U}{\partial \boldsymbol{s}_0} \right)\Bigg|_{(\xi_i,\phi(\xi_i))} = 0,$$

where

$$\boldsymbol{s}_0 = \frac{k}{c} \begin{pmatrix} 1 \\ m_* \end{pmatrix}.$$

Now we have

$$\begin{gathered} \left(-D_{rr} - \frac{N-2}{r} D_r - D_{zz} - cD_z - f'(U(r,z)) \right) \left(-\frac{\partial U}{\partial \boldsymbol{s}_0} \right) = 0, \\ -\frac{\partial U}{\partial \boldsymbol{s}_0} \geq 0 \end{gathered} \tag{4.4}$$

for $r > 0$, $z \in \mathbb{R}$. Then the Harnack inequality (Lemma 2.5) gives

$$\max_{B((\xi_i,\phi(\xi_i));R_*)} \left(-\frac{\partial U}{\partial \boldsymbol{s}_0} \right) \leq K_{R_*} \min_{B((\xi_i,\phi(\xi_i));R_*)} \left(-\frac{\partial U}{\partial \boldsymbol{s}_0} \right),$$

where

$$R_* = 1 + \frac{c}{k} b_*.$$

Using $(s_i, \phi(s_i)) \in B((\xi_i, \phi(\xi_i)); R_*)$, we get

$$\lim_{i\to\infty} \left(-\frac{\partial U}{\partial \boldsymbol{s}_0} \right)\Bigg|_{(s_i,\phi(s_i))} = 0.$$

Since the gradient of U does not vanish at $(s_i, \phi(s_i))$ by Lemma 4.1, we obtain $\lim_{i\to\infty} \phi'(s_i) = m_*$, which gives a contradiction. This completes the proof. ∎

Now we prove the following property of U.

LEMMA 4.3. *One has*

$$\lim_{s\to\infty} U(s + r, \phi(s) + z) = \Phi\left(\frac{k}{c}(z - m_* r) \right) \qquad \text{in } C^2_{\mathrm{loc}}(\mathbb{R}^2).$$

Moreover, for every $\eta \geq 0$ one has

$$\begin{aligned} &U(r + \eta, z + m_*\eta) \leq U(r, z) \qquad \text{for all } r \geq 0,\ z \in \mathbb{R}, \\ &\lim_{R\to\infty} \sup_{(r,z)\in[R,\infty)\times\mathbb{R}} (U(r, z) - U(r + \eta, z + m_*\eta)) = 0. \end{aligned}$$

Proof. Using (4.4), Lemma 2.5, and

$$\lim_{s\to\infty} \left(-\frac{\partial U}{\partial \boldsymbol{s}_0}\Bigg|_{(s,\phi(s))} \right) = 0,$$

we obtain

$$\lim_{s\to\infty} \sup_{B((s,\phi(s));R)} \left(-\frac{\partial U}{\partial \boldsymbol{s}_0} \right) = 0$$

for every $R > 0$. From this equality and (1.5) we see that U converges to a unique one-dimensional traveling front to the direction $(-m_*, 1)$ with speed k, that is,

$$\lim_{s\to\infty} \sup_{|r|+|z|\leq R} \left| U(r+s, \phi(s)+z) - \Phi\left(\frac{k}{c}(z - m_* r) \right) \right| = 0$$

for every $R > 0$. This shows the first equality of the lemma. The inequality in the lemma follows from Lemma 4.1. Combining this inequality, the first equality, and Lemma 4.1, we obtain the final equality. This completes the proof. $\square$

5. Surfaces in $\mathbb{R}^{N-1}$ with positive principal curvatures. Let $g \in C^2(S^{N-2})$ satisfy $g(\boldsymbol{\xi}) > 0$ for all $\boldsymbol{\xi} \in S^{N-2}$. We set

$$C_g = \left\{ g(\boldsymbol{\xi})\boldsymbol{\xi} \,|\, \boldsymbol{\xi} \in S^{N-2} \right\},$$
$$D_g = \left\{ r\boldsymbol{\xi} \,|\, 0 \leq r < g(\boldsymbol{\xi}), \boldsymbol{\xi} \in S^{N-2} \right\},$$

and have $C_g = \partial D_g \subset \mathbb{R}^{N-1}$. For some neighborhood of $g(\boldsymbol{\xi})\boldsymbol{\xi} \in C_g$ with $\boldsymbol{\xi} \in S^{N-2}$, we write C_g as $(\boldsymbol{y}, \psi(\boldsymbol{y}))$ with $\psi(\boldsymbol{y}^0) = 0$ and $\nabla\psi(\boldsymbol{y}^0) = \boldsymbol{0}$, where $\boldsymbol{y} = (y_1, \ldots, y_{N-2})$. Here we put $g(\boldsymbol{\xi})\boldsymbol{\xi} = (\boldsymbol{y}^0, \psi(\boldsymbol{y}^0))$ with $\boldsymbol{y}^0 \in \mathbb{R}^{N-2}$.

