# Tableau formulas for one-row Macdonald polynomials of type $C_n$

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# Tableau formula of type $A$ (Review)

Notations

- $\lambda = (\lambda_1, \lambda_2, \cdots)$: partition of $n$
- $T$: semi-standard tableau of shape $\lambda$ with entries $1, 2, \cdots, n$
- $\theta_{i,j}$: number of $j$ in $i$-th row
- $\theta^{(i)} = \theta_{1,i} + \cdots + \theta_{i,i}$ $(1 \le i \le n)$: number of $i$ in $T$

Ex. $n = 7$ $\lambda = (4, 2, 1)$ $T = $

| | | | |
|---|---|---|---|
| 1 | 1 | 1 | 2 |
| 2 | 3 | | |
| 4 | | | |

$\theta_{1,1} = 3,\ \theta_{1,2} = 1,\ \cdots$

$\theta^{(1)} = 3,\ \theta^{(2)} = 2,\ \theta^{(3)} = 1,\ \theta^{(4)} = 1,\ \theta^{(5)} = \theta^{(6)} = \theta^{(7)} = 0$

# Tableau formula of type $A$ (Review)

- $x = (x_1, \cdots, x_n)$: variables
- $P_\lambda^{(A_{n-1})}(x|q,t)$: Macdonald polynomial of type $A_{n-1}$
- $(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1})$: q-Pochhammer

## Theorem (Macdonald)

$$P_\lambda^{(A_{n-1})}(x|q,t) = \sum_T \psi_T(q,t)\, x^T \quad (T : \text{SST of shape } \lambda),$$

where

$$\psi_T(q,t) =$$

$$\prod_{k=1}^{n} \prod_{1\le i\le j\le k-1} \frac{(q^{-\lambda_i^{(k)}+\lambda_j^{(k-1)}+1}t^{-j+i-1};q)_{\theta_{i,k}}}{(q^{-\lambda_i^{(k)}+\lambda_j^{(k-1)}}t^{-j+i};q)_{\theta_{i,k}}} \frac{(q^{-\lambda_i^{(k)}+\lambda_{j+1}^{(k)}}t^{-j+i};q)_{\theta_{i,k}}}{(q^{-\lambda_i^{(k)}+\lambda_{j+1}^{(k)}+1}t^{-j+i-1};q)_{\theta_{i,k}}},$$

$$x^T = x_1^{\theta^{(1)}} x_2^{\theta^{(2)}} \cdots x_n^{\theta^{(n)}}$$

$$\left( \lambda_i^{(j)} := \textstyle\sum_{k=i}^{j} \theta_{i,k} \right)$$

# Deformed $\mathcal{W}$ algebra of type $A$ (Review)

Rank 2 case

- Heisenberg algebra $\mathfrak{h}$ over $\mathbb{F} := \mathbb{Q}(q,t)$

generators: $a_n \quad (n \in \mathbb{Z})$, relations: $[a_n, a_m] = n\dfrac{1-q^{|n|}}{1-t^{|n|}}\delta_{n+m,0}$

- Fock space $\mathcal{F}$, $\mathcal{F}^*$

$\langle 0|$, $|0\rangle$: vacuum vector, $a_n|0\rangle = 0$, $\langle 0| a_{-n} = 0 \quad (n \in \mathbb{Z}_{>0})$

- Normal order $: a_n a_{-n} := a_{-n} a_n \ (n > 0)$

- Operators on Fock space

$$\Lambda_1(z) = \exp\left(\sum_{n=1}^{\infty} \frac{1-t^{-n}}{1+(q/t)^n}\frac{a_{-n}}{n}z^n\right)\exp\left(-\sum_{n=1}^{\infty}(1-t^{-n})\frac{a_n}{n}z^{-n}\right)$$

$$\Lambda_2(z) = \exp\left(-\sum_{n=1}^{\infty} \frac{1-t^{-n}}{1+(q/t)^n}\frac{a_{-n}}{n}z^n\right)\exp\left(\sum_{n=1}^{\infty}(1-t^{-n})\frac{a_n}{n}z^{-n}\right)$$

