A generalization of Mehta-Wang determinant and Askey-Wilson polynomials

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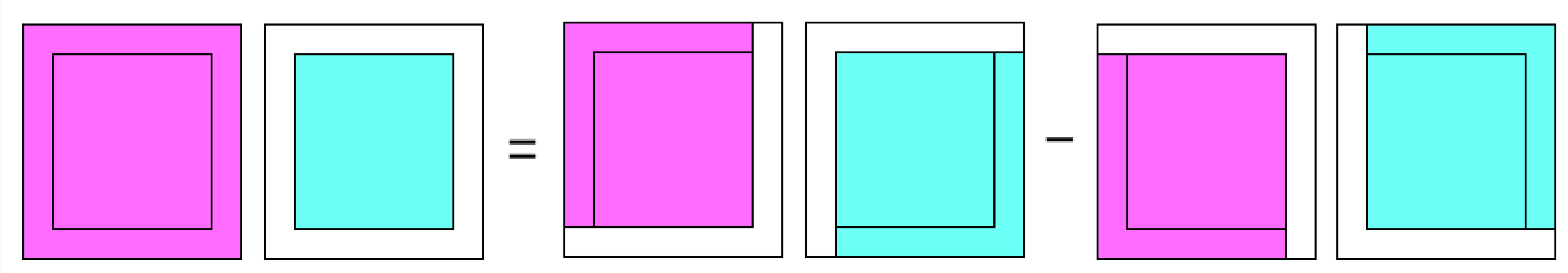
the Desnanot-Jacobi adjoint matrix theorem

If a and b are integers, we write $[a,b]=\{\,x\in\mathbb{Z}\,|\,a\le x\le b\,\}$. We also write $[n]=[1,n]$ for short. If S is a finite set and r a nonnegative integer, let $\binom{S}{r}$ denote the set of all r-element subsets of S. Let A be an $m\times n$ matrix. If $\mathrm{i}=(i_1,\dots,i_r)$ is an r-tuple of positive integers and $\mathrm{j}=(j_1,\dots,j_s)$ is an s-tuple of positive integers, then let $A_{\mathrm{j}}^{\mathrm{i}}=A_{j_1,\dots,j_s}^{i_1,\dots,i_r}$ denote the submatrix formed by selecting the row i and the column j from A.

$$\det A_{[n]}^{[n]}\det A_{[2,n-1]}^{[2,n-1]}=\det A_{[n-1]}^{[n-1]}\det A_{[2,n]}^{[2,n]}-\det A_{[n-1]}^{[2,n]}\cdot\det A_{[2,n]}^{[n-1]}$$



Notation

Throughout this paper we use the standard notation for q-series

$$(a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k),\qquad (a;q)_n=\frac{(a;q)_\infty}{(aq^n;q)_\infty}$$

for any integer n. Usually $(a;q)_n$ is called the q-shifted factorial, and we frequently use the compact notation:

$$(a_1,a_2,\dots,a_r;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n.$$

The ${}_r\phi_s$ basic hypergeometric series is defined by

$${}_r\phi_s\left(\begin{matrix}a_1,a_2,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,z\right)=\sum_{n=0}^{\infty}\frac{(a_1,a_2,\dots,a_r;q)_n}{(q,b_1,\dots,b_s;q)_n}\left((-1)^nq^{\binom{n}{2}}\right)^{1+s-r}z^n$$

Here we also use the q-Gamma function

$$\Gamma_q(z)=(1-q)^{1-z}\frac{(q;q)_\infty}{(q^z;q)_\infty},$$

the q-integer $[n]_q=\frac{1-q^n}{1-q}$ and the q-factorial $[n]_q!=\prod_{k=1}^{n}[k]_q$.