Let $\boldsymbol{\nu}(\boldsymbol{y})$ be the unit normal vector of C_g at $(\boldsymbol{y}, \psi(\boldsymbol{y}))$ pointing from D_g to $\mathbb{R}^{N-1} \backslash D_g$. We have

$$\boldsymbol{\nu}(\boldsymbol{y}) = \frac{1}{1 + |\nabla\psi(\boldsymbol{y})|^2} \begin{pmatrix} -\nabla\psi(\boldsymbol{y}) \\ 1 \end{pmatrix},$$

where

$$\nabla\psi(\boldsymbol{y}) = {}^t(D_1\psi(\boldsymbol{y}), \ldots, D_{N-2}\psi(\boldsymbol{y})).$$

The eigenvalues $\kappa_1(\boldsymbol{y}^0), \ldots, \kappa_{N-2}(\boldsymbol{y}^0)$ of the Hessian matrix

$$-D^2\psi(\boldsymbol{y}^0) = -\left(D_{ij}\psi(\boldsymbol{y}^0) \right)_{1\leq i,j\leq N-2}$$

are the principal curvatures of C_g at $(\boldsymbol{y}^0, \psi(\boldsymbol{y}^0))$. We take the basis of $\mathbb{R}^{N-1}$ as the eigenvectors of the Hessian matrix. Using this principal coordinate system, we have

$$-D^2\psi(\boldsymbol{y}^0) = \operatorname{diag}\left(\kappa_1(\boldsymbol{y}^0), \ldots, \kappa_{N-2}(\boldsymbol{y}^0) \right)$$

and

$$D_j\nu_i(\boldsymbol{y}^0) = \kappa_i(\boldsymbol{y}^0)\delta_{ij} \qquad 1 \leq i, j \leq N-2.$$

We define $\mathcal{G}$ by

$$\left\{ g \in C^2(S^{N-2}) \,| \right.$$
$$\left. \qquad g > 0, \text{ all principal curvatures of } C_g \text{ are positive at every point of } C_g \right\}.$$

For any $g \in \mathcal{G}$ and $a \geq 0$, we define $g_1 = \tau_a g$ by

$$C_{g_1} = \left\{ \boldsymbol{z} \in \mathbb{R}^{N-1} \backslash D_g \,|\, \operatorname{dist}(\boldsymbol{z}, C_g) = a \right\}.$$

See Figure 2.

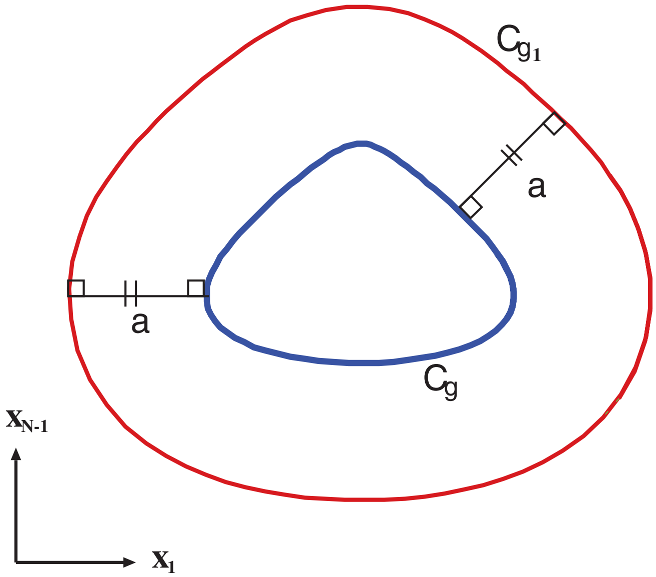

FIG. 2. *The graphs of C_g and C_{g_1}.*

Then we have the following lemma.

LEMMA 5.1. *For any $a \geq 0$, τ_a is a mapping in $\mathcal{G}$. Moreover, one has*

$$\tau_b\left(\tau_a g\right) = \tau_{b+a} g \tag{5.1}$$

for any $a \geq 0$, $b \geq 0$ and $g \in \mathcal{G}$.