# Deformed $\mathcal{W}$ algebra of type $A$ (Review)

## Definition ([FF], [AKOS])

Define the first generating field $T^{(A_{n-1})}(x,z)$ of the deformed $\mathcal{W}$ algebra of type $A_{n-1}$ with the indeterminate $x=(x_1,\cdots,x_n)$:

$$T^{(A_{n-1})}(x,z)=x_1\Lambda_1(z)+x_2\Lambda_2(z)+\cdots+x_n\Lambda_n(z)$$

Operator product expansion

- $\Lambda_i(z)$ $(i=1,\cdots,n)$: operator on Fock space
- $\Lambda_i(z)\Lambda_j(w)=\gamma_{i,j}(w,z):\Lambda_i(z)\Lambda_j(w):$,

$$\gamma_{i,j}(z,w)=\begin{cases}\dfrac{(z-tw)(z-w/t)}{(z-w)^2} & (i=j),\\ \dfrac{(z-qw)(z-w/t)(z-tw/q)}{(z-w)^3} & (i<j),\\ \dfrac{(z-w/q)(z-tw)(z-qw/t)}{(z-w)^3} & (i>j)\end{cases}$$

# Deformed $\mathcal{W}$ algebra of type $A$ (Review)

Specialization map

$\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$: partition of $n$

$$
\begin{array}{rcccl}
\varphi_\lambda^{(\zeta)}: & \mathbb{F}(z_1, \ldots, z_n) & \longrightarrow & \mathbb{F}(y) & \\
 & f(z_1, \ldots, z_n) & \mapsto & f( & y, \quad q^{-1}y, \quad \ldots, \quad q^{-(\lambda_1-1)}y, \\
 & & & & \zeta y, \quad q^{-1}\zeta y, \quad \ldots, \quad q^{-(\lambda_2-1)}\zeta y, \\
 & & & & \ldots, \\
 & & & & \zeta^{l-1}y, q^{-1}\zeta^{l-1}y, \ldots, q^{-(\lambda_l-1)}\zeta^{l-1}y \,)
\end{array}
$$

Ex. $\lambda = (4, 2, 1)$

| | | | |
|---|---|---|---|
| $y$ | $q^{-1}y$ | $q^{-2}y$ | $q^{-3}y$ |
| $\zeta y$ | $q^{-1}\zeta y$ | | |
| $\zeta^2 y$ | | | |

# Deformed $\mathcal{W}$ algebra of type $A$ (Review)

- $\langle 0 \,|:\Lambda_i(z_1)\Lambda_j(z_2):|\, 0\rangle = 1$ normalization

**Theorem ([FSSY-H])**

$$\lim_{\zeta\to t} \varphi_\lambda^{(\zeta)}(\langle 0\,|\,T(z_1)T(z_2)\cdots T(z_m)\,|\,0\rangle) = c_\lambda \cdot P_\lambda^{(A_{n-1})}(x|q,t)$$

Ex. $A_1$ case $\lambda = (2)$

$$\begin{aligned}
& T(z_1)T(z_2) \\
=\; & (x_1\Lambda_1(z_1) + x_2\Lambda_2(z_1))\,(x_1\Lambda_1(z_2) + x_2\Lambda_2(z_2)) \\
=\; & x_1^2\,\Lambda_1(z_1)\Lambda_1(z_2) + x_1x_2\,\Lambda_1(z_1)\Lambda_2(z_2) + x_1x_2\,\Lambda_2(z_1)\Lambda_1(z_2) \\
& + x_2^2\,\Lambda_2(z_1)\Lambda_2(z_2) \\
=\; & x_1^2\gamma_{1,1}(z_2,z_1):\Lambda_1(z_1)\Lambda_1(z_2): + x_1x_2\gamma_{1,2}(z_2,z_1):\Lambda_1(z_1)\Lambda_2(z_2): \\
& + x_1x_2\gamma_{2,1}(z_2,z_1):\Lambda_2(z_1)\Lambda_1(z_2): + x_2^2\gamma_{2,2}(z_2,z_1):\Lambda_2(z_1)\Lambda_2(z_2):
\end{aligned}$$

