The Combinatorics of Alternating Sign Matrices

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Plan

Tuesday

- Introduce & enumerate
  - Alternating sign matrices (ASMs)
  - Alternating sign triangles (ASTs)
  - Descending plane partitions (DPPs)
  - Totally symmetric self-complementary plane partitions (TSSCPPs)
  - Double-staircase semistandard Young tableaux

Today

- Discuss refined enumeration of ASMs with
  - Fixed values of statistics
  - Invariance under symmetry operations

- Sketch proofs for enumerations of
  - Unrestricted ASMs
  - Odd-order diagonally & antidiagonally symmetric ASMs

The story so far . . .

Introduced 4 combinatorial objects:

- **$n \times n$ alternating sign matrices (ASMs)**

  e.g. There are 7 3×3 ASMs:

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix},\ \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},\ \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},\ \begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$$

- **Order n alternating sign triangles (ASTs)**

  e.g. There are 7 order 3 ASTs:

$$\begin{pmatrix}1&0&0&0&0\\&1&0&0&\\&&1&&\end{pmatrix},\quad \begin{pmatrix}0&0&0&1&0\\&1&0&0&\\&&1&&\end{pmatrix},\quad \begin{pmatrix}0&0&0&0&1\\&1&0&0&\\&&1&&\end{pmatrix},$$

$$\begin{pmatrix}1&0&0&0&0\\&0&0&1&\\&&1&&\end{pmatrix},\quad \begin{pmatrix}0&1&0&0&0\\&0&0&1&\\&&1&&\end{pmatrix},\quad \begin{pmatrix}0&0&0&0&1\\&0&0&1&\\&&1&&\end{pmatrix},$$

$$\begin{pmatrix}0&0&1&0&0\\&1&-1&1&\\&&1&&\end{pmatrix}$$

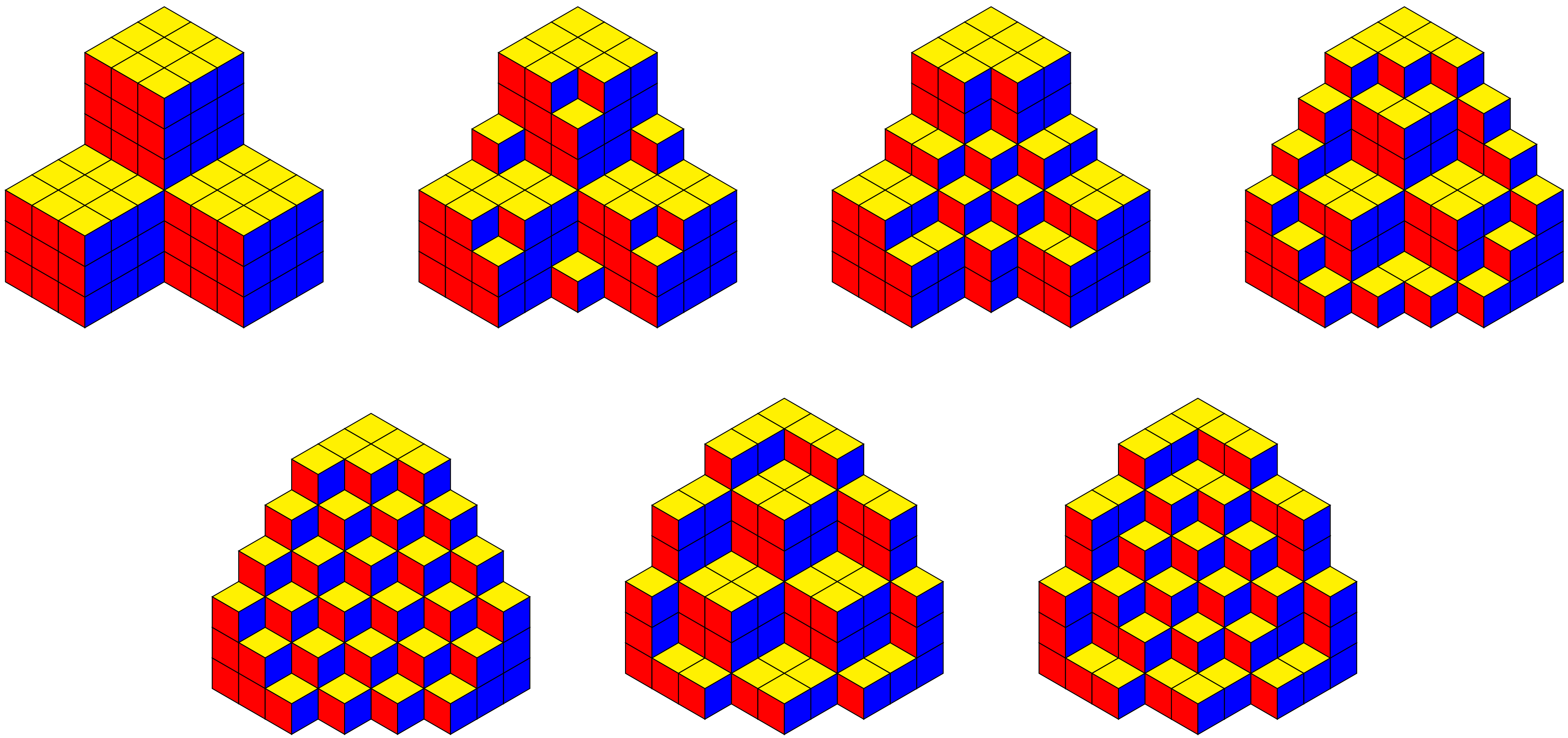
- **Order n descending plane partitions (DPPs)**

  e.g. There are 7 order 3 DPPs:

$$\emptyset,\quad 2,\quad 3,\quad 3\,1,\quad 3\,2,\quad 3\,3,\quad \begin{array}{cc} 3 & 3 \\ & 2 \end{array}$$

- **Totally symmetric self-complementary plane partitions in a $2n\times 2n\times 2n$ box**

  e.g. There are are 7 TSSCPP in a $6\times 6\times 6$ box:



- Also considered $\dfrac{\mathsf{SSYT}\big((n-1,n-1,\ldots,2,2,1,1),2n\big)}{\mathsf{SSYT}\big((2n-2,2n-4,\ldots,6,4,2),n\big)}$

$$= \mathsf{SSYT}\big((n-1,n-1,\ldots,2,2,1,1),2n\big)\,/\,3^{n(n-1)/2}$$

  where $\mathsf{SSYT}(\lambda,k) := \begin{pmatrix}\text{\# of semistandard Young tableaux of}\\ \text{shape } \lambda \text{ with entries from } \{1,2,\ldots,k\}\end{pmatrix}$

- $(n-1,n-1,\ldots,2,2,1,1)$ & $(2n-2,2n-4,\ldots,6,4,2)$ are conjugate partitions of double-staircase shape.

- e.g. for $n=3$:

$$\frac{\mathsf{SSYT}\big((2,2,1,1),6\big)}{\mathsf{SSYT}\big((4,2),3\big)} = \mathsf{SSYT}\left(\ydiagram{2,2,1,1},6\right)\Big/\mathsf{SSYT}\left(\ydiagram{4,2},3\right) = 189/3^3 = 189/27 = 7$$

Main result

- The following are all equal
  - # of $n\times n$ ASMs
  - # of order n ASTs
  - # of order n DPPs
  - # of TSSCPPs in $2n\times 2n\times 2n$ box
  - $\mathrm{SSYT}\big((n-1,n-1,\ldots,2,2,1,1),2n\big)\,/\,3^{n(n-1)/2}$
  - $\displaystyle\prod_{i=0}^{n-1}\frac{(3i+1)!}{(n+i)!}$

- Several of the equalities were conjectured long before being proved.

