「Complete intersection dimension and related topics」

参考文献

- Complete intersection dimension. Publ. Math 86 (1997)
- Remarks on a depth formula, a graded inequality and a conjecture of Auslander.

環はすべて Noeth. local ring $(R, \mathfrak{m})$, $(S, \mathfrak{n})$, R'

<u>Def.</u> (1) S is a <u>deformation</u> of R of codim r

$$\underset{\text{def}}{\Leftrightarrow} \ {}^{\exists}\underline{x} = x_1, \dots, x_r \in \mathfrak{n} \ ; \ S\text{-seq.} \quad \text{s.t. } R \cong S/(\underline{x})S$$

(2) $R \nearrow R' \nwarrow S$ is a <u>quasi-deformation</u> of R of codim r.

$\underset{\text{def}}{\Leftrightarrow}$ (i) $R \to R'$ が faithfully flat
(ii) S is a deform. of R' of codim r

(3) $M \in \operatorname{mod} R$

$$\text{CI-}\dim M := \inf\left\{ \operatorname{pd}_S M \otimes_R R' - \operatorname{pd}_S R' \;\middle|\; \underbrace{R \nearrow R' \nwarrow S}_{\text{quasi-deform.}} \right\}$$

(4) $\widetilde{R} := \begin{cases} \widehat{R} & (|k| = \infty) \\ \widehat{R[X]_{\mathfrak{m}R[X]}} & (|k| < \infty) \end{cases}$

$$\operatorname{vpd} M := \inf\left\{ \operatorname{pd}_S M \otimes_R \widetilde{R} \;\middle|\; S \text{ is a deform of } \widetilde{R} \right\}$$

<u>Rem</u>

(1) $R \to \tilde{R} \leftarrow S$: quasi-deform より

$$\mathcal{V}\mathrm{pd}\, M < \infty \Rightarrow \mathrm{CI\text{-}dim}\, M < \infty$$

$$\mathcal{V}\mathrm{pd}\, M = \mathrm{depth}\, R - \mathrm{depth}\, M + \mathrm{cx}_R M$$

(2) $\mathrm{CI\text{-}dim}\, M < \infty$

${}^{\forall} R \to R' \leftarrow S$: quasi-deform with $\mathrm{pd}_S M' < \infty$

(ただし $(\ )' = (\ ) \otimes_R R'$)

$$\begin{aligned}
\mathrm{pd}_S M' - \mathrm{pd}_S R' &= (\mathrm{depth}_S S - \mathrm{depth}_S M') \\
&\quad - (\mathrm{depth}_S S - \mathrm{depth}_S R') \\
&= \mathrm{depth}_{R'} R' - \mathrm{depth}_{R'} M' \\
&= (\mathrm{depth}_R R + \mathrm{depth}_{R'} k') - (\mathrm{depth}_R M + \mathrm{depth}_{R'} k') \\
&= \mathrm{depth}\, R - \mathrm{depth}\, M.
\end{aligned}$$

特に $\mathrm{CI\text{-}dim}_R M = \mathrm{depth}\, R - \mathrm{depth}\, M$

(3) $\mathrm{CI\text{-}dim}\, M < \infty \Leftrightarrow {}^{\exists} R \to R' \leftarrow S$; quasi-deform.

s.t. $\mathrm{pd}_S M' < \infty$

(4) $\mathrm{pd}_R M < \infty \Rightarrow \mathrm{CI}\text{-}\dim_R M < \infty$

(その1) $R \xrightarrow{=} R \xleftarrow{\cong} R$

(その2) $\forall\ R \to R' \leftarrow S$ quasi-deform に対し.

$\mathrm{pd}_S M' < \infty$ を示す.

(∵) ind. on $\mathrm{pd}_R M =: n$

・$n = 0$ のとき

$$M \cong R^r \quad (\exists r)$$

$$M' \cong (R')^r \qquad \therefore\ \mathrm{pd}_S M' < \infty$$

・$n > 0$ のとき

$$0 \to \Omega_R M \to R^r \to M \to 0 \ \text{: exact in mod } R$$

$\downarrow \otimes_R R'$

$$0 \to \underbrace{(\Omega_R M)'}_{} \to \underbrace{(R')^r}_{} \to M' \to 0 \ \text{: exact in mod } R'$$

$\mathrm{pd}_S < \infty$

<u>Cor</u>

$$\mathrm{CI}\text{-}\dim_R M = \mathrm{CI}\text{-}\dim \Omega_R M + 1$$

<u>Question</u>

$$0 \to L \to M \to N \to 0 \ \text{: ex. in mod } R$$

2つの $\mathrm{CI}\text{-}\dim < \infty \Longrightarrow$ 残りも $\mathrm{CI}\text{-}\dim < \infty$

- R : Complete Intersection のとき

$\exists R \to \hat{R} \leftarrow S$: quasi-deform where S : RLR

$\forall M \in \mathrm{mod}\, R$, $\mathrm{pd}_S M' < \infty \Rightarrow$ CI-dim $M < \infty$

<u>Example</u>

$R = \dfrac{k[[x]]}{(x^2)}$, $M = R/(x)$, $R = R \leftarrow k[[x]]$

- CI-dim $M < \infty \Rightarrow$ G-dim $M < \infty$

(∵)

Cor より CI-dim $M = 0$ としてよい.

