Leaf poset and multi-colored hook length property

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joint work with Hiroyuki Tagawa

Introduction

Definition (Poset)

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Definition (Cover)

An element a is said to be *covered* by another element b, written $a \lessdot b$, if $a < b$ and there is no element c such that $a < c < b$.

Hasse diagram

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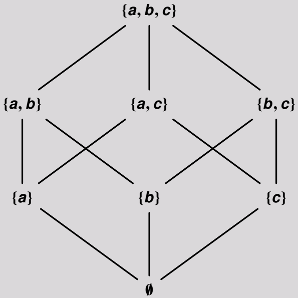
Example

Let $S = \{a, b, c\}$ be 3 element set, $P = 2^S$ the set of all subsets of S. of a (finite) set P and the order $\leq$ is defined by inclusion $\subseteq$.



(P, ω)-partition

Let P be a finite poset of cardinality p. Let $\omega : P \to [p] = \{1, \dots, p\}$ be a bijection, called a labeling of P.

Definition ((P, ω)-partition)

Let $\mathbb{N}$ denote the set of nonnegative integers.

If ω is natural, i.e., $s < t \Rightarrow \omega(s) < \omega(t)$, then a *$(P, \omega)$-partition* is just an order-reversing map $\sigma : P \to \mathbb{N}$. We then call σ simply a P-partition. Write $\mathscr{A}(P, \omega)$ for the set of all (P, ω)-partitions $\sigma : P \to \mathbb{N}$. If ω is a natural labeling, we simply write $\mathscr{A}(P)$. Let $|\sigma| = \sum_{s \in P} \sigma(s)$ denote the sum of the entries of σ.

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(P, ω)-partition

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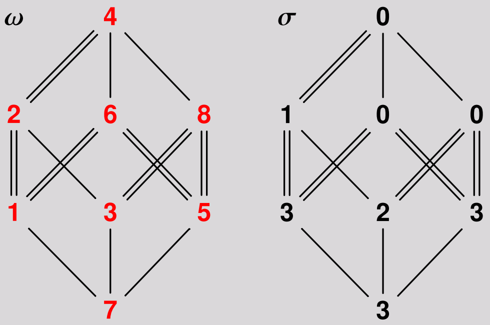
If $P = 2^{\{a,b,c\}}$ is the Boolean poset.



Hook Length Property

For a labeled poset (P, ω), we write

$$F(P, \omega; q) = \sum_{\sigma \in \mathscr{A}(P,\omega)} q^{|\sigma|},$$

which we call the *one variable generating function* of (P, ω)-partitions. When ω is natural, we write $F(P; q)$ for $F(P, \omega; q)$.

Definition

We say that P has *hook-length property* if there exists a map h from P to $\mathbb{N}$ satisfying

$$F(P; q) = \prod_{x \in P} \frac{1}{1 - q^{h(x)}}.$$

If P has hook-length property, then $h(x)$ is called the hook length of x, and h is called the *hook-length function*. A *hook-length poset* is a poset which has hook length property. The hook-length property was first defined by B. Sagan.

Colored Hook Length Property

Let (P, ω) be a labeled poset, and $z = (z_1, \dots, z_k)$ be variables. Assume there exists a sujective map $c : P \to \{1, 2, \dots, k\}$, which we call the *color function*. We write

$$F_c(P, \omega; z) = \sum_{\sigma \in \mathscr{A}(P,\omega)} z^{\sigma},$$

where $z^{\sigma} = \prod_{x \in P} z_{c(x)}^{\sigma(x)}$. We call $F_c(P, \omega; z)$ the *colored generating function* or *multi-variable generating function*.

Definition

We say that P has *k-colored hook-length property* if there exists a map h from P to $\mathbb{N}^k$ satisfying

$$F(P; q) = \prod_{x \in P} \frac{1}{1 - z^{h(x)}},$$

where $z^{h(x)} = \prod_{x \in P} z_{c(x)}^{h(x)}$. A *colored hook-length poset* is a poset which has colored hook length property.

(Shifted) diagrams

Definition

A *partiton* is a nonincreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of nonnegative integers with finitely many λ_i unequal to zero.

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$$D(\lambda) = \{ (i, j) \in \mathbb{Z}^2 \ : \ 1 \le j \le \lambda_i \}$$
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Example (The diagram and shifted diagram for $\lambda = (4, 3, 1)$)

$$D(\lambda) = \begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & & & \end{array} \qquad S(\lambda) = \begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet \\ & & \bullet & \end{array}$$

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We define the order on $D(\lambda)$ (or $S(\lambda)$) by

$$(i_1, j_1) \geq (i_2, j_2) \Leftrightarrow i_1 \leq i_2 \text{ and } j_1 \leq j_2$$

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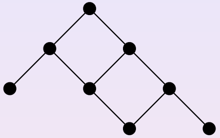
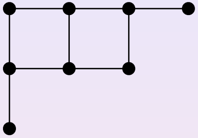
We rotate the Hasse diagram of the poset by $45°$ counterclockwise. Hence a vertex in the north-east is bigger than a vertex in south-west.

Examples

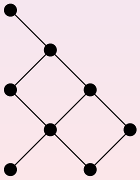
shape



$\longrightarrow$



shifted shape



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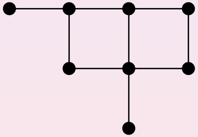

d-complete poset

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4. The ***d**-complete coloring* is important for the multivariate generating function. The content should be replaced by color for *d*-complete posets.
5. Okada defined (q, t)-weight $W_P(\pi; q, t)$ for d-compete posets.

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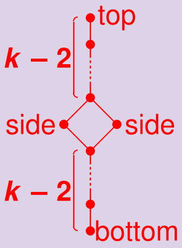
- The double-tailed diamond poset $d_k(1)$ is the poset depicted below:



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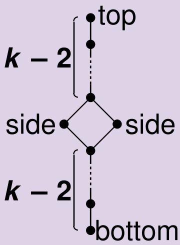

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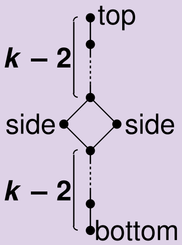

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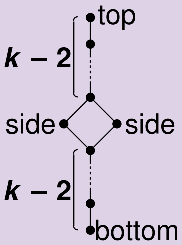

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- A d_3^--interval consists of three elements x, y and w such that w is covered by x and y.

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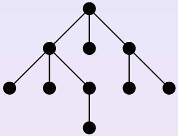
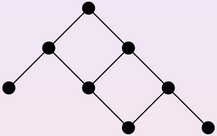
3. There are no d_k^--intervals which differ only in the minimal elements.

Examples

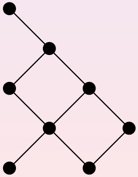
rooted tree



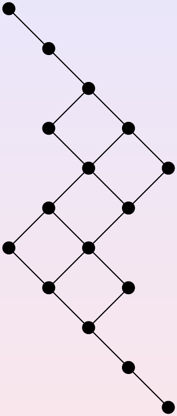
shape



shifted shape



swivel



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(a) **Any connected d-complete poset is uniquely decomposed into a slant sum of one-element posets and slant-irreducible d-complete posets.**

(b) **Slant-irreducible d-complete posets are classified into 15 families : shapes, shifted shapes, birds, insets, tailed insets, banners, nooks, swivels, tailed swivels, tagged swivels, swivel shifts, pumps, tailed pumps, near bats, bat.**

Irreducible d-complete poset

Definition (Filter)

Let S be a subset of a poset P. If S satisfies the condition

$$x \in S \text{ and } y \geq x \quad \Rightarrow y \in S$$

then S is said to be a *filter*.