Proof. First, we show $\tau_a g \in \mathcal{G}$ for $a \geq 0$ and $g \in \mathcal{G}$. In a neighborhood of $g(\boldsymbol{\xi})\boldsymbol{\xi} = (\boldsymbol{y}^0, \psi(\boldsymbol{y}^0))$, we have

$$\frac{1}{2}\sum_{j=1}^{N-2}\left(\kappa_j(\boldsymbol{y}^0) - \varepsilon\right)(y_j - y_j^0)^2 \leq -\psi(\boldsymbol{y}) \leq \frac{1}{2}\sum_{j=1}^{N-2}\left(\kappa_j(\boldsymbol{y}^0) + \varepsilon\right)(y_j - y_j^0)^2,$$

where $\boldsymbol{y}^0 = {}^t(y_1^0, \ldots, y_{N-2}^0)$ and ε is any number with

$$0 < 2\varepsilon < \min\{\kappa_1(\boldsymbol{y}^0), \ldots, \kappa_{N-2}(\boldsymbol{y}^0)\}.$$

By putting

$$\kappa_{\min} = \min\left\{\kappa_1(\boldsymbol{y}^0), \ldots, \kappa_{N-2}(\boldsymbol{y}^0)\right\},$$
$$\kappa_{\max} = \max\left\{\kappa_1(\boldsymbol{y}^0), \ldots, \kappa_{N-2}(\boldsymbol{y}^0)\right\},$$

we have

$$\frac{1}{2}\left(\kappa_{\min} - \varepsilon\right)\left|\boldsymbol{y} - \boldsymbol{y}^0\right|^2 \leq -\psi(\boldsymbol{y}) \leq \frac{1}{2}\left(\kappa_{\max} + \varepsilon\right)\left|\boldsymbol{y} - \boldsymbol{y}^0\right|^2$$

when $\boldsymbol{y}$ belongs to a neighborhood of $\boldsymbol{y}^0$.

Let r_0 and R_0 be the radii of the inscribed ball and the circumscribed ball of C_g at $g(\boldsymbol{\xi})\boldsymbol{\xi} = (\boldsymbol{y}^0, \psi(\boldsymbol{y}^0))$, respectively. Then we have

$$\left(\kappa_{\max} + \varepsilon\right)^{-1} \leq r_0 \leq R_0 \leq \left(\kappa_{\min} - \varepsilon\right)^{-1}.$$

Next let r_1 and R_1 be the radii of the inscribed ball and the circumscribed ball of C_{g_1} at $(\boldsymbol{y}^0, \psi(\boldsymbol{y}^0) + a\nu(\boldsymbol{y}^0))$, respectively. Then we have

$$r_0 + a \le r_1 \le R_1 \le R_0 + a$$

and thus

$$a + (\kappa_{\max} + \varepsilon)^{-1} \le r_1 \le R_1 \le a + (\kappa_{\min} - \varepsilon)^{-1}.$$

Now the principal curvatures $(\tilde{\kappa}_j)_{1\le j\le N-2}$ of C_{g_1} at $(\boldsymbol{y}^0, \psi(\boldsymbol{y}^0) + a\nu(\boldsymbol{y}^0))$ satisfy

$$\left(a + (\kappa_{\min} - \varepsilon)^{-1}\right)^{-1} \le \tilde{\kappa}_j \le \left(a + (\kappa_{\max} + \varepsilon)^{-1}\right)^{-1}$$

for $1 \le j \le N-2$. Sending $\varepsilon \to 0$, we obtain

$$0 < \frac{\kappa_{\min}}{1 + a\kappa_{\min}} \le \tilde{\kappa}_j \le \frac{\kappa_{\max}}{1 + a\kappa_{\max}} \qquad (1 \le j \le N-2).$$

This shows that τ_a is a mapping in $\mathcal{G}$.

Next we prove (5.1). It suffices to prove that

$$\boldsymbol{\nu}(\boldsymbol{y}^0) = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}$$

is orthogonal to the tangent space of C_{g_1} at $(\boldsymbol{y}^0, \psi(\boldsymbol{y}^0) + a\boldsymbol{\nu}(\boldsymbol{y}^0))$. Now C_{g_1} is parameterized by

$$\begin{pmatrix} \boldsymbol{y} \\ \psi(\boldsymbol{y}) \end{pmatrix} + \frac{a}{\sqrt{1 + |\nabla\psi(\boldsymbol{y})|^2}} \begin{pmatrix} -\nabla\psi(\boldsymbol{y}) \\ 1 \end{pmatrix}$$

when $\boldsymbol{y}$ belongs to a neighborhood of $\boldsymbol{y}^0$. Let $\{\boldsymbol{t}^{(j)}\}_{1\le j\le N-2}$ be the tangent vectors of C_{g_1} at $(\boldsymbol{y}^0, \psi(\boldsymbol{y}^0) + a\boldsymbol{\nu}(\boldsymbol{y}^0))$. We have

$$\begin{aligned} \boldsymbol{t}^{(j)} &= \begin{pmatrix} \boldsymbol{e}_j \\ D_j\psi(\boldsymbol{y}^0) \end{pmatrix} + \frac{a}{\sqrt{1 + |\nabla\psi(\boldsymbol{y}^0)|^2}} \begin{pmatrix} -D_j\nabla\psi(\boldsymbol{y}^0) \\ 0 \end{pmatrix} \\ &\quad - a\left(1 + |\nabla\psi(\boldsymbol{y}^0)|^2\right)^{-\frac{3}{2}} \sum_{i=1}^{N-2} D_i\psi(\boldsymbol{y}^0) D_{ij}\psi(\boldsymbol{y}^0) \begin{pmatrix} -\nabla\psi(\boldsymbol{y}^0) \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{e}_j \\ 0 \end{pmatrix} + a \begin{pmatrix} -D_j\nabla\psi(\boldsymbol{y}^0) \\ 0 \end{pmatrix}. \end{aligned}$$

