LOZENGE TILINGS WITH GAPS IN A 90° WEDGE DOMAIN WITH MIXED BOUNDARY CONDITIONS

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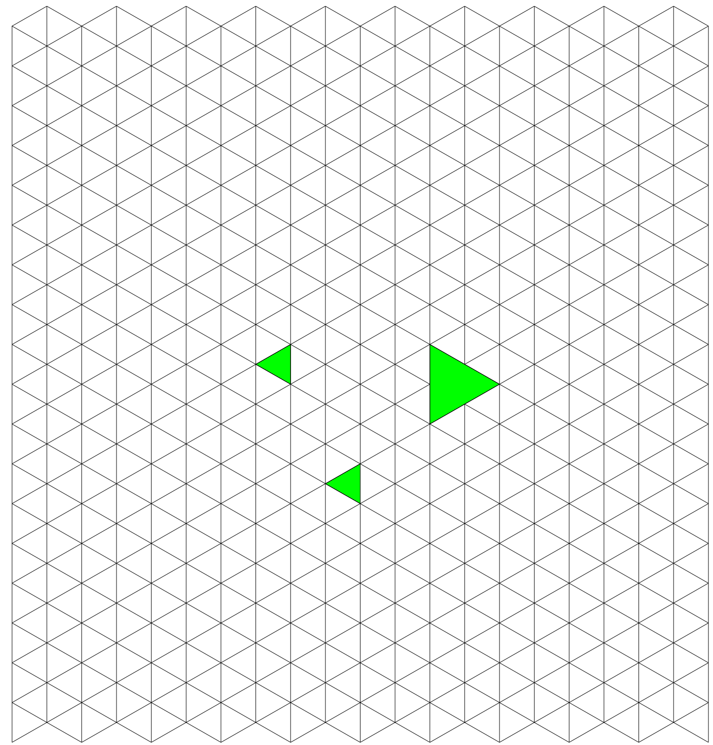


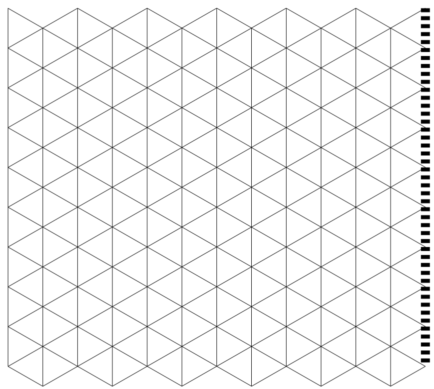
Correlation in a sea of dimers

[C, ’05–’10]

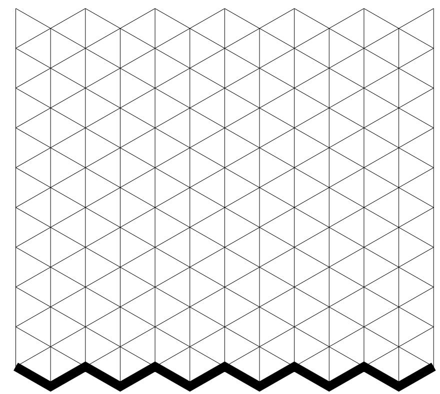
In bulk, for large separations, this is asymptotically 2D electrostatics

What about the interaction with boundary?

Two natural types:

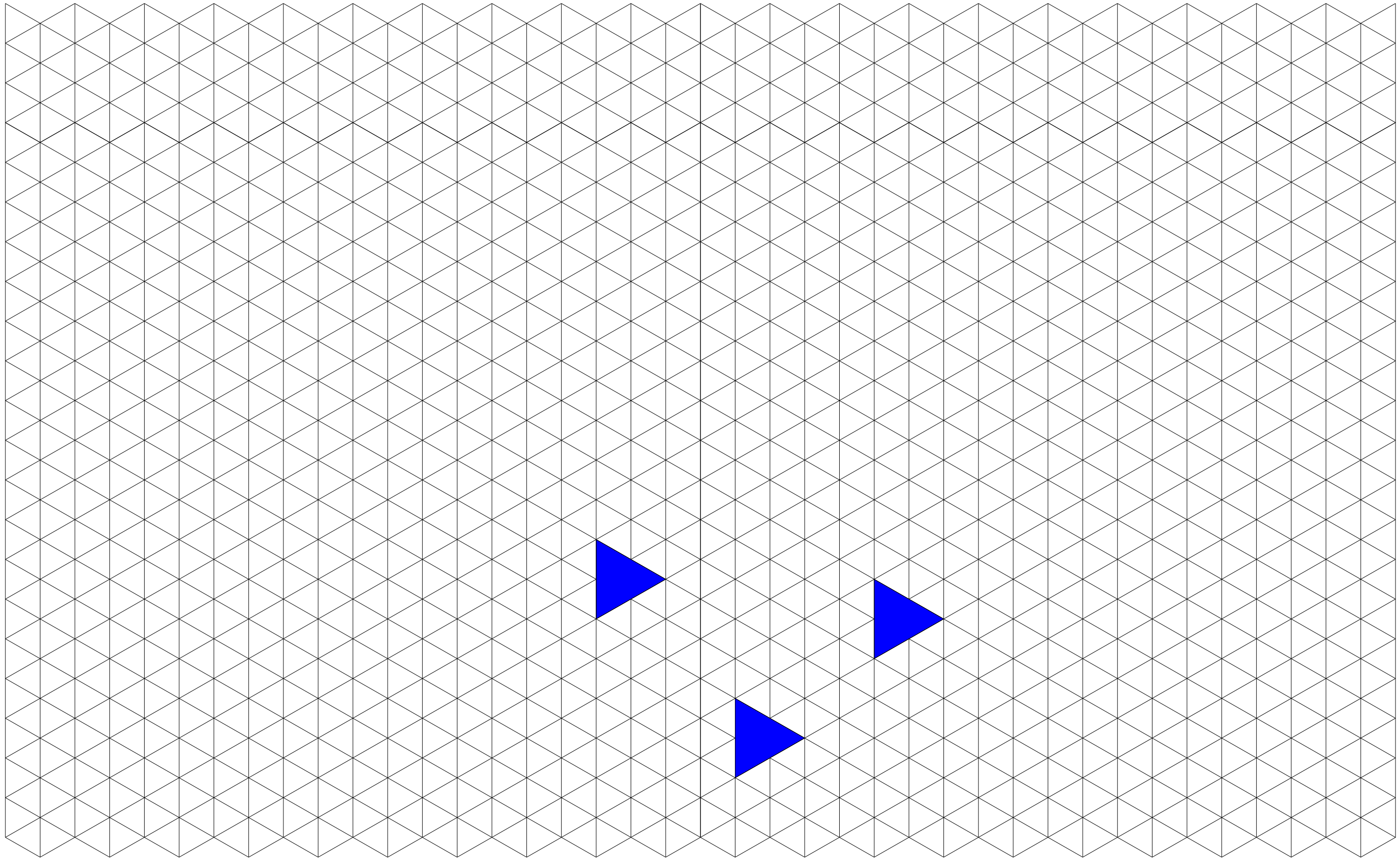


free boundary

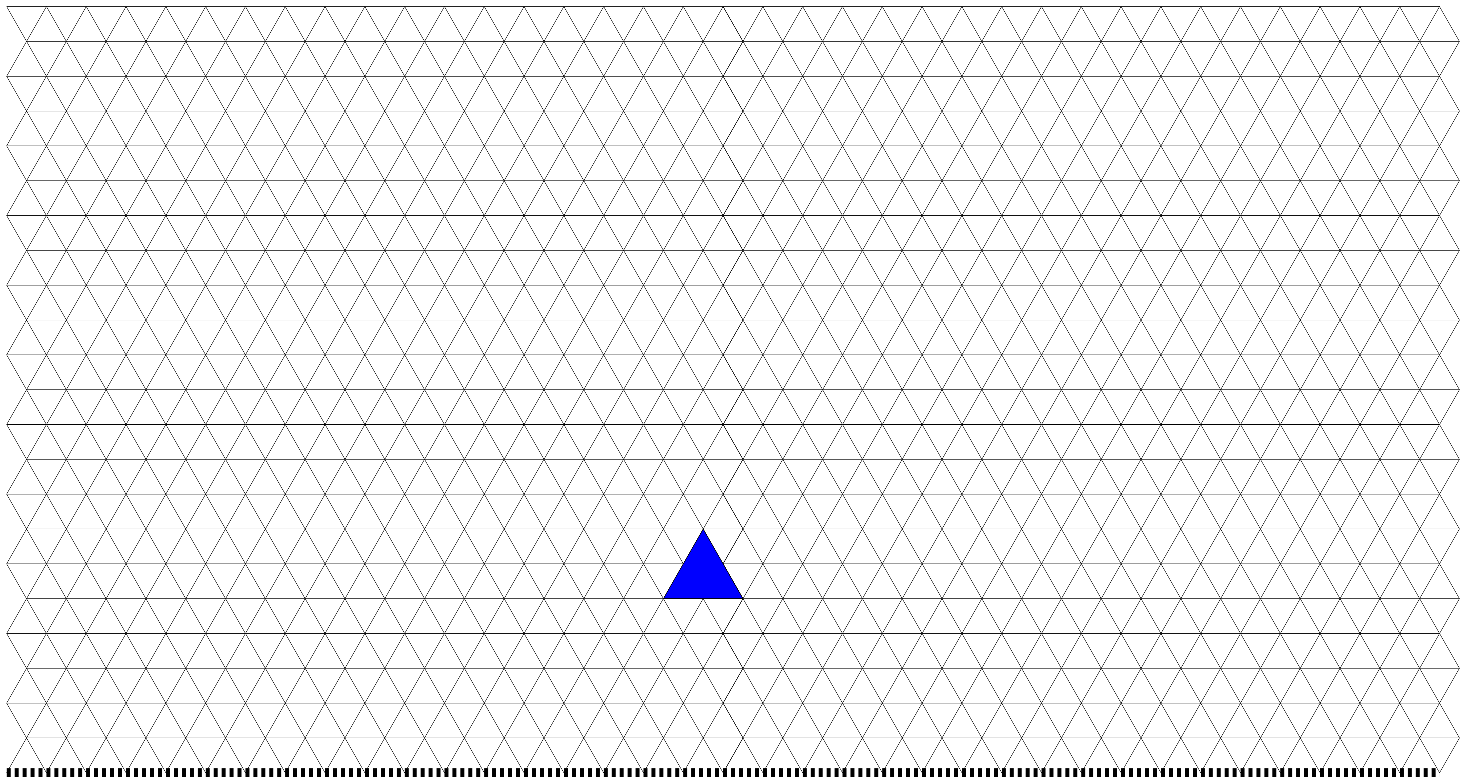


constrained boundary

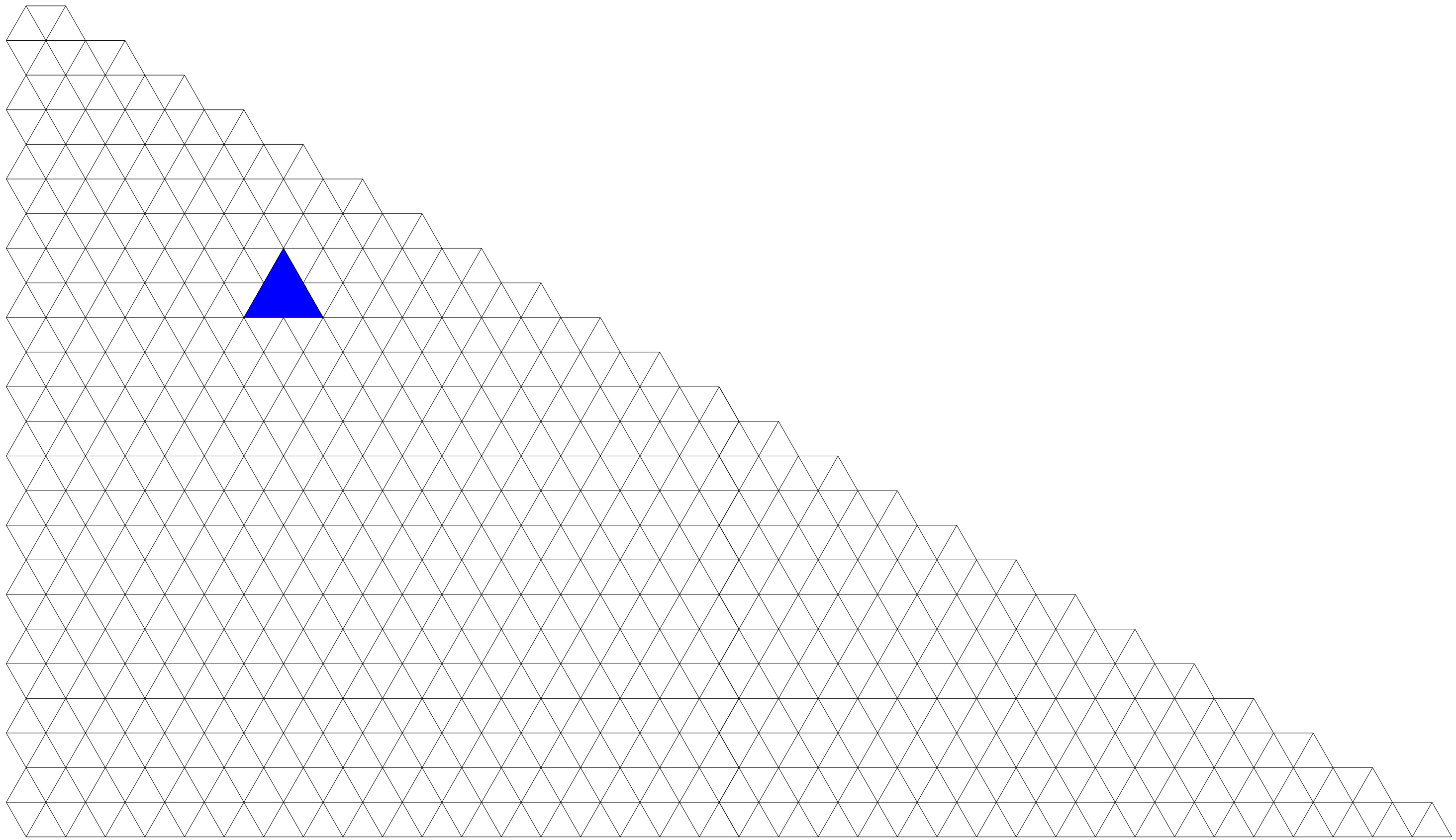
Previous examples



