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Multi-dimensional pyramidal travelling fronts in the Allen–Cahn equations

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We study travelling-front solutions of pyramidal shapes in the Allen–Cahn equation in $\mathbb{R}^N$ with $N \geqslant 3$. It is well known that two-dimensional V-form travelling fronts and three-dimensional pyramidal travelling fronts exist and are stable. The aim of this paper is to show that for $N \geqslant 4$ there exist N-dimensional pyramidal travelling fronts. We construct a supersolution and a subsolution, and find a pyramidal travelling-front solution between them. For the construction of a supersolution we use a multi-scale method.

1. Introduction

Multi-dimensional travelling fronts in the Allen–Cahn equation or the Nagumo equation have been studied recently. For example, two-dimensional V-form front solutions have been studied by Ninomiya and Taniguchi [8, 9], Hamel *et al*. [5, 6] and Haragus and Scheel [7]; cylindrically symmetric travelling fronts have been studied by Hamel *et al*. [5, 6] and Chen *et al*. [1]; three-dimensional travelling fronts with pyramidal shapes were studied in [12, 13]. The aim of this paper is to study the existence of travelling fronts with pyramidal shapes in the four-or-higher-dimensional space. A new stationary wave was found by del Pino *et al*. [2, 3] for dimension $N \geqslant 9$ (see also [11] for the non-existence in lower dimensions). Thus, the spatial dimension is crucial for the existence and non-existence of multi-dimensional travelling fronts. It is an interesting but extremely difficult problem to classify all possible travelling-front solutions in $\mathbb{R}^N$. This problem becomes more interesting and difficult as the spatial dimension N becomes higher. As a first step to approaching this problem, we consider the existence of pyramidal travelling fronts in $\mathbb{R}^N$.

In this paper we consider the parabolic equation of the form

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u + f(u), \quad \boldsymbol{x} \in \mathbb{R}^N,\ t > 0, \\ u|_{t=0} &= u_0, \qquad\qquad \boldsymbol{x} \in \mathbb{R}^N, \end{aligned}$$

where $N \geqslant 3$ and a given function u_0 is of class $\mathrm{BU}(\mathbb{R}^N)$, and Δ stands for the Laplacian $\sum_{i=1}^N \partial^2/\partial x_i^2$. Here $\mathrm{BU}(\mathbb{R}^N)$ is the space of bounded uniformly continuous functions with the supremum norm.

In one-dimensional space, let $\Phi(x - kt)$ be a travelling wave that connects two stable equilibrium states ± 1 with speed k. By putting $\mu = x - kt$, Φ satisfies

$$\left.\begin{aligned} -\Phi''(\mu) - k\Phi'(\mu) - f(\Phi(\mu)) = 0, \quad -\infty < \mu < \infty, \\ \Phi(-\infty) = 1, \qquad \Phi(\infty) = -1. \end{aligned}\right\} \tag{1.1}$$

To fix the phase we set $\Phi(0) = 0$.

The following are the assumptions on f throughout this paper:

(A1) f is of class $C^1[-1, 1]$ with $f(1) = 0$, $f(-1) = 0$, $f'(1) < 0$ and $f'(-1) < 0$;

(A2) f satisfies $\int_{-1}^{1} f > 0$;

(A3) there exists $\Phi(\mu)$ that satisfies (1.1) for some $k \in \mathbb{R}$.

In order to study the travelling-wave solutions in one direction, we adopt the moving coordinates of speed c towards the x_N-axis without loss of generality. We write $\boldsymbol{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$ and $\boldsymbol{x}' = (x_1, \dots, x_{N-1}) \in \mathbb{R}^{N-1}$, and put $s = x_N - ct$ and $u(\boldsymbol{x}, t) = w(\boldsymbol{x}', s, t)$. We denote $w(\boldsymbol{x}', s, t)$ by $w(\boldsymbol{x}, t)$ for simplicity. Then we have

$$\left.\begin{aligned} \frac{\partial w}{\partial t} - \Delta w - c\frac{\partial w}{\partial x_N} - f(w) = 0 \quad \text{in } \mathbb{R}^N,\ t > 0, \\ w|_{t=0} = u_0 \quad \text{in } \mathbb{R}^N. \end{aligned}\right\} \tag{1.2}$$

We denote the solution of this equation by $w(\boldsymbol{x}, t; u_0)$. If v is a travelling wave with speed c, it must satisfy

$$-\Delta v - c\frac{\partial v}{\partial x_N} - f(v) = 0 \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

To study this equation, we introduce a nonlinear operator

$$\mathfrak{L}[v] := -\Delta v - c\frac{\partial v}{\partial x_N} - f(v) \tag{1.4}$$

for a function $v \in C^2(\mathbb{R}^N)$. We assume $c > k$ throughout this paper, because the curvature effect is expected to accelerate the speed.

Let $n \geqslant 3$ be a given integer, and

$$m := \frac{\sqrt{c^2 - k^2}}{k} > 0. \tag{1.5}$$

Let $\{\boldsymbol{A}_j\}_{j=1}^{n}$ be a set of unit vectors in $\mathbb{R}^{N-1}$ such that $\boldsymbol{A}_i \neq \boldsymbol{A}_j$ for $i \neq j$. Then $\boldsymbol{A}_j = (A_{1,j}, \dots, A_{N-1,j}) \in \mathbb{R}^{N-1}$ satisfies

$$|\boldsymbol{A}_j|^2 = \sum_{i=1}^{N-1} (A_{i,j})^2 = 1 \quad \text{for } j = 1, \dots, n. \tag{1.6}$$

Now $(-m\boldsymbol{A}_j, 1) \in \mathbb{R}^N$ is the normal vector of $\{\boldsymbol{x} \in \mathbb{R}^N \mid x_N = m(\boldsymbol{A}_j, \boldsymbol{x}')\}$, where $(\boldsymbol{A}_j, \boldsymbol{x}')$ is the inner product of $\boldsymbol{A}_j$ and $\boldsymbol{x}'$ given by

$$(\boldsymbol{A}_j, \boldsymbol{x}') := \sum_{i=1}^{N-1} A_{i,j} x_i.$$

We set

$$\left.\begin{aligned} h_j(\boldsymbol{x}') &:= m(\boldsymbol{A}_j, \boldsymbol{x}'), \\ h(\boldsymbol{x}') &:= \max_{1\leqslant j\leqslant n} h_j(\boldsymbol{x}') = m \max_{1\leqslant j\leqslant n} (\boldsymbol{A}_j, \boldsymbol{x}') \end{aligned}\right\} \tag{1.7}$$

in $\boldsymbol{x}' \in \mathbb{R}^{N-1}$. We call $\{\boldsymbol{x} = (\boldsymbol{x}', x_{N-1}) \in \mathbb{R}^N \mid x_N = h(\boldsymbol{x}')\}$ an N-dimensional pyramid in $\mathbb{R}^N$. Setting

$$\Omega_j := \{\boldsymbol{x}' \in \mathbb{R}^{N-1} \mid h(\boldsymbol{x}') = h_j(\boldsymbol{x}')\}$$

for $j = 1, \dots, n$, we have

$$\mathbb{R}^{N-1} = \bigcup_{j=1}^{n} \Omega_j. \tag{1.8}$$

We denote the boundary of Ω_j by $\partial\Omega_j$. Now we put

$$S_j := \{\boldsymbol{x} \in \mathbb{R}^N \mid x_N = h_j(\boldsymbol{x}') \text{ for } \boldsymbol{x}' \in \Omega_j\}$$

for each j, and call $\bigcup_j^n S_j \subset \mathbb{R}^N$ the lateral faces of a pyramid. We put

$$\Gamma_j := \{\boldsymbol{x} \in \mathbb{R}^N \mid x_N = h_j(\boldsymbol{x}') \text{ for } \boldsymbol{x}' \in \partial\Omega_j\}$$

for $j = 1, \dots, n$. Then $\bigcup_{j=1}^n \Gamma_j$ represents the set of all edges of a pyramid.

For every $\boldsymbol{A}_j$ with (1.6), (1.3) has a planar front solution $\Phi(k(x_N - h_j(\boldsymbol{x}'))/c)$. We define

$$\underline{v}(\boldsymbol{x}) := \Phi\biggl(\frac{k}{c}(x_N - h(\boldsymbol{x}'))\biggr) = \max_{1\leqslant j\leqslant n} \Phi\biggl(\frac{k}{c}(x_N - h_j(\boldsymbol{x}'))\biggr). \tag{1.9}$$

Then it is easy to see that $\underline{v}$ becomes a subsolution to (1.3). We define

$$D(\gamma) := \biggl\{\boldsymbol{x} \in \mathbb{R}^N \biggm| \operatorname{dist}\biggl(\boldsymbol{x}, \bigcup_{j=1}^{n} \Gamma_j\biggr) > \gamma\biggr\} \tag{1.10}$$

for $\gamma > 0$. We will show that the function

$$\bar{v}(\boldsymbol{x}) := \Phi\biggl(\frac{\alpha x_N - \varphi(\alpha\boldsymbol{x}')}{\alpha\sqrt{1 + |\nabla\varphi(\alpha\boldsymbol{x}')|^2}}\biggr) + \varepsilon S(\alpha\boldsymbol{x}')$$

becomes a suitable supersolution with $\underline{v} < \bar{v}$ to obtain a solution of (1.3) between them by the comparison principle. Then this solution has a contour surface of a pyramidal front shape. See (4.7) below for the precise definition of $\bar{v}$.

The following theorem is the main assertion in this paper.

Theorem 1.1. *Let $c > k$, and let $\underline{v}(\boldsymbol{x})$ be given by (1.9). Under assumptions (A1)–(A3), there exists a solution $V(\boldsymbol{x})$ to (1.3) such that*

$$\lim_{\gamma\to\infty} \sup_{\boldsymbol{x}\in D(\gamma)} |V(\boldsymbol{x}) - \underline{v}(\boldsymbol{x})| = 0, \quad \underline{v}(\boldsymbol{x}) < V(\boldsymbol{x}) < 1 \text{ for all } \boldsymbol{x} \in \mathbb{R}^N, \tag{1.11}$$

and

$$\frac{\partial V}{\partial x_N}(\boldsymbol{x}) < 0 \quad \text{for all } \boldsymbol{x} \in \mathbb{R}^N. \tag{1.12}$$

Most of this paper is devoted to constructing a supersolution $\bar{v}$. We introduce a small positive parameter α and rescale the spatial variable as $\boldsymbol{\xi} = \alpha \boldsymbol{x}$. In §§ 2 and 3, we take a suitable positive function $S(\boldsymbol{\xi})$ that is used to construct the supersolution $\bar{v}$. This function takes positive values near edges $\bigcup_{j=1}^{n} \Gamma_j$ and decays to 0 as $\boldsymbol{\xi}$ moves away from the edges. On the other hand, the first term of $\bar{v}$ converges to a planar front travelling with a slower speed $k \in (c, \infty)$ as $\alpha \to +0$. If we use the moving coordinate of a speed c, we expect that $w(\boldsymbol{x}, t; \bar{v})$ is monotone decreasing in $t > 0$. This suggests that $\bar{v}$ is a supersolution of (1.3). We carry out this argument and prove the main theorem in § 4.

2. Preliminaries

In this section we make preparations. We state known results for travelling fronts, and prepare to construct supersolutions for pyramidal travelling-front solutions.

Lemma 2.1 (Fife and McLeod [4]). *Under assumptions (A1) and (A3), $\Phi(\mu)$ as in (1.1) satisfies*

$$\Phi'(\mu) < 0 \quad \text{for all } \mu \in \mathbb{R},$$
$$\max\{|\Phi'(\mu)|, |\Phi''(\mu)|, |\mu\Phi'(\mu)|\} \leqslant K_0 \exp(-\kappa_0 |\mu|).$$

Here K_0, κ_0 are positive constants.

