Examples of degenerations of Cohen-Macaulay modules

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This is a joint work with Yuji Yoshino.

§0. Introduction

Notations.

- k: an algebraically closed field of characteristic 0.
- R: a (commutative Noetherian) k-algebra.
- $\mathrm{mod}(R)$: the category of finitely generated R-modules.

Definition ("Degeneration").

Let M, $N \in \mathrm{mod}(R)$. M **degenerates** to $N \Leftrightarrow \exists(V, tV, k)$: D.V.R. that is a k-algebra and $\exists Q$: a finitely generated $R \otimes_k V$-module such that

(1) Q is V-flat.

(2) $Q/tQ \cong N$ as an R-module.

(3) $Q[1/t] \cong M \otimes_k V[1/t]$ as an $R \otimes_k V[1/t]$-module.

Theorem 1 [Yoshino, 2004].

T.F.A.E. for M and $N \in \mathrm{mod}(R)$.

(1) M degenerates to N.

(2) $\exists$ a short exact sequence of finitely generated R-modules:

$$0 \longrightarrow Z \xrightarrow{\begin{pmatrix} \varphi \\ \psi \end{pmatrix}} M \oplus Z \longrightarrow N \longrightarrow 0,$$

such that ψ is nilpotent, *i.e.* $\psi^n = 0$ for $n \gg 1$.

Remark (The degeneration given by an extension).

Assume that there is an exact sequence $0 \to L \xrightarrow{p} M \xrightarrow{q} N \to 0$. Then M degenerates to $L \oplus N$. In fact, we have

$$0 \longrightarrow L \xrightarrow{\begin{pmatrix} p \\ 0 \end{pmatrix}} M \oplus L \xrightarrow{\begin{pmatrix} q, 0 \\ 0, 1 \end{pmatrix}} N \oplus L \longrightarrow 0.$$

[Bongartz, 1996]

If R is a representation directed k-algebra (e.g. path algebra of Dynkin quiver). $\Rightarrow \forall$ Minimal degenerations come from degenerations given by an extension.

[Yoshino, 2002]

Let $(R, \mathfrak{m}, k)$ be a complete Cohen-Macaulay local ring which is of finite representation type. If R is a 1-dim. domain or 2-dim., $\Rightarrow \forall$ Degenerations of maximal Cohen-Macaulay R-modules are obtained by AR sequences.

Idea for the proofs.

The order relations for finitely generated modules w.r.t. degenerations, extensions, and AR sequences.

§1. First example

In the rest of this talk, we consider

R is a commutative Noetherian k-algebra.

Remark (for the later use).

Let M and $N \in \mathrm{mod}(R)$. Suppose that M degenerates to N. Then,

(1) $[M] = [N]$ in $K_0(\mathrm{mod}(R))$.

(2) $\mathcal{F}_i^R(M) \supseteq \mathcal{F}_i^R(N)$ for all $i \geq 0$, where $\mathcal{F}_i^R(M)$: the i th Fitting ideal of M.

Definition (The "deg" order and the "ext" order).

- $M \leq_{deg} N \Leftrightarrow \exists L_0 \cong M, L_1, \cdots, L_r \cong N$ such that L_i degenerates to L_{i+1} for all i.
- M degenerates by an extension to $N \Leftrightarrow \exists 0 \to U \to M \to V \to 0$ in $\mathrm{mod}(R)$ such that $N \cong U \oplus V$.
- $M \leq_{ext} N \Leftrightarrow \exists L_0 \cong M, L_1, \cdots, L_r \cong N$ such that L_i degenerates by an extension to L_{i+1} for all i.

Remark.

- "$\leq_{deg}$" and "$\leq_{ext}$" are partial orders.
- If R is Artinian, the degeneration is transitive. Namely, L degenerates to M and M degenerates to $N \Rightarrow L$ degenerates to N. (We do not know whether this property holds or not in general.) If M degenerates by an extension to N, the transitivity property holds.
- If $M \leq_{ext} N \Rightarrow M$ degenerates to N.

- Let $R = k[[x]]$ and M be an R-module with length n. Since $M \cong R/(x^{p_1}) \oplus \cdots \oplus R/(x^{p_n})$ where $\Sigma_{i=1}^{n} p_i = n$, we have the finite presentation of M:

$$0 \longrightarrow R^k \xrightarrow{\begin{pmatrix} x^{p_1} & & \\ & \ddots & \\ & & x^{p_n} \end{pmatrix}} R^k \longrightarrow M \longrightarrow 0.$$

  where $p_1 \geq p_2 \geq \cdots \geq p_n \geq 0$.
- $p_M := (p_1, p_2. \cdots, p_n)$.

Definition (The dominance order for partitions).

Let $p = (p_1, p_2, \cdots, p_n)$ and $q = (q_1, q_2, \cdots, q_n)$ be partitions of n. Namely $p_1 \geq p_2 \geq \cdots \geq p_n \geq 0$ and $\Sigma_{i=1}^{n} p_i = n$.

$$p \geq_{dom} q \Leftrightarrow \Sigma_{i=1}^{j} p_i \geq \Sigma_{i=1}^{j} q_i \text{ for all } 1 \leq j \leq n.$$

Proposition 2.