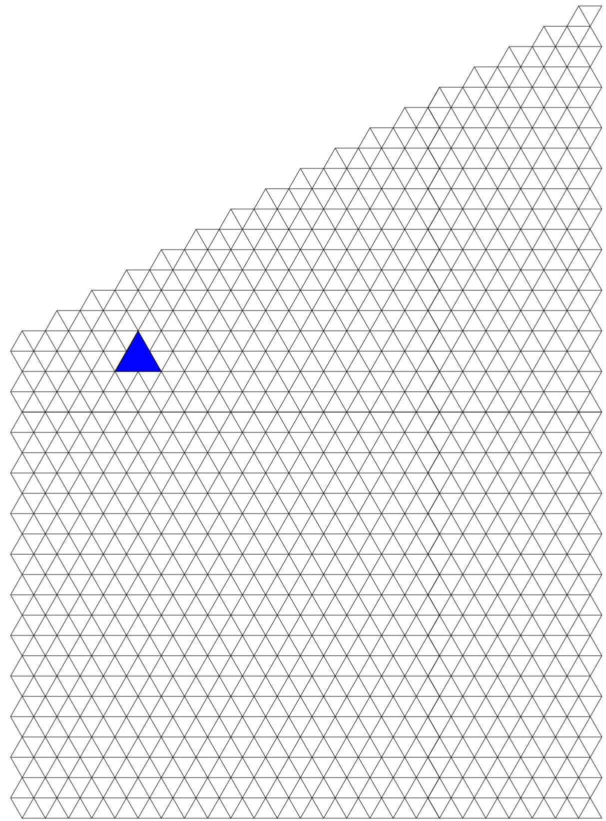
“straight line” constrained boundary



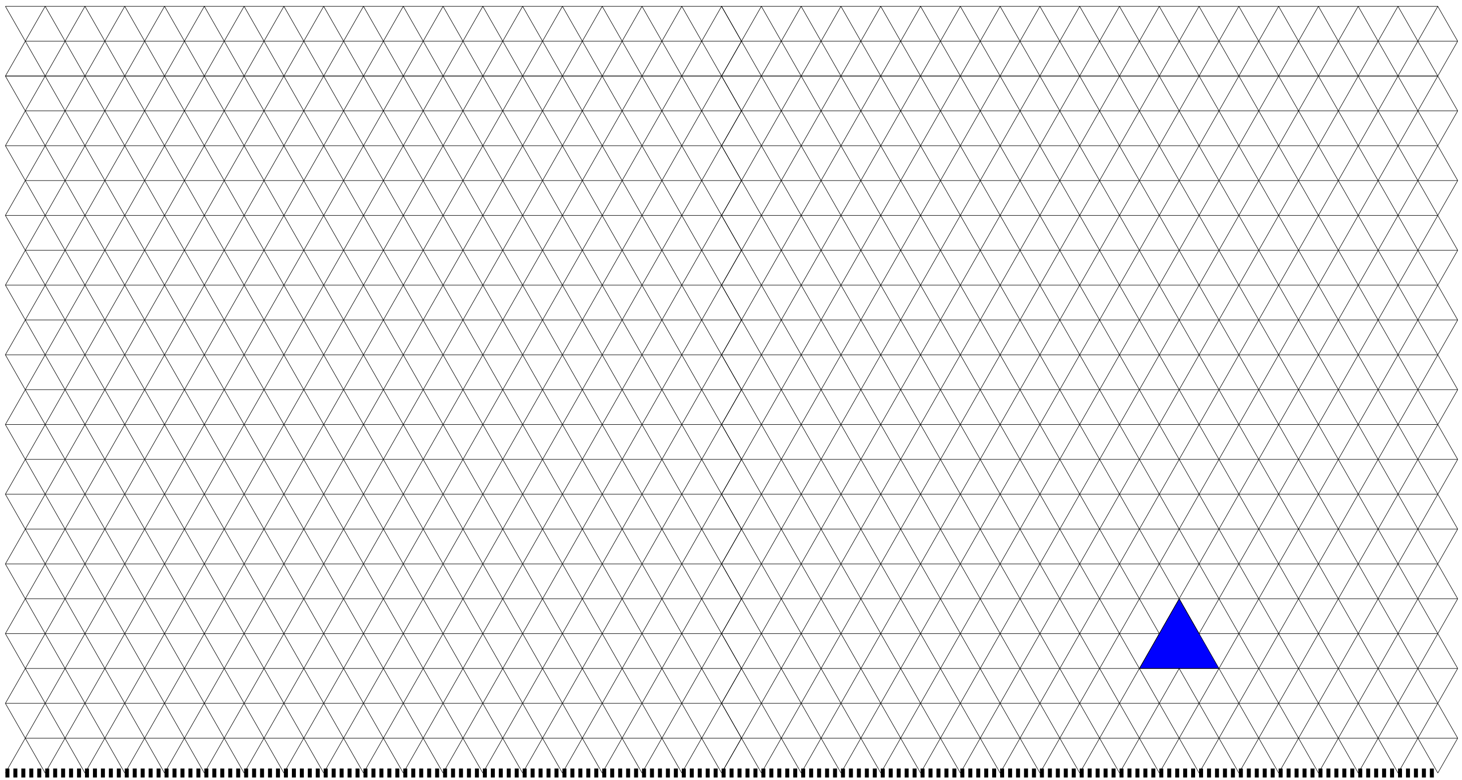
straight line free boundary