$\forall$
$$\begin{tikzcd}[column sep=small] & R' & \\ R \arrow[ur] & & S \arrow[ul] \end{tikzcd}$$
: quasi-deform with $\mathrm{pd}_S M' < \infty$

このとき

$$\text{G-dim}_R M = 0 \Leftrightarrow \text{G-dim}_{R'} M' = 0$$

$R = R'$ としてよい.

$\boxed{\text{設定}}$

(S, n) : local ring.

$\underline{x} = x_1, \dots, x_r \in n$: S-seq.

$R = S/(\underline{x})S$

$M \in \mathrm{mod}\, R$, depth M = depth R

$\mathrm{pd}_S M < \infty$

$(\underline{x})\cdot M = 0$ より $\operatorname{grade}_S M \geq r$

一方、$\operatorname{pd}_S M = \operatorname{depth} S - \operatorname{depth} M$

$= \operatorname{depth} S - \operatorname{depth} R = r$

$\therefore$ M は perfect S-mod of grade r.

$\operatorname{Ext}_R^i(M, R) \cong \operatorname{Ext}_S^{i+r}(M, S) = 0 \quad (i>0)$

$\operatorname{Ext}_S^r(M, S)$ も perfect S-module なので

$$\operatorname{Ext}_S^{i+r}(\operatorname{Ext}_S^r(M, S), S) \cong \begin{cases} M & i=0 \\ 0 & i>0 \end{cases}$$

$$\begin{array}{c} \wr\Vert \\ \operatorname{Ext}_R^i(\operatorname{Hom}_R(M, R), R) \end{array}$$

$\therefore$ $G\text{-}\dim_R M = 0$ //

__Def__

$M \in \operatorname{mod} R$

$\beta_i^R(M)$: i-th Betti-number

$\quad \Vert$

$\dim_k \operatorname{Ext}_R^i(M, k)$

$\operatorname{cx}_R M := \inf\{ d \geq 0 \mid {}^\exists r \in \mathbb{R} \text{ s.t. } \beta_i^R(M) \leq r \cdot i^{d-1} \}$

$({}^\forall i \gg 0 \text{ or } {}^\forall i > 0)$

__Rem__

$\operatorname{pd}_R M < \infty \iff \operatorname{cx}_R M = 0$

Ex

$S = k[[x]]$

$R = S/(f)$

$M \in CM(R)$: non free

$$\begin{tikzcd} 0 \arrow[r] & S^n \arrow[r, "A"] \arrow[d, "f"'] & S^n \arrow[r] \arrow[d, "f"] \arrow[dl, "2B"'] & M \arrow[r] \arrow[d, "f=0"] & 0 \\ 0 \arrow[r] & S^n \arrow[r, "A"] & S^n \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

$AB = BA = fE_n$

$$\Rightarrow \begin{cases} \cdot\ cx_S M = 0 \\ \cdot\ cx_R M = 1 \end{cases}$$

Thm

S : deform of codim r

$M \in \mathrm{mod}\, R$

$cx_S M \leq cx_R M \leq cx_S M + r$

Lem

$R = S/xS$ $\quad x$: S-reg. elem.

$M, N \in \mathrm{mod}\, R$ このとき 次の long exact seq. が存在する.

$$0 \to \mathrm{Ext}_R^1(M,N) \to \mathrm{Ext}_S^1(M,N) \to \mathrm{Hom}_R(M,N)$$

$$\to \mathrm{Ext}_R^2(M,N) \to \mathrm{Ext}_S^2(M,N) \to \mathrm{Ext}_R^1(M,N)$$

$$\to \cdots$$

$$\to \mathrm{Ext}_R^i(M,N) \to \mathrm{Ext}_S^i(M,N) \to \mathrm{Ext}_R^{i-1}(M,N) \to$$

(proof of Thm)　Lem を $N = k$ として適用する.

- $\mathrm{cx}_S M \leq \mathrm{cx}_R M$

  $\mathrm{cx}_R M = d$ とすると. $\beta_i^R \leq r \cdot i^{d-1}$ $(i \gg 0)$

  $i \gg 0$, $\beta_i^S \leq \beta_i^R + \beta_{i-1}^R$

  $$\leq r \cdot i^{d-1} + r(i-1)^{d-1}$$

  $$\leq r \cdot i^{d-1} + r \cdot i^{d-1} = 2r \cdot i^{d-1}$$

  $$\therefore \mathrm{cx}_S M \leq d$$

- $\mathrm{cx}_R M \leq \mathrm{cx}_S M + 1$

  $\beta_0^R = \beta_0^S$

  $\beta_1^R \leq \beta_1^S$

  $\beta_2^S + \beta_0^R + \beta_1^R \geqslant \beta_2^R + \beta_1^S$ より

  $\beta_2^R \leq \beta_2^S + \beta_0^R + \beta_1^R - \beta_1^S \leq \beta_2^S + \beta_0^S$