- No bijective proofs currently known for equality involving any of the pairs of combinatorial objects.

Refined Enumeration

- Many further results and conjectures are known for #'s of ASMs, ASTs, DPPs or TSSCPPs with fixed values of statistics and/or invariance under symmetry operations.

- An example involving statistics is equality of all the following:
  - $\dfrac{(n+k-1)!\,(2n-k-2)!}{k!\,(n-k-1)!\,(2n-2)!}\displaystyle\prod_{i=0}^{n-2}\frac{(3i+1)!}{(n+i-1)!}$
  - # of $n \times n$ ASMs with the 1 of the first row in column $k+1$
  - # of order n ASTs with (# of 1's on left boundary) + (# of 0's on right boundary in columns with sum 1) $= k+1$
  - # of order n DPPs with k parts equal to n
  - # of TSSCPPs in $2n \times 2n \times 2n$ box with k "maximal" entries in "fundamental region"

  (Mills, Robbins, Rumsey 1982; Zeilberger 1996; Razumov, Stroganov, Zinn-Justin 2007; Fischer)

- Again, several of the equalities were conjectured long before being proved.

- Again, no bijective proofs currently known.

- An example involving symmetry operations is equality of all the following:
  - $\prod_{i=1}^{n} \frac{(6i-2)!}{(2n+2i)!}$
  - # of $(2n+1)\times(2n+1)$ vertically symmetric ASMs
  - # of $2n\times 2n$ diagonally symmetric ASMs with only 0's on diagonal
  - # of order $2n+1$ vertically symmetric ASTs
  - # of order $2n+1$ DPPs whose associated rhombus tiling is invariant under reflection in a line bisecting two sides of hexagon
  - # of TSSCPPs in $(4n+2)\times(4n+2)\times(4n+2)$ box invariant under certain composition of local involutions

  (Mills, Robbins, Rumsey 1987; Kuperberg 2002; Ishikawa 2006; RB, Fischer)

- Again, several of the equalities were conjectured long before being proved.

- Again, no bijective proofs currently known.

"Classical" ASM Statistics

For an $n \times n$ ASM A

- Bulk statistics:

  Minus$(A) :=$ # of -1's in A

  $$\text{Inv}(A) := \sum_{\substack{1 \le i < i' \le n \\ 1 \le j' \le j \le n}} A_{ij} \, A_{i'j'} \;=\; \sum_{i,j=1}^{n} \Big(\sum_{j'=1}^{j} A_{ij'}\Big)\Big(\sum_{i'=1}^{i-1} A_{i'j}\Big)$$

  $=$ # of "inversions" in A

- Boundary statistics:

  TOP$(A) :=$ column of the 1 in top row of A

  RIGHT$(A) :=$ row of the 1 in right-most column of A

  BOTTOM$(A) :=$ column of the 1 in bottom row of A

  LEFT$(A) :=$ row of the 1 in left-most column of A

- e.g. $A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$

  $\Rightarrow$ Minus$(A) = 3$, Inv$(A) = 5$,

  TOP$(A) = 4$, RIGHT$(A) = 5$, BOTTOM$(A) = 4$, LEFT$(A) = 3$

- Enumerative results for $n \times n$ ASMs with fixed values of classical statistics:
  - Minus: *Le Gac 2011*
  - Inv: *RB 2008*
  - TOP: *Zeilberger 1996*
  - TOP & BOTTOM: *Colomo, Pronko 2005; Stroganov 2006*
  - TOP & LEFT: *Stroganov 2006*
  - Minus, Inv & TOP: *RB, Di Francesco, Zinn–Justin 2012*
  - Minus, Inv, TOP & BOTTOM: *RB, Di Francesco, Zinn–Justin 2013*
  - TOP, BOTTOM, LEFT & RIGHT: *Ayyer, Romik 2013*
  - Minus, Inv, TOP, BOTTOM, LEFT & RIGHT: *RB 2013*

- In some, but not all cases, statistics with same enumerative properties as classical ASM statistics are known for ASTs, DPPs or TSSCPPs.

- This already seen for TOP, for which statistics are known for ASTs, DPPs & TSSCPPs.

- For each of ASTs & DPPs (but not TSSCPPs) two statistics are known which have the same *joint* enumerative properties as Minus & Inv for ASMs.
  (RB, Di Francesco, Zinn-Justin 2012; Ayyer, RB, Fischer 2016)

- Another important example of refined enumeration is # of $n \times n$ ASMs whose associated "fully packed loop configuration" has a fixed "link pattern".
  (Razumov, Stroganov 2004; Cantini, Sportiello 2011)

ASM Symmetry Classes

- Symmetry group of square is dihedral group $D_4 = \{\mathcal{I}, \mathcal{V}, \mathcal{H}, \mathcal{D}, \mathcal{A}, \mathcal{R}_{\pi/2}, \mathcal{R}_\pi, \mathcal{R}_{3\pi/2}\}$, with $\mathcal{I} =$ identity,
  $\mathcal{V}$, $\mathcal{H}$, $\mathcal{D}$, $\mathcal{A} =$ reflection in vertical, horizontal, diagonal, antidiagonal axes,
  $\mathcal{R}_{\pi/2}$, $\mathcal{R}_\pi$, $\mathcal{R}_{3\pi/2} =$ counterclockwise quarter-turn, half-turn, three-quarter turn rotation.

- D_4 has 10 subgroups: $\{\mathcal{I}\}$, $\{\mathcal{I},\mathcal{V}\} \approx \{\mathcal{I},\mathcal{H}\}$, $\{\mathcal{I},\mathcal{V},\mathcal{H},\mathcal{R}_\pi\}$, $\{\mathcal{I},\mathcal{R}_\pi\}$, $\{\mathcal{I},\mathcal{R}_{\pi/2},\mathcal{R}_\pi,\mathcal{R}_{3\pi/2}\}$, $\{\mathcal{I},\mathcal{D},\mathcal{A},\mathcal{R}_\pi\}$ $\{\mathcal{I},\mathcal{D}\} \approx \{\mathcal{I},\mathcal{A}\}$ & D_4 (where $\approx$ is conjugacy).

- D_4 has natural action on set of $n \times n$ ASMs.

