Automorphism groups and Picard groups of additive full subcategories

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This is a joint work with Yuji Yoshino.

Notations

- k : a commutative ring.
- A : a commutative k-algebra.

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- $\mathcal{C}$: an additive full subcategory which contains A as an object.

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Since A is a k-algebra, $\mathcal{C}$ has a structure of k-category.

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- A functor $F : \mathcal{C} \to \mathcal{C}$ is a k-linear functor if $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(F(X), F(Y))$ is a k-linear map.
- A covariant functor $F : \mathcal{C} \to \mathcal{C}$ is an automorphism of $\mathcal{C}$ if it gives an auto-equivalence of $\mathcal{C}$.

Definition

- The k-linear automorphisms group of $\mathcal{C}$;

$$\mathrm{Aut}_k(\mathcal{C}) := \{F : \mathcal{C} \to \mathcal{C} \mid \begin{array}{l} k\text{-linear} \\ \text{automorphism of } \mathcal{C} \end{array} \}/\cong$$

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- Multiplication ⇒ Composition of functors.
- Identity element ⇒ Identity functor.

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♢ Motivated example ♢

Theorem 1

$(A, \mathfrak{m})$: a CM local k-algebra where k: a field.
$\mathrm{CM}(A)$: the category of MCM A-modules.
Suppose A has only an isolated singularity.
Then,

$$\mathrm{Aut}_k(\mathrm{CM}(A)) \cong \begin{cases} \mathrm{Aut}_{k\text{-alg}}(A) & (\text{if } \dim A \neq 2), \\ \mathrm{Aut}_{k\text{-alg}}(A) \ltimes \mathrm{C}\ell(A) & (\text{if } \dim A = 2). \end{cases}$$

Here, we mean

- $\mathrm{Aut}_{k\text{-alg}}(A)$: the group of k-algebra automorphisms.
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- presentation of a k-linear automorphism of $\mathcal{C}$.
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Functor induced by k-algebra automorphisms

- For $\sigma \in \mathrm{Aut}_{k\text{-alg}}(A)$,

$$\sigma_* : A\text{-Mod} \longrightarrow A\text{-Mod}$$

$$M \longmapsto \sigma_* M \quad \Rightarrow \begin{cases} \sigma_* M = M & \text{as } k\text{-modules}, \\ a \circ m = \sigma^{-1}(a)m & \text{with } A\text{-action}. \end{cases}$$

$$f : M \to N \longmapsto \sigma_* f : \sigma_* M \to \sigma_* N$$

$$\Rightarrow \ \sigma_* f(m) = f(m).$$

σ_* : **covariant, k-linear** $\Rightarrow \sigma_* \in \mathrm{Aut}_k(A\text{-Mod})$

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Definition

- $\mathcal{C}$ is stable under k-algebra automorphisms if

$$\forall \sigma \in \mathrm{Aut}_{k\text{-alg}}(A),\ \sigma_*(\mathcal{C}) \subseteq \mathcal{C}.$$

(e.g. $A\text{-Mod}$, $\mathrm{CM}(A)$.)

Lemma 2.

If $\mathcal{C}$ is stable under $\mathrm{Aut}_{k\text{-alg}}(A)$, then

$$\mathrm{Aut}_{k\text{-alg}}(A) \hookrightarrow \mathrm{Aut}_k(\mathcal{C}); \quad [\sigma] \hookrightarrow [\sigma_*].$$

By this Lemma,

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Key Theorem

Theorem 3.

For $\forall F \in \mathrm{Aut}_k(\mathcal{C})$,
there exist $\sigma \in \mathrm{Aut}_{k\text{-alg}}(A)$ and $N \in \mathcal{C}$ s.t.

$$F(-) \cong \sigma_* \circ \mathrm{Hom}_A(N, -) \mid_{\mathcal{C}} .$$

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Picard group of additive categories

Definition

- We define $\mathrm{Pic}(\mathcal{C})$ as

$$\{M \in \mathcal{C} \mid \mathrm{Hom}_A(M, -)|_{\mathcal{C}} \text{ gives an auto-equivalence}\}/\cong .$$

- For $[M], [N] \in \mathrm{Pic}(\mathcal{C})$, $\exists\, L \in \mathcal{C}$ s.t.

$$\mathrm{Hom}_A(M, -)|_{\mathcal{C}} \circ \mathrm{Hom}_A(N, -)|_{\mathcal{C}} \cong \mathrm{Hom}_A(L, -)|_{\mathcal{C}}.$$

$$\Rightarrow [M] \cdot [N] = [L].$$

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Remark.