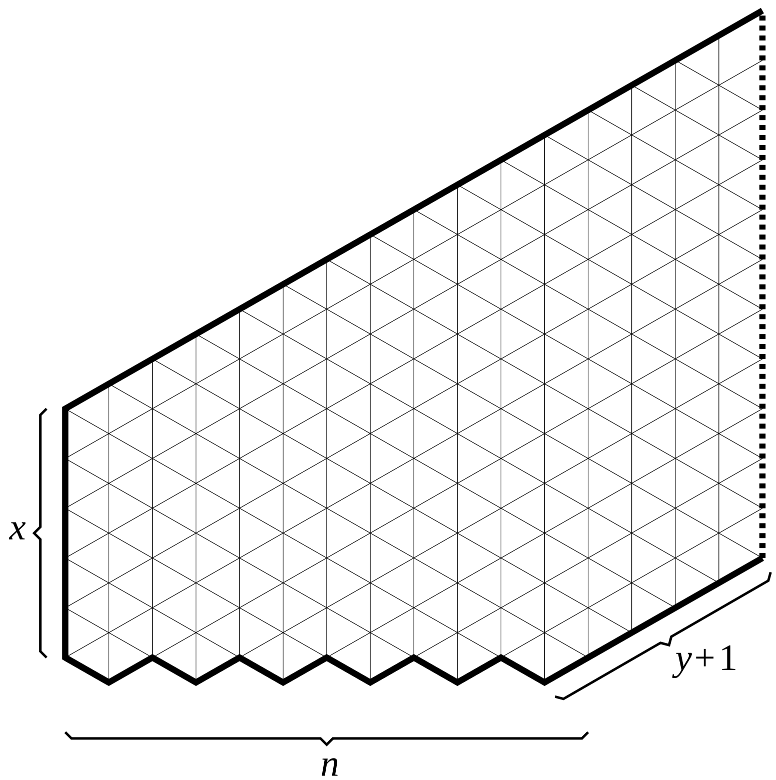
60° angle, constrained boundary



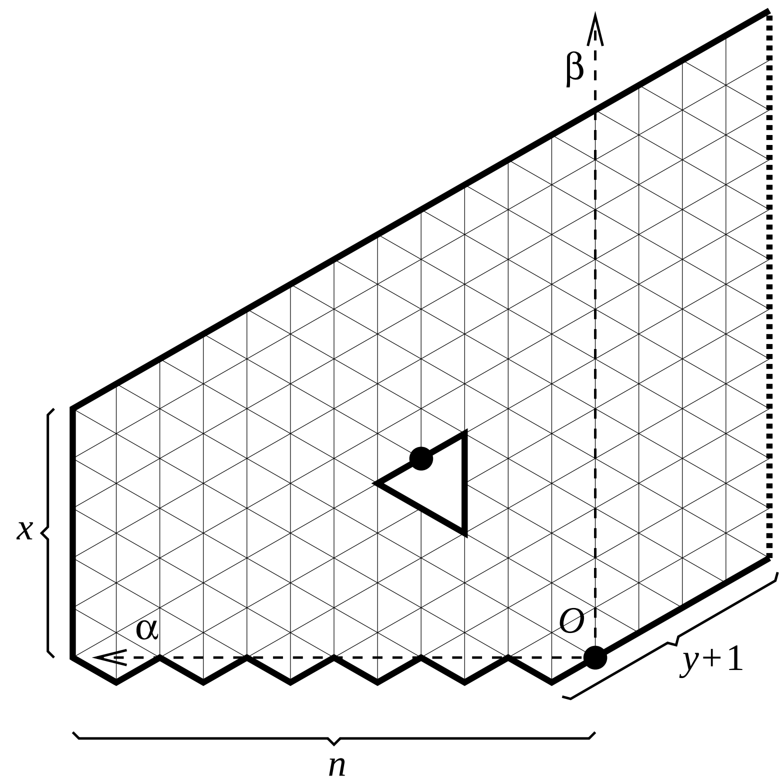
120° angle, constrained boundary



Current talk: 90° angle, mixed boundary