Definition (Askey-Wilson polynomials)

The Askey-Wilson polynomials (or q-Wilson polynomials) are defined by

$$p_n(x;a,b,c,d;q)=\frac{(ab,ac,ad;q)_n}{a^n}{}_4\phi_3\left[\begin{matrix}q^{-n},abcdq^{n-1},ae^{\iota\theta},ae^{-\iota\theta}\\ ab,ac,ad\end{matrix};q,q\right]$$

where ϕ is a basic hypergeometric function and $x=\cos\theta$ and $(a;q)_n$ is the q-Pochhammer symbol. Further we write

$$P_n(z;a,b,c,d;q)=p_n((z+z^{-1})/2;a,b,c,d,q).$$

The Mehta-Wang determinant

$$\det\bigl((a+j-i)\Gamma(b+i+j)\bigr)_{0\le i,j\le n-1}=D_n\prod_{i=0}^{n-1}i!\Gamma(b+i),$$

where D_n satisfies the three term recurrence relation

$$D_{-1}=0,\quad D_0=1,\quad D_{n+1}=aD_n+n(b+n-1)D_{n-1},$$

which can be considered as the recurrence relation for a special case of the Meixner-Pollaczek polynomials.

The Nishizawa determinant

For $a,b\in\mathbb{C}$, we have

$$\det\bigl([a+j-i]_q\Gamma_q(b+i+j)\bigr)_{0\le i,j\le n-1}=q^{na+n(n-1)b/2+n(n-1)(2n-7)/6}D_{n,q}\prod_{k=0}^{n-1}[k]_q!\cdot\Gamma_q(b+k),$$

where $D_{n,q}$ satisfies the recurrence relation

$$D_{-1,q}=0,\quad D_{0,q}=1,\quad D_{n+1,q}=q^{-a+n}[a]_qD_{n,q}+q^{-a-b}[n]_q[b+n-1]_qD_{n-1,q}.$$

Theorem (Our First Determinant)

Let a, b and c be parameters, and let $n\ge1$ and r be integers. Then we have

$$\begin{aligned}&\det\left((q^i-cq^j)\frac{(a;q)_{i+j+r}}{(abq;q)_{i+j+r}}\right)_{0\le i,j\le n-1}\\&=(-1)^na^{\frac{n(n-3)}{2}}q^{\frac{n(n-1)(n-2)}{3}+\frac{n(n-3)r}{2}}(abcq^r;q^2)_n\prod_{k=1}^{n}\frac{(q;q)_{k-1}(a;q)_{k+r}(bq;q)_{k-2}}{(abq;q)_{k+n+r-2}}\\&\quad\times{}_4\phi_3\left(\begin{matrix}q^{-n},a^{\frac12}c^{\frac12}q^{\frac r2},-a^{\frac12}c^{\frac12}q^{\frac r2},abq^{n+r-1}\\ aq^r,a^{\frac12}b^{\frac12}c^{\frac12}q^{\frac r2},-a^{\frac12}b^{\frac12}c^{\frac12}q^{\frac r2}\end{matrix};q,q\right)\\&=(-\iota)^na^{\frac{n(n-2)}{2}}c^{\frac n2}q^{\frac{n(n-1)(n-2)}{3}+\frac{n(n-2)r}{2}}\prod_{k=1}^{n}\frac{(q;q)_{k-1}(a;q)_{k+r-1}(bq;q)_{k-2}}{(abq;q)_{k+n+r-2}}\\&\quad\times p_n\left(0;a^{\frac12}c^{\frac12}q^{\frac r2}\iota,-a^{\frac12}c^{-\frac12}q^{\frac r2}\iota,b^{\frac12}\iota,-b^{\frac12}\iota;q\right).\end{aligned}$$

Theorem (A Quadratic Relation)

Let $r,s\ge0$, $a,b,c,d,q\in\mathbb{C}$, $E_r=(e_1,e_2,\dots,e_r)\in\mathbb{C}^r$, $F_s=(f_1,f_2,\dots,f_s)\in\mathbb{C}^s$. Then we have