There exists a positive constant δ_* with $0 < \delta_* < \frac{1}{4}$ and

$$-f'(s) > \beta \quad \text{if } |s+1| < 2\delta_* \text{ or } |s-1| < 2\delta_*,$$

where

$$\beta = \tfrac{1}{2} \min\{-f'(-1), -f'(1)\} > 0.$$

Since the unit vectors $\{\boldsymbol{A}_j\}_{j=1}^{n}$ satisfy $\boldsymbol{A}_i \neq \boldsymbol{A}_j$ for $i \neq j$, we have

$$-1 \leqslant (\boldsymbol{A}_i, \boldsymbol{A}_j) < 1 \quad \text{for } i \neq j.$$

For $j, i, i' \in \{1, \dots, n\}$ with $j \neq i$, $j \neq i'$ and $i \neq i'$, let $\theta_{i,i'}(j)$ be the angle between $\boldsymbol{A}_j - \boldsymbol{A}_i$ and $\boldsymbol{A}_j - \boldsymbol{A}_{i'}$. Then we have $0 < \theta_{i,i'}(j) < \pi$ and

$$(\boldsymbol{A}_j - \boldsymbol{A}_i, \boldsymbol{A}_j - \boldsymbol{A}_{i'}) = |\boldsymbol{A}_j - \boldsymbol{A}_i||\boldsymbol{A}_j - \boldsymbol{A}_{i'}| \cos\theta_{i,i'}(j).$$

We define

$$\theta_{\min} := \min\{\theta_{i,i'}(j) \mid j \neq i,\ j \neq i',\ i \neq i'\},$$
$$\theta_{\max} := \max\{\theta_{i,i'}(j) \mid j \neq i,\ j \neq i',\ i \neq i'\}$$

and have $0 < \theta_{\min} \leqslant \theta_{\max} < \pi$.

Let $\tilde{\rho}(r) \in C^{\infty}[0, \infty)$ be a function with the following properties:

$$\tilde{\rho}(r) > 0, \quad \tilde{\rho}_r(r) \leqslant 0 \quad \text{for } r \geqslant 0,$$
$$\tilde{\rho}(r) = \begin{cases} 1 & \text{if } r > 0 \text{ is small enough,} \\ \mathrm{e}^{-r} & \text{if } r > 0 \text{ is large enough, say } r > R_0, \end{cases}$$
$$\int_{\mathbb{R}^{N-1}} \tilde{\rho}(|\boldsymbol{x}'|)\, \mathrm{d}\boldsymbol{x}' = 1.$$

We assume $R_0 > 1$ without loss of generality. We have

$$\int_{\mathbb{R}^{N-1}} \tilde{\rho}(|\boldsymbol{x}'|)\,\mathrm{d}\boldsymbol{x}' = \frac{(N-1)\pi^{(N-1)/2}}{\Gamma((N+1)/2)} \int_0^\infty r^{N-2}\tilde{\rho}(r)\,\mathrm{d}r,$$

where Γ is the Gamma function.

We put $\rho(\boldsymbol{x}') := \tilde{\rho}(|\boldsymbol{x}'|)$. Then $\rho : \mathbb{R}^{N-1} \to \mathbb{R}$ belongs to $C^\infty(\mathbb{R}^{N-1})$ and satisfies

$$\int_{\mathbb{R}^{N-1}} \rho(\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}' = 1, \qquad (\rho * h_j)(\boldsymbol{x}') = h_j(\boldsymbol{x}')$$

for all $\boldsymbol{x}' \in \mathbb{R}^{N-1}$ and $j = 1, 2, \ldots, n$. Here $\rho * h_j$ implies the convolution of ρ and h_j given by

$$\rho * h_j(\boldsymbol{x}') := \int_{\mathbb{R}^{N-1}} \rho(\boldsymbol{y}')h_j(\boldsymbol{x}' - \boldsymbol{y}')\,\mathrm{d}\boldsymbol{y}'.$$

For all non-negative integers $j_1, j_2, \ldots, j_{N-1}$ with $0 \leqslant \sum_{p=1}^{N-1} j_p \leqslant 3$, we have

$$|\mathrm{D}_1^{j_1}\mathrm{D}_2^{j_2}\cdots\mathrm{D}_{N-1}^{j_{N-1}}\rho(\boldsymbol{x}')| \leqslant M_1\rho(\boldsymbol{x}') \quad \text{for all } \boldsymbol{x}' \in \mathbb{R}^{N-1}, \tag{2.1}$$

where $M_1 > 0$ is a constant.

We put $\varphi := \rho * h$, that is,

$$\varphi(\boldsymbol{x}') := \int_{\mathbb{R}^{N-1}} \rho(\boldsymbol{x}' - \boldsymbol{y}')h(\boldsymbol{y}')\,\mathrm{d}\boldsymbol{y}' = \int_{\mathbb{R}^{N-1}} \rho(\boldsymbol{y}')h(\boldsymbol{x}' - \boldsymbol{y}')\,\mathrm{d}\boldsymbol{y}' \tag{2.2}$$

for $\boldsymbol{x}' \in \mathbb{R}^{N-1}$. We call $x_N = \varphi(\boldsymbol{x}')$ a mollified pyramid for a pyramid $x_N = h(\boldsymbol{x}')$. We put

$$S(\boldsymbol{x}') := \frac{c}{\sqrt{1 + |\nabla\varphi(\boldsymbol{x}')|^2}} - k, \tag{2.3}$$

where $\nabla\varphi(\boldsymbol{x}') = (\partial\varphi/\partial x_1, \ldots, \partial\varphi/\partial x_{N-1})$. We denote $(\partial/\partial x_p)^i$ by D_p^i. We have the following lemma.

Lemma 2.2. *Let φ and S be as in (2.2) and (2.3), respectively. For any fixed $(i_1, \ldots, i_{N-1}) \neq (0, \ldots, 0)$ with $i_p \geqslant 0$ $(p = 1, \ldots, N-1)$, one has*

$$\sup_{\boldsymbol{x}' \in \mathbb{R}^{N-1}} |\mathrm{D}_1^{i_1}\mathrm{D}_2^{i_2}\cdots\mathrm{D}_{N-1}^{i_{N-1}}\varphi(\boldsymbol{x}')| < \infty.$$

One has

$$h(\boldsymbol{x}') < \varphi(\boldsymbol{x}') \leqslant h(\boldsymbol{x}') + m\int_{\mathbb{R}^{N-1}} |\boldsymbol{y}'|\rho(\boldsymbol{y}')\,\mathrm{d}\boldsymbol{y}',$$
$$|\nabla\varphi(\boldsymbol{x}')| < m, \qquad 0 < S(\boldsymbol{x}') \leqslant c - k$$

for all $\boldsymbol{x}' \in \mathbb{R}^{N-1}$.

Proof. We have

$$\int_{\mathbb{R}^{N-1}} |\boldsymbol{x}'|\rho(\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}' = \frac{(N-1)\pi^{(N-1)/2}}{\Gamma((N+1)/2)} \int_0^\infty r^{N-1}\tilde{\rho}(r)\,\mathrm{d}r.$$

Without loss of generality we assume $i_1 \geqslant 1$ and have

$$\begin{aligned}\mathrm{D}_1^{i_1-1}\mathrm{D}_2^{i_2}\cdots\mathrm{D}_{N-1}^{i_{N-1}}\varphi(\boldsymbol{x}') &= \int_{\mathbb{R}^{N-1}} \mathrm{D}_1^{i_1-1}\mathrm{D}_2^{i_2}\cdots\mathrm{D}_{N-1}^{i_{N-1}}\rho(\boldsymbol{x}'-\boldsymbol{y}')h(\boldsymbol{y}')\,\mathrm{d}\boldsymbol{y}' \\ &= \int_{\mathbb{R}^{N-1}} \mathrm{D}_1^{i_1-1}\mathrm{D}_2^{i_2}\cdots\mathrm{D}_{N-1}^{i_{N-1}}\rho(\boldsymbol{y}')h(\boldsymbol{x}'-\boldsymbol{y}')\,\mathrm{d}\boldsymbol{y}'.\end{aligned}$$

Then we get

$$\mathrm{D}_1^{i_1}\mathrm{D}_2^{i_2}\cdots\mathrm{D}_{N-1}^{i_{N-1}}\varphi(\boldsymbol{x}') = \int_{\mathbb{R}^{N-1}} \mathrm{D}_1^{i_1-1}\mathrm{D}_2^{i_2}\cdots\mathrm{D}_{N-1}^{i_{N-1}}\rho(\boldsymbol{y}')\mathrm{D}_1 h(\boldsymbol{x}'-\boldsymbol{y}')\,\mathrm{d}\boldsymbol{y}',$$

and thus

$$|\mathrm{D}_1^{i_1}\mathrm{D}_2^{i_2}\cdots\mathrm{D}_{N-1}^{i_{N-1}}\varphi(\boldsymbol{x}')| \leqslant m\int_{\mathbb{R}^{N-1}} |\mathrm{D}_1^{i_1-1}\mathrm{D}_2^{i_2}\cdots\mathrm{D}_{N-1}^{i_{N-1}}\rho(\boldsymbol{y}')|\,\mathrm{d}\boldsymbol{y}' < \infty.$$

Using $\rho > 0$, $h_j(\boldsymbol{x}') \leqslant h(\boldsymbol{x}')$, $\rho * h_j(\boldsymbol{x}') = h_j(\boldsymbol{x}')$ and $h_j(\boldsymbol{x}') \not\equiv h(\boldsymbol{x}')$, we have

$$h_j(\boldsymbol{x}') = \rho * h_j(\boldsymbol{x}') < \rho * h(\boldsymbol{x}') = \varphi(\boldsymbol{x}').$$

Thus, we get $h(\boldsymbol{x}') = \max_{1\leqslant j\leqslant n} h_j(\boldsymbol{x}') < \varphi(\boldsymbol{x}')$ for all $\boldsymbol{x}' \in \mathbb{R}^{N-1}$. Since we have

$$|h_j(\boldsymbol{x}'-\boldsymbol{y}') - h_j(\boldsymbol{x}')| \leqslant m|\boldsymbol{y}'|$$

for all $j = 1, 2, \ldots, n$, we get

$$|h(\boldsymbol{x}'-\boldsymbol{y}') - h(\boldsymbol{x}')| \leqslant m|\boldsymbol{y}'|.$$

Using this inequality and

$$\int_{\mathbb{R}^{N-1}} \rho(\boldsymbol{y}')\,\mathrm{d}\boldsymbol{y}' = 1,$$

we obtain

$$\begin{aligned}\varphi(\boldsymbol{x}') - h(\boldsymbol{x}') &= \int_{\mathbb{R}^{N-1}} h(\boldsymbol{x}'-\boldsymbol{y}')\rho(\boldsymbol{y}')\,\mathrm{d}\boldsymbol{y}' - h(\boldsymbol{x}')\int_{\mathbb{R}^{N-1}} \rho(\boldsymbol{y}')\,\mathrm{d}\boldsymbol{y}' \\ &\leqslant \int_{\mathbb{R}^{N-1}} |h(\boldsymbol{x}'-\boldsymbol{y}') - h(\boldsymbol{x}')|\rho(\boldsymbol{y}')\,\mathrm{d}\boldsymbol{y}' \\ &\leqslant m\int_{\mathbb{R}^{N-1}} |\boldsymbol{y}'|\rho(\boldsymbol{y}')\,\mathrm{d}\boldsymbol{y}'.\end{aligned}$$

Next we have

$$(\nabla\varphi)(\boldsymbol{x}') = \int_{\mathbb{R}^{N-1}} \rho(\boldsymbol{y}')(\nabla h)(\boldsymbol{x}'-\boldsymbol{y}')\,\mathrm{d}\boldsymbol{y}' = m\sum_{j=1}^{n}\int_{\Omega_j} \rho(\boldsymbol{x}'-\boldsymbol{y}')\,\mathrm{d}\boldsymbol{y}'\boldsymbol{A}_j.$$

Here we used $\nabla h(\boldsymbol{x}') = m\boldsymbol{A}_j$ if $\boldsymbol{x}' \in \Omega_j$ for $j = 1, 2, \ldots, n$. Thus, we have an inequality

$$|(\nabla\varphi)(\boldsymbol{x}')| < m\sum_{j=1}^{n}\int_{\Omega_j} \rho(\boldsymbol{x}'-\boldsymbol{y}')\,\mathrm{d}\boldsymbol{y}' = m.$$

Since $S < c$ is valid, we obtain $0 < S \leqslant c - k$. This completes the proof. □

The following proposition plays a key role in this paper.