$D_{n,x,y}$: $n = 6, x = 5, y = 4$

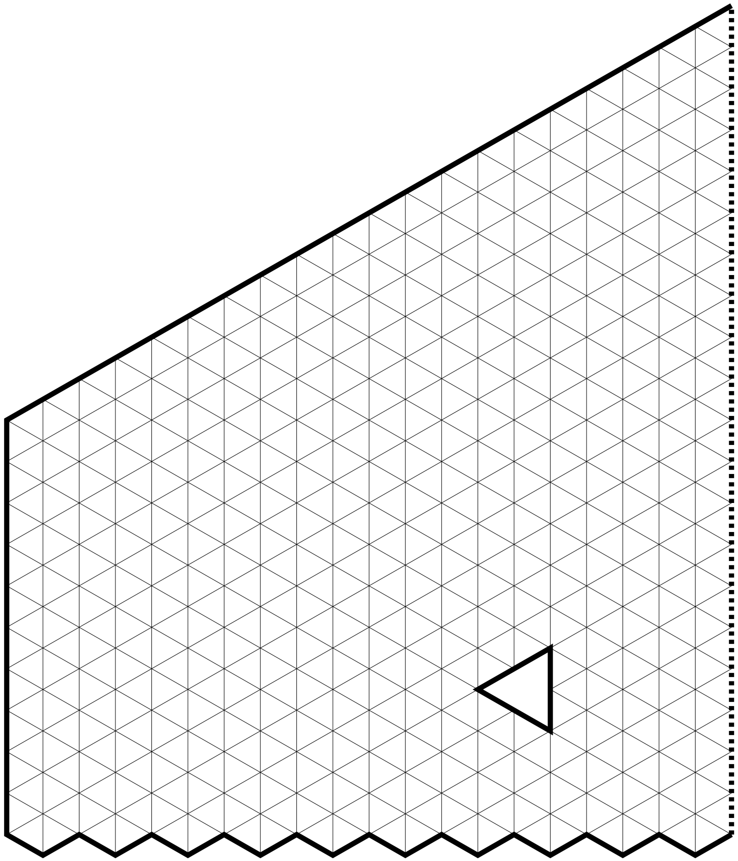


$D_{n,x,y}(\alpha,\beta)$: $n = 6, x = 5, y = 4, \alpha = 2, \beta = 4$

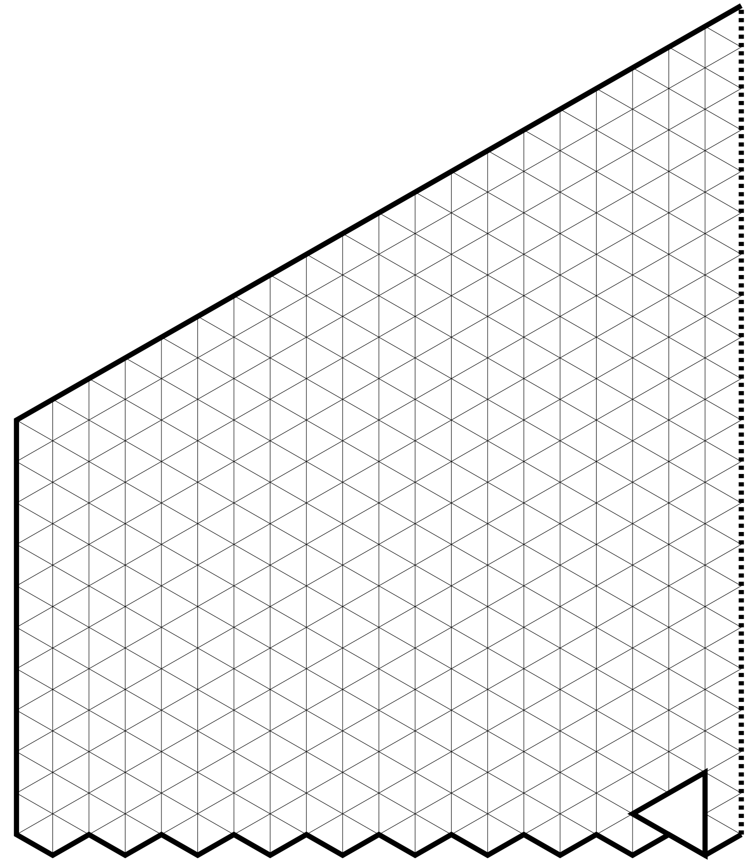