Here $\boldsymbol{e}_j \in \mathbb{R}^{N-2}$ has 1 for the jth element and 0 for other elements. Then we find $(\boldsymbol{t}^j, \boldsymbol{\nu}(\boldsymbol{y}^0)) = 0$ for $1 \le j \le N-2$. This completes the proof of Lemma 5.1. ∎

Now we define an equivalence relation $g_1 \sim g_2$ for $g_1, g_2 \in \mathcal{G}$. We define $g_1 \sim g_2$ if and only if one has either $g_1 = \tau_a g_2$ or $g_2 = \tau_a g_1$ for some $a \ge 0$. We will show that $\mathcal{G}/\sim$ gives a traveling front of (1.1) in section 6.

6. A traveling front associated with a surface C_g. In this section we give a proof to Theorem 1.1 in section 1, as follows.

Proof of Theorem 1.1. We construct a weak subsolution and a weak supersolution and show the existence of $\widetilde{U}$ between them. Let U be a cylindrically symmetric traveling front solution to (1.5). We define a weak supersolution $\overline{V}(\boldsymbol{x})$ as

$$\overline{V}(\boldsymbol{x}) = \min_{\boldsymbol{\xi}\in S^{N-2}} U(|\boldsymbol{x}' - g(\boldsymbol{\xi})\boldsymbol{\xi}|, x_N) \qquad \text{for } (\boldsymbol{x}', x_N) \in \mathbb{R}^N.$$

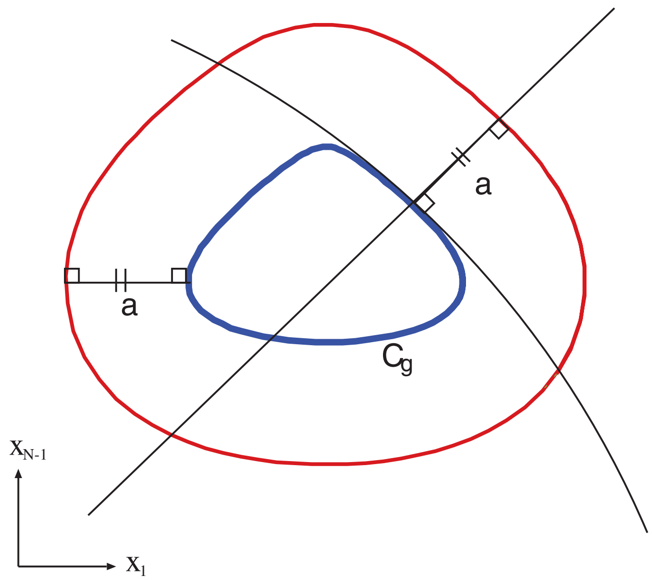

FIG. 3. *A circumscribed ball of* C_g.

Let $\{\kappa_j(\boldsymbol{\xi})\}_{1\le j\le N-2}$ denote the principal curvatures of C_g at $g(\boldsymbol{\xi})\boldsymbol{\xi}$ for $\boldsymbol{\xi} \in S^{N-2}$. By the assumption, we have

$$\min_{1\le j\le N-2} \kappa_j(\boldsymbol{\xi}) > 0 \qquad \text{for all } \boldsymbol{\xi} \in S^{N-2}.$$

We choose $\eta > 0$ large enough such that we have

$$\eta > \max_{1\le j\le N-2} \max_{\boldsymbol{\xi}\in S^{N-2}} \frac{1}{\kappa_j(\boldsymbol{\xi})}$$

and $\overline{D_g}$ is included in the closure of a circumscribed ball of C_g at $g(\boldsymbol{\xi})\boldsymbol{\xi}$ with radius η for every $\boldsymbol{\xi} \in S^{N-2}$. See Figure 3. Let $\boldsymbol{\nu}(\boldsymbol{\xi})$ be the unit normal vector of C_g at $g(\boldsymbol{\xi})\boldsymbol{\xi}$ pointing from D_g to $\mathbb{R}^{N-1}\backslash D_g$ for $\boldsymbol{\xi} \in S^{N-2}$. A set $\{(g(\boldsymbol{\xi})\boldsymbol{\xi}, 0) \,|\, \boldsymbol{\xi} \in S^{N-2}\}$ lies in a cone