PROPOSITION 2.3. *One has*

$$0 < \inf_{\boldsymbol{x}' \in \mathbb{R}^{N-1}} \frac{\varphi(\boldsymbol{x}') - h(\boldsymbol{x}')}{S(\boldsymbol{x}')} \leqslant \sup_{\boldsymbol{x}' \in \mathbb{R}^{N-1}} \frac{\varphi(\boldsymbol{x}') - h(\boldsymbol{x}')}{S(\boldsymbol{x}')} < \infty.$$

For every integer $j_p \geqslant 0$ *for* $p = 1, \dots, N-1$ *with* $2 \leqslant \sum_{p=1}^{N-1} j_p \leqslant 3$,

$$\sup_{\boldsymbol{x}' \in \mathbb{R}^{N-1}} \left| \frac{\mathrm{D}_1^{j_1} \mathrm{D}_2^{j_2} \cdots \mathrm{D}_{N-1}^{j_{N-1}} \varphi(\boldsymbol{x}')}{S(\boldsymbol{x}')} \right| < \infty$$

holds true.

We give the proof of this proposition in § 3. In preparation we study $\varphi(\boldsymbol{x}') - h(\boldsymbol{x}')$. We set

$$\tilde{\varphi}(\boldsymbol{x}') := \varphi(\boldsymbol{x}') - h(\boldsymbol{x}') \quad \text{for all } \boldsymbol{x}' \in \mathbb{R}^{N-1}. \tag{2.4}$$

For each $1 \leqslant j \leqslant n$ we have

$$\tilde{\varphi}(\boldsymbol{x}') = \varphi(\boldsymbol{x}') - h_j(\boldsymbol{x}') = \rho * (h - h_j) \quad \text{for all } \boldsymbol{x}' \in \Omega_j. \tag{2.5}$$

We put $q(\boldsymbol{x}') := \max\{-x_1, 0\}$ for $\boldsymbol{x}' = (x_1, \dots, x_{N-1}) \in \mathbb{R}^{N-1}$. We set $\boldsymbol{e}_1 = (1, 0, \dots, 0) \in \mathbb{R}^{N-1}$ and introduce P as

$$\begin{aligned} P(x_1) &:= (\rho * q)(x_1 \boldsymbol{e}_1) \\ &= \int_{\mathbb{R}^{N-1}} \rho(\boldsymbol{y}') \max\{(\boldsymbol{y}', \boldsymbol{e}_1) - x_1, 0\} \, \mathrm{d}\boldsymbol{y}' \\ &= -\int_{\mathbb{R}^{N-2}} \left(\int_{x_1}^{\infty} \rho(y_1, \boldsymbol{y}'')(x_1 - y_1) \, \mathrm{d}y_1 \right) \mathrm{d}\boldsymbol{y}'' \\ &> 0 \end{aligned} \tag{2.6}$$

for $x_1 \in \mathbb{R}$. Since

$$P(x_1) = -\int_{\mathbb{R}^{N-2}} \left(x_1 \int_{x_1}^{\infty} \rho(y_1, \boldsymbol{y}'') \, \mathrm{d}y_1 - \int_{x_1}^{\infty} y_1 \rho(y_1, \boldsymbol{y}'') \, \mathrm{d}y_1 \right) \mathrm{d}\boldsymbol{y}'',$$

we have

$$P'(x_1) = -\int_{\mathbb{R}^{N-2}} \left(\int_{x_1}^{\infty} \rho(y_1, \boldsymbol{y}'') \, \mathrm{d}y_1 \right) \mathrm{d}\boldsymbol{y}'' < 0, \tag{2.7}$$

$$\begin{aligned} P''(x_1) &= \int_{\mathbb{R}^{N-2}} \rho(x_1, \boldsymbol{y}'') \, \mathrm{d}\boldsymbol{y}'' \\ &= \int_{\mathbb{R}^{N-2}} \tilde{\rho}\Big(\sqrt{x_1^2 + |\boldsymbol{y}''|^2}\Big) \, \mathrm{d}\boldsymbol{y}'' > 0, \\ P'''(x_1) &= \int_{\mathbb{R}^{N-2}} \frac{x_1}{\sqrt{x_1^2 + |\boldsymbol{y}''|^2}} \tilde{\rho}_r\Big(\sqrt{x_1^2 + |\boldsymbol{y}''|^2}\Big) \, \mathrm{d}\boldsymbol{y}'' \leqslant 0. \end{aligned}$$

In particular, we have

$$P''(x_1) = \int_{\mathbb{R}^{N-2}} \exp\left(-\sqrt{x_1^2 + |\boldsymbol{y}''|^2}\right) \mathrm{d}\boldsymbol{y}'', \tag{2.8}$$

$$P'''(x_1) = -\int_{\mathbb{R}^{N-2}} \frac{x_1}{\sqrt{x_1^2 + |\boldsymbol{y}''|^2}} \exp\left(-\sqrt{x_1^2 + |\boldsymbol{y}''|^2}\right) \mathrm{d}\boldsymbol{y}'', \tag{2.9}$$

if $x_1 > 0$ is large enough. We have

$$P'(x) > -1 \quad \text{for } x \geqslant 0 \tag{2.10}$$

from (2.7).

Now we have the following lemma.

Lemma 2.4. *Let P be as in (2.6). Then one has*

$$\lim_{x\to\infty} \frac{P(x)}{\frac{1}{2}\Gamma(\frac{1}{2}N-1)(2x)^{(N/2)-1}\mathrm{e}^{-x}} = 1. \tag{2.11}$$

Moreover,

$$\lim_{x\to\infty} \frac{|P^{(i)}(x)|}{P(x)} = 1, \quad 0 < \inf_{x\geqslant 1} \frac{|P^{(i)}(x)|}{P(x)} \leqslant \sup_{x\geqslant 1} \frac{|P^{(i)}(x)|}{P(x)} < +\infty$$

hold true for all i with $1 \leqslant i \leqslant 3$.

Proof. We use the polar coordinates in $\mathbb{R}^{N-1}$ given by

$$\begin{aligned}\Psi&(r, \theta_1, \dots, \theta_{N-2}) \\ &:= \left(r\cos\theta_1, r\sin\theta_1\cos\theta_1, \dots, r\left(\prod_{i=1}^{N-3}\sin\theta_i\right)\cos\theta_{N-2}, r\left(\prod_{i=1}^{N-3}\sin\theta_i\right)\sin\theta_{N-2}\right)\end{aligned}$$

for $(\theta_1, \dots, \theta_{N-2}) \in [0,\pi]^{N-3} \times [0, 2\pi]$. We put $I_0 := [0,\pi]^{N-3} \times [0, 2\pi]$. Next we have

$$P''(x) = \frac{(N-2)\pi^{(N-2)/2}}{\Gamma(\frac{1}{2}N)} \int_0^\infty \exp\left(-\sqrt{x^2 + r^2}\right) r^{N-3}\,\mathrm{d}r \tag{2.12}$$

if $x > R_0$. Putting $r = \sqrt{s^2 + 2sx}$, we have

$$\begin{aligned}\int_0^\infty &\exp\left(-\sqrt{x^2 + r^2}\right) r^{N-3}\,\mathrm{d}r \\ &= \int_0^\infty \mathrm{e}^{-x-s}(s^2 + 2sx)^{(N-4)/2}(s + x)\,\mathrm{d}s \\ &= \tfrac{1}{2}(2x)^{(N/2)-1}\mathrm{e}^{-x} \int_0^\infty \mathrm{e}^{-s}\left(s + \frac{s^2}{2x}\right)^{(N-4)/2}\left(1 + \frac{s}{x}\right)\mathrm{d}s.\end{aligned}$$

By the Lebesgue convergence theorem we have

$$\lim_{x\to\infty} \int_0^\infty \mathrm{e}^{-s}\left(s + \frac{s^2}{2x}\right)^{(N-4)/2}\left(1 + \frac{s}{x}\right)\mathrm{d}s = \int_0^\infty s^{(N/2)-2}\mathrm{e}^{-s}\,\mathrm{d}s = \Gamma(\tfrac{1}{2}N - 1).$$

Thus, we get

$$\lim_{x\to\infty} \frac{P''(x)}{\frac{1}{2}\Gamma(\frac{1}{2}N-1)(2x)^{(N/2)-1}\mathrm{e}^{-x}} = 1.$$

Similarly, we calculate

$$\begin{aligned}P'''(x) &= -\int_{\mathbb{R}^{N-2}} \frac{x}{\sqrt{x^2+|\boldsymbol{y}''|^2}} \exp\left(-\sqrt{x^2+|\boldsymbol{y}''|^2}\right) \mathrm{d}\boldsymbol{y}'' \\ &= -\frac{(N-2)\pi^{(N-2)/2}}{\Gamma(\frac{1}{2}N)} \int_0^\infty \frac{x}{\sqrt{x^2+r^2}} \exp\left(-\sqrt{x^2+r^2}\right) r^{N-3}\,\mathrm{d}r\end{aligned}$$

if $x > R_0$.

Putting $r = \sqrt{s^2+2sx}$ again, we have

$$P'''(x) = -\int_0^\infty x\mathrm{e}^{-x-s}(s^2+2sx)^{(N-4)/2}\,\mathrm{d}s$$

if $x > R_0$. Thus, we get

$$\lim_{x\to\infty} \frac{-P'''(x)}{\frac{1}{2}\Gamma(\frac{1}{2}N-1)(2x)^{(N/2)-1}\mathrm{e}^{-x}} = 1.$$

Thus, we obtain

$$\lim_{x\to\infty} \frac{-P'''(x)}{P''(x)} = 1.$$

Now the Cauchy mean-value theorem gives

$$\frac{P''(x)}{P'(x)} = \frac{P'''(x')}{P''(x')}$$

for some $x' > x$. This yields

$$\lim_{x\to\infty} \frac{P''(x)}{-P'(x)} = 1.$$

Similarly, we have

$$\lim_{x\to\infty} \frac{-P'(x)}{P(x)} = \lim_{x\to\infty} \frac{P''(x)}{-P'(x)} = 1.$$

Thus, we obtain

$$\lim_{x\to\infty} \frac{-P'(x)}{P(x)} = \lim_{x\to\infty} \frac{P''(x)}{-P'(x)} = \lim_{x\to\infty} \frac{-P'''(x)}{P''(x)} = 1.$$

Using these equalities, we obtain

$$0 < \inf_{x\geqslant 1} \frac{|P^{(i)}(x)|}{P(x)} \leqslant \sup_{x\geqslant 1} \frac{|P^{(i)}(x)|}{P(x)} < +\infty.$$

□

We choose $\tilde{\eta} \in C^\infty[0,\infty)$ with

$$\begin{aligned} \tilde{\eta}(r) &= 1 \quad \text{if } 0 \leqslant r \leqslant 1, \\ 0 < \tilde{\eta}(r) &< 1 \quad \text{if } 1 < r < 2, \\ \tilde{\eta}(r) &= 0 \quad \text{if } \qquad r \geqslant 2. \end{aligned}$$

We put $\eta(\boldsymbol{x}') := \tilde{\eta}(|\boldsymbol{x}'|)$ for $\boldsymbol{x}' \in \mathbb{R}^{N-1}$.