- $\mathrm{Pic}(\mathcal{C})$ is an abelian group.
- $\mathrm{Pic}(\mathcal{C}) \cong \mathrm{Aut}_A(\mathcal{C}) \subseteq \mathrm{Aut}_k(\mathcal{C})$.

Now, we give a structure theorem of $\mathrm{Aut}_k(\mathcal{C})$.

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♣ Main Theorem ♣

Theorem 5.

Let A be a k-algebra. Let $\mathcal{C}$ be stable under $\mathrm{Aut}_{k\text{-alg}}(A)$.
Then we have an isomorphism

$$\mathrm{Aut}_k(\mathcal{C}) \cong \mathrm{Aut}_{k\text{-alg}}(A) \ltimes \mathrm{Pic}(\mathcal{C}).$$

Examples of $\mathrm{Pic}(\mathcal{C})$

- The classical Picard group of the ring A is

$$\mathrm{Pic}\, A := \{\text{invertible } A\text{-modules}\}/\cong .$$

The multiplication $\Rightarrow$ $[M] \cdot [N] = [M \otimes N]$.

Example 6.

$A\text{-Mod} \supset A\text{-mod} :=$ { f.g.A-modules }

$\mathrm{Proj}(A) :=$ { projective A-modules }

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If $\mathcal{C}$ is one of these $\Rightarrow$ $\mathrm{Pic}(\mathcal{C}) = \mathrm{Pic}\ A$.

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Proposition 7.

A: Krull domain.

M: A-lattice of $\mathrm{rank}\ n$, (i.e. $L \subseteq M \subseteq L'$ where $L, L' \cong A^n$.)

$\mathrm{Ref}(A) :=$ { reflexive A-lattices }

$\mathrm{ref}(A) :=$ { f.g. refl. A-lattices } $\subseteq \mathrm{Ref}(A)$.

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$$\mathrm{Pic}(\mathrm{Ref}(A)) = \mathrm{C}\ell(A).$$

If A is Noetherian $\Rightarrow \mathrm{Pic}(\mathrm{ref}(A)) = \mathrm{C}\ell(A)$.

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Proposition 8.

A : a comm. k-algebra.
$\mathcal{F} \subseteq A\text{-}\mathrm{Mod}$: an additive full subcat. which contains A.
Suppose that $\mathcal{F}$ is closed under submodules, direct products and extensions.

(i.e. $\exists \mathcal{T} \subseteq A\text{-}\mathrm{Mod}$ s.t. $(\mathcal{T}, \mathcal{F})$ is a torsion pair.)

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e.g.

- In case A is an integral domain, $\mathrm{Tf}(A) := \{\text{torsion free } A\text{-mod.}\}$, $\mathrm{tf}(A) := \{\text{f.g. torsion free } A\text{-mod.}\}$.
- In case $(A, \mathfrak{m})$ is a Noeth. s.t. depth $A \geq 1$, $d^{\geq 1}(A) := \{\ M \in A\text{-}\mathrm{mod} \mid \mathrm{depth}\ M \geq 1\}$.

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♣ Structure Theorem ♣

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(Proof of Theorem 1.)

Combine with Theorem 6,

- The case $\dim A = 0$, $\mathrm{CM}(A) = A\text{-}\mathrm{mod} \Rightarrow$ trivial.
- The case $\dim A = 1$, $\mathrm{CM}(A) = d^{\geq 1}(A) \Rightarrow$ trivial.
- The case $\dim A = 2$, $\mathrm{CM}(A) = \mathrm{ref}(A) \Rightarrow \mathrm{C}\ell(A)$.
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Theorem 9.

$(A, \mathfrak{m})$: a CM k-algebra with $\dim A = d$.

Suppose A is regular in codimension 2.

(i.e. A_p is regular for $\forall p$: prime ideal with $\mathrm{ht}(p) = 2$.)

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- N. Hiramatsu, and Y. Yoshino, *Automorphism groups and Picard groups of additive full subcategories*. Math. Scand., to appear, http://www.math.okayama-u.ac.jp/~yoshino/pdffiles/picardgp1.pdf
- Y. Yoshino, *Cohen-Macaulay Modules over Cohen-Macaulay Rings*, London Mathematical Society Lecture Note Series **146**. Cambridge University Press, Cambridge, 1990. viii+177 pp.

Thank you for your attention.

ご清聴ありがとうございました.