$$\{(\boldsymbol{x}', x_N) \in \mathbb{R}^N \,|\, x_N + m_*\eta \ge m_*|\boldsymbol{x}' - g(\boldsymbol{\xi}_1)\boldsymbol{\xi}_1 + \eta\boldsymbol{\nu}(\boldsymbol{\xi}_1)|\}$$

for all $\boldsymbol{\xi}_1 \in S^{N-2}$. Then, from Lemma 4.1, we have

$$U(|\boldsymbol{x}' - g(\boldsymbol{\xi}_1)\boldsymbol{\xi}_1 + \eta\boldsymbol{\nu}(\boldsymbol{\xi}_1)|, x_N + m_*\eta) \le \min_{\boldsymbol{\xi}\in S^{N-2}} U(|\boldsymbol{x}' - g(\boldsymbol{\xi})\boldsymbol{\xi}|, x_N)$$

for all $(\boldsymbol{x}', x_N) \in \mathbb{R}^N$, $\boldsymbol{\xi} \in S^{N-2}$, and $\boldsymbol{\xi}_1 \in S^{N-2}$. Thus we get

$$\max_{\boldsymbol{\xi}_1\in S^{N-2}} U(|\boldsymbol{x}' - g(\boldsymbol{\xi}_1)\boldsymbol{\xi}_1 + \eta\boldsymbol{\nu}(\boldsymbol{\xi}_1)|, x_N + m_*\eta) \le \min_{\boldsymbol{\xi}\in S^{N-2}} U(|\boldsymbol{x}' - g(\boldsymbol{\xi})\boldsymbol{\xi}|, x_N)$$

for all $(\boldsymbol{x}', x_N) \in \mathbb{R}^N$. Now we define a weak subsolution $\underline{V}(\boldsymbol{x})$ as

$$\underline{V}(\boldsymbol{x}', x_N) = \max_{\boldsymbol{\xi}\in S^{N-2}} U\left(|\boldsymbol{x}' - g(\boldsymbol{\xi})\boldsymbol{\xi} + \eta\boldsymbol{\nu}(\boldsymbol{\xi})|, x_N + m_*\eta\right) \text{ for all } (\boldsymbol{x}', x_N) \in \mathbb{R}^N \tag{6.1}$$

and have

$$\underline{V}(\boldsymbol{x}', x_N) \leq \overline{V}(\boldsymbol{x}', x_N) \qquad \text{for all } (\boldsymbol{x}', x_N) \in \mathbb{R}^N.$$

Taking $\mu > 0$ large enough, we find

$$U(|\boldsymbol{x}'|, x_N + \mu) < \underline{V}(\boldsymbol{x}', x_N) \leq \overline{V}(\boldsymbol{x}', x_N) < U(|\boldsymbol{x}'|, x_N - \mu) \tag{6.2}$$

for all $(\boldsymbol{x}', x_N) \in \mathbb{R}^N$. From Lemma 4.3 we have

$$\lim_{R\to\infty} \sup_{|\boldsymbol{x}'|\geq R, x_N\in\mathbb{R}} (U(|\boldsymbol{x}' - g(\boldsymbol{\xi})\boldsymbol{\xi}|, x_N) - U(|\boldsymbol{x}' - g(\boldsymbol{\xi})\boldsymbol{\xi} + \eta\boldsymbol{\nu}(\boldsymbol{\xi})|, x_N + m_*\eta)) = 0$$

for each $\boldsymbol{\xi} \in S^{N-2}$. Using this fact and (6.2), we obtain

$$\lim_{R\to\infty} \sup_{|\boldsymbol{x}'|+|x_N|\geq R} \left(\overline{V}(\boldsymbol{x}', x_N) - \underline{V}(\boldsymbol{x}', x_N)\right) = 0. \tag{6.3}$$

Using $w_t(\boldsymbol{x}, t; \underline{V}) \geq 0$ for $t > 0$, we define $\widetilde{U}$ as

$$\widetilde{U}(\boldsymbol{x}) = \lim_{t\to\infty} w(\boldsymbol{x}, t; \underline{V}) \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^N.$$

Combining this convergence and (6.3), we get

$$\lim_{t\to\infty} \sup_{\boldsymbol{x}\in\mathbb{R}^N} |w(\boldsymbol{x}, t; \underline{V}) - \widetilde{U}(\boldsymbol{x})| = 0. \tag{6.4}$$

Then $\widetilde{U}$ satisfies (1.6) with

$$\underline{V}(\boldsymbol{x}) < \widetilde{U}(\boldsymbol{x}) < \overline{V}(\boldsymbol{x}) \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^N.$$