Let $a > 0$ be any given constant. For any $\zeta \in L^\infty(\mathbb{R}^{N-1})$ with

$$\left.\begin{aligned} \zeta(0) = 0, \\ \operatorname{supp}\zeta \subset \{(x_1,\dots,x_{N-1}) \mid x_1 \leqslant 0\}, \\ |\zeta(\boldsymbol{x}')| \leqslant |\boldsymbol{x}'| \quad \text{for all } \boldsymbol{x}' \in \mathbb{R}^{N-1}, \end{aligned}\right\} \tag{2.13}$$

we set

$$\begin{aligned} Q(x;a,\zeta) &:= \int_{\mathbb{R}^{N-1}} \rho(x\boldsymbol{e}_1 - \boldsymbol{y}')\eta\left(\frac{\boldsymbol{y}'}{ax}\right)\zeta(\boldsymbol{y}')\,\mathrm{d}\boldsymbol{y}' \qquad \text{for } x > 0, \\ R(x;a,\zeta) &:= \int_{\mathbb{R}^{N-1}} \rho(x\boldsymbol{e}_1 - \boldsymbol{y}')\left(1 - \eta\left(\frac{\boldsymbol{y}'}{ax}\right)\right)\zeta(\boldsymbol{y}')\,\mathrm{d}\boldsymbol{y}' \quad \text{for } x > 0. \end{aligned}$$

Then we have

$$P(x) = Q(x;a,q) + R(x;a,q) \quad \text{for all } x > 0.$$

Lemma 2.5. *Let $a > 0$ be any given constant. Let $\zeta \in L^\infty(\mathbb{R}^{N-1})$ be a given function with (2.13). Then there exists $K(a) > 0$ depending on a such that one has*

$$\max_{i=0,1,2,3} |R^{(i)}(x;a,\zeta)| \leqslant K(a)\exp\left(-\frac{1+\sqrt{1+a^2}}{2}x\right) \quad \textit{for all } x > 0.$$

Hence, lemma 2.4 holds true by replacing $P(x)$ by $Q(x;a,q)$.

Proof. The latter statement follows from the former one and lemma 2.4. It suffices to prove the former statement.

We use the polar coordinate in $\mathbb{R}^{N-1}$ given by

$$\boldsymbol{x}' = \Psi(r,\theta_1,\dots,\theta_{N-2}) \quad \text{for } (\theta_1,\dots,\theta_{N-2}) \in [0,\pi]^{N-3} \times [0,2\pi].$$

Here $I_0 = [0,\pi]^{N-3} \times [0,2\pi]$. We have

$$0 \leqslant \zeta(\boldsymbol{y}') \leqslant ax \quad \text{if } \boldsymbol{y}' \in B(0;ax)$$

for $x > 0$. Here $B(0;ax)$ is a ball in $\mathbb{R}^N$ whose centre is the origin and whose radius is ax. Then we get

$$0 < R(x;a,\zeta) \leqslant 2\pi^{N-2}\int_{ax}^{\infty} \exp\left(-\sqrt{x^2+r^2}\right)r^{N-1}\,\mathrm{d}r \quad \text{if } r > R_0.$$

We obtain

$$\begin{aligned} 0 &< \int_{ax}^{\infty} \exp\bigl(-\sqrt{x^2+r^2}\bigr) r^{N-1}\,\mathrm{d}r \\ &= \int_{(\sqrt{1+a^2}-1)x}^{\infty} \mathrm{e}^{-x-s}(s^2+2sx)^{(N-2)/2}(s+x)\,\mathrm{d}s \\ &\leqslant \biggl(\frac{a^2+2a+4}{a^2}\biggr)^{N/2} \mathrm{e}^{-x} \int_{(\sqrt{1+a^2}-1)x}^{\infty} s^{N-1}\mathrm{e}^{-s}\,\mathrm{d}s. \end{aligned}$$

Here we used $(\sqrt{1+a^2}-1)x < s$, which gives $x \leqslant a^{-2}(2+a)s$. We also get

$$\begin{aligned} 0 &< \int_{(\sqrt{1+a^2}-1)x}^{\infty} s^{N-1}\mathrm{e}^{-s}\,\mathrm{d}s \\ &\leqslant \exp\biggl(-\frac{\sqrt{1+a^2}-1}{2}x\biggr)\int_0^{\infty} s^{N-1}\mathrm{e}^{-s/2}\,\mathrm{d}s \\ &\leqslant 2^N(N-1)!\exp\biggl(-\frac{\sqrt{1+a^2}-1}{2}x\biggr). \end{aligned}$$

Thus, we obtain

$$\lim_{x\to\infty} R(x;a,\zeta)\exp\biggl(\frac{1+\sqrt{1+a^2}}{2}x\biggr) < \infty.$$

Next we estimate derivatives of $R(x;a,\zeta)$. Let $i=1,2,3$. We have

$$R^{(i)}(x;a,\zeta) = \int_{\mathbb{R}^{N-1}} \mathrm{D}_x^i\biggl(\rho(x\boldsymbol{e}_1-\boldsymbol{y}')\biggl(1-\eta\biggl(\frac{\boldsymbol{y}'}{ax}\biggr)\biggr)\biggr)\zeta(\boldsymbol{y}')\,\mathrm{d}\boldsymbol{y}' \quad \text{for } x>0.$$

By direct calculation, we obtain

$$\sup_{x\geqslant 1,\boldsymbol{y}'\in\mathbb{R}^{N-1}} \biggl|\mathrm{D}_x^i\biggl(1-\tilde{\eta}\biggl(\frac{|\boldsymbol{y}'|}{ax}\biggr)\biggr)\biggr| \leqslant M_2 \tag{2.14}$$

for all $i=0,1,2,3$. Here $M_2>0$ is a constant depending only on $\tilde{\eta}$ and is independent of a. We have

$$\mathrm{D}_x^i\biggl(1-\tilde{\eta}\biggl(\frac{|\boldsymbol{y}'|}{ax}\biggr)\biggr) = 0 \quad \text{for all } \boldsymbol{y}'\in B(0;ax).$$

Using (2.1) yields

$$|R^{(i)}(x,a,\zeta)| \leqslant 2^i M_1 M_2 \int_{\mathbb{R}^{N-1}\setminus B(0;ax)} \rho(x\boldsymbol{e}_1-\boldsymbol{y}')|\zeta(\boldsymbol{y}')|\,\mathrm{d}\boldsymbol{y}'.$$

We obtain

$$\int_{\mathbb{R}^{N-1}\setminus B(0;ax)} \rho(x\boldsymbol{e}_1-\boldsymbol{y}')|\zeta(\boldsymbol{y}')|\,\mathrm{d}\boldsymbol{y}' \leqslant 2\pi^{N-2}\int_{ax}^{\infty}\exp\bigl(-\sqrt{x^2+r^2}\bigr)r^{N-1}\,\mathrm{d}r$$

if $r>R_0$. From the argument stated above we obtain the estimates of the derivatives of $R^{(i)}(x;a,\zeta)$. This completes the proof. □

3. Proof of proposition 2.3

We study Ω_j for each $1 \leqslant j \leqslant n$. Without loss of generality we can assume that $\boldsymbol{A}_j = \boldsymbol{e}_{N-1}$, where $\boldsymbol{e}_{N-1} := (0, \dots, 0, 1) \in \mathbb{R}^{N-1}$. For any $i \neq j$, we have $A_{N-1,i} < 1$ from $\boldsymbol{A}_i \neq \boldsymbol{e}_{N-1}$. Now $h_j(\boldsymbol{x}') \geqslant h_i(\boldsymbol{x}')$ is equivalent to

$$x_{N-1} \geqslant (1 - A_{N-1,i})^{-1} \sum_{p=1}^{N-2} A_{p,i} x_p$$

for every $i \neq j$. Then from the definition of Ω_j, we have

$$\Omega_j = \left\{ \boldsymbol{x}' \in \mathbb{R}^{N-1} \;\middle|\; x_{N-1} \geqslant \max_{i \neq j} \frac{\sum_{p=1}^{N-2} A_{p,i} x_p}{1 - A_{N-1,i}} \right\} \tag{3.1}$$

if $\boldsymbol{A}_j = \boldsymbol{e}_{N-1}$. This fact implies that Ω_j is an $(N-1)$-dimensional pyramid in $\mathbb{R}^{N-1}$ for every $j \in \{1, \dots, n\}$. In particular, Ω_j is a convex set in $\mathbb{R}^{N-1}$. The lateral faces of Ω_j are given by

$$\partial \Omega_j = \bigcup_{i \in A'(j)} (\Omega_j \cap \Omega_i).$$

Here $A'(j)$ is a subset of $\{1, \dots, n\} \setminus \{j\}$. We determine $A(j)$ as the minimum of such an $A'(j)$, that is,

$$A(j) := \bigcap \left\{ A'(j) \;\middle|\; \partial \Omega_j = \bigcup_{i \in A'(j)} (\Omega_j \cap \Omega_i) \right\}.$$

We see that $i \in A(j)$ is equivalent to $j \in A(i)$. We call Ω_i and Ω_j are adjacent if and only if $i \in A(j)$.

Lemma 3.1. *If $\boldsymbol{A}_j = \boldsymbol{e}_{N-1}$, one has*

$$\Omega_j = \left\{ \boldsymbol{x}' \in \mathbb{R}^{N-1} \;\middle|\; x_{N-1} \geqslant \max_{i \in A(j)} \frac{\sum_{p=1}^{N-2} A_{p,i} x_p}{1 - A_{N-1,i}} \right\}.$$

Proof. We obtain this lemma from the definition of $A(j)$ and (3.1). □

A set $A(j)$ has at least two elements. For any $i \in A(j)$, $\mathcal{H}_i$ is given by

$$x_{N-1} = (1 - A_{N-1,i})^{-1} \sum_{p=1}^{N-2} A_{p,i} x_p.$$

The normal vector of $\mathcal{H}_i$ is given by $\boldsymbol{e}_{N-1} - \boldsymbol{A}_i$. For each $i, i' \in A(j)$ with $i \neq i'$, let $\theta_{i,i'}$ be the angle between $\mathcal{H}_i$ and $\mathcal{H}_{i'}$ with $0 < \theta_{i,i'} < \pi$. We have

$$0 < \theta_{\min} \leqslant \min_{i,i' \in A(j), i \neq i'} \theta_{i,i'} \leqslant \max_{i,i' \in A(j), i \neq i'} \theta_{i,i'} \leqslant \theta_{\max} < \pi.$$

Let $\boldsymbol{x}' \in \Omega_j$; the length of a perpendicular onto $\mathcal{H}_i$ is given by

$$\frac{(\boldsymbol{A}_j - \boldsymbol{A}_i, \boldsymbol{x}')}{|\boldsymbol{A}_j - \boldsymbol{A}_i|} = (\boldsymbol{b}_{i,j}, \boldsymbol{x}') \geqslant 0 \tag{3.2}$$

for each $i \in A(j)$. Here we set

$$\boldsymbol{b}_{i,j} := \frac{\boldsymbol{A}_j - \boldsymbol{A}_i}{|\boldsymbol{A}_j - \boldsymbol{A}_i|} \in \mathbb{R}^{N-1}$$

for all $i \neq j$. The foot of this perpendicular is given by

$$\boldsymbol{g}_i(\boldsymbol{x}') := \boldsymbol{x}' - (\boldsymbol{b}_{i,j}, \boldsymbol{x}')\boldsymbol{b}_{i,j} \in \mathbb{R}^{N-1}$$

for all $i \neq j$.