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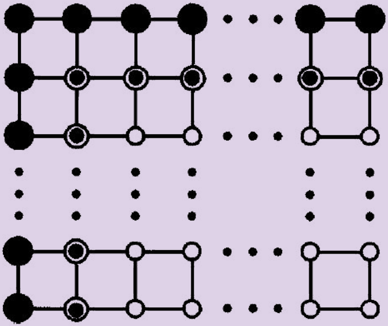
3. **1) Shapes, 2) Shifted shapes, 3) Birds, 4) Insets, 5) Tailed insets, 6) Banners, 7) Nooks, 8) Swivels, 9) Tailed swivels, 10) Tagged swivels, 11) Swivel shifteds, 12) Pumps, 13) Tailed pumps, 14) Near bats, 15) Bat**

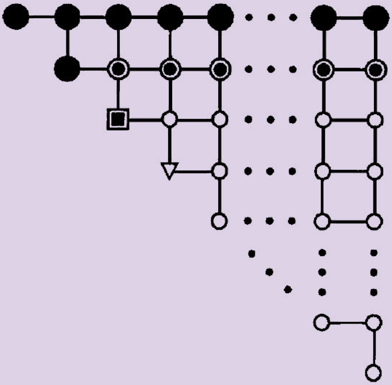
Definition (Shapes)

1) Shapes



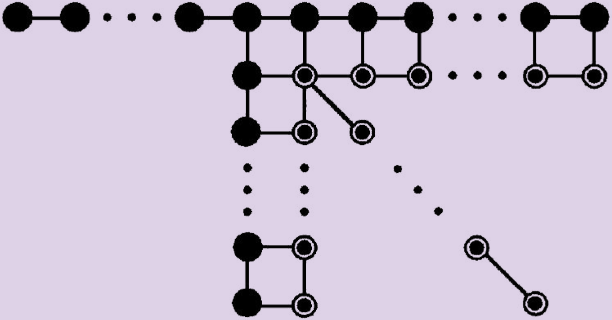
Definition (Shifted shapes)

2) Shifted shapes



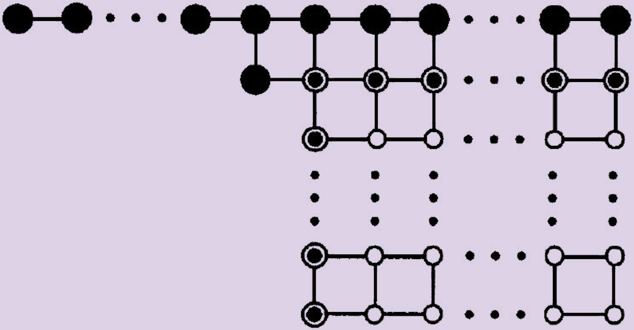
Definition (Birds)

3) Birds



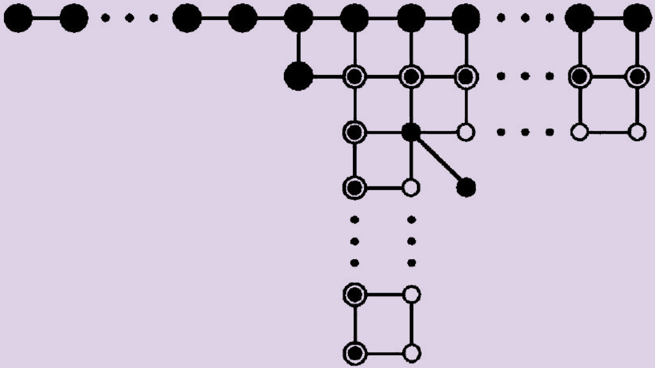
Definition (Insets)

4) Insets



Definition (Tailed insets)

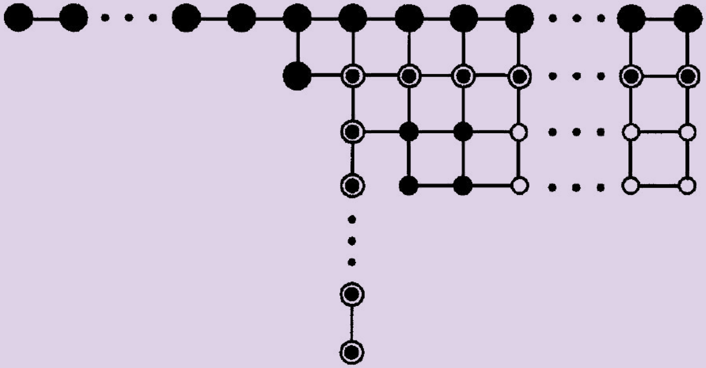
5) Tailed insets



Banners

Definition (Banners)

6) Banners



Nooks

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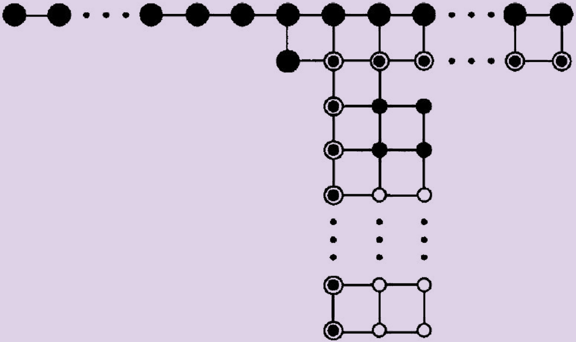
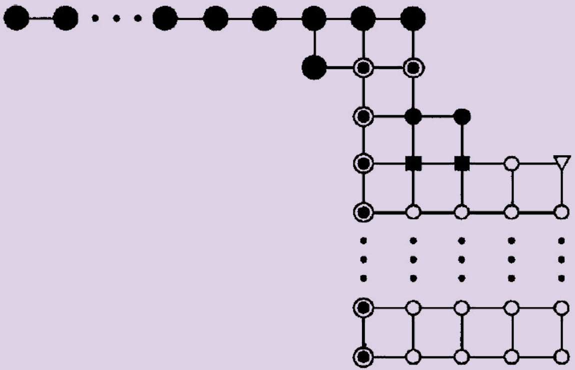
7) Nooks



Definition (Swivels)

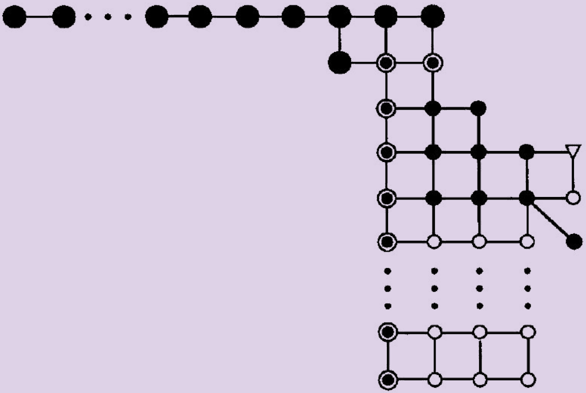
8) Swivels



Tailed swivels

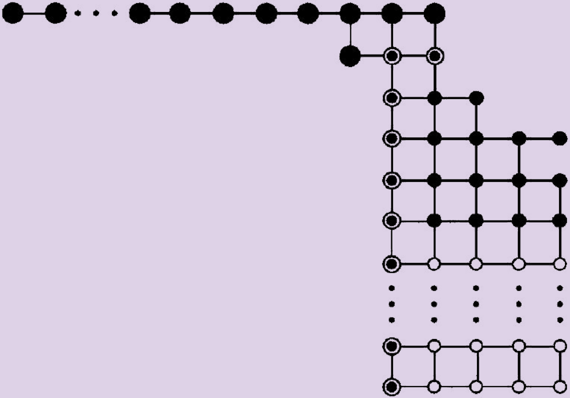
Definition (Tailed swivels)

9) Tailed swivels



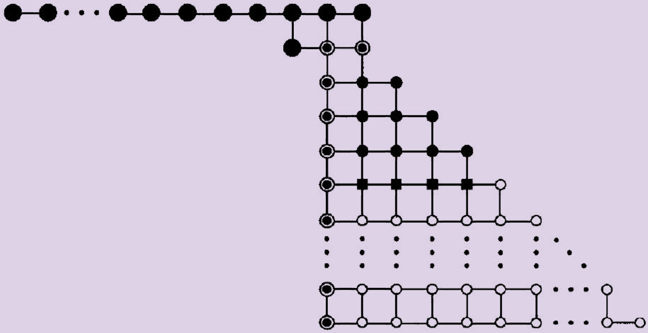
Definition (Tagged swivels)

10) Tagged swivels



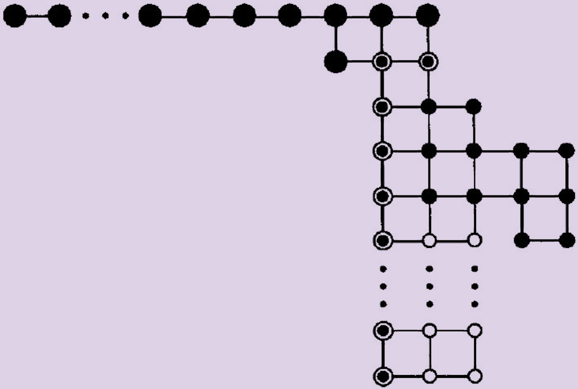
Definition (Swivel shifteds)

11) Swivel shifteds



Definition (Pumps)