Then we have

$$U(|\boldsymbol{x}'|, x_N + \mu) < \widetilde{U}(\boldsymbol{x}', x_N) < U(|\boldsymbol{x}'|, x_N - \mu) \qquad \text{for all } (\boldsymbol{x}', x_N) \in \mathbb{R}^N. \tag{6.5}$$

From the definition of $\widetilde{U}$ we get

$$-\frac{\partial \widetilde{U}}{\partial \boldsymbol{t}} > 0 \quad \text{in } \mathbb{R}^N. \tag{6.6}$$

Here $\boldsymbol{t}$ is as in Lemma 2.2. From (6.3) we get (1.7). If $g_1 \not\sim g_2$, we find

$$\widetilde{U}_2(x_1, \ldots, x_{N-1}, x_N) \not\equiv \widetilde{U}_1(x_1, \ldots, x_{N-1}, x_N - \zeta)$$

for any $\zeta \in \mathbb{R}$.

Finally, we show the uniqueness. First, assume that we have another $\widetilde{U}_1$ satisfying (1.6) and (1.7) for the same g. For any $\delta \in (0, \delta_*)$, we take $\lambda > 0$ large enough and have

$$\widetilde{U}(\boldsymbol{x}', x_N + \lambda) - \delta \leq \widetilde{U}_1(\boldsymbol{x}', x_N) \leq \widetilde{U}(\boldsymbol{x}', x_N - \lambda) + \delta.$$

Then we get

$$\widetilde{U}(\boldsymbol{x}', x_N + \lambda + \sigma\delta(1 - e^{-\beta t})) - \delta e^{-\beta t} \leq \widetilde{U}_1(\boldsymbol{x}', x_N) \leq \widetilde{U}(\boldsymbol{x}', x_N - \lambda - \sigma\delta(1 - e^{-\beta t})) + \delta e^{-\beta t}.$$

Now, from Lemma 4.1 and (6.5), $\widetilde{U}(\boldsymbol{x}', x_N \mp \sigma\delta(1-e^{-\beta t})) \pm \delta e^{-\beta t}$ become a supersolution and a subsolution, respectively, if $\sigma > 0$ is large enough. Sending $t \to \infty$, we find

$$\widetilde{U}(\boldsymbol{x}', x_N + \lambda + \sigma\delta) \le \widetilde{U}_1(\boldsymbol{x}', x_N) \le \widetilde{U}(\boldsymbol{x}', x_N - \lambda - \sigma\delta),$$

that is,

$$\widetilde{U}(\boldsymbol{x}', x_N) \le \widetilde{U}_1(\boldsymbol{x}', x_N - \lambda - \sigma\delta) \le \widetilde{U}(\boldsymbol{x}', x_N - 2\lambda - 2\sigma\delta).$$

Now we put $V(\boldsymbol{x}', x_N) = \widetilde{U}_1(\boldsymbol{x}', x_N - \lambda - \sigma\delta)$ and have

$$\widetilde{U}(\boldsymbol{x}', x_N) \le V(\boldsymbol{x}', x_N) \le \widetilde{U}(\boldsymbol{x}', x_N - 2\lambda - 2\sigma\delta).$$

Thus we can define

$$\Lambda = \inf\left\{\lambda \ge 0 \,|\, V(\boldsymbol{x}', x_N) \le \widetilde{U}(\boldsymbol{x}', x_N - \lambda)\right\}.$$

Then we have $\Lambda \ge 0$ and

$$\widetilde{U}(\boldsymbol{x}', x_N) \le V(\boldsymbol{x}', x_N) \le \widetilde{U}(\boldsymbol{x}', x_N - \Lambda).$$

If $\Lambda = 0$, we have $\widetilde{U} \equiv \widetilde{U}_1$ using $\widetilde{U}(\boldsymbol{x}', x_N) \equiv \widetilde{U}_1(\boldsymbol{x}', x_N - \lambda - \sigma\delta)$ and (1.7). Assume $\Lambda > 0$ and get a contradiction. Then the strong maximum principle gives

$$\widetilde{U}(\boldsymbol{x}', x_N) < V(\boldsymbol{x}', x_N) < \widetilde{U}(\boldsymbol{x}', x_N - \Lambda).$$

Using (4.1), (6.5), and the Schauder estimate, we have

$$\lim_{R\to\infty} \sup\left\{|D_N\widetilde{U}(\boldsymbol{x}', x_N)| \;\middle|\; |x_N - \phi(\boldsymbol{x}')| \ge R\right\} = 0.$$

Taking $R > 0$ large enough, we get

$$2\sigma \sup\left\{|D_N\widetilde{U}(\boldsymbol{x}', x_N)| \;\middle|\; |x_N - \phi(\boldsymbol{x}')| \ge R - \Lambda - 1\right\} < 1.$$