Now we return to studying $\tilde{\varphi}_j(\boldsymbol{x}')$ and $S(\boldsymbol{x}')$. By the definition of S we have

$$S(\boldsymbol{x}') = \frac{k^2(m^2 - |\nabla\varphi(\boldsymbol{x}')|^2)}{\sqrt{1+|\nabla\varphi(\boldsymbol{x}')|^2}(c + k\sqrt{1+|\nabla\varphi(\boldsymbol{x}')|^2})}. \tag{3.3}$$

By lemma 2.2, we have

$$\begin{aligned} 0 &< m^2 - |\nabla\varphi|^2 \\ &= m^2 - |\nabla(\tilde{\varphi} + h_j)|^2 \\ &= m^2 - |\nabla\tilde{\varphi}|^2 - |\nabla h_j|^2 - 2(\nabla\tilde{\varphi}, \nabla h_j) \\ &= m^2 - |\nabla\tilde{\varphi}|^2 - m^2 - 2m(\boldsymbol{A}_j, \nabla\tilde{\varphi}) \\ &= -|\nabla\tilde{\varphi}|^2 - 2m(\boldsymbol{A}_j, \nabla\tilde{\varphi}) \end{aligned} \tag{3.4}$$

for all $\boldsymbol{x}' \in \Omega_j$. Then using lemma 2.2, we have

$$\frac{k^3}{2c^2}(-2m(\boldsymbol{A}_j, \nabla\tilde{\varphi}) - |\nabla\tilde{\varphi}|^2) \leqslant S(\boldsymbol{x}') \leqslant \frac{k^2}{c+k}(-2m(\boldsymbol{A}_j, \nabla\tilde{\varphi}) - |\nabla\tilde{\varphi}|^2)$$

for all $\boldsymbol{x}' \in \Omega_j$.

We study $S(\boldsymbol{x}')$ when $\boldsymbol{x}'$ lies near $\bigcup_{j=1}^n \partial\Omega_j$.

LEMMA 3.2. *For any given $b > 0$ one has*

$$0 < \inf\left\{S(\boldsymbol{x}') \,\middle|\, \operatorname{dist}\left(\boldsymbol{x}', \bigcup_{j=1}^n \partial\Omega_j\right) \leqslant b\right\}.$$

Proof. Let $b > 0$ be given arbitrarily. Using lemma 3.1, we have

$$\inf_{\boldsymbol{x}' \in \Omega_j} \int_{\Omega_j} \rho(\boldsymbol{x}' - \boldsymbol{y}')\,\mathrm{d}\boldsymbol{y}' > 0, \qquad \inf_{\operatorname{dist}(\boldsymbol{x}', \Omega_i) \leqslant b} \int_{\Omega_i} \rho(\boldsymbol{x}' - \boldsymbol{y}')\,\mathrm{d}\boldsymbol{y}' > 0$$

for all $i, j \in \{1, \ldots, n\}$. Without loss of generality we can assume $\boldsymbol{x}' \in \Omega_j$ for some j and $\operatorname{dist}(\boldsymbol{x}', \Omega_i) \leqslant b$ for some $i \in A(j)$.

By lemma 2.2 and its proof we have

$$\nabla\varphi(\boldsymbol{x}') = m \sum_{j=1}^n \int_{\Omega_j} \rho(\boldsymbol{x}' - \boldsymbol{y}')\,\mathrm{d}\boldsymbol{y}'\boldsymbol{A}_j.$$

Using the inequalities stated above, we obtain

$$0 < \inf \left\{ m - |\nabla \varphi| \;\middle|\; \operatorname{dist}\left(\boldsymbol{x}', \bigcup_{j=1}^{n} \Gamma_j \leqslant b \right) \right\}.$$

Using this inequality and (3.3), we complete the proof. □

We study $\tilde{\varphi}$ in Ω_j. Without loss of generality we assume $\boldsymbol{A}_j = \boldsymbol{e}_{N-1}$. Let H be the Heaviside function given by

$$H(x) := \begin{cases} 1, & x \geqslant 0, \\ 0, & x < 0. \end{cases}$$

For each $i \in A(j)$ we set

$$\chi_i(\boldsymbol{x}') := H\left((1 - A_{N-1,i})^{-1} \sum_{p=1}^{N-2} A_{p,i} x_p - x_{N-1} \right)$$

for all $\boldsymbol{x}' = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$. We set $A(j) = \{i_1, \ldots, i_\ell\}$, where $1 \leqslant \ell \leqslant n$. We have

$$0 = (h - h_j) \prod_{\kappa=1}^{\ell} (1 - \chi_\kappa) \quad \text{in } \mathbb{R}^{N-1}.$$

We set

$$1 - \prod_{\kappa=1}^{\ell} (1 - \chi_\kappa) = \sum_{\kappa=1}^{\ell} \chi_\kappa p_\kappa(\chi_1, \ldots, \chi_\ell),$$

where p_κ is a polynomial. The degree of p_κ is no greater than ℓ.

Then we have

$$0 = (h - h_j)\left(1 - \sum_{\kappa=1}^{\ell} \chi_\kappa p_\kappa(\chi_1, \ldots, \chi_\ell) \right)$$

and thus

$$h - h_j = \sum_{\kappa=1}^{\ell} \chi_\kappa p_\kappa(\chi_1, \ldots, \chi_\ell)(h - h_j). \tag{3.5}$$

For any $\kappa \in \{1, \ldots, \ell\}$ we set

$$c_\kappa := \max\{ |p_\kappa(\chi_1, \ldots, \chi_\ell)| \mid \chi_i = 0 \text{ or } 1 \text{ for all } 1 \leqslant i \leqslant \ell \}.$$

We put $b_* := \max\{1, \max_{1 \leqslant \kappa \leqslant \ell} c_\kappa\}$ and obtain $1 \leqslant b_* < \infty$.

Lemma 3.3. *Let $\boldsymbol{A}_j = \boldsymbol{e}_{N-1}$. One has*

$$|h(\boldsymbol{x}') - h_j(\boldsymbol{x}')| \leqslant b_* \sum_{i \in A(j)} (h(\boldsymbol{x}') - h_j(\boldsymbol{x}'))\chi_i(\boldsymbol{x}') \quad \textit{for all } \boldsymbol{x}' \in \mathbb{R}^{N-1}.$$

Proof. We obtain $0 \leqslant h - h_j \leqslant h$. Then we get this lemma immediately by (3.5). □

We set

$$a_* := \tfrac{1}{4}\sin\tfrac{1}{2}\theta_{\min}\cot\tfrac{1}{2}\theta_{\max},$$

and obtain $0 < a_* < \infty$.

LEMMA 3.4. *For any $\boldsymbol{x}'$ that lies in the interior of Ω_j, one has*

$$\overline{B(\boldsymbol{g}_i(\boldsymbol{x}'); 2a_*(\boldsymbol{b}_{i,j}, \boldsymbol{x}'))} \subset \Omega_i \cup \Omega_j$$

with respect to any $i \in A(j)$ with $\operatorname{dist}(\boldsymbol{x}', \partial\Omega) = \operatorname{dist}(\boldsymbol{x}', \mathcal{H}_i)$.

Proof. For any $s \in A(j)$ with $s \neq i$ we have

$$\mathcal{H}_s \cap \overline{\mathcal{H}_i \cap B(\boldsymbol{g}_i(\boldsymbol{x}'); \cot(\tfrac{1}{2}\theta_{\max})(\boldsymbol{b}_{i,j}, \boldsymbol{x}'))} = \emptyset.$$

Thus, we obtain

$$\overline{\mathcal{H}_i \cap B(\boldsymbol{g}_i(\boldsymbol{x}'); \cot(\tfrac{1}{2}\theta_{\max})(\boldsymbol{b}_{i,j}, \boldsymbol{x}'))} \subset \Omega_i \cap \Omega_j.$$

Then we get

$$\overline{B(\boldsymbol{g}_i(\boldsymbol{x}'); \sin(\tfrac{1}{2}\theta_{\min})\cot(\tfrac{1}{2}\theta_{\max})(\boldsymbol{b}_{i,j}, \boldsymbol{x}'))} \subset \Omega_i \cup \Omega_j.$$

This completes the proof. □

Proof of proposition 2.3. It suffices to prove this proposition by assuming $\boldsymbol{x}' \in \Omega_j$ with $\operatorname{dist}(\boldsymbol{x}', \partial\Omega_j) \geqslant 1$ due to lemma 2.2 and lemma 3.2. For $\boldsymbol{x}' \in \Omega_j$ let

$$C(j) := \{i \in A(j) \mid \operatorname{dist}(\boldsymbol{x}', \partial\Omega_j) = \operatorname{dist}(\boldsymbol{x}', \mathcal{H}_i)\}.$$

There exists a constant $\nu > 1$ such that we have

$$\nu \operatorname{dist}(\boldsymbol{x}', \mathcal{H}_i) \leqslant \operatorname{dist}(\boldsymbol{x}', \mathcal{H}_s)$$

for all $\boldsymbol{x}' \in \Omega_j$ and all $s \in A(j) \setminus C(j)$. We set

$$\lambda := (\boldsymbol{b}_{i,j}, \boldsymbol{x}') = \operatorname{dist}(\boldsymbol{x}', \mathcal{H}_i) \quad \text{for } i \in C(j).$$

Note that λ is independent of $i \in C(j)$.

It suffices to prove this proposition by assuming that $C(j)$ remains unchanged, say C_0, as $|\boldsymbol{x}'| \to +\infty$. Here C_0 is a subset of $\{1, \ldots, n\}$. We set

$$\Delta_j := \{\boldsymbol{x}' \in \Omega_j \mid \operatorname{dist}(\boldsymbol{x}', \partial\Omega_j) \geqslant 1, \{i \in A(j) \mid \operatorname{dist}(\boldsymbol{x}', \partial\Omega_j) = \operatorname{dist}(\boldsymbol{x}', \mathcal{H}_i)\} = C_0\}.$$

We assume $\boldsymbol{x}' \in \Delta_j$. We have

$$B(\boldsymbol{g}_i(\boldsymbol{x}'); 2a_*\lambda) \subset \Omega_i \cap \Omega_j$$

for all $i \in C(j)$ and $\boldsymbol{x}' \in \Delta_j$.