# Deformed $\mathcal{W}$ algebra of type $A_n$ (Review)

Note that $\varphi_\lambda^{(\zeta)}(\gamma_{2,1}(z_2, z_1)) = 0$,

$$\lim_{\zeta\to t} \varphi_\lambda^{(\zeta)}(\langle 0 \,|\, T(z_1)T(z_2) \,|\, 0\rangle)$$
$$= \frac{(q-t)(q-1/t)}{(1-q)^2}\left(x_1^2 + x_2^2 + \frac{(1-t)(1-q^2)}{(1-q)(1-qt)}x_1x_2\right)$$
$$= \frac{(q-t)(q-1/t)}{(1-q)^2} P_{(2)}^{(A_1)}(x|q,t)$$

In general, for $i > j$, $\lim_{\zeta\to t} \varphi_\lambda^{(\zeta)}(\gamma_{i,j}(z_2, z_1)) = 0$. The coefficient of $:\Lambda_i(z_1)\Lambda_j(z_2):$ is 0 for $i > j$.

When we identify $:\Lambda_i(z_1)\Lambda_j(z_2):$ with $\boxed{i}\boxed{j}$, the coefficient of $\boxed{i}\boxed{j}$ which does not satisfy the condition of semi-standard tableau is 0 by the specialization map $\varphi_\lambda^{(\zeta)}$.

# Macdonald polynomial of type $C$

## Definition of Koornwinder polynomial

- $\mathbb{K} = \mathbb{Q}(a^{\frac{1}{2}}, b^{\frac{1}{2}}, c^{\frac{1}{2}}, d^{\frac{1}{2}}, q^{\frac{1}{2}}, t^{\frac{1}{2}})$: basic field
- $x = (x_1, \cdots, x_n)$ $(x^{-1} = (x_1^{-1}, \cdots, x_n^{-1}))$: variables
- $T_{q^{\pm}, x_i}$: difference operator

$$T_{q^{\pm}, x_i} f(x_1, \cdots, x_i, \cdots, x_n) = f(x_1, \cdots, q^{\pm} x_i, \cdots, x_n)$$

- $\mathcal{D}_x = \sum_{i=1}^{n} \mathcal{A}_i^{+}(x)(T_{q,x_i} - 1) + \sum_{i=1}^{n} \mathcal{A}_i^{-}(x)\Big(T_{q^{-1},x_i} - 1\Big),$

where

$$\mathcal{A}_i^{+}(x) = \frac{(1 - a x_i)(1 - b x_i)(1 - c x_i)(1 - d x_i)}{(abcdq^{-1})^{\frac{1}{2}} t^{n-1} (1 - x_i^2)(1 - q x_i^2)} \times \prod_{\substack{1 \le j \le n, \\ j \ne i}} \frac{1 - t x_i x_j}{1 - x_i x_j} \frac{1 - t x_i / x_j}{1 - x_i / x_j},$$

$$\mathcal{A}_i^{-}(x) = \mathcal{A}_i^{+}(x^{-1}) \quad (i = 1, \ldots, n)$$

# Macdonald polynomial of type $C$

- $m_\mu(x)$: monomial (Laurent) polynomial

Ex. (n=3) $m_{(2)}(x) = x_1^2 + x_2^2 + x_3^2 + 1/x_1^2 + 1/x_2^2 + 1/x_3^2$

(invariant under the actions $x_i \leftrightarrow x_j$ and $x_i \leftrightarrow 1/x_i$)

**Theorem (Koornwinder)**

Koornwinder polynomial $P_\lambda(x) = P_\lambda(x|a,b,c,d|q,t)$ with partition $\lambda = (\lambda_1, \cdots, \lambda_n)$ is uniquely characterized by the two conditions:

1. $P_\lambda(x) = m_\lambda(x) + \sum_{\mu<\lambda} c_{\lambda\mu} m_\mu(x)$ for some $c_{\lambda\mu} \in \mathbb{K}$,

2. $\mathcal{D}_x P_\lambda(x) =$

$$\left( \sum_{i=1}^{n} \left( \alpha t^{n-i} q^{\lambda_i} + \alpha^{-1} t^{-n+i} q^{-\lambda_i} \right) - \sum_{i=1}^{n} \left( \alpha t^{n-i} + \alpha^{-1} t^{-n+i} \right) \right) P_\lambda(x),$$

where $\alpha = (abcdq^{-1})^{1/2}$.