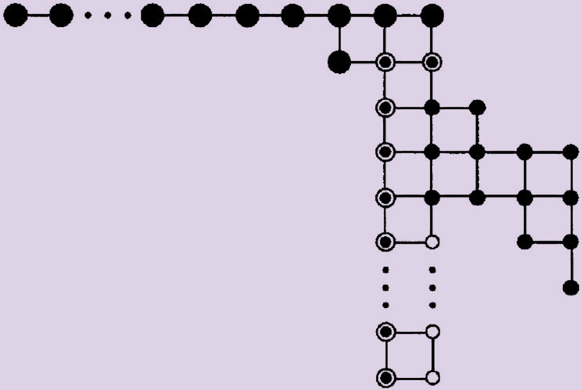
12) Pumps



Tailed pumps

Definition (Tailed pumps)

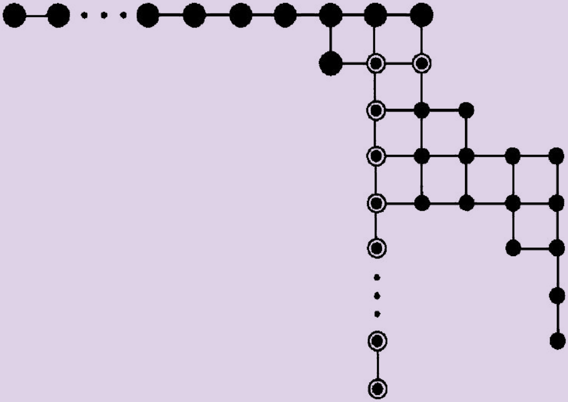
13) Tailed pumps



Near bats

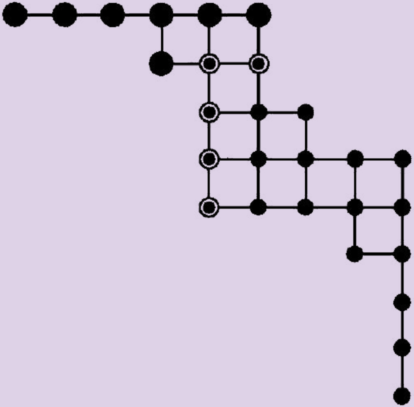
Definition (Near bats)

14) Near bats



Definition (Bat)

15) Bat



Colored hook length property of d-complete posets

Theorem (Peterson-Proctor)

***d*-complete poset has the colored hook-length property.**

Colored hook length property of d-complete posets

Theorem (Peterson-Proctor)

***d*-complete poset has the colored hook-length property.**

Remark

Recently, Jan Soo Kim and Meesue Yoo gave a proof of the hook-length property by *q*-integral.

Leaf Posets

Contents of this section

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1. We define 6 family of posets, which we call the basic leaf posets. (It is not possible to define “irreducibility”.)

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Leaf Posets

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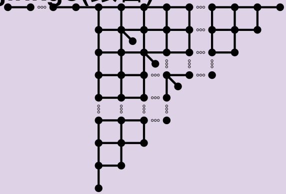
5. **The colored hook-length property of the basic leaf posets reduces to the Schur function identities.**

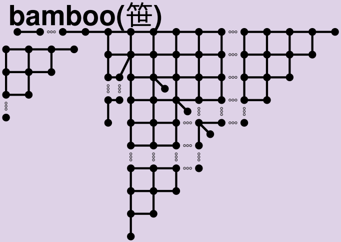
Definition

Definition

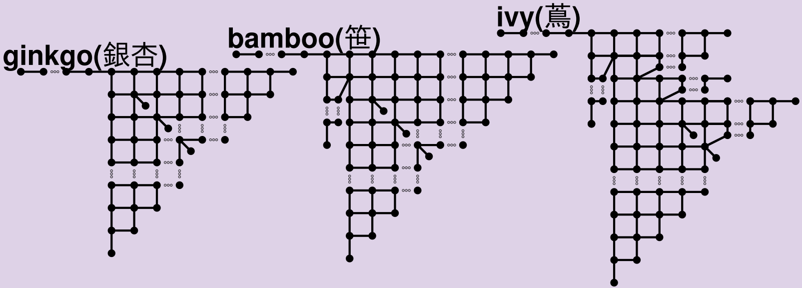
ginkgo(銀杏)



Definition



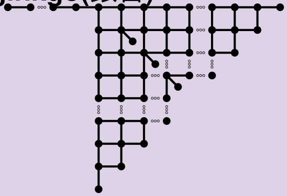
Definition



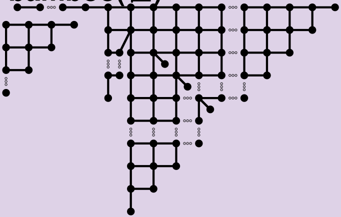
Basic Leaf Posets

Definition

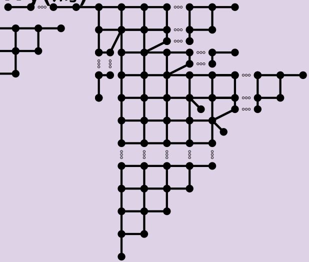
ginkgo(銀杏)



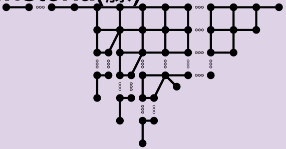
bamboo(笹)



ivy(蔦)

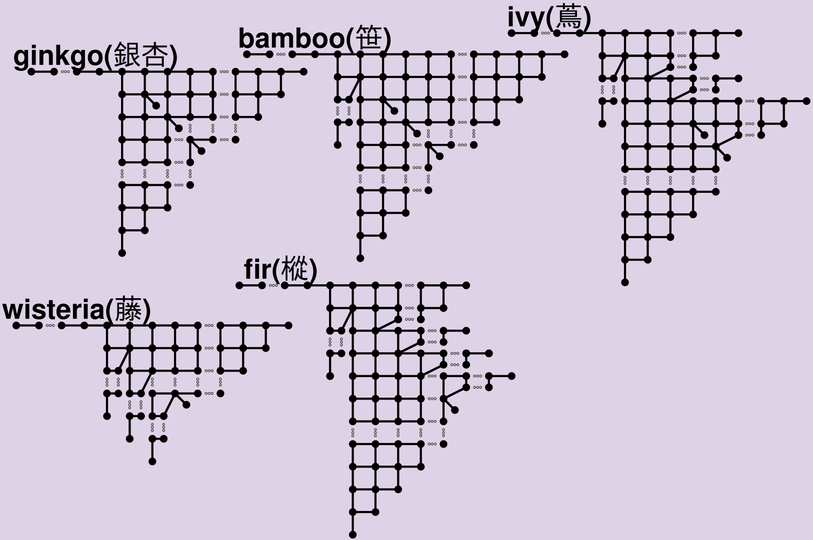


wisteria(藤)



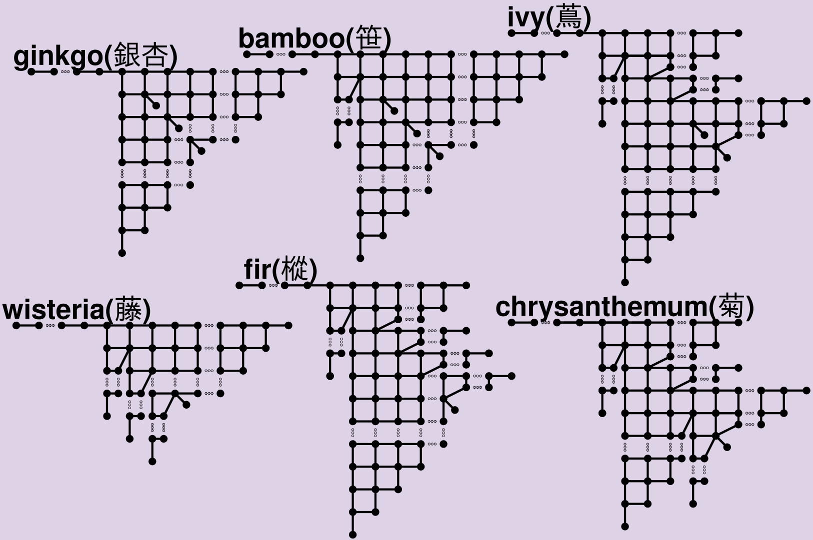
Basic Leaf Posets

Definition



Basic Leaf Posets

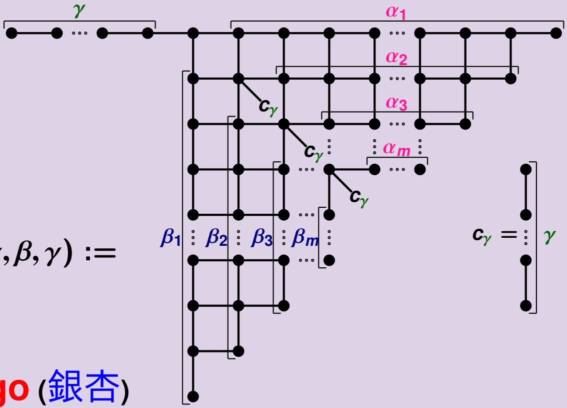
Definition



Definition

(i) $m \geq 2$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$: strict partitions