$$\begin{aligned}&(a-b)(a-c)(bc-d)(1-d)\\&\times{}_{r+4}\phi_{s+3}\left[\begin{matrix}a^{-1}bc,bcq^{-2},c,dq^{-1},E_r\\ aq^{-1},bq^{-1},bcd^{-1},F_s\end{matrix};q,z\right]{}_{r+4}\phi_{s+3}\left[\begin{matrix}a^{-1}bc,bc,c,dq,E_rq\\ aq,bq,bcd^{-1},F_sq\end{matrix};q,q^{s-r}z\right]\\&=(a-d)(1-b)(1-c)(bc-ad)\\&\quad\times{}_{r+4}\phi_{s+3}\left[\begin{matrix}a^{-1}bc,bcq^{-2},cq^{-1},d,E_r\\ aq^{-1},b,bcd^{-1}q^{-1},F_s\end{matrix};q,z\right]{}_{r+4}\phi_{s+3}\left[\begin{matrix}a^{-1}bc,bc,cq,d,E_rq\\ aq,b,bcd^{-1}q,F_sq\end{matrix};q,q^{s-r}z\right]\\&\quad-(1-a)(b-d)(c-d)(a-bc)\\&\quad\times{}_{r+4}\phi_{s+3}\left[\begin{matrix}a^{-1}bcq^{-1},bcq^{-2},c,d,E_r\\ a,bq^{-1},bcd^{-1}q^{-1},F_s\end{matrix};q,z\right]{}_{r+4}\phi_{s+3}\left[\begin{matrix}a^{-1}bcq,bc,c,d,E_rq\\ a,bq,bcd^{-1}q,F_sq\end{matrix};q,q^{s-r}z\right].\end{aligned}$$

- Nishizawa's theorem is the case where

$$c=d=x=0,\quad a=q^{(a+b)/2}\iota,\quad b=-q^{(b-a)/2}\iota$$

in (1).

- Our first determinant is the case where

$$x=0,\quad a=a^{1/2}c^{1/2}q^{r/2}\iota,\quad b=-a^{1/2}c^{-1/2}q^{r/2}\iota,\quad c=b^{1/2}\iota\quad d=-b^{1/2}\iota$$

in (1).

Proposition (Catalan Determinants)

Hankel determinants of combinatorial numbers such as Catalan numbers

$$C_n=\frac{1}{n+1}\binom{2n}{n}$$

have been attracted many researchers in relation to combinatorial arguments of lattice paths. Viennot has proven

$$\det\left(C_{i+j+r}\right)_{0\le i,j\le n-1}=\prod_{1\le i\le j\le r-1}\frac{i+j+2n}{i+j},$$

and Krattenthaler has obtained the following general formula. Let n be a positive integer and $k_0,k_1,\dots,k_{n-1}$ non-negative integers. Then

$$\det\left(C_{k_i+j}\right)_{0\le i,j\le n-1}=\prod_{0\le i\le j\le n-1}(k_j-k_i)\prod_{i=0}^{n-1}\frac{(i+n)!(2k_i)!}{(2i)!k_i!(k_i+n)!}.$$

Proposition (More General Formulas)

The moments of the little q-Jacobi polynomials are defined by

$$\mu_n=\frac{(aq;q)_n}{(abq^2;q)_n}.$$

In [2] we have proven the Hankel determinant identity

$$\det\left(\mu_{i+j+r}\right)_{0\le i,j\le n-1}=a^{\frac{n(n-1)}{2}}q^{\frac{n(n-1)(2n-1)}{6}+\frac{n(n-1)r}{2}}\prod_{k=1}^{n}\frac{(q,bq;q)_{k-1}(aq;q)_{k+r-1}}{(abq^2;q)_{k+n+r-2}}$$

and the following general formula. Let n be a positive integer, and $k_0,\dots,k_{n-1}$ nonnegative integers. Then we have

$$\det\left(\mu_{k_i+j}\right)_{0\le i,j\le n-1}=a^{\binom n2}q^{\binom{n+1}{3}}\prod_{i=0}^{n-1}\frac{(aq;q)_{k_i}}{(abq^2;q)_{k_i+n-1}}\prod_{0\le i<j\le n-1}(q^{k_i}-q^{k_j})\prod_{i=0}^{n-1}(bq;q)_i.$$

Theorem (Our Second Determinant)