Let $R = k[[x]]$ and M and N be R-modules with length n. Then,

$$M \text{ degenerates to } N \Leftrightarrow p_M \geq_{dom} p_N.$$

In particular, M degenerates by extensions to N. Hence,

$$M \leq_{deg} N \Leftrightarrow p_M \geq_{dom} p_N \Leftrightarrow M \leq_{ext} N.$$

Proof.

$(\Rightarrow)$ $\mathcal{F}_i^R(M) = (x^{p_{i+1}+\cdots p_n}) \supseteq \mathcal{F}_i^R(N) = (x^{q_{i+1}+\cdots q_n})$.

$(\Leftarrow)$ Let $p_M = (a, b) \geq_{dom} p_N = (a-1, b+1)$ where $a \geq b+2$. Then we have a short exact sequence:

$$0 \longrightarrow R/(x^{a-1}) \xrightarrow{(x,1)} R/(x^a) \oplus R/(x^b) \longrightarrow R/(x^{b+1}) \longrightarrow 0.$$

Therefore, M degenerates by an extension to N. $\square$

§2. Second example

Theorem 3.

Let $R = k[[x_0, x_1, \cdots .x_d]]/(x_0^{n+1} + x_1^2 + \cdots + x_d^2)$ where d is even and let M and N be maximal Cohen-Macaulay R-modules. Suppose that M degenerate to N. Then,

$$M \text{ degenerates by extensions to } N, \text{ i.e., } M \leq_{ext} N.$$

- A: a commutative Gorenstein ring.
- $\mathrm{CM}(A)$: the category of maximal Cohen-Macaulay A-modules.
- $\underline{\mathrm{CM}}(A)$: the stable category of $\mathrm{CM}(A)$. Note that $\underline{\mathrm{CM}}(A)$ has a structure of triangulated category.
- By Knörrer's periodicity,

$$\underline{\mathrm{CM}}(k[[x_0, x_1, \cdots .x_d]]/(x_0^{n+1} + x_1^2 + \cdots + x_d^2)) \cong \underline{\mathrm{CM}}(k[[x_0]]/(x_0^{n+1})),$$

where d is even.

Strategy: The stable analogue of degenerations.

Theorem 4 [Yoshino, 2010].

Let $(R, \mathfrak{m}, k)$ be a complete Gor. local k-algebra and M and $N \in \mathrm{CM}(R)$.

(1) $R^m \oplus M$ degenerates to $R^n \oplus N$ for some $m, n \in \mathbb{N}$.

(2) $\exists$ a triangle in $\underline{\mathrm{CM}}(R)$:

$$\underline{Z} \xrightarrow{\begin{pmatrix} \underline{\varphi} \\ \underline{\psi} \end{pmatrix}} \underline{M} \oplus \underline{Z} \longrightarrow \underline{N} \longrightarrow \underline{Z}[1]$$

such that $\underline{\psi}$ is a nilpotent.

(3) $\underline{M}$ stably degenerates to $\underline{N}$.

(4) $\exists X \in \mathrm{CM}(R)$ and $m, n \in \mathbb{N}$ such that $M \oplus R^m \oplus X$ degenerates to $N \oplus R^n \oplus X$.

In general, (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) hold.
If R is Artinian, (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) hold.
If R is an isolated singularity, (2) $\Leftrightarrow$ (3) holds.

Definition (The "triangle" order).

- $\underline{M}$ stably degenerates by a triangle to $\underline{N}$ $\Leftrightarrow$ $\exists$ triangles $\underline{U} \to \underline{M} \to \underline{V} \to \underline{U}[1]$ in $\underline{\mathrm{CM}}(R)$ such that $\underline{N} \cong \underline{U} \oplus \underline{V}$.
- $\underline{M} \leq_{tri} \underline{N} \Leftrightarrow \exists \underline{L_0} \cong \underline{M}, \underline{L_1}, \cdots, \underline{L_r} \cong \underline{N}$ such that $\underline{L_i}$ stably degenerates by a triangle to $\underline{L_{i+1}}$ for all i.

Proposition 5.

Let $(R, \mathfrak{m}, k)$ be a complete Gor. local ring and M and $N \in \mathrm{CM}(R)$. Assume $[M] = [N]$ in $K_0(\mathrm{mod}(R))$. Then,

$$\underline{M} \leq_{tri} \underline{N} \Leftrightarrow M \leq_{ext} N.$$

Proof of Theorem 3.