Using Lemma 4.3, we take $h \in (0, 1/(2\sigma))$ small enough and find

$$V(\boldsymbol{x}', x_N) \le \widetilde{U}(\boldsymbol{x}', x_N - \Lambda + 2h\sigma) \qquad \text{if } |x_N - \phi(\boldsymbol{x}')| \ge R - \Lambda - 1.$$

If $|x_N - \phi(\boldsymbol{x}')| \ge R - \Lambda - 1$, we have

$$\begin{aligned}\widetilde{U}(\boldsymbol{x}', x_N - \Lambda + 2h\sigma) - V(\boldsymbol{x}', x_N) &> \widetilde{U}(\boldsymbol{x}', x_N - \Lambda + 2h\sigma) - \widetilde{U}(\boldsymbol{x}', x_N - \Lambda)\\ &= 2h\sigma \int_1^1 D_N\widetilde{U}(\boldsymbol{x}', x_N - \Lambda + 2h\sigma\theta)\, d\theta \ge -h.\end{aligned}$$

Combining the two inequalities stated above, we find

$$V(\boldsymbol{x}', x_N) \le \widetilde{U}(\boldsymbol{x}', x_N - \Lambda + 2h\sigma) + h \qquad \text{for all } (\boldsymbol{x}', x_N) \in \mathbb{R}^N,$$

which yields

$$V(\boldsymbol{x}', x_N) \le \widetilde{U}(\boldsymbol{x}', x_N - \Lambda + 2h\sigma - h\sigma(1 - e^{-\beta t})) + he^{-\beta t}.$$

Sending $t \to \infty$, we get

$$V(\boldsymbol{x}', x_N) \le \widetilde{U}(\boldsymbol{x}', x_N - \Lambda + h\sigma).$$

This contradicts the definition of Λ. This gives $\Lambda = 0$ and the uniqueness of $\widetilde{U}$. Finally, if $g_1 \sim g_2$, the definition of $\widetilde{U}$ gives (1.8). This completes the proof of Theorem 1.1. ◻

Now we state the stability of $\widetilde{U}$ as follows.

COROLLARY 6.1 (stability). *Let $\underline{V}$ and $\widetilde{U}$ be as in* (6.1) *and Theorem* 1.1, *respectively. Let a bounded and uniformly continuous function u_0 satisfy*

$$\lim_{R\to\infty} \sup_{|\boldsymbol{x}|\ge R} |u_0(\boldsymbol{x}) - \widetilde{U}(\boldsymbol{x})| = 0,$$
$$\underline{V}(\boldsymbol{x}) \le u_0(\boldsymbol{x}) \le 1 \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^N.$$

Then one has

$$\lim_{t\to\infty} \sup_{\boldsymbol{x}\in\mathbb{R}^N} |w(\boldsymbol{x}, t; u_0) - \widetilde{U}(\boldsymbol{x})| = 0.$$

Proof. For $a \ge 0$, we introduce

$$v_1(\boldsymbol{x}', x_N) = \min_{\boldsymbol{\xi}\in S^{N-2}} U(|\boldsymbol{x}' - \tau_a g(\boldsymbol{\xi})\boldsymbol{\xi}|, x_N - m_* a)$$

and have

$$\lim_{R\to\infty} \sup_{|\boldsymbol{x}|\ge R} |v_1(\boldsymbol{x}) - \underline{V}(\boldsymbol{x})| = 0, \quad \lim_{R\to\infty} \sup_{|\boldsymbol{x}|\ge R} |v_1(\boldsymbol{x}) - \widetilde{U}(\boldsymbol{x})| = 0$$

by using Lemma 4.3. Let $\delta \in (0, \delta_*)$ be given arbitrarily. Using Lemma 4.1, we take $a > 0$ large enough such that we get

$$U(|\boldsymbol{x}' - g(\boldsymbol{\xi})\boldsymbol{\xi} + \eta\boldsymbol{\nu}(\boldsymbol{\xi})|, x_N + m_*\eta) - \delta \le u_0(\boldsymbol{x}) \le U(|\boldsymbol{x}' - \tau_a g(\boldsymbol{\xi})\boldsymbol{\xi}|, x_N - m_* a) + \delta \tag{6.7}$$

for all $(\boldsymbol{x}', x_N) \in \mathbb{R}^N$ and $\boldsymbol{\xi} \in S^{N-2}$. Then we have

$$\begin{aligned} &U(|\boldsymbol{x}' - g(\boldsymbol{\xi})\boldsymbol{\xi} + \eta\boldsymbol{\nu}(\boldsymbol{\xi})|, x_N + m_*\eta + \sigma\delta(1 - e^{-\beta t})) - \delta e^{-\beta t} \\ &\qquad \le w(\boldsymbol{x}, t; u_0) \le U(|\boldsymbol{x}' - \tau_a g(\boldsymbol{\xi})\boldsymbol{\xi}|, x_N - m_* a - \sigma\delta(1 - e^{-\beta t})) + \delta e^{-\beta t}. \end{aligned}$$