$G(\alpha, \beta, \gamma) :=$

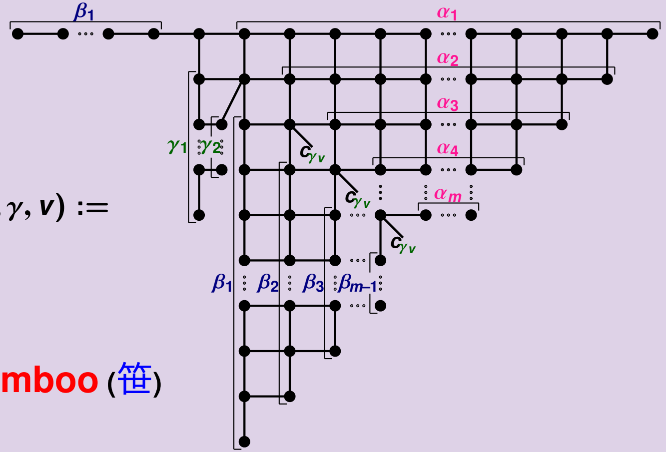


ginkgo (銀杏)

Definition

(ii) $m \geq 2$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_{m-1})$, $\gamma = (\gamma_1, \gamma_2)$: strict partition, $v = 1, 2$

$B(\alpha, \beta, \gamma, v) :=$

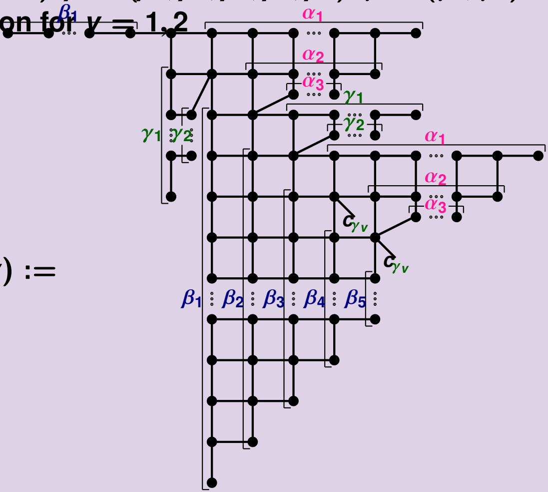


bamboo (笹)

Definition

(iii) $\alpha = (\alpha_1, \alpha_2, \alpha_3), \beta = (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5), \gamma = (\gamma_1, \gamma_2)$: strict partition for $v = 1, 2$

$$I(\alpha, \beta, \gamma, v) :=$$

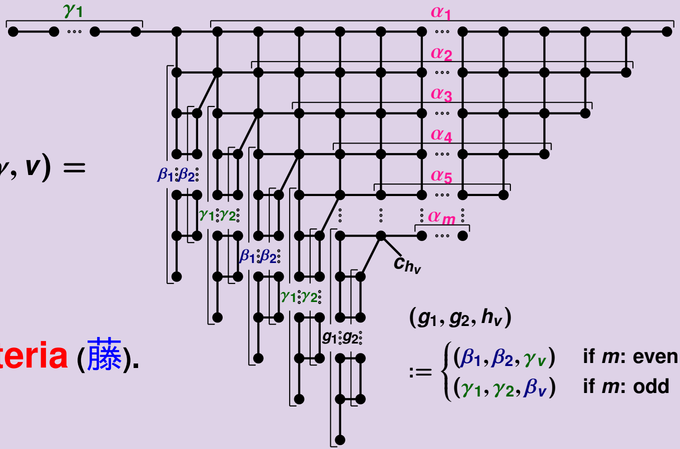


ivy (蔦)

Definition

(iv) $m \geq 2$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, $\beta = (\beta_1, \beta_2)$, $\gamma = (\gamma_1, \gamma_2)$: strict partition

$$W(\alpha, \beta, \gamma, v) =$$



$$(g_1, g_2, h_v) := \begin{cases} (\beta_1, \beta_2, \gamma_v) & \text{if } m\text{: even} \\ (\gamma_1, \gamma_2, \beta_v) & \text{if } m\text{: odd} \end{cases}$$

wisteria (藤).

Definition

(v) $m \geq 3$,

$\alpha = (\alpha_1, \alpha_2, \alpha_3)$,

$\beta = (\beta_1, \beta_2, \ldots, \beta_{m-1})$,

$\gamma = (\gamma_1, \gamma_2)$

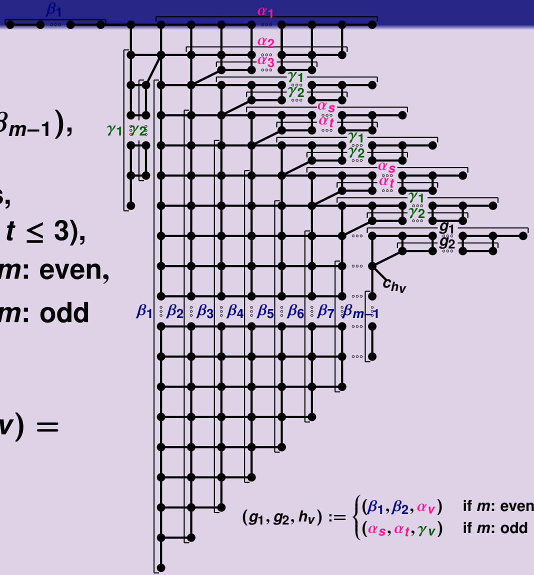
: strict partitions,

$s, t \geq 1$ $(1 \leq s < t \leq 3)$,

$$v = \begin{cases} s \text{ or } t & \text{if } m\text{: even,} \\ 1 \text{ or } 2 & \text{if } m\text{: odd} \end{cases}$$

$$F(\alpha, \beta, \gamma, s, t, v) =$$

fir (樅).

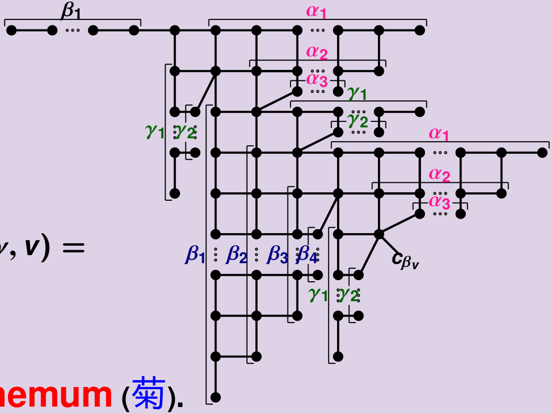


$$(g_1, g_2, h_v) := \begin{cases} (\beta_1, \beta_2, \alpha_v) & \text{if } m\text{: even} \\ (\alpha_s, \alpha_t, \gamma_v) & \text{if } m\text{: odd} \end{cases}$$

Definition

(vi) $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$ and $\gamma = (\gamma_1, \gamma_2)$: strict partitions, $\delta \geq 0$ for $v = 1, 2, 3, 4$

$C(\alpha, \beta, \gamma, v) =$



chrysanthemum (菊).