Let a, b and c be parameters. Let n be a positive integer, and $k=(k_1,\dots,k_n)$ be an n-tuple of positive integers. Then we have

$$\begin{aligned}&\det\left((q^{k_i-1}-cq^{j-1})\frac{(a;q)_{k_i+j-2}}{(abq;q)_{k_i+j-2}}\right)_{1\le i,j\le n}\\&=a^{\frac{n(n-3)}{2}}q^{\frac{n(n^2-6n+11)}{6}}\prod_{i=1}^{n}\frac{(a;q)_{k_i-1}(bq;q)_{i-2}}{(abq;q)_{k_i+n-2}}\prod_{1\le i<j\le n}(q^{k_i-1}-q^{k_j-1})\\&\quad\times\sum_{\nu=0}^{n}(-1)^{n-\nu}(abcq^{2\nu};q^2)_{n-\nu}(ac;q^2)_\nu\,R_{n,\nu}(k,a,1,b,1;q),\end{aligned}$$

where

$$\begin{aligned}&R_{n,\nu}(k,a,b,c,d;q)\\&=\sum_{(i,j)}q^{\sum_{l=1}^{n-\nu}i_l-n+\nu}\prod_{l=1}^{n-\nu}(1-abq^{k_{i_l}-i_l+l+\nu-1})\prod_{l=1}^{\nu}(1-abcdq^{k_{j_l}+j_l-l+\nu-2}).\end{aligned}$$

Here the sum on the right-hand side runs over all pairs (i,j) such that $[n]$ is a disjoint union of $i=\{i_1,\dots,i_{n-\nu}\}\in\binom{[n]}{n-\nu}$ and $j=\{j_1,\dots,j_\nu\}\in\binom{[n]}{\nu}$ (i.e., $i\cup j=[n]$ and $i\cap j=\emptyset$). Hereafter, we also use the convention that $R_{n,\nu}(k,a,b;q)$ is 0 unless $0\le\nu\le n$.

Theorem (Our Third Determinants)

If we define $B_{i,j}=B_{i,j}(x;a,b,c,d;q)$ by

$$\begin{aligned}B_{i,j}=\bigl(&cq+dq+aq^j-acdq^j+bq^i-bcdq^i-abcq^{i+j-1}-abdq^{i+j-1}\\&-2qx+2abcdxq^{i+j-1}\bigr)\frac{(ab;q)_{i+j-2}}{(abcd;q)_{i+j-1}}\end{aligned}$$

then we obtain

$$\begin{aligned}&\det\left(B_{i,j}\right)_{1\le i,j\le n}\\&=\frac{(-1)^na^{\frac{n(n-1)}{2}}b^{\frac{n(n-1)}{2}}q^{\frac{n(n^2-3n+5)}{3}}}{(abcd;q)_{2n-1}}p_n(x;a,b,c,d;q)\prod_{j=1}^{n-1}\frac{(ab,cd;q)_j(q;q)_j}{(abcd;q)_{n+j-1}}.\end{aligned}\qquad(1)$$

Further we obtain the following generalization. Let n be a positive integer, and $k=(k_1,\dots,k_n)$ be an n-tuple of positive integers. Then we have

$$\begin{aligned}&\det\left(B_{k_i,j}\right)_{1\le i,j\le n}\\&=a^{\frac{n(n-3)}{2}}b^{\frac{n(n-1)}{2}}q^{\frac{n(n^2-6n+11)}{6}}\prod_{i=1}^{n}\frac{(ab;q)_{k_i-1}(cd;q)_{i-1}}{(abcd;q)_{k_i+n-1}}\prod_{1\le i<j\le n}(q^{k_i-1}-q^{k_j-1})\\&\quad\times\sum_{\nu=0}^{n}(-1)^{n-\nu}(acq^\nu,adq^\nu;q)_{n-\nu}(az,az^{-1};q)_\nu R_{n,\nu}(k,a,b,c,d;q),\end{aligned}\qquad(2)$$

where $x=\frac{z+z^{-1}}{2}$. Here the sum on the right-hand side runs over all pairs (i,j) such that $[n]$ is a disjoint union of $i=\{i_1,\dots,i_{n-\nu}\}\in\binom{[n]}{n-\nu}$ and $j=\{j_1,\dots,j_\nu\}\in\binom{[n]}{\nu}$ (i.e., $i\cup j=[n]$ and $i\cap j=\emptyset$). We also use the convention that $R_{n,\nu}(k,a,b,c,d;q)$ is 0 unless $0\le\nu\le n$.

- (1) can be proven by the Desnanot-Jacobi adjoint matrix theorem and the quadratic relation.