We have

$$1 - \sum_{i \in C(j)} \eta\left(\frac{\boldsymbol{y}' - \boldsymbol{g}_i(\boldsymbol{x}')}{a_*\lambda}\right) = 0 \quad \text{if } \boldsymbol{y}' \in \bigcup_{i \in C(j)} B(\boldsymbol{g}_i(\boldsymbol{x}'); a_*\lambda).$$

For any $\boldsymbol{x}' \in \Omega_j$ with $\operatorname{dist}(\boldsymbol{x}', \partial\Omega_j) \geqslant 1$ we have $\tilde{\varphi}(\boldsymbol{x}') = I(\boldsymbol{x}') + J(\boldsymbol{x}')$, where

$$I(\boldsymbol{x}') := \int_{\mathbb{R}^{N-1}} \rho(\boldsymbol{x}' - \boldsymbol{y}') \sum_{i \in C(j)} \eta\left(\frac{\boldsymbol{y}' - \boldsymbol{g}_i(\boldsymbol{x}')}{a_* \lambda}\right)(h(\boldsymbol{y}') - h_j(\boldsymbol{y}'))\,\mathrm{d}\boldsymbol{y}',$$

$$J(\boldsymbol{x}') := \int_{\mathbb{R}^{N-1}} \rho(\boldsymbol{x}' - \boldsymbol{y}')\left(1 - \sum_{i \in C(j)} \eta\left(\frac{\boldsymbol{y}' - \boldsymbol{g}_i(\boldsymbol{x}')}{a_* \lambda}\right)\right)(h(\boldsymbol{y}') - h_j(\boldsymbol{y}'))\,\mathrm{d}\boldsymbol{y}'.$$

We have

$$I(\boldsymbol{x}') = m \sum_{i \in C(j)} |\boldsymbol{A}_i - \boldsymbol{A}_j| Q(\lambda; a_*, q). \tag{3.6}$$

Using lemma 3.3, we obtain

$$\begin{aligned}
|J(\boldsymbol{x}')| &\leqslant 2mb_* \sum_{i \in C(j)} \int_{\mathbb{R}^{N-1}} \rho(\boldsymbol{x}' - \boldsymbol{y}')\left(1 - \sum_{i \in C(j)} \eta\left(\frac{\boldsymbol{y}' - \boldsymbol{g}_i(\boldsymbol{x}')}{a_* \lambda}\right)\right) \\
&\qquad\qquad \times \chi_i(\boldsymbol{y}') \frac{h(\boldsymbol{y}') - h_j(\boldsymbol{y}')}{2m}\,\mathrm{d}\boldsymbol{y}' \\
&\quad + 2mb_* \sum_{i \in A(j) \setminus C(j)} \int_{\mathbb{R}^{N-1}} \rho(\boldsymbol{x}' - \boldsymbol{y}')\chi_i(\boldsymbol{y}') \frac{h(\boldsymbol{y}') - h_j(\boldsymbol{y}')}{2m}\,\mathrm{d}\boldsymbol{y}' \\
&\leqslant 2mb_* \sum_{i \in C(j)} R(\lambda; a_*, q) + 2mb_* \sum_{i \in A(j) \setminus C(j)} P(\lambda; a_*, q) \\
&\leqslant 2mb_* nK(a_*) \exp\left(-\frac{1 + \sqrt{1 + (a_*)^2}}{2}\lambda\right) + 2mb_* nP(\nu\lambda; a_*, q).
\end{aligned}$$

Putting

$$a_0 := \min\left\{\nu, \frac{1 + \sqrt{1 + (a_*)^2}}{2}\right\} \in (1, \infty),$$

we have

$$|J(\boldsymbol{x}')| \leqslant M_3 \mathrm{e}^{-a_0 \lambda},$$

where $M_3 > 0$ is a constant.

By (2.14) we have

$$\left|\left(\frac{\partial}{\partial x_1}\right)^{j_1} \cdots \left(\frac{\partial}{\partial x_{N-1}}\right)^{j_{N-1}} \eta\left(\frac{\boldsymbol{y}' - \boldsymbol{g}_i(\boldsymbol{x}')}{a_* \lambda}\right)\right| \leqslant M_2 \tag{3.7}$$

for $j_1 \geqslant 0, \dots, j_{N-1} \geqslant 0$ with $\sum_{p=1}^{N-1} j_p \leqslant 3$.

Using (2.1) and (3.7), we have

$$\begin{aligned}
&|\mathrm{D}_1^{j_1} \cdots \mathrm{D}_{N-1}^{j_{N-1}} J(\boldsymbol{x}')| \\
&\qquad \leqslant 8M_1 M_2 \int_{\mathbb{R}^{N-1} \setminus \bigcup_{i \in C(j)} B(\boldsymbol{g}_i(\boldsymbol{x}'); a_* \lambda)} \rho(\boldsymbol{x}' - \boldsymbol{y}')(h(\boldsymbol{y}') - h_j(\boldsymbol{y}'))\,\mathrm{d}\boldsymbol{y}'.
\end{aligned}$$

Now we have

$$
\begin{aligned}
&\int_{\mathbb{R}^{N-1}\setminus\bigcup_{i\in C(j)} B(\boldsymbol{g}_i(\boldsymbol{x}');a_*\lambda)} \rho(\boldsymbol{x}'-\boldsymbol{y}')(h(\boldsymbol{y}')-h_j(\boldsymbol{y}'))\,\mathrm{d}\boldsymbol{y}' \\
&\qquad \leqslant 2mb_* \sum_{i\in C(j)} \int_{\mathbb{R}^{N-1}\setminus B(\boldsymbol{g}_i(\boldsymbol{x}');a_*\lambda)} \rho(\boldsymbol{x}'-\boldsymbol{y}')\chi_i(\boldsymbol{y}')\frac{h(\boldsymbol{y}')-h_j(\boldsymbol{y}')}{2m}\,\mathrm{d}\boldsymbol{y}' \\
&\qquad\qquad + 2mb_* \sum_{i\in A(j)\setminus C(j)} \int_{\mathbb{R}^{N-1}} \rho(\boldsymbol{x}'-\boldsymbol{y}')\chi_i(\boldsymbol{y}')\frac{h(\boldsymbol{y}')-h_j(\boldsymbol{y}')}{2m}\,\mathrm{d}\boldsymbol{y}' \\
&\qquad \leqslant 2mb_*nR(\lambda;a_*,q) + 2mb_*nP(\nu\lambda;a_*,q).
\end{aligned}
$$

Thus, we obtain

$$
|\mathrm{D}_1^{j_1}\cdots\mathrm{D}_{N-1}^{j_{N-1}}J(\boldsymbol{x}')| \leqslant M_4\mathrm{e}^{-a_0\lambda} \tag{3.8}
$$

for all $\lambda>0$, $j_1\geqslant 0,\dots,j_{N-1}\geqslant 0$ with $\sum_{p=1}^{N-1} j_p\leqslant 3$. From lemma 2.5, and (3.3), (3.6) and (3.8), we obtain

$$
\lim_{\lambda\to\infty}\ \sup_{\boldsymbol{x}'\in\Delta_j,(\boldsymbol{b}_{i,j},\boldsymbol{x}')=\lambda} \frac{S(\boldsymbol{x}')}{\sum_{i\in C(j)}(1-(\boldsymbol{A}_i,\boldsymbol{A}_j))Q(\lambda;a_*,q)} = \frac{2m^2k^2}{c+k} \tag{3.9}
$$

and

$$
|\mathrm{D}_1^{j_1}\cdots\mathrm{D}_{N-1}^{j_{N-1}}\tilde{\varphi}(\boldsymbol{x}')| \leqslant M_5Q(\lambda;a_*,q) \quad \text{for } \boldsymbol{x}'\in\Delta_j,(\boldsymbol{b}_{i,j},\boldsymbol{x}')=\lambda>0. \tag{3.10}
$$

Here $M_5>0$ is a constant, and $j_1\geqslant 0,\dots,j_{N-1}\geqslant 0$ satisfy $\sum_{p=1}^{N-1} j_p\leqslant 3$. This completes the proof of proposition 2.3. □

4. Proof of theorem 1.1

In this section we construct a pyramidal travelling-front solution and prove theorem 1.1 by constructing a supersolution and a subsolution and by using the methods of [8, 12].

We put $\boldsymbol{\xi}=(\xi_1,\dots,\xi_N)\in\mathbb{R}^N$ and $\boldsymbol{\xi}'=(\xi_1,\dots,\xi_{N-1})\in\mathbb{R}^{N-1}$. We study $\xi_N=\varphi(\boldsymbol{\xi}')$.

For $\alpha\in(0,1)$ we consider the graph of

$$
\left\{\boldsymbol{x}\in\mathbb{R}^N \;\middle|\; x_N=\frac{1}{\alpha}\varphi(\alpha\boldsymbol{x}')\right\}. \tag{4.1}
$$

Later we will choose $\alpha>0$ to be small enough. We note that

$$
\frac{1}{\alpha}h(\alpha\boldsymbol{x}')=h(\boldsymbol{x}').
$$

Putting $\boldsymbol{\xi}=\alpha\boldsymbol{x}$, we have $\xi_N=\varphi(\boldsymbol{\xi}')$. For a given constant $\boldsymbol{b}'\in\mathbb{R}^{N-1}$, the tangent plane of (4.1) at $(\boldsymbol{b}',(1/\alpha)\varphi(\alpha\boldsymbol{b}'))$ is expressed by

$$
-(\nabla\varphi(\alpha\boldsymbol{b}'),(\boldsymbol{x}'-\boldsymbol{b}'))+x_N-\frac{1}{\alpha}\varphi(\boldsymbol{b}')=0. \tag{4.2}
$$

Here we denote $(\partial/\partial\xi_1, \partial/\partial\xi_2, \ldots, \partial/\partial\xi_{N-1})$ by ∇. The length of the perpendicular from $\boldsymbol{b} = (\boldsymbol{b}', b_N) \in \mathbb{R}^N$ onto the tangent plane is given by

$$\frac{|b_N - \varphi(\alpha\boldsymbol{b}')/\alpha|}{\sqrt{1 + |\nabla\varphi(\alpha\boldsymbol{b}')|^2}}.$$

We set

$$\hat{\mu} := \frac{x_N - \varphi(\alpha\boldsymbol{x}')/\alpha}{\sqrt{1 + |\nabla\varphi(\alpha\boldsymbol{x}')|^2}} = \frac{1}{\alpha}\frac{\xi_N - \varphi(\boldsymbol{\xi}')}{\sqrt{1 + |\nabla\varphi(\boldsymbol{\xi}')|^2}}, \tag{4.3}$$

and obtain

$$\frac{\partial\hat{\mu}}{\partial x_N} = \frac{1}{\sqrt{1 + |\nabla\varphi(\boldsymbol{\xi}')|^2}}, \qquad \frac{\partial^2\hat{\mu}}{\partial x_N^2} = 0.$$

We have

$$\frac{\partial\hat{\mu}}{\partial x_i} = -\frac{1}{\sqrt{1 + |\nabla\varphi|^2}}\frac{\partial\varphi}{\partial\xi_i} + \alpha\hat{\mu}F_i, \qquad \frac{\partial^2\hat{\mu}}{\partial x_i^2} = \alpha G_i + \alpha^2\hat{\mu}H_i,$$

where

$$F_i(\boldsymbol{\xi}') := \sqrt{1 + |\nabla\varphi|^2}\frac{\partial}{\partial\xi_i}\left(\frac{1}{\sqrt{1 + |\nabla\varphi|^2}}\right), \tag{4.4}$$

$$G_i(\boldsymbol{\xi}') := -\frac{\partial}{\partial\xi_i}\left(\frac{1}{\sqrt{1 + |\nabla\varphi|^2}}\frac{\partial\varphi}{\partial\xi_i}\right) - \frac{F_i(\boldsymbol{\xi}')}{\sqrt{1 + |\nabla\varphi|^2}}\frac{\partial\varphi}{\partial\xi_i}, \tag{4.5}$$

$$H_i(\boldsymbol{\xi}') := \frac{\partial F_i}{\partial\xi_i} + (F_i(\boldsymbol{\xi}'))^2 \tag{4.6}$$

for $i = 1, \ldots, N-1$. We define

$$\bar{v}(\boldsymbol{x}) = \Phi(\hat{\mu}) + \sigma(\boldsymbol{x}'), \tag{4.7}$$

where $\hat{\mu}$ is as in (4.3) and $\sigma(\boldsymbol{x}') := \varepsilon S(\alpha\boldsymbol{x}')$. Here we will fix $\varepsilon > 0$ in (4.9). We have

$$\frac{\partial\bar{v}}{\partial x_N} = \frac{1}{\sqrt{1 + |\nabla\varphi|^2}}\Phi'(\hat{\mu}), \qquad \frac{\partial^2\bar{v}}{\partial x_N^2} = \frac{1}{1 + |\nabla\varphi|^2}\Phi''(\hat{\mu}).$$