Goal of This Talk

Property of leaf posets

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Property of leaf posets

1. **Any *d*-complete poset is a leaf poset.**

Goal of This Talk

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   1. **1) Shapes, 3) Birds $\subseteq$ Ginkgo**

Goal of This Talk

Property of leaf posets

1. Any d-complete poset is a leaf poset.
   1. 1) Shapes, 3) Birds $\subseteq$ Ginkgo
   2. 2) Shifted shapes, 6) Banners $\subseteq$ Wisteria

Goal of This Talk

Property of leaf posets

1. Any d-complete poset is a leaf poset.
   1. 1) Shapes, 3) Birds $\subseteq$ Ginkgo
   2. 2) Shifted shapes, 6) Banners $\subseteq$ Wisteria
   3. 5) Tailed insets, 4) Insets $\subseteq$ Bamboo

Goal of This Talk

Property of leaf posets

1. **Any *d*-complete poset is a leaf poset.**
   1. **1) Shapes, 3) Birds $\subsetneq$ Ginkgo**
   2. **2) Shifted shapes, 6) Banners $\subsetneq$ Wisteria**
   3. **5) Tailed insets, 4) Insets $\subsetneq$ Bamboo**
   4. **7) Nooks, 9) Tailed swivels, 10) Tagged swivels, 11) Swivel shifteds $\subsetneq$ Fir**

Goal of This Talk

Property of leaf posets

1. Any d-complete poset is a leaf poset.
    1. 1) Shapes, 3) Birds $\subsetneq$ Ginkgo
    2. 2) Shifted shapes, 6) Banners $\subsetneq$ Wisteria
    3. 5) Tailed insets, 4) Insets $\subsetneq$ Bamboo
    4. 7) Nooks, 9) Tailed swivels, 10) Tagged swivels, 11) Swivel shifteds $\subsetneq$ Fir
    5. 8) Swivels $\subsetneq$ Ivy

Goal of This Talk

Property of leaf posets

1. Any *d*-complete poset is a leaf poset.
   1. 1) Shapes, 3) Birds ⊊ Ginkgo
   2. 2) Shifted shapes, 6) Banners ⊊ Wisteria
   3. 5) Tailed insets, 4) Insets ⊊ Bamboo
   4. 7) Nooks, 9) Tailed swivels, 10) Tagged swivels, 11) Swivel shifteds ⊊ Fir
   5. 8) Swivels ⊊ Ivy
   6. 12) Pumps, 13) Tailed pumps, 14) Near bats, 15) Bat ⊊ Chrysanthemum

Goal of This Talk

Property of leaf posets

1. Any *d*-complete poset is a leaf poset.
   1. 1) Shapes, 3) Birds $\subseteq$ Ginkgo
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   6. 12) Pumps, 13) Tailed pumps, 14) Near bats, 15) Bat $\subseteq$ Chrysanthemum

Theorem

A leaf poset has multi-colored hook length property.

Schur Function

Definition (Schur Function)

If $\lambda = (\lambda_1, \dots, \lambda_n)$ is a partition of length $\leq n$, then

$$s_\lambda(x_1, \dots, x_n) = \frac{\begin{vmatrix} x_1^{\lambda_1+n-1} & \dots & x_1^{\lambda_n} \\ \vdots & \ddots & \vdots \\ x_n^{\lambda_1+n-1} & \dots & x_n^{\lambda_n} \end{vmatrix}}{\begin{vmatrix} x_1^{n-1} & \dots & 1 \\ \vdots & \ddots & \vdots \\ x_n^{n-1} & \dots & 1 \end{vmatrix}}.$$

The Schur functions are the irreducible characters of the polynomial representations of the General Linear Group.

Theorem (Cauchy's formula)

If n is a positive integer, then

$$\sum_{\lambda} s_{\lambda}(x_1,\dots,x_n)s_{\lambda}(y_1,\dots,y_n) = \prod_{i=1}^{n}\prod_{j=1}^{n}\frac{1}{1-x_i y_j}.$$

Symmetric Functions

Theorem (Cauchy's formula)

If n is a positive integer, then

$$\sum_{\lambda} s_\lambda(x_1,\dots,x_n)s_\lambda(y_1,\dots,y_n) = \prod_{i=1}^{n}\prod_{j=1}^{n}\frac{1}{1-x_iy_j}.$$

Proposition

If n is a positive integer, then

$$\prod_{j=1}^{n}\frac{1}{1-tx_i} = \sum_{r\geq 0} h_r(x_1,\dots,x_n)t^n,$$

$$\prod_{j=1}^{n}(1+tx_i) = \sum_{r=0}^{n} e_r(x_1,\dots,x_n)t^n$$

where h_r is the complete symmetric function and e_r is the elementary symmetric function.

Theorem (Pieri's rule)

If n is a positive integer and μ is a partition, then

$$s_\mu(x_1,\dots,x_n)h_r(x_1,\dots,x_n) = \sum_{\lambda} s_\lambda(x_1,\dots,x_n),$$

where the sum runs over all partitions λ such that λ/μ is horizontal r-strip.

Symmetric Functions

Theorem (Pieri's rule)

If n is a positive integer and μ is a partition, then

$$s_\mu(x_1,\dots,x_n)h_r(x_1,\dots,x_n) = \sum_{\lambda} s_\lambda(x_1,\dots,x_n),$$

where the sum runs over all partitions λ such that λ/μ is horizontal r-strip.

Theorem (Littlewood's formula)

If n is a positive integer, then

$$\sum_{\nu} s_\nu(x_1,\dots,x_n) = \prod_{1\le i<j\le n} \frac{1}{1-x_ix_j}$$

where the sum runs over all partitions ν such that ν' are even partitions.

Pre-Leaf Poset

Definition

If λ is a strict partition with length $p = \ell(\lambda)$, let

$$P(\lambda) = \{(i,j) \mid 1 \le i \le p \text{ and } i \le j \le i + \lambda_i\}.$$

We say $x = (i,j) \ge y = (i',j')$ in $P(\lambda)$ if $i \le i'$ and $j \le j'$.

Pre-Leaf Poset

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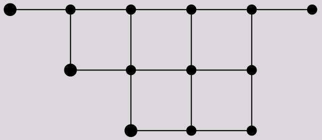
Example $P(\lambda)$

If $\lambda = (5,3,2)$ then $P(\lambda)$ is as follows:



Definition (Pre-Leaf Poset)

Let $\lambda^{(k)}$ $(k = 1, \dots, m)$ be strict partitions with $\ell\left(\lambda^{(k)}\right) = p^{(k)}$, and let $s^{(k)}$ be positive integers. Let

$$n = \max\{s^{(k)} + p^{(k)} - 1 | k = 1, \dots, m\},$$
$$C = \{(i, i) | 1 \leq i \leq n\}.$$

Let $P\left[(\lambda^{(k)}, s^{(k)})_{1 \leq k \leq m}\right]$ denote the set obtained by identifying $(s^{(k)} + i - 1, s^{(k)} + i - 1)$ in C and (i, i) in $P(\lambda^{(k)})$.

Pre-Leaf Poset

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Definition (Order)

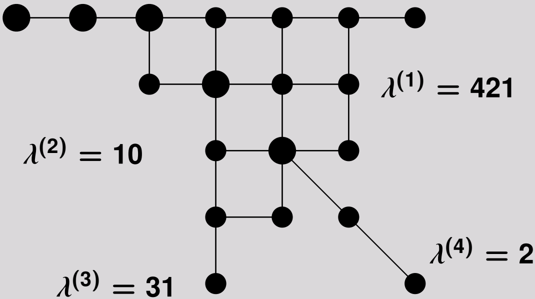
We say $x = (i, j) \ge y = (i', j')$ in $P\left[(\lambda^{(k)}, s^{(k)})_{1 \le k \le m}\right]$ if x and y are both in some $\lambda^{(i)}$ and $x \ge y$, or, $x \in C$ and $y \in \lambda^{(k)}$ and $i = j \le i'$. We call $P\left[(\lambda^{(k)}, s^{(k)})_{1 \le k \le m}\right]$ the *pre-leaf poset associated with* $(\lambda^{(k)}, s^{(k)})_{1 \le k \le m}$. We call C the *central chain* of *length* n.

Example (Pre-Leaf Poset)

Example (Pre-Leaf Poset)

If $(\lambda^{(1)}, s^{(1)}) = (421, 3)$, $(\lambda^{(2)}, s^{(2)}) = (10, 3)$, $(\lambda^{(3)}, s^{(3)}) = (31, 4)$, and $(\lambda^{(4)}, s^{(4)}) = (2, 5)$, then we have

Pre-Leaf Poset $P\left[(\lambda^{(k)}, s^{(k)})_{1\le k\le 4}\right]$



Notation

Definition

If λ is a strict partition, then we define the weight $w_{P(\lambda)}$ of $P(\lambda)$ by

$$w_{P(\lambda)}(i,j) := \begin{cases} p_i & \text{if } i = j, \\ q_{j-i} & \text{if } i < j \end{cases}.$$

Notation

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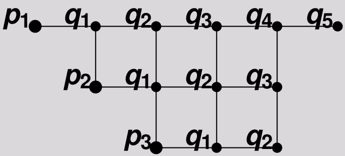
Examle $w_{P(\lambda)}$

If $\lambda = (5,3,2)$ then $w_{P(\lambda)}$ is as follows:



Notation

Definition

If $\mathbf{q} = (\dots, q_{-1}, q_0, q_1, q_2 \dots)$ be variables, then we use the notation:

$$\mathbf{q}_{[k,l]} = \prod_{i=k}^{l} q_i = q_k \cdots q_l,$$

$$(\mathbf{q})_n = \prod_{k=1}^{n} \left(1 - \prod_{i=1}^{k} \mathbf{q}_i\right)$$

$$= (1 - q_1)(1 - q_1 q_2) \cdots (1 - q_1 \cdots q_k),$$

$$\langle \mathbf{q} \rangle_n = \prod_{k=1}^{n} \left(1 - \prod_{i=k}^{n} \mathbf{q}_i\right)$$

$$= (1 - q_n)(1 - q_{n-1} q_n) \cdots (1 - q_1 \cdots q_k).$$

Especially we write $\mathbf{q}_{[k]}$ for $\mathbf{q}_{[1,k]}$.