- $\mathrm{M}_f(D)$: # tilings of D with tiles allowed to protrude across free boundary portions

•: $\omega_c(\alpha, \beta)$ (correlation of the gap with the corner):

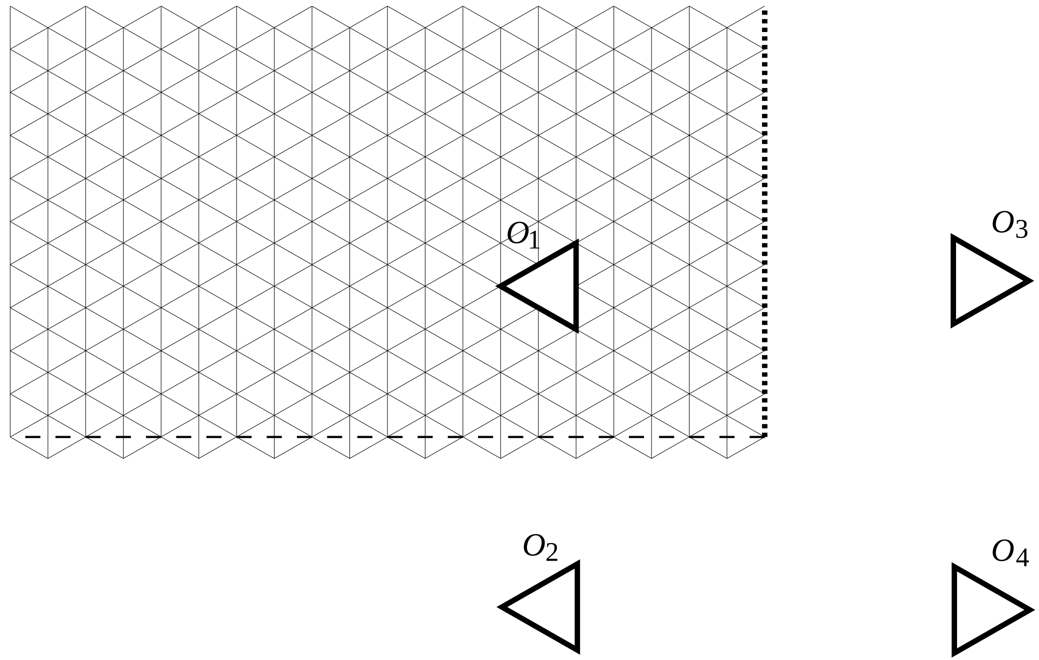
$$\omega_c(\alpha, \beta) := \lim_{n \to \infty} \frac{\mathrm{M}_f(D_{n,n,0}(\alpha, \beta))}{\mathrm{M}_f(D_{n,n,0}(1, 1))}$$