  $\vdots$

  $$\beta_i^R \leq \beta_i^S + \beta_{i-2}^S + \cdots + \begin{cases} \beta_1^S & i : \text{odd} \\ \beta_0^S & i : \text{even}. \end{cases}$$

  $$\leq \beta_i^S + \beta_{i-1}^S + \cdots + \beta_0^S$$

  ($\mathrm{cx}_S M = d$ とすると $\beta_j^S \leq r \cdot j^{d-1}$)

  $$\leq r \cdot i^{d-1} + r(i-1)^{d-1} + \cdots + r + \beta_0^S$$

  $$\leq r \cdot i^{d-1} + \cdots + r \cdot i^{d-1} + \beta_0$$

  $$\leq r \cdot i^d + \beta_0 \leq r' \cdot i^d \quad (\exists r')$$

  $$\therefore \mathrm{cx}_R M \leq \mathrm{cx}_S M + 1$$

$$\mathrm{CI\text{-}dim}_R\, k < \infty \Rightarrow {}^{\exists} R \to R' \leftarrow S \ : \text{quasi deform.}$$
$$\text{s.t. } \mathrm{pd}_S\, k' < \infty$$
$$\Rightarrow \underset{\underset{\mathrm{cx}_R\, k}{\parallel}}{\mathrm{cx}_{R'}\, k'} \le \mathrm{cx}_S\, k' + \mathrm{pd}_S\, R' < \infty$$
$$\Longrightarrow R : \text{complete intersection}$$

[Gulliksen]

<u>Def.</u>

$M, N \in \mathrm{mod}\, R$

$P_R(M, N) := \sup \{ n \mid \mathrm{Ext}_R^n(M, N) \neq 0 \}$

<u>Lem.</u>

S is a deformation of codim r

$M, N \in \mathrm{mod}\, R$

$$P_R(M, N) < \infty \Rightarrow \underset{\underset{P_R(M,N) + r}{\parallel}}{P_S(M, N)} < \infty$$

☺ ind on r. $r = 1$ としてよい.

先の lem より従う. //

<u>Thm</u>

$M, N \neq 0 \in R\text{-mod}$.

$\text{CI-dim}\, M < \infty$, $P_R(M, N) < \infty$

$\Rightarrow P_R(M, N) = \operatorname{depth} R - \operatorname{depth} M$.

∵ $\text{CI-dim}\, M < \infty$ より

$$\exists \quad R \nearrow R' \nwarrow S \quad : \text{quasi-deform. s.t. } \operatorname{pd}_S M' < \infty$$

$$\begin{cases} P_R(M, N) = P_{R'}(M', N') \\ \operatorname{depth} R - \operatorname{depth} M = \operatorname{depth} R' - \operatorname{depth} M' \end{cases}$$

よって $R = R'$ としてよい.

$\operatorname{pd}_S M =: n$ とおく. $P_R(M, N) \leq n$.

$$0 \to S^{\beta_n} \xrightarrow{\varphi} S^{\beta_{n-1}} \to \cdots \to S^{\beta_0} \to M \to 0$$

min. free. resol of M.

$$\begin{array}{ccccc} \operatorname{Hom}_S(S^{\beta_{n-1}}, N) & \xrightarrow{\varphi^*} & \operatorname{Hom}_S(S^{\beta_n}, N) & \to & \operatorname{Ext}_S^n(M, N) \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} & & \\ N^{\beta_{n-1}} & & N^{\beta_n} & & \end{array}$$

もし $\operatorname{Ext}_S^n(M, N) = 0$

$$\Rightarrow \begin{array}{c} \operatorname{Im} \varphi^* = N^{\beta_n} \\ \cap \\ \mathfrak{m} N^{\beta_n} \end{array} \Rightarrow N = 0 \quad \text{by Nakayama.}$$

$\therefore \operatorname{Ext}_S^n(M, N) \neq 0$

$$\rho_S(M,N) = \mathrm{pd}_S M$$

$$\rho_R(M,N) = \rho_S(M,N) - r = \mathrm{pd}_S N - r$$
$$= \mathrm{depth}\, S - r - \mathrm{depth}\, M$$
$$= \mathrm{depth}\, R - \mathrm{depth}\, M \qquad //$$

<u>Thm</u>

$$\left.\begin{array}{l} \mathrm{CI\text{-}dim}\, M < \infty \\ \rho_R(M,M) = 0 \end{array}\right\} \Rightarrow M : \mathrm{free}$$

(証明)

$\mathrm{CI\text{-}dim}\, M < \infty$, $\rho_R(M.M) = 0$ より

$\mathrm{depth}\, R = \mathrm{depth}\, M$.

$\mathrm{G\text{-}dim}\, M = 0 \Rightarrow \rho_R(M,R) = 0$

$$\therefore \underset{\displaystyle \mathrm{depth}\, R - \mathrm{depth}\, M = 0}{\underset{\shortparallel}{\rho_R(M, \Omega M)}} < \infty$$

$\therefore \mathrm{Ext}^1(M, \Omega M) = 0 \qquad \therefore M : \mathrm{free}$

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