Generating Function

Definition

Let P be a poset. If w is a weight of P and $\sigma \in \mathscr{A}(P)$, we write

$$w^\sigma = \prod_{x \in P} w(x)^{\sigma(x)}, \quad F(P; w) = \sum_{\sigma \in \mathscr{A}(P)} w^\sigma.$$

Generating Function

Definition

Let P be a poset. If w is a weight of P and $\sigma \in \mathscr{A}(P)$, we write

$$w^{\sigma} = \prod_{x \in P} w(x)^{\sigma(x)}, \quad F(P; w) = \sum_{\sigma \in \mathscr{A}(P)} w^{\sigma}.$$

Theorem

Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be a strict partition, and $x_1, \dots, x_m \in \mathbb{Z}$ be integers such that $0 \le x_1 \le x_2 \le \cdots \le x_m$. Then we have

$$\sum_{\substack{\varphi \in \mathscr{A}(P(\lambda)) \\ \sigma(i,i) = x_i (1 \le i \le m)}} w_{P(\lambda)}^{\sigma} = \frac{\prod_{i=1}^{m} p_i^{x_i} \prod_{1 \le i < j \le m} (1 - \mathbf{q}_{[\lambda_j + 1, \lambda_i]})}{\prod_{i=1}^{m} \langle \mathbf{q} \rangle_{\lambda_i}} \times s_{(x_m, x_{m-1}, \dots, x_1)}(\mathbf{q}_{[\lambda_1]}, \dots, \mathbf{q}_{[\lambda_m]}).$$

Weight of Pre-Leaf Poset

Definition (Weight)

Let $\lambda^{(k)}$ be strict partitions with $\ell\left(\lambda^{(k)}\right) = p^{(k)}$, and let $s^{(k)}$ be positive integers for $k = 1, \dots, m$. Let $P = P\left[(\lambda^{(k)}, s^{(k)})_{1 \le k \le m}\right]$ be the pre-leaf poset associated with $(\lambda^{(k)}, s^{(k)})_{1 \le k \le m}$, and let $\mathrm{q}^{(k)} = (q_i^{(k)})_{1 \le i \le \lambda_1}$ be variables associated with each diagonal of $\lambda^{(k)}$, and $\mathrm{p} = (p_i)_{1 \le i \le n}$ be variables associated with the central chain C. We write

$$w\left[(\mathrm{q}^{(k)})_{1 \le k \le m}, \mathrm{p}\right]$$

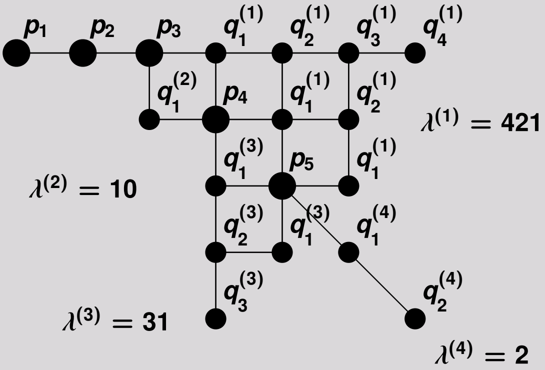
for this weight.

Weight of Pre-Leaf Poset

Example (Weight of Pre-Leaf Poset)

If $(\lambda^{(1)}, s^{(1)}) = (421, 3)$, $(\lambda^{(2)}, s^{(2)}) = (10, 3)$, $(\lambda^{(3)}, s^{(3)}) = (31, 4)$, and $(\lambda^{(4)}, s^{(4)}) = (2, 5)$, then we have

Pre-Leaf Poset $P\left[(\lambda^{(k)}, s^{(k)})_{1\le k\le 4}\right]$



Generating Function

Theorem

Let $P = P\left[(\lambda^{(k)}, s^{(k)})_{1 \le k \le m}\right]$ be the pre-leaf poset associated with $(\lambda^{(k)}, s^{(k)})_{1 \le k \le m}$, and let $\mathrm{q}^{(k)} = (q_i^{(k)})_{1 \le i \le \lambda_1}$ be variables associated with each diagonal of λ, and $\mathrm{p} = (p_i)_{1 \le i \le n}$ be variables associated with the central chain C.

$$\sum_{\sigma \in \mathscr{A}(P)} w^{\sigma}_{P(\lambda)} = \frac{\prod_{k=1}^{m} \prod_{1 \le i < j \le p^{(k)}} (1 - \mathrm{q}_{[\lambda_j^{(k)}+1, \lambda_i^{(k)}]})}{\prod_{k=1}^{m} \prod_{i=1}^{p^{(k)}} \langle \mathrm{q} \rangle_{\lambda_i^{(k)}}}$$

$$\times \sum_{\lambda=(\lambda_1,\dots,\lambda_n)} \prod_{i=1}^{n} p_i^{\lambda_{n+1-i}} \prod_{k=1}^{m} s_{\lambda[n+2-s_k-p^{(k)}, n+1-s_k]}(\mathrm{q}^{(k)}_{[\lambda_1^{(k)}]}, \dots, \mathrm{q}^{(k)}_{[\lambda^{(k)}_{p^{(k)}}]}).$$

where $\lambda[i,j]$ stands for $(\lambda_i, \dots, \lambda_j)$.

Schur Function Identities

Lemma

[ginkgo]

$$\sum_{\lambda=(\lambda_1,\lambda_2,\dots,\lambda_m)\in\mathscr{P}} w^{\lambda_m} s_\lambda(x_1,\dots,x_m) s_\lambda(y_1,\dots,y_m)$$

$$= \frac{1-\prod_{i=1}^m x_i y_i}{\left(1-w\prod_{i=1}^m x_i y_i\right)\prod_{i,j=1}^m (1-x_i y_j)}.$$

Lemma

[ginkgo]

$$\sum_{\lambda=(\lambda_1,\lambda_2,\dots,\lambda_m)\in\mathscr{P}} w^{\lambda_m} s_\lambda(x_1,\dots,x_m) s_\lambda(y_1,\dots,y_m) = \frac{1-\prod_{i=1}^m x_i y_i}{(1-w\prod_{i=1}^m x_i y_i)\prod_{i,j=1}^m (1-x_i y_j)}.$$

[bamboo]

$$\sum_{\lambda\in\mathscr{P}} w^{\lambda_m} s_{(\lambda_1,\dots,\lambda_{m-1})}(x_1,\dots,x_{m-1}) s_{(\lambda_{m-1},\lambda_m)}(1,z_2) s_\lambda(y_1,\dots,y_m)$$
$$= \frac{\prod_{i=1}^{m-1}(1-z_2 x_i \prod_{k=1}^{m-1} x_k \prod_{k=1}^m y_k)}{(1-wz_2\prod_{k=1}^{m-1} x_k \prod_{k=1}^m y_k)\prod_{i=1}^{m-1}\prod_{j=1}^m (1-x_i y_j)}$$
$$\times \frac{1}{\prod_{i=1}^m (1-y_i^{-1} z_2 \prod_{k=1}^{m-1} x_k \prod_{k=1}^m y_k)}.$$