Since

$$\frac{\partial\bar{v}}{\partial x_i} = \Phi'(\hat{\mu})\frac{\partial\hat{\mu}}{\partial x_i} + \frac{\partial\sigma}{\partial x_i}, \qquad \frac{\partial^2\bar{v}}{\partial x_i^2} = \Phi''(\hat{\mu})\left(\frac{\partial\hat{\mu}}{\partial x_i}\right)^2 + \Phi'(\hat{\mu})\frac{\partial^2\hat{\mu}}{\partial x_i^2} + \frac{\partial^2\sigma}{\partial x_i^2}$$

and

$$\begin{aligned}\sum_{i=1}^{N-1}\left(\frac{\partial^2\hat{\mu}}{\partial x_i^2}\right)^2 &= \sum_{i=1}^{N-1}\left(-\frac{1}{\sqrt{1 + |\nabla\varphi|^2}}\frac{\partial\varphi}{\partial\xi_i} + \alpha\hat{\mu}F_i\right)^2 \\ &= \frac{|\nabla\varphi|^2}{1 + |\nabla\varphi|^2} + \sum_{i=1}^{N-1}\left(\frac{-2\alpha\hat{\mu}F_i}{\sqrt{1 + |\nabla\varphi|^2}}\frac{\partial\varphi}{\partial\xi_i} + \alpha^2\hat{\mu}^2(F_i)^2\right)\end{aligned}$$

for $i = 1, \ldots, N-1$, we have

$$\begin{aligned}\sum_{i=1}^{N-1} \frac{\partial^2 \bar{v}}{\partial x_i^2} &= \varPhi'(\hat{\mu}) \sum_{i=1}^{N-1} \frac{\partial^2 \hat{\mu}}{\partial x_i^2} + \varPhi''(\hat{\mu}) \sum_{i=1}^{N-1} \left(\frac{\partial^2 \hat{\mu}}{\partial x_i^2}\right)^2 + \sum_{i=1}^{N-1} \frac{\partial^2 \sigma}{\partial x_i^2} \\ &= \varPhi'(\hat{\mu}) \sum_{i=1}^{N-1} (\alpha G_i + \alpha^2 \hat{\mu} H_i) + \varPhi''(\hat{\mu}) \left(\frac{|\nabla\varphi|^2}{1+|\nabla\varphi|^2}\right) \\ &\quad + \varPhi''(\hat{\mu}) \sum_{i=1}^{N-1} \left(\frac{-2\alpha\hat{\mu}F_i}{\sqrt{1+|\nabla\varphi|^2}} \frac{\partial\varphi}{\partial\xi_i} + \alpha^2\hat{\mu}^2 (F_i)^2\right) + \sum_{i=1}^{N-1} \frac{\partial^2 \sigma}{\partial x_i^2}.\end{aligned}$$

We calculate $\mathfrak{L}[\bar{v}]$ as

$$\begin{aligned}\mathfrak{L}[\bar{v}] &= -\sum_{i=1}^{N} \frac{\partial^2 \bar{v}}{\partial x_i^2} - c\frac{\partial \bar{v}}{\partial x_N} - f(\bar{v}) \\ &= -\varPhi'(\hat{\mu}) \sum_{i=1}^{N-1} (\alpha G_i + \alpha^2 \hat{\mu} H_i) - \varPhi''(\hat{\mu}) \left(\frac{|\nabla\varphi|^2}{1+|\nabla\varphi|^2}\right) \\ &\quad + \varPhi''(\hat{\mu}) \sum_{i=1}^{N-1} \left(\frac{2\alpha\hat{\mu}F_i}{\sqrt{1+|\nabla\varphi|^2}} \frac{\partial\varphi}{\partial\xi_i} - \alpha^2\hat{\mu}^2 (F_i)^2\right) - \sum_{i=1}^{N-1} \frac{\partial^2 \sigma}{\partial x_i^2} \\ &\quad - \frac{1}{1+|\nabla\varphi|^2} \varPhi''(\hat{\mu}) - \frac{c}{\sqrt{1+|\nabla\varphi|^2}} \varPhi'(\hat{\mu}) - f(\bar{v}) \\ &= -\varPhi''(\hat{\mu}) - \frac{c}{\sqrt{1+|\nabla\varphi|^2}} \varPhi'(\hat{\mu}) - f(\varPhi(\hat{\mu}) + \sigma) - \varPhi'(\hat{\mu}) \sum_{i=1}^{N-1} (\alpha G_i + \alpha^2 \hat{\mu} H_i) \\ &\quad + \varPhi''(\hat{\mu}) \sum_{i=1}^{N-1} \left(\frac{2\alpha\hat{\mu}F_i}{\sqrt{1+|\nabla\varphi|^2}} \frac{\partial\varphi}{\partial\xi_i} - \alpha^2\hat{\mu}^2 (F_i)^2\right) - \varepsilon\alpha^2 \sum_{i=1}^{N-1} \frac{\partial^2 S}{\partial \xi_i^2}.\end{aligned}$$

We define

$$\begin{aligned}Y(\boldsymbol{\xi}', \mu; \varepsilon, \alpha) &:= -\varPhi'(\mu) \sum_{i=1}^{N-1} (G_i + \alpha\mu H_i) \\ &\quad + \varPhi''(\mu) \sum_{i=1}^{N-1} \left(\frac{2\mu F_i}{\sqrt{1+|\nabla\varphi|^2}} \frac{\partial\varphi}{\partial\xi_i} - \alpha\mu^2 (F_i)^2\right) - \varepsilon\alpha \sum_{i=1}^{N-1} \frac{\partial^2 S}{\partial \xi_i^2}.\end{aligned}$$

Thus, we have

$$\mathfrak{L}[\bar{v}] = -\varPhi''(\hat{\mu}) - \frac{c}{\sqrt{1+|\nabla\varphi|^2}} \varPhi'(\hat{\mu}) - f(\varPhi(\hat{\mu}) + \sigma) + \alpha Y(\boldsymbol{\xi}', \hat{\mu}; \varepsilon, \alpha).$$

Using the first equality of (1.1),

$$\sigma \int_0^1 f'(\varPhi(\hat{\mu}) + s\sigma)\, \mathrm{d}s = f(\varPhi(\hat{\mu}) + \sigma) - f(\varPhi(\hat{\mu}))$$

and (2.3), we then obtain

$$\mathfrak{L}[\bar{v}] = -\Phi'(\hat{\mu})S(\boldsymbol{\xi}') - \sigma \int_0^1 f'(\Phi(\hat{\mu} + s\sigma))\,\mathrm{d}s + \alpha Y(\boldsymbol{\xi}', \hat{\mu}; \varepsilon, \alpha).$$

We have

$$|Y(\boldsymbol{\xi}', \mu; \varepsilon, \alpha)| \leqslant \max\{|\Phi'(\mu)|, |\mu\Phi'(\mu)|, |\mu\Phi''(\mu)|, |\mu^2\Phi''(\mu)|\} \times \left(\sum_{i=1}^{N-1} \left(|G_i| + |H_i| + 2|F_i| + |F_i|^2 + \left| \frac{\partial^2 S}{\partial \xi_i^2} \right| \right) \right)$$

for $0 < \alpha < 1$. Using lemmas 2.1 and 2.2, we have

$$\sum_{i=1}^{N-1} \left(|G_i| + |H_i| + 2|F_i| + |F_i|^2 + \left| \frac{\partial^2 S}{\partial \xi_i^2} \right| \right) \leqslant \nu_0 \sum_{2 \leqslant \sum_{p=1}^{N-1} i_p \leqslant 3} |\mathrm{D}_1^{i_1} \mathrm{D}_2^{i_2} \cdots \mathrm{D}_{N-1}^{i_{N-1}} \varphi(\boldsymbol{\xi}')|$$

with a constant $\nu_0 > 0$ for all $\boldsymbol{\xi}' \in \mathbb{R}^{N-1}$. Using proposition 2.3, there exists $\nu_* > 0$ satisfying

$$\frac{|Y(\boldsymbol{\xi}', \mu; \varepsilon, \alpha)|}{S(\boldsymbol{\xi}')} < \nu_*$$

for all $\boldsymbol{\xi}' \in \mathbb{R}^{N-1}$, $\mu \in \mathbb{R}$, $\varepsilon \in (0,1)$ and $\alpha \in (0,1)$. Constants ν_0 and ν_* are independent of $\boldsymbol{\xi}'$, $\alpha \in (0,1)$ and $\varepsilon \in (0,1)$. We continue to calculate $\mathfrak{L}[\bar{v}]$ as

$$\mathfrak{L}[\bar{v}] = S(\boldsymbol{\xi}') \left(-\Phi'(\hat{\mu}) - \varepsilon \int_0^1 f' \left(\Phi(\hat{\mu}) + s\sigma \right) \mathrm{d}s + \alpha \frac{Y(\boldsymbol{\xi}', \hat{\mu}; \varepsilon, \alpha)}{S(\boldsymbol{\xi}')} \right).$$

Thus, we have

$$\mathfrak{L}[\bar{v}] \geqslant S(\boldsymbol{\xi}') \left(-\Phi'(\hat{\mu}) + \varepsilon \int_0^1 (-f'(\Phi(\hat{\mu}) + s\sigma))\,\mathrm{d}s - \alpha\nu_* \right). \tag{4.8}$$

We set

$$\omega := \inf_{\boldsymbol{x}' \in \mathbb{R}^{N-1}} \frac{\varphi(\boldsymbol{x}') - h(\boldsymbol{x}')}{S(\boldsymbol{x}')} \in (0, \infty)$$

by using proposition 2.3. Now we choose ε small enough to satisfy

$$0 < \varepsilon < \min\left\{ \frac{1}{2}, \frac{\delta_*}{c}, \frac{4K_0}{\mathrm{e}k\kappa_0}, \frac{\min_{-1+\delta_* \leqslant \Phi(p) \leqslant 1-\delta_*}(-\Phi'(p))}{4\max_{|s| \leqslant 1+\delta_*} |f'(s)|} \right\}. \tag{4.9}$$

Then we choose α small enough to satisfy

$$0 < \alpha < \min\left\{ \frac{1}{2}, \frac{\varepsilon\beta}{\nu_*}, \frac{\min_{-1+\delta_* \leqslant \Phi(p) \leqslant 1-\delta_*}(-\Phi'(p))}{4\nu_*}, \frac{k^2\omega\kappa_0}{2c\log(4K_0/\mathrm{e}k\kappa_0\varepsilon)} \right\}. \tag{4.10}$$

Now we show that $\bar{v}$ is a supersolution and is larger than our subsolution.

LEMMA 4.1. *Assume ε and α satisfy (4.9) and (4.10), respectively. Let $\underline{v}$ and $\bar{v}$ be as in (1.9) and (4.7), respectively. Then*

$$\mathfrak{L}[\bar{v}] > 0 \quad \textit{in } \mathbb{R}^N$$

holds true. Moreover, one has $\underline{v}(\boldsymbol{x}) < \bar{v}(\boldsymbol{x})$ in $\mathbb{R}^N$.

Proof. By lemma 2.2 and (4.9) we have

$$|s\varepsilon S| \leqslant s\varepsilon c \leqslant \delta_* \quad \text{for all } s \in [0,1].$$

We consider the case of $\Phi(\hat{\mu}) < -1+\delta_*$ or $1-\delta_* < \Phi(\hat{\mu})$. We have $\Phi(\hat{\mu}) + s\varepsilon S \leqslant -1+2\delta_*$ or $1-2\delta_* \leqslant \Phi(\hat{\mu}) + s\varepsilon S$. Since $\sigma = \varepsilon S$, we have

$$\Phi(\hat{\mu}) + s\sigma \leqslant -1+2\delta_* \text{ or } 1-2\delta_* \leqslant \Phi(\hat{\mu}) + s\sigma.$$

From the definition of δ_* we have

$$\int_0^1 -f'(\Phi(\hat{\mu}) + s\sigma)\,\mathrm{d}s > \beta.$$

Combining this inequality, $-\Phi'(\hat{\mu}) > 0$ and (4.8), we have

$$\mathfrak{L}[\bar{v}] \geqslant S(\boldsymbol{\xi}')(\varepsilon\beta - \alpha\nu_*) > 0.$$

Next, we consider the case of $-1+\delta_* \leqslant \Phi(\hat{\mu}) \leqslant 1-\delta_*$. We have

$$-\Phi'(\hat{\mu}) \geqslant \min_{-1+\delta_* \leqslant \Phi(p) \leqslant 1-\delta_*} (-\Phi'(p))$$

and

$$-\varepsilon \int_0^1 f'(\Phi + \sigma s)\,\mathrm{d}s \geqslant -\varepsilon \max_{|s|\leqslant 1+\delta_*} |f'(s)|.$$

Combing these inequalities and (4.8), we have

$$\mathfrak{L}[\bar{v}] \geqslant S(\boldsymbol{\xi}')\biggl(\min_{-1+\delta_* \leqslant \Phi(p) \leqslant 1-\delta_*} (-\Phi'(p)) - \varepsilon \max_{|s|\leqslant 1+\delta_*} |f'(s)| - \alpha\nu_*\biggr) > 0.$$

Thus, $\bar{v}$ is a supersolution.