- Consider # of $n \times n$ ASMs invariant under action of a subgroup of D_4.
  Gives 8 symmetry classes:
  1. Unrestriced $\{\mathcal{I}\}$: product formula known (*Kuperberg 1996, Zeilberger 1996*)
  2. Vertically symmetric $\{\mathcal{I},\mathcal{V}\}$: n even empty;
     n odd product formula known (*Kuperberg 2002*)
  3. Vertically & horizontally symmetric $\{\mathcal{I},\mathcal{V},\mathcal{H},\mathcal{R}_\pi\}$: n even empty;
     n odd product formula known (*Okada 2006*)
  4. Half-turn symmetric $\{\mathcal{I},\mathcal{R}_\pi\}$: n even product formula known *(Kuperberg 2002)*;
     n odd product formula known *(Razumov, Stroganov 2006)*

5. Quarter-turn symmetric $\{\mathcal{I}, \mathcal{R}_{\pi/2}, \mathcal{R}_{\pi}, \mathcal{R}_{3\pi/2}\}$:
   $n \equiv 0 \bmod 4$ product formula known *(Kuperberg 2002)*;
   $n \equiv 1, 3 \bmod 4$ product formula known *(Razumov, Stroganov 2006)*;
   $n \equiv 2 \bmod 4$ empty

6. Diagonally & antidiagonally symmetric $\{\mathcal{I}, \mathcal{D}, \mathcal{A}, \mathcal{R}_{\pi}\}$:
   n odd product formula known *(RB, Fischer, Konvalinka 2017)*; (More details soon!)
   no formula currently known for n even

7. Diagonally symmetric $\{\mathcal{I}, \mathcal{D}\}$: no formula currently known

8. Totally symmetric D_4: n even empty;
   no formula currently known for n odd

- In some cases, symmetry operations with same enumerative properties as ASM symmetry operations are known for ASTs, DPPs or TSSCPPs.

- This already seen for vertical reflection, for which operations are known for ASTs, DPPs & TSSCPPs.

- In some cases, results or conjectures involving invariance under symmetry operations *and* fixed values of statistics are known.

Cyclic Sieving & Homomesy for ASMs

- The set $\{n\times n \text{ ASMs}\}$, cyclic group $\{\mathcal{I},\mathcal{R}_{\pi/2},\mathcal{R}_{\pi},\mathcal{R}_{3\pi/2}\}$ & function $f_n(q)=\prod_{i=0}^{n-1}\frac{[3i+1]_q!}{[n+i]_q!}$ exhibit the *cyclic sieving phenomenon*. *(Stanton)*

- i.e. $f_n(1)=(\#\ n\times n$ ASMs), $f_n(-1)=(\#$ half-turn symmetric $n\times n$ ASMs) & $f_n(i)=(\#$ quarter-turn symmetric $n\times n$ ASMs)

- This is an unusual cyclic sieving example since $\{\mathcal{I},\mathcal{R}_{\pi/2},\mathcal{R}_{\pi},\mathcal{R}_{3\pi/2}\}$ acts on $n\times n$ ASMs, but $f_n(q)$ is not a generating function for $n\times n$ ASMs with any currently-known statistic. Instead, it is only known that $f_n(q)$ is a generating function for order n DPPs,
$f_n(q)=\sum_{\text{order } n \text{ DPPs } \pi} q^{\sum_{ij}\pi_{ij}}$ (& no order 4 cyclic action on DPPs is currently known).

- The set $\{n\times n \text{ ASMs}\}$, group G & statistic $2\,\mathrm{Inv}+\mathrm{Minus}$ exhibit *homomesy* with value $n(n-1)/2$, for $G=\{\mathcal{I},\mathcal{V}\}$, $\{\mathcal{I},\mathcal{H}\}$, $\{\mathcal{I},\mathcal{R}_{\pi/2},\mathcal{R}_{\pi},\mathcal{R}_{3\pi/2}\}$ or D_4. *(RB, Roby)*

- i.e. average of $2\,\mathrm{Inv}+\mathrm{Minus}$ over any orbit of G on $\{n\times n \text{ ASMs}\}$ is $n(n-1)/2$.

- Certain cyclic sieving phenomena & homomesies also observed for ASTs, DPPs & TSSCPPs.

Sketch of Proof of ASM Enumeration Formula

ASM: square matrix in which

- each entry is 0, 1 or -1
- each row & column contains at least one 1
- along each row & column, the nonzero entries alternate in sign, starting & ending with a 1

$$(\#\text{ of } n\times n \text{ ASMs}) = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$$

Proof method *(Kuperberg 1996)*

(1) Apply bijection between $n\times n$ ASMs & configurations of statistical mechanical *six-vertex model* on $n\times n$ square with *domain-wall boundary conditions*.

(2) Introduce parameter-dependent weights & consider weighted sum over all configurations of model, i.e., generating function or *partition function*.

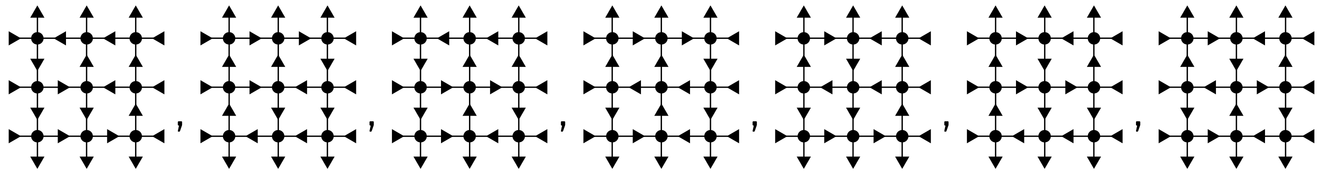
(3) Use *Yang–Baxter equation* & other properties to obtain *Izergin–Korepin formula* for partition function as $n\times n$ determinant.

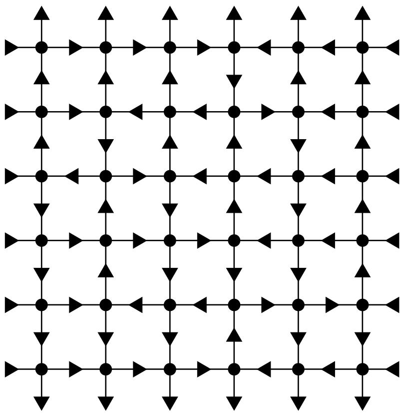
(4) Evaluate determinant at certain values of parameters for which all weights are 1.

Configurations of Six-Vertex Model on Square with Domain-Wall Boundary Conditions (DWBC)

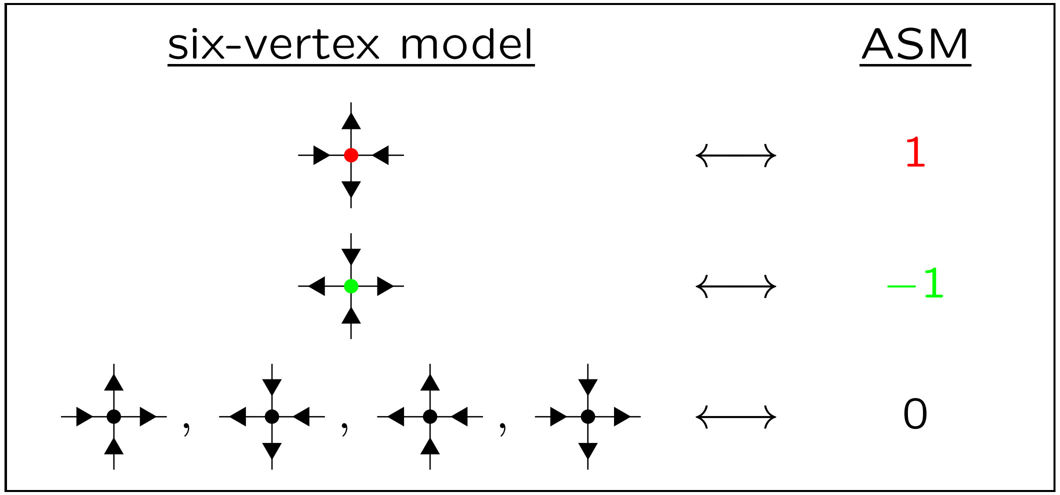
- Let

$$6\text{VDW}(n) = \left\{ \begin{array}{l} \text{edge orientations} \\ \text{of } n \times n \text{ grid} \end{array} \;\middle|\; \begin{array}{l} \bullet \text{ 2 in \& 2 out arrows at each degree 4 vertex} \\ \quad (\Rightarrow \text{6 cases}) \\ \bullet \text{ all arrows out at top \& bottom boundaries} \\ \bullet \text{ all arrows in at left \& right boundaries} \end{array} \right\}$$