# Macdonald polynomial of type $C$

**Proposition (Koornwinder)**

$$P_\lambda^{(C_n)}(x|b;q,t) = P_\lambda(x|-b^{1/2}, b^{1/2}, -(qb)^{1/2}, (qb)^{1/2}|q,t)$$

$$P_\lambda^{(D_n)}(x|q,t) = P_\lambda^{(C_n)}(x|1;q,t)$$

# Tableau formula of type $C$

- $I = \{1, 2, \cdots, n, \overline{n}, \overline{n-1}, \cdots, \overline{1}\}$ : index set with ordering

$1 \prec 2 \prec \cdots \prec n \prec \overline{n} \prec \overline{n-1} \prec \cdots \prec \overline{1}$

**Definition (Deformed $\mathcal{W}$ algebra of type $C_n$ [FR])**

- $\Lambda_i(z)\Lambda_j(w) = \gamma_{i,j}(z,w) : \Lambda_i(z)\Lambda_j(w) :$

- $$\gamma_{i,j}(z,w) = \begin{cases} 1 & (i = j), \\ \gamma(w/z) & (i < j, j \neq \overline{i}) \\ \gamma(z/w) & (i > j, \overline{j} \neq i) \\ \gamma(w/z)\gamma(qt^{i-n-1}w/z) & (i < j, j = \overline{i}) \\ \gamma(z/w)\gamma(qt^{i-n-1}z/w) & (i > j, \overline{j} = i), \end{cases}$$

  where $\gamma(z) = \dfrac{(1 - z/q)(1 - q/tz)}{(1 - z)(1 - z/t)}$

- $T(x,z) = x_1\Lambda_1(z) + \cdots + x_n\Lambda_n(z) + \dfrac{1}{x_n}\Lambda_{\overline{n}}(z) + \cdots + \dfrac{1}{x_1}\Lambda_{\overline{1}}(z)$

# Tableau formula of type $C$

## Tableau of type $C_n$ for one-row type [KN]

- $I = \{1, 2, \cdots, n, \overline{n}, \overline{n-1}, \cdots, \overline{1}\}$ : index set with ordering

$1 \prec 2 \prec \cdots \prec n \prec \overline{n} \prec \overline{n-1} \prec \cdots \prec \overline{1}$

- rule: the entries $i \in I$ into a one-row Young diagram are arranged in the weakly increasing manner

Ex. $T =$ 

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 3 | $\overline{3}$ | $\overline{1}$ |

(size $r = 6, \; n = 3$)

- $\theta_k$: number of $k$ into $T$

( $T \leftrightarrow (\theta_1, \theta_2, \cdots, \theta_{\overline{1}})$: 1:1-correspondence )

- $\theta_1 + \theta_2 + \cdots + \theta_{\overline{1}} = r$ $\quad r$: size of tableau

# Tableau formula of type $C$

- $T_{(r)}(x) := \langle 0| : T(z)T(q^{-1}z)\cdots T(q^{-(r-1)}z) : |0\rangle$

**Theorem ([FNSS-H])**

$$T_{(r)}(x) = P_{(r)}^{(C_n)}(x|t^2/q;q,t)$$

**Theorem ([FNSS-H])**

$$\begin{aligned}
&P_{(r)}^{(C_n)}(x|t^2/q;q,t) \\
=&\frac{(q;q)_r}{(t;q)_r} \sum_{\theta_1+\cdots+\theta_n+\theta_{\bar{n}}+\cdots+\theta_{\bar{1}}=r} \prod_{k\in I} \frac{(t;q)_{\theta_k}}{(q;q)_{\theta_k}} \\
&\times \prod_{1\le k\le n} \frac{(t^{n-k+1}q^{\theta_k+\cdots+\theta_{\overline{k+1}}};q)_{\theta_{\bar{k}}}(t^{n-k+2}q^{\theta_{k+1}+\cdots+\theta_{\overline{k+1}}-1};q)_{\theta_{\bar{k}}}}{(t^{n-k+2}q^{\theta_k+\cdots+\theta_{\overline{k+1}}-1};q)_{\theta_{\bar{k}}}(t^{n-k+1}q^{\theta_{k+1}+\cdots+\theta_{\overline{k+1}}};q)_{\theta_{\bar{k}}}} \\
&\times x_1^{\theta_1-\theta_{\bar{1}}}\cdots x_n^{\theta_n-\theta_{\bar{n}}}
\end{aligned}$$