Sending $t \to \infty$, we get

$$\begin{aligned} &\max_{\boldsymbol{\xi}\in S^{N-2}} U(|\boldsymbol{x}' - g(\boldsymbol{\xi})\boldsymbol{\xi} + \eta\boldsymbol{\nu}(\boldsymbol{\xi})|, x_N + m_*\eta + \sigma\delta) \\ &\qquad \le \liminf_{t\to\infty} w(\boldsymbol{x}, t; u_0) \le \limsup_{t\to\infty} w(\boldsymbol{x}, t; u_0) \\ &\qquad\qquad \le \min_{\boldsymbol{\xi}\in S^{N-2}} U(|\boldsymbol{x}' - \tau_a g(\boldsymbol{\xi})\boldsymbol{\xi}|, x_N - m_* a - \sigma\delta) \end{aligned}$$

for all $(\boldsymbol{x}', x_N) \in \mathbb{R}^N$. Taking the left-hand side and the right-hand side as initial values of (1.3), we find

$$\widetilde{U}(\boldsymbol{x}', x_N + \sigma\delta) \le \liminf_{t\to\infty} w(\boldsymbol{x}, t; u_0) \le \limsup_{t\to\infty} w(\boldsymbol{x}, t; u_0) \le \widetilde{U}(\boldsymbol{x}', x_N - \sigma\delta)$$

from Theorem 1.1. Since we can choose δ arbitrarily small, we complete the proof. ◻

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REFERENCES

[1] D. G. Aronson and H. F. Weinberger, *Nonlinear diffusion in population genetics*, Partial Differential Equations and Related Topics, J. A. Goldstein, ed., Lecture Notes in Math., 446 (1975), pp. 5–49.

[2] D. G. Aronson and H. F. Weinberger, *Multidimensional nonlinear diffusion arising in population genetics*, Adv. Math., 30 (1978), pp. 33–76.

[3] X. Chen, *Existence, uniqueness, and asymptotic stability of traveling waves in nonlocal evolution equations*, Adv. Differential Equations, 2 (1997), pp. 125–160.

[4] P. C. Fife and J. B. McLeod, *The approach of solutions of nonlinear diffusion equations to travelling front solutions*, Arch. Ration. Mech. Anal., 65 (1977), pp. 335–361.

[5] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 1983.

[6] F. Hamel, R. Monneau, and J.-M. Roquejoffre, *Existence and qualitative properties of multidimensional conical bistable fronts*, Discrete Contin. Dyn. Syst., 13 (2005), pp. 1069–1096.

[7] F. Hamel, R. Monneau, and J.-M. Roquejoffre, *Asymptotic properties and classification of bistable fronts with Lipschitz level sets*, Discrete Contin. Dyn. Syst., 14 (2006), pp. 75–92.

[8] M. Haragus and A. Scheel, *Corner defects in almost planar interface propagation*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 23 (2006), pp. 283–329.

[9] Y. I. Kanel', *Certain problems on equations in the theory of burning*, Soviet. Math. Dokl., 2 (1961), pp. 48–51.

[10] Y. I. Kanel', *Stabilization of solutions of the Cauchy problem for equations encountered in combustion theory*, Mat. Sb. (N.S.), 59 (1962), pp. 245–288.

[11] Y. Kurokawa and M. Taniguchi, *Multi-dimensional pyramidal traveling fronts in the Allen–Cahn equations*, Proc. Roy. Soc. Edinburgh Sect. A, 141 (2011), pp. 1031–1054.

[12] R. Monneau, J.-M. Roquejoffre, and V. Roussier-Michon, *Travelling graphs for the forced mean curvature motion in an arbitrary space dimension*, Ann. Sci. Éc. Norm. Supér. (4), 46 (2013), pp. 217–248.

[13] H. Ninomiya and M. Taniguchi, *Existence and global stability of traveling curved fronts in the Allen–Cahn equations*, J. Differential Equations, 213 (2005), pp. 204–233.

[14] H. Ninomiya and M. Taniguchi, *Global stability of traveling curved fronts in the Allen–Cahn equations*, Discrete Contin. Dyn. Syst., 15 (2006), pp. 819–832.

[15] M. Taniguchi, *Traveling fronts of pyramidal shapes in the Allen–Cahn equations*, SIAM J. Math. Anal., 39 (2007), pp. 319–344.

[16] M. Taniguchi, *The uniqueness and asymptotic stability of pyramidal traveling fronts in the Allen–Cahn equations*, J. Differential Equations, 246 (2009), pp. 2103–2130.

[17] M. Taniguchi, *Multi-dimensional traveling fronts in bistable reaction-diffusion equations*, Discrete Contin. Dyn. Syst., 32 (2012), pp. 1011–1046.