- e.g. $6\text{VDW}(3) = \{$  $\}$

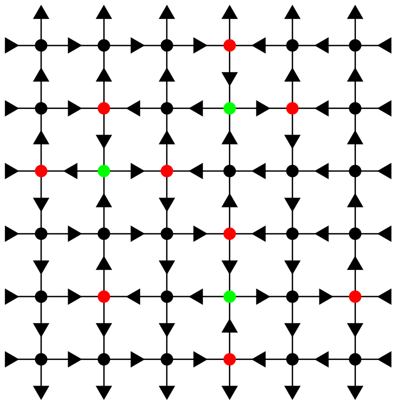
- e.g.  $\in 6\text{VDW}(6)$

Six-Vertex Model Configuration – ASM Bijection



- Gives bijection between 6VDW(n) & $\{n \times n$ ASMs$\}$.
  (Elkies, Kuperberg, Larsen, Propp 1992)

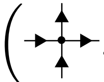
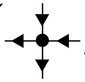
- e.g.

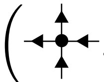
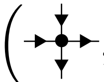


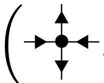
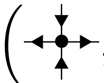
$$\longleftrightarrow \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Vertex Weights

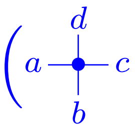
- $\sigma(x) := x - x^{-1}$
- $u =$ local parameter
- $q =$ global parameter

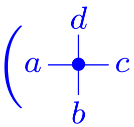
- $W\Big($$, u\Big) = W\Big($$, u\Big) = \frac{\sigma(q^2 u)}{\sigma(q^4)}$

  $W\Big($$, u\Big) = W\Big($$, u\Big) = \frac{\sigma(q^2 u^{-1})}{\sigma(q^4)}$

  $W\Big($$, u\Big) = W\Big($$, u\Big) = 1$

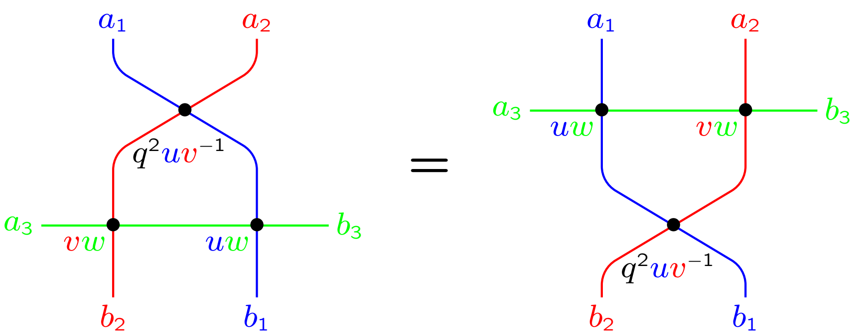
- At $u = 1$ & $q = e^{i\pi/6}$: $W(c, 1)\big|_{q=e^{i\pi/6}} = 1$ for all c

- At $u = q^{\pm 2}$: $W\Big($

$, q^{-2}\Big) = \delta_{ab}\,\delta_{cd}$, $W\Big($

$, q^{2}\Big) = \delta_{ad}\,\delta_{bc}$

- Yang–Baxter Equation:

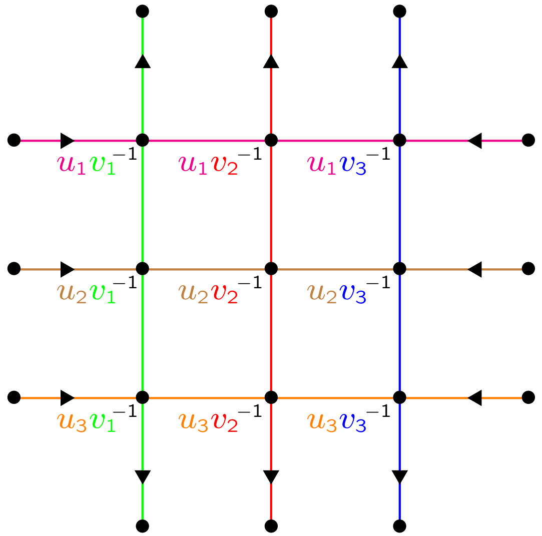


Partition (Generating) Function

- $Z(u_1,\ldots,u_n,v_1,\ldots,v_n) := \sum_{C\in 6\mathrm{VDW}(n)} \prod_{i,j=1}^{n} W(C_{ij}, u_i v_j^{-1})$

  [where C_{ij} = local configuration at vertex in row i & column j of grid]

- e.g. $Z(u_1,u_2,u_3,v_1,v_2,v_3) =$



- Therefore $Z(\underbrace{1,\ldots,1}_{2n})\big|_{q=e^{i\pi/6}} = |6\mathrm{VDW}(n)| =$ (# of $n\times n$ ASMs)

Izergin–Korepin Formula

$$Z(u_1,\ldots,u_n,v_1,\ldots,v_n)$$
$$= \frac{\prod_{i,j=1}^{n} \sigma(q^2 u_i v_j^{-1})\,\sigma(q^2 u_i^{-1} v_j)}{\sigma(q^4)^{n(n-1)} \prod_{1\le i<j\le n} \sigma(u_i u_j^{-1})\sigma(v_i^{-1} v_j)} \det_{1\le i,j\le n} \left(\frac{1}{\sigma(q^2 u_i v_j^{-1})\,\sigma(q^2 u_i^{-1} v_j)} \right)$$

(Izergin 1987)

Proof outline:

- Show that a function $X(u_1,\ldots,u_n,v_1,\ldots,v_n)$ which satisfies the following properties is uniquely determined:

(1) $X(u_1,v_1)=1$

(2) $X(u_1,\ldots,u_n,v_1,\ldots,v_n)$ a Laurent polynomial in u_1 of lower degree $\ge -n+1$, upper degree $\le n-1$

(3) $X(u_1,\ldots,u_n,v_1,\ldots,v_n)|_{u_1 v_1^{-1}=q^{\pm 2}} = \frac{\prod_{i=2}^{n} \sigma(q^{\pm 2} u_1 v_i^{-1})\sigma(q^{\pm 2} u_i v_1^{-1})}{\sigma(q^4)^{2n-2}} X(u_2,\ldots,u_n,v_2,\ldots,v_n)$

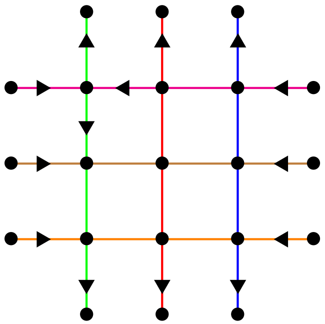
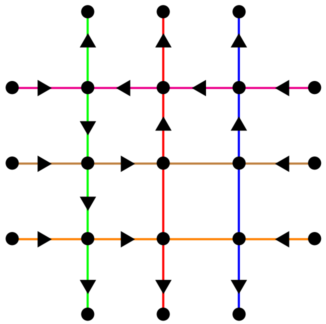
(4) $X(u_1,\ldots,u_n,v_1,\ldots,v_n)$ symmetric in $v_1,\ldots,v_n$

- Show that LHS & RHS of Izergin–Korepin formula both satisfy all of these properties.