# Tableau formula of type $C$

Tableau of type $D_n$ for one-row type [KN]

- $I = \{1, 2, \cdots, n, \overline{n}, \overline{n-1}, \cdots, \overline{1}\}$ : index set with ordering

$$1 \prec 2 \prec \cdots \prec n-1 \begin{array}{ccc} \prec & n & \succ \\ \succ & \overline{n} & \prec \end{array} \overline{n-1} \prec \cdots \prec \overline{1}$$

- rule: the entries $i \in I$ into a one-row Young diagram are arranged in the weakly increasing manner and $n$ and $\overline{n}$ are not arranged simultaneously

Ex. $T = \boxed{1}\boxed{2}\boxed{3}\boxed{3}\boxed{\overline{2}}\boxed{\overline{1}}$ (size $r = 6,\ n = 3$)

- $\theta_k$: number of $k$ into $T$

- $\theta_1 + \theta_2 + \cdots + \theta_{\overline{1}} = r$ and $\theta_n \theta_{\overline{n}} = 0$ $r$: size of tableau

# Tableau formula of type $C$

**Theorem ([FNSS-H])**

$$
\begin{aligned}
&P^{(D)}_{(r)}(x|q,t)\\
&= \frac{(q;q)_r}{(t;q)_r} \sum_{\substack{\theta_1+\theta_2+\cdots+\theta_{\overline{1}}=r \\ \theta_n\theta_{\overline{n}}=0}} \prod_{k\in I} \frac{(t;q)_{\theta_k}}{(q;q)_{\theta_k}}\\
&\quad \times \prod_{1\le l\le n-1} \frac{(q^{\theta_{l+1}+\theta_{l+2}+\cdots+\theta_{\overline{l+1}}} t^{n-l};q)_{\theta_{\overline{l}}} (q^{\theta_l+\theta_{l+1}+\cdots+\theta_{\overline{l+1}}+1} t^{n-l-1};q)_{\theta_{\overline{l}}}}{(q^{\theta_{l+1}+\theta_{l+2}+\cdots+\theta_{\overline{l+1}}+1} t^{n-l-1};q)_{\theta_{\overline{l}}} (q^{\theta_l+\theta_{l+1}+\cdots+\theta_{\overline{l+1}}} t^{n-l};q)_{\theta_{\overline{l}}}}\\
&\quad \times x_1^{\theta_1-\theta_{\overline{1}}} x_2^{\theta_2-\theta_{\overline{2}}} \cdots x_n^{\theta_n-\theta_{\overline{n}}}
\end{aligned}
$$

# Tableau formula of type $C$

$b$: generic

## Theorem ([FNSS-H])

Set $\theta := \min(\theta_n, \theta_{\bar{n}})$. We have

$$
\begin{aligned}
& P^{(C_n)}_{(r)}(x|b;q,t) \\
= & \frac{(q;q)_r}{(t;q)_r} \sum_{\theta_1+\theta_2+\cdots+\theta_{\bar{1}}=r} \prod_{k\in I\setminus\{n,\bar{n}\}} \frac{(t;q)_{\theta_k}}{(q;q)_{\theta_k}} \frac{(t;q)_{|\theta_n-\theta_{\bar{n}}|}}{(q;q)_{|\theta_n-\theta_{\bar{n}}|}} \\
& \times \prod_{1\le l\le n-1} \left( \frac{(t^{n-l-1}q^{\theta_l+\cdots+\theta_{n-1}+|\theta_n-\theta_{\bar{n}}|+\theta_{\overline{n-1}}+\cdots+\theta_{\overline{l+1}}+1};q)_{\theta_{\bar{l}}}}{(t^{n-l}q^{\theta_l+\cdots+\theta_{n-1}+|\theta_n-\theta_{\bar{n}}|+\theta_{\overline{n-1}}+\cdots+\theta_{\overline{l+1}}};q)_{\theta_{\bar{l}}}} \right. \\
& \qquad \left. \times \frac{(t^{n-l}q^{\theta_{l+1}+\cdots\theta_{n-1}+|\theta_n-\theta_{\bar{n}}|+\theta_{\overline{n-1}}+\cdots+\theta_{\overline{l+1}}};q)_{\theta_{\bar{l}}}}{(t^{n-l-1}q^{\theta_{l+1}+\cdots+\theta_{n-1}+|\theta_n-\theta_{\bar{n}}|+\theta_{\overline{n-1}}+\cdots+\theta_{\overline{l+1}}+1};q)_{\theta_{\bar{l}}}} \right) \\
& \times \frac{(b;q)_\theta (t^n q^{r-2\theta};q)_{2\theta}}{(q;q)_\theta (bt^{n-1}q^{r-\theta};q)_\theta (t^{n-1}q^{r-2\theta+1};q)_\theta} x_1^{\theta_1-\theta_{\bar{1}}} x_2^{\theta_2-\theta_{\bar{2}}} \cdots x_n^{\theta_n-\theta_{\bar{n}}}
\end{aligned}
$$