Lemma

[ivy]

$$\sum_{\lambda=(\lambda_1,\lambda_2,\dots,\lambda_6)\in\mathscr{P}} \begin{array}{l} w^{\lambda_6} s_{(\lambda_1,\lambda_2,\lambda_3)}(x_1,x_2,x_3) s_{(\lambda_3,\lambda_4)}(1,z_2) s_{(\lambda_4,\lambda_5,\lambda_6)}(x_1,x_2,x_3) \\ \times s_{(\lambda_1,\dots,\lambda_5)}(y_1,\dots,y_5) s_{(\lambda_5,\lambda_6)}(1,z_2) \end{array}$$

$$= \frac{1}{(1 - wz_2^2 \prod_{k=1}^3 x_k^2 \prod_{k=1}^5 y_k) \prod_{i=1}^3 \prod_{j=1}^5 (1 - x_i y_j)}$$

$$\times \frac{\prod_{i=1}^5 (1 - y_i z_2^2 \prod_{k=1}^3 x_k^2 \prod_{k=1}^5 y_k)}{\prod_{i=1}^3 (1 - x_i^{-1} z_2^2 \prod_{k=1}^3 x_k^2 \prod_{k=1}^5 y_k)}$$

$$\times \frac{1}{\prod_{1\le i<j\le 5} (1 - y_i^{-1} y_j^{-1} z_2 \prod_{k=1}^3 x_k \prod_{k=1}^5 y_k)}.$$

Lemma

[wisteria]

$$\sum_{\lambda=(\lambda_1,\lambda_2,\dots,\lambda_{2m})\in\mathscr{P}} w^{\lambda_{2m}} s_\lambda(y_1,\dots,y_{2m}) \prod_{i=1}^{m} s_{(\lambda_{2i-1},\lambda_{2i})}(x_1,x_2) \prod_{i=1}^{m-1} s_{(\lambda_{2i},\lambda_{2i+1})}(1,z_2)$$

$$= \frac{(1-z_2^{m-1}\prod_{k=1}^{2} x_k^m \prod_{k=1}^{2m} y_k)(1-z_2^{m}\prod_{k=1}^{2} x_k^m \prod_{k=1}^{2m} y_k)}{(1-wz_2^{m-1}\prod_{k=1}^{2} x_k^m \prod_{k=1}^{2m} y_k)\prod_{i=1}^{2}\prod_{j=1}^{2m}(1-x_iy_j)\prod_{1\le i<j\le 2m}(1-y_iy_jz_2\prod_{k=1}^{2} x_k)}.$$

$$\sum_{\lambda=(\lambda_1,\lambda_2,\dots,\lambda_{2m+1})\in\mathscr{P}} w^{\lambda_{2m+1}} s_\lambda(y_1,\dots,y_{2m+1}) \prod_{i=1}^{m} s_{(\lambda_{2i-1},\lambda_{2i})}(x_1,x_2) \prod_{i=1}^{m} s_{(\lambda_{2i},\lambda_{2i+1})}(1,z_2)$$

$$= \frac{\prod_{i=1}^{2}(1-x_iz_2^{m}\prod_{k=1}^{2} x_k^m \prod_{k=1}^{2m+1} y_k)}{(1-wz_2^{m}\prod_{k=1}^{2} x_k^m \prod_{k=1}^{2m+1} y_k)\prod_{i=1}^{2}\prod_{j=1}^{2m+1}(1-x_iy_j)\prod_{1\le i<j\le 2m+1}(1-y_iy_jz_2\prod_{k=1}^{2} x_k)}.$$

Lemma

[fir]

$$\sum_{\lambda=(\lambda_1,\lambda_2,\dots,\lambda_{2m})\in\mathscr{P}} \begin{array}{c} w^{\lambda_{2m}} s_{(\lambda_1,\dots,\lambda_{2m-1})}(y_1,\dots,y_{2m-1}) s_{(\lambda_{2m-2},\lambda_{2m-1},\lambda_{2m})}(z_1,z_2,z_3) \\ \times \prod_{i=1}^{m} s_{(\lambda_{2i-1},\lambda_{2i})}(x_1,x_2) \prod_{i=1}^{m-2} s_{(\lambda_{2i},\lambda_{2i+1})}(1,z_2) \end{array}$$

$$= \frac{1}{(1-wz_2^{m-1}z_3\prod_{k=1}^{2}x_k^m\prod_{k=1}^{2m-1}y_k)\prod_{i=1}^{2}\prod_{j=1}^{2m-1}(1-x_iy_j)\prod_{1\le i<j\le 2m-1}(1-y_iy_jz_2\prod_{k=1}^{2}x_k)}$$

$$\times \frac{\prod_{i=1}^{2m-1}(1-y_iz_2^{m-1}z_3\prod_{k=1}^{2}x_k^m\prod_{k=1}^{2m-1}y_k)}{\prod_{i=1}^{2}(1-x_iz_2^{m-1}z_3\prod_{k=1}^{2}x_k^{m-1}\prod_{k=1}^{2m-1}y_k)\prod_{i=1}^{2m-1}(1-y_i^{-1}z_2^{m-2}z_3\prod_{k=1}^{2}x_k^{m-1}\prod_{k=1}^{2m-1}y_k)}.$$

$$\sum_{\lambda=(\lambda_1,\lambda_2,\dots,\lambda_{2m+1})\in\mathscr{P}} \begin{array}{c} w^{\lambda_{2m+1}} s_{(\lambda_1,\dots,\lambda_{2m})}(y_1,\dots,y_{2m}) s_{(\lambda_{2m-1},\lambda_{2m},\lambda_{2m+1})}(x_1,x_2,x_3) \\ \times \prod_{i=1}^{m} s_{(\lambda_{2i},\lambda_{2i+1})}(1,z_2) \prod_{i=1}^{m-1} s_{(\lambda_{2i-1},\lambda_{2i})}(x_1,x_2) \end{array}$$

$$= \frac{1}{(1-wx_3z_2^m\prod_{k=1}^{2}x_k^m\prod_{k=1}^{2m}y_k)\prod_{i=1}^{2}\prod_{j=1}^{2m}(1-x_iy_j)\prod_{1\le i<j\le 2m}(1-y_iy_jz_2\prod_{k=1}^{2}x_k)}$$

$$\times \frac{\prod_{i=1}^{2m}(1-x_3y_iz_2^m\prod_{k=1}^{2}x_k^m\prod_{k=1}^{2m}y_k)}{\prod_{i=1}^{2}(1-x_3x_iz_2^m\prod_{k=1}^{2}x_k^{m-1}\prod_{k=1}^{2m}y_k)\prod_{i=1}^{2m}(1-x_3y_i^{-1}z_2^{m-1}\prod_{k=1}^{2}x_k^{m-1}\prod_{k=1}^{2m}y_k)}.$$

Lemma

[chrysanthemum]

$$\sum_{\lambda=(\lambda_1,\dots,\lambda_6)\in\mathscr{P}} \begin{array}{l} w^{\lambda_6} s_{(\lambda_1,\lambda_2)}(x_1,x_2) s_{(\lambda_2,\dots,\lambda_5)}(1,z_2,z_3,z_4) s_{(\lambda_5,\lambda_6)}(x_1,x_2) \\ \times s_{(\lambda_1,\lambda_2,\lambda_3)}(y_1,y_2,y_3) s_{(\lambda_3,\lambda_4)}(x_1,x_2) s_{(\lambda_4,\lambda_5,\lambda_6)}(y_1,y_2,y_3) \end{array}$$

$$= \frac{(1-\prod_{k=1}^{2} x_k^3 \prod_{k=1}^{3} y_k^2 \prod_{k=2}^{4} z_k) \prod_{j=2}^{4}(1-z_j \prod_{k=1}^{2} x_k^3 \prod_{k=1}^{3} y_k^2 \prod_{k=2}^{4} z_k)}{(1-w\prod_{k=1}^{2} x_k^3 \prod_{k=1}^{3} y_k^2 \prod_{k=2}^{4} z_k) \prod_{i=1}^{2}\prod_{j=1}^{3}(1-x_i y_j) \prod_{i=1}^{3}(1-y_i \prod_{k=1}^{2} x_k^2 \prod_{k=1}^{3} y_k \prod_{k=2}^{4} z_k)}$$

$$\times \frac{1}{\prod_{j=2}^{4}\prod_{i=1}^{2}(1-x_i z_j^{-1} \prod_{k=1}^{2} x_k \prod_{k=1}^{3} y_k \prod_{k=2}^{4} z_k) \prod_{j=2}^{4}\prod_{i=1}^{3}(1-y_i^{-1} z_j \prod_{k=1}^{2} x_k \prod_{k=1}^{3} y_k)}.$$