Examples of parts of proof:

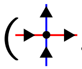
- Reduction of $Z(u_1, u_2, u_3, v_1, v_2, v_3)$ at $u_1 {v_1}^{-1} = q^{-2}$:

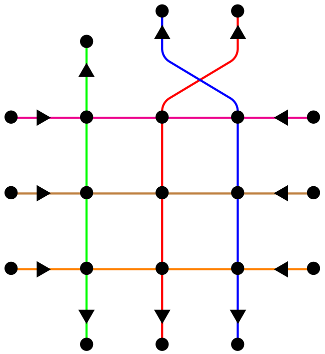
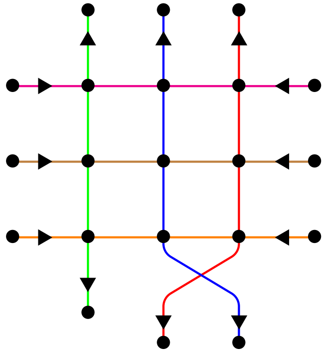
$$Z(u_1, u_2, u_3, v_1, v_2, v_3)|_{u_1 {v_1}^{-1} = q^{-2}}$$

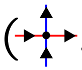
$=$  $=$ 

$$= \frac{\sigma(q^{-2} u_1 {v_2}^{-1}) \sigma(q^{-2} u_1 {v_3}^{-1}) \sigma(q^{-2} u_2 {v_1}^{-1}) \sigma(q^{-2} u_3 {v_1}^{-1})}{\sigma(q^4)^4} Z(u_2, u_3, v_2, v_3) \qquad \square$$

- Symmetry of $Z(u_1, u_2, u_3, v_1, v_2, v_3)$ in v_2 & v_3:

$W($$, q^2 {v_2}^{-1} v_3)\, Z(u_1, u_2, u_3, v_1, v_2, v_3)$

$=$  $\overset{\text{YBE}}{=}$ 

$= W($$, q^2 {v_2}^{-1} v_3)\, Z(u_1, u_2, u_3, v_1, v_3, v_2) \qquad \square$

Partition Function at $q = e^{i\pi/6}$

$$Z(u_1,\ldots,u_n,v_1,\ldots,v_n)\Big|_{q=e^{i\pi/6}} = 3^{-n(n-1)/2}\left(\prod_{i=1}^{n} u_i v_i\right)^{-n+1} s_{(n-1,n-1,\ldots,2,2,1,1)}(u_1^2,\ldots,u_n^2,v_1^2,\ldots,v_n^2)$$

(Okada 2006)

<u>Proof outline:</u>

- Substitute $q = e^{i\pi/6}$ into determinantal expression for partition function.
- Apply identity which converts certain $n \times n$ determinant to $2n \times 2n$ determinant. *(Okada 1998)*
- Use $s_\lambda(x_1,\ldots,x_k) = \det_{1\le i,j\le k}\left(x_i^{\lambda_j+k-j}\right) \Big/ \prod_{1\le i<j\le k}(x_i - x_j)$.

Final Step

- Set $u_1 = \ldots = u_n = v_1 = \ldots = v_n = 1$ & recall $Z(\underbrace{1,\ldots,1}_{2n})\Big|_{q=e^{i\pi/6}} = (\#\text{ of } n \times n \text{ ASMs})$.
- Obtain $(\#\text{ of } n\times n\text{ ASMs}) = 3^{-n(n-1)/2}\,\mathrm{SSYT}\big((n-1,n-1,\ldots,2,2,1,1),2n\big) = \prod_{i=0}^{n-1}\frac{(3i+1)!}{(n+i)!}$

as required.

Diagonally and Antidiagonally Symmetric Alternating Sign Matrices (DASASMs)

DASASM: ASM which is invariant under

- reflection in the diagonal
- reflection in the antidiagonal

- i.e. invariant under action of subgroup $\{\mathcal{I}, \mathcal{D}, \mathcal{A}, \mathcal{R}_\pi\}$ of D_4

e.g. $$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Observations:
- any DASASM uniquely determined by its entries in a triangle bounded by diagonal and antidiagonal
- central entry of an odd-order DASASM is ± 1

Number D_n of $(2n-1)\times(2n-1)$ DASASMs
& numbers $D_n^{\pm}$ of $(2n-1)\times(2n-1)$ DASASMs with central entry ± 1

n=1

$$\begin{pmatrix} 1 \end{pmatrix}$$

$$\Rightarrow \quad D_1 = 1$$

$$\& \quad \frac{D_1^-}{D_1^+} = \frac{0}{1}$$

n=2

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\Rightarrow \quad D_2 = 3$$

$$\& \quad \frac{D_2^-}{D_2^+} = \frac{1}{2}$$

n=3

$$
\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},
\begin{pmatrix}0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0\end{pmatrix},
\begin{pmatrix}0&0&0&0&1\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\1&0&0&0&0\end{pmatrix},
\begin{pmatrix}1&0&0&0&0\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\0&0&0&0&1\end{pmatrix},
$$

$$
\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&1&0&0\\0&0&0&0&1\\0&0&0&1&0\end{pmatrix},
\begin{pmatrix}0&0&0&1&0\\0&0&0&0&1\\0&0&1&0&0\\1&0&0&0&0\\0&1&0&0&0\end{pmatrix},
\begin{pmatrix}0&1&0&0&0\\1&-1&0&1&0\\0&0&1&0&0\\0&1&0&-1&1\\0&0&0&1&0\end{pmatrix},
\begin{pmatrix}0&0&0&1&0\\0&1&0&-1&1\\0&0&1&0&0\\1&-1&0&1&0\\0&1&0&0&0\end{pmatrix},
$$

$$
\begin{pmatrix}0&0&1&0&0\\0&1&-1&1&0\\1&-1&1&-1&1\\0&1&-1&1&0\\0&0&1&0&0\end{pmatrix},
\begin{pmatrix}1&0&0&0&0\\0&0&1&0&0\\0&1&-1&1&0\\0&0&1&0&0\\0&0&0&0&1\end{pmatrix},
\begin{pmatrix}0&0&0&0&1\\0&0&1&0&0\\0&1&-1&1&0\\0&0&1&0&0\\1&0&0&0&0\end{pmatrix},
\begin{pmatrix}0&0&1&0&0\\0&1&0&0&0\\1&0&-1&0&1\\0&0&0&1&0\\0&0&1&0&0\end{pmatrix},
$$

$$
\begin{pmatrix}0&0&1&0&0\\0&0&0&1&0\\1&0&-1&0&1\\0&1&0&0&0\\0&0&1&0&0\end{pmatrix},
\begin{pmatrix}0&1&0&0&0\\1&-1&1&0&0\\0&1&-1&1&0\\0&0&1&-1&1\\0&0&0&1&0\end{pmatrix},
\begin{pmatrix}0&0&0&1&0\\0&0&1&-1&1\\0&1&-1&1&0\\1&-1&1&0&0\\0&1&0&0&0\end{pmatrix}
$$

$$\Rightarrow \quad D_3 = 15$$

$$\& \quad \frac{D_3^-}{D_3^+} = \frac{6}{9} = \frac{2}{3}$$

General Case

of $(2n-1)\times(2n-1)$ DASASMs: $D_n = \prod_{i=0}^{n-1} \frac{(3i)!}{(n+i-1)!} = 1, 3, 15, 126, 1782, \ldots$