# Proof for Tableau formula of type $C$

- $I = \{1, 2, \cdots, n, \overline{n}, \overline{n-1}, \cdots, \overline{1}\}$: index set

**Definition ([L])**

$$G_r(x; q, t) = \sum_{\theta_1+\theta_2+\cdots+\theta_{\overline{1}}=r} \prod_{i\in I} \frac{(t;q)_{\theta_i}}{(q;q)_{\theta_i}} x_1^{\theta_1-\theta_{\overline{1}}} x_2^{\theta_2-\theta_{\overline{2}}} \cdots x_n^{\theta_n-\theta_{\overline{n}}},$$

where $r \in \mathbb{N}$ and $\theta_i$, $\theta_{\overline{i}} \geq \mathbb{Z}_{\geq 0}$ $(i = 1, 2, \cdots, n)$

# Proof for Tableau formula of type $C$

**Theorem ([NS-H], (Conjecture 1, Conjecture 2 [L]))**

$$P^{(C_n)}_{(r)}(x|b;q,t) = \frac{(q;q)_r}{(t;q)_r}\sum_{i=0}^{[r/2]}\Bigl(G_{r-2i}(x;q,t) \times t^i \frac{(b/t;q)_i(t^n q^{r-i};q)_i}{(q;q)_i(bt^{n-1}q^{r-i};q)_i}\frac{1-t^n q^{r-2i}}{1-t^n q^{r-1}}\Bigr)$$

$$P^{(D_n)}_{(r)}(x|q,t) = \frac{(q;q)_r}{(t;q)_r}\sum_{i=0}^{[r/2]}\Bigl(G_{r-2i}(x;q,t) \times t^i \frac{(1/t;q)_i(t^n q^{r-i};q)_i}{(q;q)_i(t^{n-1}q^{r-i};q)_i}\frac{1-t^n q^{r-2i}}{1-t^n q^{r-1}}\Bigr)$$

# Proof for Tableau formula of type $C$

## Transformation formula for $P^{(C_n)}_{(r)}(x|t^2/q;q,t)$

### Theorem ([FNSS-H])

Let $n \in \mathbb{Z}_{\geq 2}$. Fix $K, m_1, m_2, \cdots, m_n \in \mathbb{Z}_{\geq 0}$ arbitrarily. Set $m_{l,n} := \sum_{k=l}^{n} m_k$, $\phi_{l,n} := \sum_{k=l}^{n} \phi_k$. We have

$$
\begin{aligned}
&\sum_{\substack{\phi_1,\phi_2,\cdots,\phi_n,i\geq 0\\ \phi_1+\phi_2+\cdots+\phi_n+i=K}} \prod_{1\leq k\leq n} \frac{(t;q)_{\phi_k}(t;q)_{\phi_k+m_k}}{(q;q)_{\phi_k}(q;q)_{\phi_k+m_k}}\\
\times\quad & (t^2/q)^i \frac{(t^{-1}q;q)_i (t^n q^{2K+m_{1,n}-2i};q)_i}{(q;q)_i (t^{n+1} q^{2K+m_{1,n}-2i};q)_i}\\
\times\quad & \prod_{1\leq l\leq n} \frac{(t^{n-l+1} q^{\phi_l+\phi_{l+1,n}+m_{l,n}};q)_{\phi_l} (t^{n-l+2} q^{2\phi_{l+1,n}+m_{l+1,n}-1};q)_{\phi_l}}{(t^{n-l+2} q^{\phi_l+\phi_{l+1,n}+m_{l,n}-1};q)_{\phi_l} (t^{n-l+1} q^{2\phi_{l+1,n}+m_{l+1,n}};q)_{\phi_l}}\\
=\quad & \sum_{\substack{\phi_1,\phi_2,\cdots,\phi_n\geq 0\\ \phi_1+\phi_2+\cdots+\phi_n=K}} \prod_{1\leq j\leq n} \frac{(t;q)_{\phi_j}(t;q)_{\phi_j+m_j}}{(q;q)_{\phi_j}(q;q)_{\phi_j+m_j}}.
\end{aligned}
$$

