PICARD GROUPS OF TORSION FREE CLASSES

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Let k be a commutative ring and A be a commutative k-algebra. We denote by A-Mod the module category over A. The notion of Picard group of additive full subcategory appears in [3]. It defined as an automorphism group of such a subcategory. In this note, we investigate the picard groups of torsion free classes in A-Mod.

Let $\mathfrak{F}$ be an additive full subcategory of A-Mod which is closed under submodules, direct products and extensions. Note from this assumption, there exists an additive full subcategory $\mathfrak{T}$ in A-Mod such that $(\mathfrak{T}, \mathfrak{F})$ is a torsion pair. The following proposition is well-known (see [1]).

Proposition 1. *Let $(\mathfrak{T}, \mathfrak{F})$ be a torsion pair in A-*Mod *and M be an A-module. There exists a short exact sequence*

$$0 \longrightarrow t(M) \longrightarrow M \longrightarrow M/t(M) \longrightarrow 0$$

with $t(M) \in \mathfrak{T}$ and $M/t(M) \in \mathfrak{F}$.

We denote by Pic A the ordinary Picard group of the ring A. Recall that an A-module M is called invertible if there is an A-module M' such that $M \otimes_A M' \cong A$ as A-modules. Then Pic A consists of all the isomorphism classes of invertible A-modules, and the multiplication in Pic A is defined by tensor product over A.

Lemma 2. *Let A be a commutative ring and let $\mathfrak{C}$ be a full subcategory of A-*Mod *that is additively closed and $A \in \mathfrak{C}$. Then the classes of invertible A-modules are elements of* $\mathrm{Pic}(\mathfrak{C})$*, and hence* Pic A *is naturally a subgroup of* $\mathrm{Pic}(\mathfrak{C})$.

Proposition 3. *Let A be a commutative ring. Let $\mathfrak{F}$ be an additive full subcategory of A-*Mod *which contains A as an object. Suppose that $\mathfrak{F}$ is closed under submodules, direct products and extensions. Then we have the equalities*

$$\mathrm{Pic}(\mathfrak{F}) = \mathrm{Pic}\ A.$$

Proof. Since $\mathfrak{F}$ is additively closed, the inclusion Pic $A \subseteq \mathrm{Pic}(\mathfrak{F})$ holds by Lemma 2.

To prove the other inclusion, assume $[M] \in \mathrm{Pic}(\mathfrak{F})$. We have only to show that M is an invertible A-module. Take an A-module $[N] \in \mathrm{Pic}(\mathfrak{F})$ such that $\mathrm{Hom}_A(M \otimes_A N, -) \cong 1$ as functors on $\mathfrak{F}$. We note that $\mathrm{Hom}_A(N, A) \cong M$. We define the torsion-free tensor product by

$$M \bar{\otimes}_A N := (M \otimes_A N)/t(M \otimes_A N).$$

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It is easy to see that every A-homomorphism from $M \otimes_A N$ to an element of $\mathfrak{F}$ factors through the natural surjection $M \otimes_A N \to M \bar{\otimes}_A N$. Hence the above isomorphism of functors induces an isomorphism

$$\mathrm{Hom}_A(M\bar{\otimes}_A N, -) \cong 1,$$

as functors on $\mathfrak{F}$. Since $M\bar{\otimes}_A N$ is an object in $\mathfrak{F}$, it follows from Yoneda's lemma that $M\bar{\otimes}_A N \cong A$ as A-modules. Note from this isomorphism that M has rank one.

Now we prove the following claim.

Claim: *Assume that $M\bar{\otimes}_A N \cong A$ for torsion-free A-modules M and N. Then M and N are projective as A-modules.*

In fact, take an element $\sum_{i=1}^{r} m_i \otimes n_i \in M \otimes_A N$ which maps to 1 by the natural epimorphism

$$\pi \ : \ M \otimes_A N \to M\bar{\otimes}_A N \cong A.$$

We define an A-linear homomorphism $g_i : A \to M$ by $g_i(a) = am_i$. And we also define an A-linear homomorphism $g = \Sigma^m g_i : A \to M$. Tensoring N of this morphism and composition with π, we have a following splitting epimorphism

$$\tilde{g} \ : \ \oplus^m N \xrightarrow{g\otimes N} M \otimes N \xrightarrow{\ \pi\ } A.$$

Hence, we have

$$\oplus^m N \cong A \oplus K,$$

where K is a kernel of $\tilde{g}$. Tensoring M and applying $\mathrm{Hom}_A(-, A)$ to this, we have the following isomorphism.

$$\mathrm{Hom}_A(\oplus^m(N \otimes_A M), A) \cong \mathrm{Hom}_A(M, A) \oplus \mathrm{Hom}_A(K \otimes_A M, A).$$

Since LHS is isomorphic to $\oplus^m A$ and $\mathrm{Hom}_A(M, A) \cong N$, we see that N is a direct summand of free A-module. Thus, N is projective. Also M so is. This completes the proof of the claim.

Now from the claim we have that $M\otimes N$ is a projective A-modules, especially submodule of direct sums of A. Since $\mathfrak{F}$ is additive and closed under submodule, $M \otimes N$ is an element of $\mathfrak{F}$. Therefore, $M \otimes N \cong M\bar{\otimes}_A N \cong A$, which shows that M is an invertible A-module. □

Example 4. The following subcategories of A-Mod satisfy the assumption of Proposition 3.

(1) A-Mod itself and the full subcategory A-mod consiting of all finitely generated A-modules.
(2) the full subcategories $\mathrm{Tf}(A)$ (resp. $\mathrm{tf}(A)$) consisting of all torsion-free A-modules (resp. all finitely generated torsion-free A-modules) in case A is an integral domain (see [1, 3]).
(3) the full subcategory $d^{\geq i}(A)$ which consists of all the finitely generated A-modules M with depth $M \geq i$ when A is a Noetherian local ring with depth $A \geq i$, where i is any natural number (see [3]).
(4) the full subcategory of (I, J)-torsion free A-modules for ideals I, J of A in case A is a Noetherian ring (see [2]).

References

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