- Recursion: $\binom{2n-1}{n} D_{n+1} = \binom{3n}{n} D_n$
- $\prod_{i=0}^{n-1} \frac{(3i)!}{(n+i-1)!} = 3^{-(n-1)(n-2)/2}\,\mathrm{SSYT}\big((n-1, n-2, n-2, \ldots, 2, 2, 1, 1), 2n-1\big)$
- Conjecture: *Robbins 1985*
- Proof: *RB, Fischer, Konvalinka 2017 (Adv. Math.* **315***, 324–365)*

#'s of $(2n-1)\times(2n-1)$ DASASMs with central entry ± 1: $\frac{D_n^-}{D_n^+} = \frac{n-1}{n}$

- Conjecture: *Stroganov 2008*
- Proof: *RB, Fischer, Konvalinka 2017*

Sketch of Proof of Odd-Order DASASM Formula

(1) Obtain bijection between $(2n+1)\times(2n+1)$ DASASMs & configurations of six-vertex model on certain triangle.

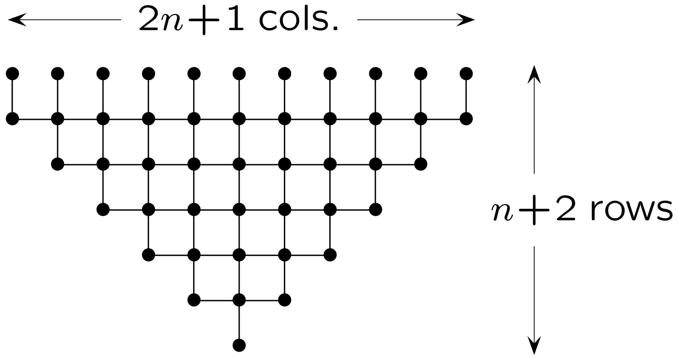
(2) Introduce bulk weights, *boundary weights* & associated partition function.

(3) Use Yang–Baxter equation, *reflection equation* & other properties to prove formula for partition function involving *sum* of two $(n+1)\times(n+1)$ determinants.

(4) Evaluate determinantal formula at values of parameters for which all weights are 1.

Configurations of Six-Vertex Model on Triangle

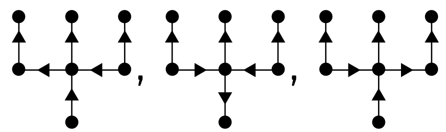
- Let $\mathcal{T}_n$ =



$6\mathrm{VT}(n) = \left\{ \begin{array}{l} \text{orientations of} \\ \text{edges of } \mathcal{T}_n \end{array} \right|$

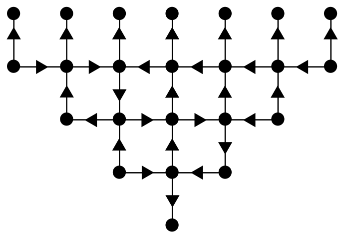
- 2 in & 2 out arrows at each degree 4 vertex ($\Rightarrow$ 6 cases)
- no restriction at degree 2 vertices ($\Rightarrow$ 4 cases)
- all arrows up on top edges
- no restriction on single bottom edge ($\Rightarrow$ 2 cases) $\}$

- e.g. $6\mathrm{VT}(1) = \{$



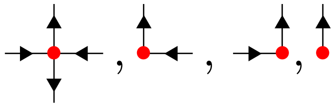
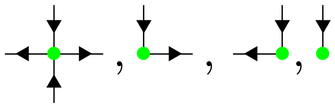
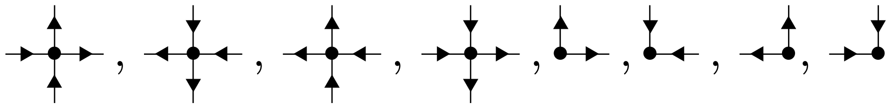
$\}$

- e.g.

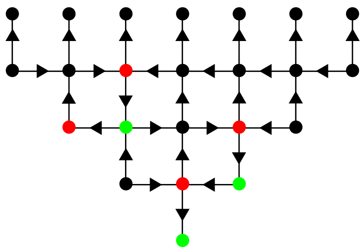


$\in 6\mathrm{VT}(3)$

Six-Vertex Model Configuration – DASASM Bijection

six-vertex model		DASASM
	$\longleftrightarrow$	1
	$\longleftrightarrow$	-1
	$\longleftrightarrow$	0

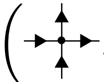
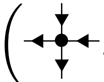
- Also use reflections in diagonal and antidiagonal
- Gives bijection between 6VT(n) & $\{(2n+1)\times(2n+1)$ DASASMs$\}$
- e.g.

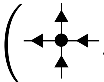
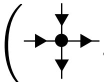


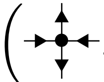
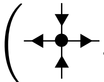
$$\longleftrightarrow \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ & 1 & -1 & 0 & 1 & 0 & \\ & & 0 & 1 & -1 & & \\ & & & -1 & & & \end{pmatrix} \longleftrightarrow \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

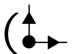
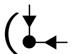
Vertex Weights

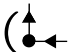
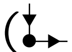
- $\sigma(x) := x - x^{-1}$
- $u =$ local parameter
- $q =$ global parameter

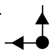
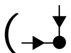
- Bulk weights: $W\Big($$, u\Big) = W\Big($$, u\Big) = \frac{\sigma(q^2 u)}{\sigma(q^4)}$

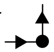
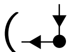
  $W\Big($$, u\Big) = W\Big($$, u\Big) = \frac{\sigma(q^2 u^{-1})}{\sigma(q^4)}$

  $W\Big($$, u\Big) = W\Big($$, u\Big) = 1$

- Left boundary weights: $W($$, u) = W($$, u) = \frac{\sigma(q\,u)}{\sigma(q)}$

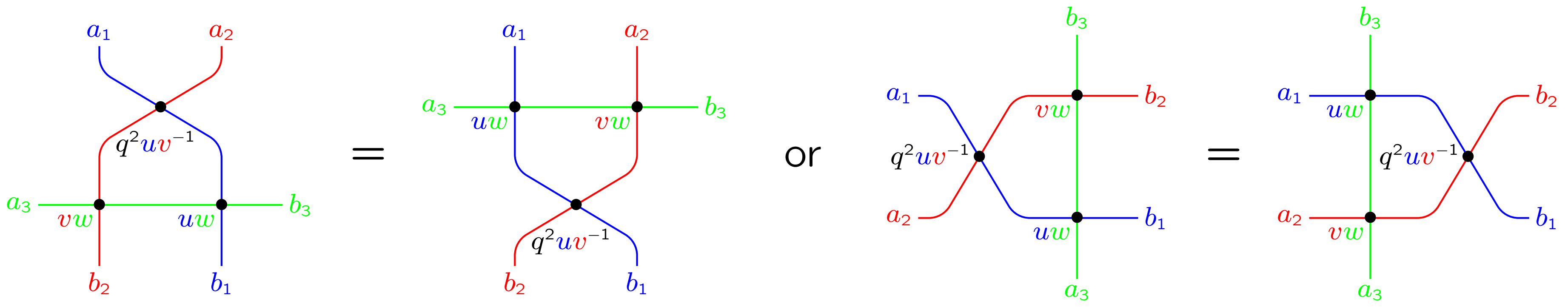
  $W($$, u) = W($$, u) = 1$

- Right boundary weights: $W($$, u) = W($$, u) = \frac{\sigma(q\,u^{-1})}{\sigma(q)}$