# Proof for Tableau formula of type $C$

## Transformation formula for $P^{(D_n)}_{(r)}(x|q,t)$ and $P^{(C_n)}_{(r)}(x|b;q,t)$

### Theorem ([FNSS-H])

Let $n \in \mathbb{Z}_{\geq 2}$. Fix $K, m_1, m_2, \cdots, m_n \in \mathbb{Z}_{\geq 0}$ arbitrarily. Set $m_{l,n} := \sum_{k=l}^{n} m_k$, $\phi_{l,n} := \sum_{k=l}^{n} \phi_k$. We have

$$\sum_{\substack{\phi_1,\phi_2,\cdots,\phi_{n-1},i\geq 0\\ \phi_1+\phi_2+\cdots+\phi_{n-1}+i=K}} \Bigg( \prod_{1\leq l\leq n-1} \frac{(t;q)_{\phi_l}(t;q)_{\phi_l+m_l}}{(q;q)_{\phi_l}(q;q)_{\phi_l+m_l}}$$

$$\times \quad \frac{(t^{n-l-1}q^{\phi_l+2\phi_{l+1,n-1}+m_{l,n}+1};q)_{\phi_l}(t^{n-l}q^{2\phi_{l+1,n-1}+m_{l+1,n}};q)_{\phi_l}}{(t^{n-l}q^{\phi_l+2\phi_{l+1,n-1}+m_{l,n}};q)_{\phi_l}(t^{n-l-1}q^{2\phi_{l+1,n-1}+m_{l+1,n}+1};q)_{\phi_l}}\Bigg)$$

$$\times \quad \frac{(t;q)_{m_n}}{(q;q)_{m_n}} \frac{(t;q)_i(t^nq^{2K+m_{1,n}-2i};q)_i}{(q;q)_i(t^{n-1}q^{2K+m_{1,n}-2i+1};q)_i}$$

$$= \sum_{\substack{\phi_1,\phi_2,\cdots,\phi_{n-1},\phi_n\geq 0\\ \phi_1+\phi_2+\cdots+\phi_n=K}} \prod_{1\leq j\leq n} \frac{(t;q)_{\phi_j}(t;q)_{\phi_j+m_j}}{(q;q)_{\phi_j}(q;q)_{\phi_j+m_j}}$$

[AKOS] H. Awata, H. Kubo, S. Odake and J. Shiraishi, Quantum $\mathcal{W}_N$ algebras and Macdonald polynomials

[FF] B. Feigin and E. Frenkel, Quantum $\mathcal{W}$-algebras and elliptic algebras

[FHSSY] B. Feigin, A. Hoshino, J. Shibahara, J. Shiraishi and S. Yanagida, *Kernel function and quantum algebras*

[FHNSS] B. Feigin, A. Hoshino, M. Noumi, J. Shibahara and J. Shiraishi, Tableau Formulas for One-Row Macdonald Polynomials of Types $C_n$ and $D_n$

[FR] E. Frenkel and N. Reshetikhin, Deformations of $\mathcal{W}$-algebras associated to simple Lie algebras

[HNS] A. Hoshino, M. Noumi and J. Shiraishi, Some transformation formulas associated with Askey-Wilson polynomials and Lassalle's formulas for Macdonald-Koornwinder polynomials

[K] T. Koornwinder, Askey-Wilson polynomials for root systems of type $BC$

[L] M. Lassalle, Some conjectures for Macdonald polynomials of type $B$, $C$, $D$

[M] I. G. Macdonald, Symmetric Functions and Hall Polynomials (2nd ed.)