A Proof of the Schur Function Identities

Schur Function Indentity

Lemma

If m is a nonnegative integer, then

$$\sum_{\lambda=(\lambda_1,\dots,\lambda_{2m+1})} s_\lambda(y_1,\dots,y_{2m+1}) s_{(\lambda_{2m+1})}(x_1,x_2)$$
$$\times \prod_{i=1}^{m} s_{(\lambda_{2i-1},\lambda_{2i})}(x_1,x_2) \prod_{i=1}^{m} s_{(\lambda_{2i},\lambda_{2i+1})}(1,z_2)$$
$$= \frac{1}{\prod_{i=1}^{2}\prod_{j=1}^{2m+1}(1-x_i y_j)\prod_{1\le i<j\le 2m+1}\left(1-y_i y_j z_2 \prod_{k=1}^{2} x_k\right)}$$

By the Littlewood formula, we have

$$\prod_{1\le i<j\le 2m+1} \frac{1}{1-ty_iy_j} = \sum_{\nu} t^{|\nu|/2} s_\nu(y),$$

where the sum rons over all ν with ν' even. Hence we can write $\nu = (\nu_1, \nu_1, \nu_2, \nu_2, \dots, \nu_m, \nu_m)$, where

$$\nu_1 \ge \nu_2 \ge \cdots \ge \nu_m \ge 0.$$

By the Pieri rule, we obtain

$$\frac{1}{\prod_{j=1}^{2m+1}(1-x_1y_j)} \cdot R_1 = \sum_{\mu,\nu} s_\mu(y) x_1^{\sum_{k=1}^{m}(\mu_k-\nu_k)+\mu_{m+1}} \prod_{k=1}^{m} (x_1x_2)^{\nu_k} \prod_{k=1}^{m} z_2^{\nu_k},$$

where the sum on the right-hand side μ runs over all partitions such that $\mu = (\mu_1, \nu_1, \mu_2, \nu_2, \dots, \mu_m, \nu_m, \mu_{m+1})$ with

$$\mu_1 \ge \nu_1 \ge \mu_2 \ge \nu_2 \ge \cdots \ge \mu_m \ge \nu_m \ge 0.$$

Proof

Here we write

$$|\mu| - |\nu| = \sum_{k=1}^{m} (\mu_k - \nu_k) + \mu_{m+1} = \mu_1 - \nu_1 + \cdots + \mu_m - \nu_m + \mu_{m+1}$$

in short. We use the Pieri rule again and obtain

$$R = \frac{1}{\prod_{i=1}^{2} \prod_{j=1}^{2m+1} (1 - x_i y_j)} \cdot R_1 = \sum_{\lambda,\mu,\nu} s_\lambda(y) x_1^{|\mu|-|\nu|} x_2^{|\lambda|-|\mu|} \prod_{k=1}^{m} (x_1 x_2)^{\nu_k} \prod_{k=1}^{m} z_2^{\nu_k},$$

where λ in the sum in the right-hand side is of the form $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{2m}, \lambda_{2m+1})$ with

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \nu_1 \geq \cdots \geq \lambda_{2m-1} \geq \mu_m \geq \lambda_{2m} \geq \nu_m \geq \lambda_{2m+1} \geq \mu_{m+1} \geq 0.$$

Here we write

$$|\lambda| - |\mu| = \sum_{k=1}^{m+1} (\lambda_{2k-1} - \mu_k) + \sum_{k=1}^{m} (\lambda_{2k} - \nu_k).$$

Proof

Note that

$$\sum_{\substack{\mu_k \\ \lambda_{2k-1}\geq\mu_k\geq\lambda_{2k}}} x_1^{\mu_k-\nu_k} x_2^{\lambda_{2k-1}-\mu_k+\lambda_{2k}-\nu_k} (x_1x_2)^{\nu_k} = s_{(\lambda_{2k-1},\lambda_{2k})}(x_1,x_2)$$

holds for $k = 1, 2, \ldots, m$. Similarly,

$$\sum_{\substack{\mu_k \\ \lambda_{2m+1}\geq\mu_{m+1}\geq 0}} x_1^{\mu_{m+1}} x_2^{\lambda_{2m+1}-\mu_{m+1}} = s_{(\lambda_{2m+1})}(x_1,x_2)$$

holds. Meanwhile, it is also easy to see that

$$\sum_{\substack{\nu_k \\ \lambda_{2k}\geq\nu_k\geq\lambda_{2k+1}}} z_2^{\nu_k} = s_{(\lambda_{2k},\lambda_{2k+1})}(1,z_2)$$

holds for $k = 1, 2, \ldots, m$. From these identities we coclude thar

$$\text{RHS} = \sum_{\lambda} s_\lambda(y) \prod_{k=1}^{m} s_{(\lambda_{2k-1},\lambda_{2k})}(x_1,x_2) \prod_{k=1}^{m} s_{(\lambda_{2k},\lambda_{2k+1})}(1,z_2)$$

Wisteria Identity

Theorem

If m is nonnegative integer, then we have

$$\sum_{\lambda=(\lambda_1,\lambda_2,\dots,\lambda_{2m})} w^{\lambda_{2m}} s_\lambda(y_1,\dots,y_{2m}) \prod_{i=1}^{m} s_{(\lambda_{2i-1},\lambda_{2i})}(x_1,x_2) \prod_{i=1}^{m-1} s_{(\lambda_{2i},\lambda_{2i+1})}(1,z_2)$$
$$= \frac{\left(1 - z_2^{m-1} \prod_{k=1}^{2} x_k^m \prod_{k=1}^{2m} y_k\right)}{\left(1 - w z_2^{m-1} \prod_{k=1}^{2} x_k^m \prod_{k=1}^{2m} y_k\right)}$$
$$\times \frac{\left(1 - z_2^{m} \prod_{k=1}^{2} x_k^m \prod_{k=1}^{2m} y_k\right)}{\prod_{i=1}^{2} \prod_{j=1}^{2m} (1 - x_i y_j) \prod_{1\le i<j\le 2m} \left(1 - y_i y_j z_2 \prod_{k=1}^{2} x_k\right)}.$$

Proof

First we assume $w = 0$. If we put $x = (x_1, x_2)$, $y = (y_1, \dots, y_{2m})$, $z = (1, z_2)$, $X = \prod_{k=1}^{2} x_k$, $Y = \prod_{k=1}^{2m} y_k$, then the above identity reads

$$\frac{1}{\left(1 - z_2^{m-1} X^m Y\right)\left(1 - z_2^m X^m Y\right)} \sum_{\nu=(\mu_1,\dots,\mu_{2m-1})} s_\mu(y) s_{(\mu_{2m-1})}(x)$$

$$\times \prod_{i=1}^{m-1} s_{(\mu_{2i-1},\mu_{2i})}(x) \prod_{i=1}^{m-1} s_{(\mu_{2i},\mu_{2i+1})}(z)$$

$$= \frac{1}{\prod_{i=1}^{2} \prod_{j=1}^{2m} (1 - x_i y_j) \prod_{1 \le i < j \le 2m} (1 - y_i y_j z_2 X)}$$

The left-hand side of this identity equals

$$L = \frac{1}{1 - z_2^m X^m Y} \sum_{t \ge 0} \sum_{\mu} s_{\mu + t^{2m}}(y)\, s_{(\mu_{2m-1}+t,t)}(x)$$

$$\times \prod_{i=1}^{m-1} s_{(\mu_{2i-1}+t,\mu_{2i}+t)}(x) \prod_{i=1}^{m-1} s_{(\mu_{2i}+t,\mu_{2i+1}+t)}(z)$$

Proof

$$L = \sum_{u\geq 0}\sum_{t\geq 0}\sum_{\mu} z_2^u\, s_{\mu+(t+u)^{2m}}(y)\, s_{(\mu_{2m-1}+t+u,t+u)}(x)$$
$$\times \prod_{i=1}^{m-1} s_{(\mu_{2i-1}+t+u,\mu_{2i}+t+u)}(x) \prod_{i=1}^{m-1} s_{(\mu_{2i}+t+u,\mu_{2i+1}+t+u)}(z)$$

If we set $\lambda_i = \mu_i + t + u$ $(i = 1, \dots, 2m-1)$, $\lambda_{2m} = t + u$, then we obtain $\sum_{u=0}^{\lambda_{2m}} z_2^u = s_{(\lambda_{2m})}(1, z_2)$, which implies

$$L = \sum_{\lambda} s_\lambda(y) \prod_{i=1}^{m} s_{(\lambda_{2i-1},\lambda_{2i})}(x) \prod_{i=1}^{m-1} s_{(\lambda_{2i},\lambda_{2i+1})}(z) \cdot s_{(\lambda_{2m})}(z).$$

This is true if we set $y_{2m+1} = 0$ in the identity of the above formula. The general case follows immediately from the $w = 0$ case. This compete the proof. $\square$

Thank you!