Let $R = k[[x_0, x_1, \cdots .x_d]]/(x_0^{n+1} + x_1^2 + \cdots + x_d^2)$ and $S = k[[x_0]]/(x_0^{n+1})$. Note that $\Omega : \underline{\mathrm{CM}}(R) \cong \underline{\mathrm{CM}}(S)$ as triangulated categories. Let M, $N \in \mathrm{CM}(R)$ and suppose that M degenerates to N. Then,

$$\begin{aligned}
&\Rightarrow \ \underline{M} \text{ stably degenerates to } \underline{N} \\
&\Rightarrow \ \underline{\Omega(M)} \text{ stably degenerates to } \underline{\Omega(N)} \\
&\Rightarrow \ \Omega(M) \oplus S^m \text{ degenerates to } \Omega(N) \oplus S^n \text{ by Theorem 4} \\
&\Leftrightarrow \ \Omega(M) \oplus S^m \leq_{ext} \Omega(N) \oplus S^n \text{ by Proposition 2} \\
&\Leftrightarrow \ \underline{\Omega(M)} \leq_{tri} \underline{\Omega(N)} \\
&\Leftrightarrow \ \underline{M} \leq_{tri} \underline{N} \\
&\Leftrightarrow \ M \leq_{ext} N \text{ by Proposition 5.}
\end{aligned}$$

□

Example ($d = n = 2$)

Let $R = k[[x_0, x_1, x_2]]/(x_0^3 + x_1^2 + x_2^2)$ and let $\mathfrak{p}$ and $\mathfrak{q}$ be $(x_0, x_1 - \sqrt{-1}\, x_2)$ and $(x_0^2, x_1 + \sqrt{-1}\, x_2)$ respectively. The Hasse diagram of degenerations of maximal Cohen-Macaulay R-modules of rank 3 is the following:

$$
\begin{tikzcd}
\mathfrak{p}^3 \arrow[dr, dash] & & \mathfrak{q}^3 \arrow[dl, dash] \\
& R \oplus \mathfrak{p} \oplus \mathfrak{q} \arrow[d, dash] & \\
& R^3, &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{p}^2 \oplus \mathfrak{q} \arrow[d, dash] \\
R \oplus \mathfrak{q}^2 \arrow[d, dash] \\
R^2 \oplus \mathfrak{p},
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{p} \oplus \mathfrak{q}^2 \arrow[d, dash] \\
R \oplus \mathfrak{p}^2 \arrow[d, dash] \\
R^2 \oplus \mathfrak{q}.
\end{tikzcd}
$$

§3. Extended orders

Let $(R, \mathfrak{m}, k)$ be a complete CM local k-algebra.

Remark (Cancellation properties).

- $M \oplus L \leq_{deg} (\leq_{ext}) N \oplus L \not\Rightarrow M \leq_{deg} (\leq_{ext}) N$ for $\forall L \in \mathrm{CM}(R)$.
- $M^n \leq_{deg} (\leq_{ext}) N^n \not\Rightarrow M \leq_{deg} (\leq_{ext}) N$ for $\forall n \in \mathbb{N}$.

Definition (The "DEG" order).

The relation "$\leq_{DEG}$" between MCM R-modules is a partial order generated by the following rules:

(1) If $M \leq_{deg} N \Rightarrow M \leq_{DEG} N$.

(2) $M \leq_{DEG} N \Leftrightarrow M \oplus L \leq_{DEG} N \oplus L$ for $\forall L \in \mathrm{CM}(R)$.

(3) $M \leq_{DEG} N \Leftrightarrow M^n \leq_{DEG} N^n$ for $\forall n \in \mathbb{N}$.

Definition (The "EXT" order).

The relation "$\leq_{EXT}$" between MCM R-modules is a partial order generated by the following rules:

(1) If $M \leq_{ext} N \Rightarrow M \leq_{EXT} N$.

(2) $M \leq_{EXT} N \Leftrightarrow M \oplus L \leq_{EXT} N \oplus L$ for $\forall L \in \mathrm{CM}(R)$.

(3) $M \leq_{EXT} N \Leftrightarrow M^n \leq_{EXT} N^n$ for $\forall n \in \mathbb{N}$.

Definition (The "AR" order).

We also define the order "$\leq_{AR}$" on MCM R-modules as a partial order generated by:

(1) If $0 \to Y \to E \to X \to 0$: AR sequence in $\mathrm{CM}(R) \Rightarrow E \leq_{AR} X \oplus Y$.

(2) $M \leq_{AR} N \Leftrightarrow M \oplus L \leq_{AR} N \oplus L$ for $\forall L \in \mathrm{CM}(R)$.

(3) $M \leq_{AR} N \Leftrightarrow M^n \leq_{AR} N^n$ for $\forall n \in \mathbb{N}$.

Theorem 6.

If R is of finite representation type, then

$$M \leq_{AR} N \Leftrightarrow M \leq_{EXT} N \Leftrightarrow M \leq_{DEG} N$$

for $M, N \in \mathrm{CM}(R)$.

Difficulty: To show "$M \leq_{EXT} N \Rightarrow M \leq_{AR} N$".

[Yoshino, 2002]

Let $(R, \mathfrak{m}, k)$ be of finite representation type. If R is an integral domain of dim. 1 or of dim. 2, then

$$M \leq_{AR} N \Leftrightarrow M \leq_{EXT} N \Leftrightarrow M \leq_{DEG} N \ \Leftrightarrow M \leq_{hom} N$$

for $M,\ N \in \mathrm{CM}(R)$.

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Thank you for your attention.

ご清聴ありがとうございました.