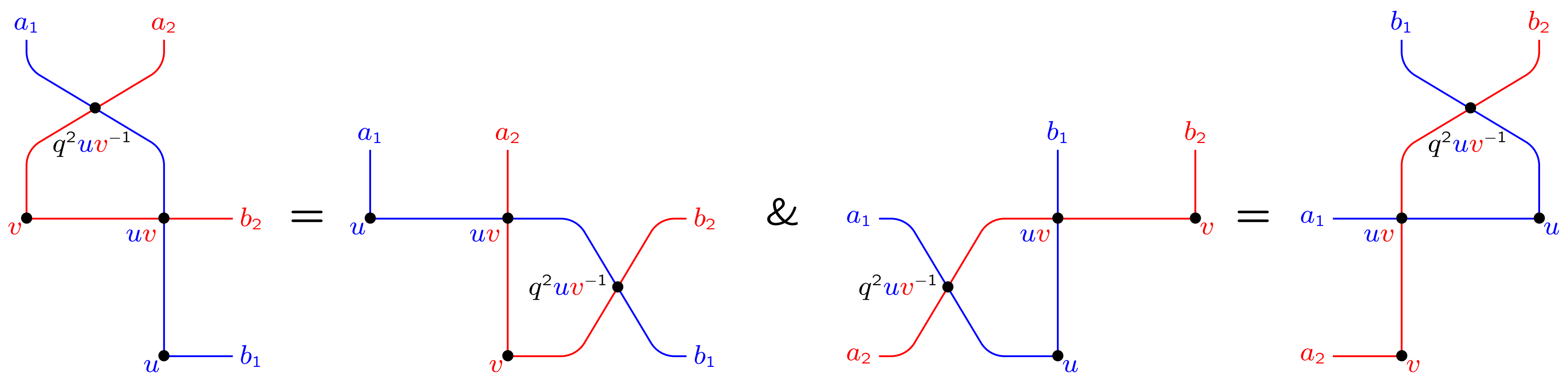
  $W($$, u) = W($$, u) = 1$

- Weights at $u = 1$ & $q = e^{i\pi/6}$: $W(c, 1)\big|_{q=e^{i\pi/6}} = 1$ for all c

- Yang–Baxter equation



- Left & right reflection equations



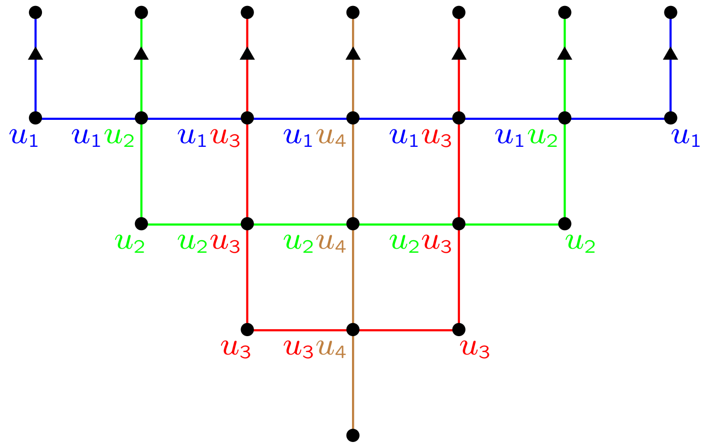
Odd-Order DASASM Partition Function

- $Z(u_1,\ldots,u_{n+1})$

$$:= \sum_{C\in 6\mathsf{VT}(n)} \prod_{i=1}^{n} W(C_{ii},u_i) \left(\prod_{j=i+1}^{2n+1-i} W(C_{ij}, u_i\, u_{\min(j,2n+2-j)}) \right) W(C_{i,2n+1-i},u_i)$$

[where C_{ij} = local configuration at vertex in row i & column j of $\mathcal{T}_n$]

- e.g. $Z(u_1,u_2,u_3,u_4) =$



- Therefore $Z(\underbrace{1,\ldots,1}_{n+1})\Big|_{q=e^{i\pi/6}} = |6\mathsf{VT}(n)| =$ (# of $(2n+1)\times(2n+1)$ DASASMs)

Sum of Determinants Formula for Partition Function

$$Z(u_1,\ldots,u_{n+1})$$

$$= \frac{\sigma(q^2)^n}{\sigma(q)^{2n}\,\sigma(q^4)^{n^2}} \prod_{i=1}^{n} \frac{\sigma(u_i)\sigma(qu_i)\sigma(qu_i^{-1})\sigma(q^2u_iu_{n+1})\sigma(q^2u_i^{-1}u_{n+1}^{-1})}{\sigma(u_iu_{n+1}^{-1})} \prod_{1\le i<j\le n}\left(\frac{\sigma(q^2u_iu_j)\sigma(q^2u_i^{-1}u_j^{-1})}{\sigma(u_iu_j^{-1})}\right)^2$$

$$\times \left(\det_{1\le i,j\le n+1}\left(\begin{cases} \frac{q^2+q^{-2}+u_i^2+u_j^{-2}}{\sigma(q^2u_iu_j)\,\sigma(q^2u_i^{-1}u_j^{-1})}, & i\le n \\ \frac{u_{n+1}-1}{u_j^2-1}, & i=n+1 \end{cases}\right) + \det_{1\le i,j\le n+1}\left(\begin{cases} \frac{q^2+q^{-2}+u_i^{-2}+u_j^{2}}{\sigma(q^2u_iu_j)\,\sigma(q^2u_i^{-1}u_j^{-1})}, & i\le n \\ \frac{u_{n+1}^{-1}-1}{u_j^{-2}-1}, & i=n+1 \end{cases}\right)\right)$$

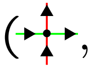
Proof outline:

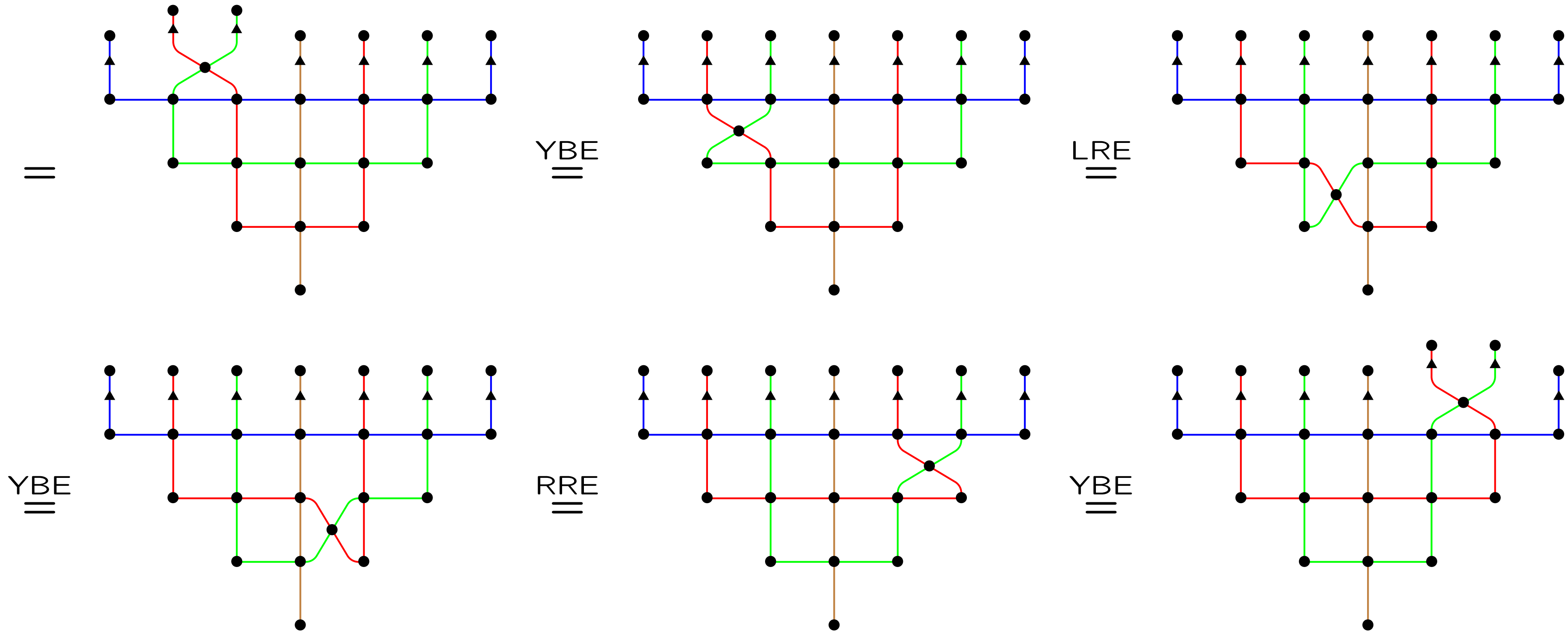
- Show that a function $X(u_1,\ldots,u_{n+1})$ which satisfies the following properties is uniquely determined:
  (1) $X(u_1)=1$
  (2) $X(u_1,\ldots,u_{n+1})$ a Laurent polynomial in u_{n+1} of lower degree $\ge -n$, upper degree $\le n$
  (3) $X(u_1,\ldots,u_{n+1})|_{u_1u_{n+1}=q^2} = \frac{\sigma(qu_1)\,(\sigma(qu_1^{-1})+\sigma(q))\prod_{i=2}^{n}\sigma(q^2u_1u_i)\sigma(q^2u_iu_{n+1})}{\sigma(q)^2\,\sigma(q^4)^{2n-2}}\,X(u_2,\ldots,u_n,u_1)$
  (4) $X(u_1,\ldots,u_{n+1})$ symmetric in $u_1,\ldots,u_n$
  (5) $X(u_1^{-1},\ldots,u_{n+1}^{-1})=X(u_1,\ldots,u_{n+1})$
  (6) $X(u_1,\ldots,u_{n+1})$ even in u_i, for $i=1,\ldots,n$
- Show that LHS & RHS of required formula both satisfy all of these properties.

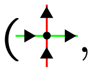
Part of proof:

Use Yang–Baxter and reflection equations (YBE, LRE, RRE) to show that $Z(u_1, \ldots, u_{n+1})$ is symmetric in u_i and u_{i+1}, $i = 1, \ldots, n-1$.

e.g. $n = 3$ & $i = 2$:

$W($$, q^2 u_2{}^{-1} u_3)\, Z(u_1, u_2, u_3, u_4)$



$= \; W($$, q^2 u_2{}^{-1} u_3)\, Z(u_1, u_3, u_2, u_4)$ □

Partition Function at $q = e^{i\pi/6}$

$$Z(u_1,\ldots,u_{n+1})\big|_{q=e^{i\pi/6}} = 3^{-n(n-1)/2}\Big(\tfrac{u_{n+1}^n}{u_{n+1}+1}\, s_{(n,n-1,n-1,\ldots,2,2,1,1)}(u_1^2,u_1^{-2},\ldots,u_n^2,u_n^{-2},u_{n+1}^{-2}) + \tfrac{u_{n+1}^{-n}}{u_{n+1}^{-1}+1}\, s_{(n,n-1,n-1,\ldots,2,2,1,1)}(u_1^2,u_1^{-2},\ldots,u_n^2,u_n^{-2},u_{n+1}^{2})\Big)$$

Proof outline:

- Substitute $q = e^{i\pi/6}$ into determinantal expression for partition function.
- Apply identity which converts certain $(n+1)\times(n+1)$ determinant to $(2n+2)\times(2n+2)$ determinant. *(Okada 1998)*
- Use $s_\lambda(x_1,\ldots,x_k) = \det_{1\le i,j\le k}\Big(x_i^{\lambda_j+k-j}\Big)\Big/\prod_{1\le i<j\le k}(x_i - x_j)$.

Final Step

- Set $u_1 = \ldots = u_{n+1} = 1$ & recall $Z(\underbrace{1,\ldots,1}_{n+1})\big|_{q=e^{i\pi/6}} = (\#\text{ of }(2n+1)\times(2n+1)\text{ DASASMs})$.
- Obtain $((2n+1)\times(2n+1)\text{ DASASMs}) = 3^{-n(n-1)/2}\,\text{SSYT}\big((n,n-1,n-1,\ldots,2,2,1,1),2n+1\big)$

$$= \prod_{i=0}^{n}\frac{(3i)!}{(n+i)!}\quad\text{as required.}$$

of Odd-Order DASASMs with Fixed Central Entry

$$\frac{\#\text{ of }(2n+1)\times(2n+1)\text{ DASASMs with central entry }-1}{\#\text{ of }(2n+1)\times(2n+1)\text{ DASASMs with central entry }1} = \frac{n}{n+1}$$

Proof outline:

- Introduce partition functions $Z_\pm(u_1,\dots,u_{n+1})$ for $(2n+1)\times(2n+1)$ DASASMs with central entry ± 1.

- Show that $Z_\pm(u_1,\dots,u_{n+1}) = \frac{1}{2}\bigl(Z(u_1,\dots,u_n,u_{n+1}) \pm (-1)^n Z(u_1,\dots,u_n,-u_{n+1})\bigr)$.

- Use previous results for $Z(u_1,\dots,u_{n+1})$.

Features of Odd-Order DASASM Generating Function

- Only a *single* set $u_1, \ldots, u_{n+1}$ of parameters used.

- Last parameter u_{n+1} plays *special* role.

- Yang–Baxter *and* reflection equation needed (with certain boundary weights not previously used for ASM enumeration).

- Partition function formula involves *sum* of two determinantal terms.

Final Messages

- ASMs, ASTs, DPPs & TSSCPPs are intriguing combinatorial objects.

- Many results have been proved.

- Many aspects are still not properly understood.