Now we prove the latter statement. It suffices to prove that

$$\Phi\biggl(\frac{k}{c}(x_N - h_j(\boldsymbol{x}'))\biggr) < \bar{v}(\boldsymbol{x}) \tag{4.11}$$

for any fixed j. If we have

$$\hat{\mu} \leqslant \frac{k}{c}(x_N - h_j(\boldsymbol{x}')),$$

we obtain

$$\Phi\biggl(\frac{k}{c}(x_N - h_j(\boldsymbol{x}'))\biggr) \leqslant \Phi(\hat{\mu}) < \bar{v}(\boldsymbol{x})$$

using $\Phi' < 0$. Thus, it suffices to consider the case when

$$\frac{k}{c}(x_N - h_j(\boldsymbol{x}')) < \hat{\mu}.$$

Substituting the definition of $\hat{\mu}$ into this inequality, we obtain

$$\frac{k}{c}(x_N - h_j(\boldsymbol{x}')) < \frac{x_N - \varphi(\boldsymbol{\xi}')/\alpha}{\sqrt{1+|\nabla\varphi|^2}} = \frac{x_N - h_j(\boldsymbol{x}') + (h_j(\boldsymbol{x}') - \varphi(\boldsymbol{\xi}')/\alpha)}{\sqrt{1+|\nabla\varphi|^2}}$$

and thus

$$\frac{c}{\alpha}\biggl(\frac{\varphi(\boldsymbol{\xi}')-h_j(\boldsymbol{\xi}')}{\sqrt{1+|\nabla\varphi|^2}}\biggr) < \biggl(\frac{c}{\sqrt{1+|\nabla\varphi|^2}}-k\biggr)(x_N-h_j(\boldsymbol{x}')).$$

By (2.3), we have

$$\frac{c}{\alpha}\biggl(\frac{1}{\sqrt{1+|\nabla\varphi|^2}}\biggr)\biggl(\frac{\varphi(\boldsymbol{\xi}')-h_j(\boldsymbol{\xi}')}{S(\boldsymbol{\xi}')}\biggr) < x_N-h_j(\boldsymbol{x}').$$

By the definition of ω and lemma 2.2 we have

$$\frac{k\omega}{\alpha} < \frac{c}{\alpha}\frac{\omega}{\sqrt{1+|\nabla\varphi|^2}} < x_N-h_j(\boldsymbol{x}'). \tag{4.12}$$

Since

$$h_j(\boldsymbol{x}') = \frac{1}{\alpha}h_j(\boldsymbol{\xi}') \leqslant \frac{1}{\alpha}h(\boldsymbol{\xi}') \leqslant \frac{1}{\alpha}\varphi(\boldsymbol{\xi}'),$$

we have

$$\begin{aligned}\bar{v}(\boldsymbol{x})-\Phi\biggl(\frac{k}{c}(x_N-h_j(\boldsymbol{x}'))\biggr) &= \Phi\biggl(\frac{x_N-\frac{1}{\alpha}\varphi(\boldsymbol{\xi}')}{\sqrt{1+|\nabla\varphi|^2}}\biggr)+\varepsilon S(\boldsymbol{\xi}')-\Phi\biggl(\frac{k}{c}(x_N-h_j(\boldsymbol{x}'))\biggr)\\ &\geqslant \Phi\biggl(\frac{x_N-h_j(\boldsymbol{x}')}{\sqrt{1+|\nabla\varphi|^2}}\biggr)-\Phi\biggl(\frac{k}{c}(x_N-h_j(\boldsymbol{x}'))\biggr)+\varepsilon S(\boldsymbol{\xi}').\end{aligned}$$

Since

$$\begin{aligned}&\Phi\biggl(\frac{x_N-h_j(\boldsymbol{x}')}{\sqrt{1+|\nabla\varphi|^2}}\biggr)-\Phi\biggl(\frac{k}{c}(x_N-h_j(\boldsymbol{x}'))\biggr)\\ &\quad= \frac{(x_N-h_j(\boldsymbol{x}'))S(\boldsymbol{\xi}')}{c}\int_0^1 \Phi'\biggl(\biggl(\frac{\theta}{\sqrt{1+|\nabla\varphi|^2}}+\frac{k}{c}(1-\theta)\biggr)(x_N-h_j(\boldsymbol{x}'))\biggr)\,\mathrm{d}\theta,\end{aligned}$$

we have

$$\begin{aligned}&\bar{v}(\boldsymbol{x})-\Phi\biggl(\frac{k}{c}(x_N-h_j(\boldsymbol{x}'))\biggr)\\ &\qquad\geqslant \frac{(x_N-h_j(\boldsymbol{x}'))S(\boldsymbol{\xi}')}{c}\\ &\qquad\quad\times\int_0^1 \Phi'\biggl(\biggl(\frac{\theta}{\sqrt{1+|\nabla\varphi|^2}}+\frac{k}{c}(1-\theta)\biggr)(x_N-h_j(\boldsymbol{x}'))\biggr)\,\mathrm{d}\theta+\varepsilon S(\boldsymbol{\xi}').\end{aligned}$$

By lemma 2.2 and (4.12), we have

$$\frac{k}{c} \leqslant \frac{\theta}{\sqrt{1+|\nabla\varphi|^2}}+\frac{k}{c}(1-\theta) \leqslant 1.$$

Then we have

$$\begin{aligned}\bar{v}(\boldsymbol{x})-\Phi\biggl(\frac{k}{c}(x_N-h_j(\boldsymbol{x}'))\biggr) &\geqslant -\frac{S(\boldsymbol{\xi}')}{c}\sup_{|\mu|\geqslant k^2\omega/c\alpha}\biggl|\frac{c}{k}\mu\Phi'(\mu)\biggr|+\varepsilon S(\boldsymbol{\xi}')\\ &= S(\boldsymbol{\xi}')\biggl(\varepsilon-\frac{1}{k}\sup_{|\mu|\geqslant k^2\omega/c\alpha}|\mu\Phi'(\mu)|\biggr).\end{aligned}$$

By virtue of lemma 2.1 and (4.10) we have

$$\frac{1}{k}\sup_{|\mu|\geqslant k^2\omega/c\alpha}|\mu\Phi'(\mu)| < \tfrac{1}{2}\varepsilon.$$

Then we get

$$\bar{v}(\boldsymbol{x}) - \Phi\biggl(\frac{k}{c}(x_N - h_j(\boldsymbol{x}'))\biggr) > \tfrac{1}{2}\varepsilon S(\boldsymbol{\xi}') \geqslant 0$$

and obtain (4.11). This completes the proof. □

Now we prove the main theorem in this paper.

Proof of theorem 1.1. Recall (1.9) and (4.7), and consider solutions of (1.2) given by $w(\boldsymbol{x}, t; \underline{v})$ and $w(\boldsymbol{x}, t; \bar{v})$, respectively. Since $\underline{v}$ is a subsolution and $\bar{v}$ is a supersolution, we have

$$\underline{v} \leqslant w(\boldsymbol{x}, t; \underline{v}) \leqslant w(\boldsymbol{x}, t; \bar{v}) \leqslant \bar{v}$$

for $\boldsymbol{x} \in \mathbb{R}^N$ and $t \geqslant 0$ by using [10, theorem 3.4]. Then

$$V(\boldsymbol{x}) := \lim_{t\to\infty} w(\boldsymbol{x}, t; \underline{v}) \tag{4.13}$$

exists in $L^\infty(\mathbb{R}^N)$ with

$$\underline{v}(\boldsymbol{x}) < V(\boldsymbol{x}) < \bar{v}(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \mathbb{R}^N.$$

This $V(\boldsymbol{x})$ is a solution of (1.3). See [10, theorem 3.6] for detailed arguments.

Now we prove (1.11). For any given $\varepsilon > 0$ we prove

$$\sup_{\boldsymbol{x}\in D(\gamma)} (\bar{v}(\boldsymbol{x}) - \underline{v}(\boldsymbol{x})) < 2\varepsilon \tag{4.14}$$

if $\gamma > 0$ is large enough. We use the same ε as in (4.7). Assume the contrary. Then there exist sequences $(\gamma_i)_{i=1}^\infty \subset \mathbb{R}$ and $(\boldsymbol{x}_i)_{i=1}^\infty \subset \mathbb{R}^N$ such that we have

$$\lim_{i\to\infty} \gamma_i = \infty, \qquad \boldsymbol{x}_i \in D(\gamma_i), \tag{4.15}$$

and

$$\biggl|\Phi(\hat{\mu}_i) - \Phi\biggl(\frac{k}{c}(x_{N,i} - h(\boldsymbol{x}_i'))\biggr)\biggr| \geqslant \varepsilon, \tag{4.16}$$

where $\boldsymbol{x}_i = (x_{1,i}, \dots, x_{N,i})$ and $\boldsymbol{x}_i' = (x_{1,i}, \dots, x_{N-1,i})$ for $i \in \mathbb{N}$. Here we put $\boldsymbol{\xi}_i' = \alpha\boldsymbol{x}_i'$ and

$$\begin{aligned}
\hat{\mu}_i &:= \frac{1}{\alpha}\frac{\xi_{N,i} - \varphi(\boldsymbol{\xi}_i')}{\sqrt{1 + |\nabla\varphi(\boldsymbol{\xi}_i')|^2}} \\
&= \frac{x_{N,i} - h(\boldsymbol{x}_i') - (\varphi(\boldsymbol{\xi}_i') - h(\boldsymbol{\xi}_i))/\alpha}{\sqrt{1 + |\nabla\varphi(\boldsymbol{\xi}_i')|^2}} \\
&= \frac{x_{N,i} - h(\boldsymbol{x}_i') - \tilde{\varphi}(\boldsymbol{\xi}')/\alpha}{\sqrt{1 + |\nabla\varphi(\boldsymbol{\xi}_i')|^2}}.
\end{aligned}$$

From (3.9) and (3.10) we have

$$\lim_{i\to\infty} |\tilde{\varphi}(\boldsymbol{\xi}_i)| = 0, \qquad |\nabla\varphi(\boldsymbol{\xi}'_i)| = m, \qquad \lim_{i\to\infty} S(\boldsymbol{\xi}'_i) = 0.$$

Then we obtain

$$\lim_{i\to\infty} \left| \hat{\mu}_i - \frac{k}{c}(x_{N,i} - h(\boldsymbol{x}'_i)) \right| = 0.$$

This implies

$$\lim_{i\to\infty} \left| \Phi(\hat{\mu}_i) - \Phi\left(\frac{k}{c}(x_{N,i} - h(\boldsymbol{x}'_i)) \right) \right| = 0.$$

This contradicts (4.16). This completes the proof of theorem 1.1. □

For the uniqueness and stability of pyramidal travelling fronts, see [13] in $\mathbb{R}^3$. In $\mathbb{R}^N$ ($N \geqslant 4$) those properties are left to be an interesting open problem.

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