$D_{10,10,0}(3,4)$.



$D_{10,10,0}(1,1)$.



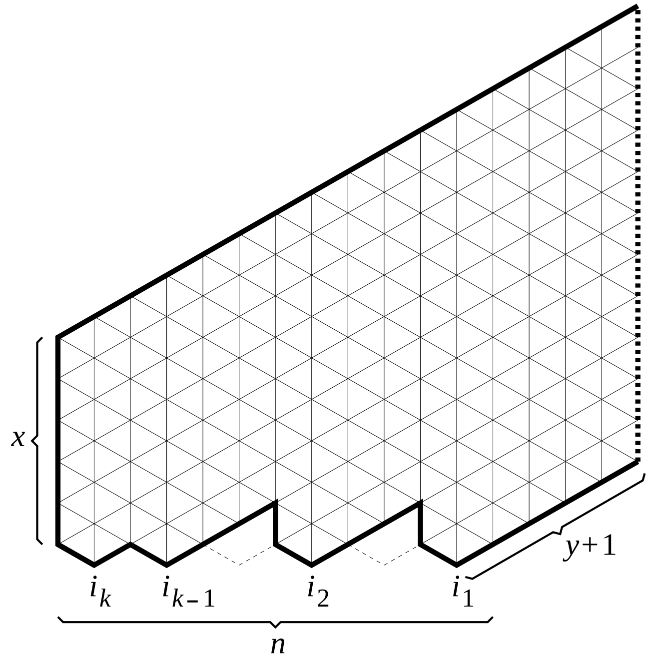
The gap and its three images for $\alpha = 3$, $\beta = 4$

The main result of this talk:

THEOREM. *Let q be a fixed positive rational number. As α and β approach infinity so that $\alpha = q\beta$, we have*

$$\omega_c(\alpha, \beta) \sim \frac{16}{3\pi R q \sqrt{q^2 + \frac{1}{3}}} \sim \frac{32}{\pi} \sqrt{\frac{\mathrm{d}(O_1, O_2)\,\mathrm{d}(O_3, O_4)}{\mathrm{d}(O_1, O_3)\,\mathrm{d}(O_1, O_4)\,\mathrm{d}(O_2, O_3)\,\mathrm{d}(O_2, O_4)}},$$

where d *is the Euclidean distance.*



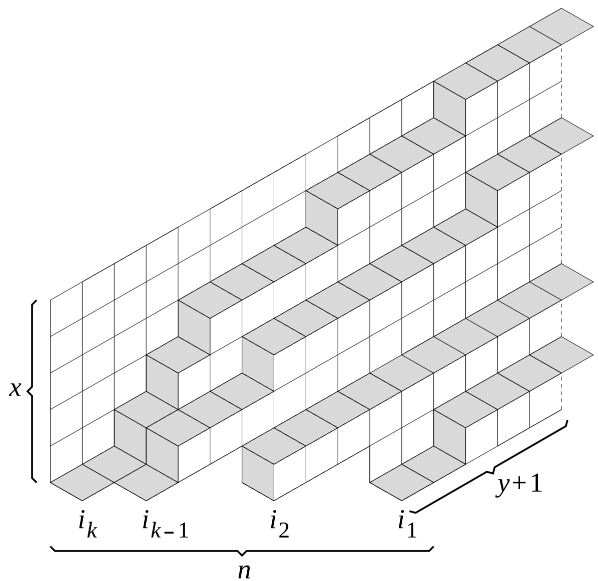
$D^{i_1,\ldots,i_k}_{n,x,y}$ for $n = 6$, $x = 5$, $y = 4$, $k = 4$, $i_1 = 1$, $i_2 = 3$, $i_3 = 5$, $i_4 = 6$.

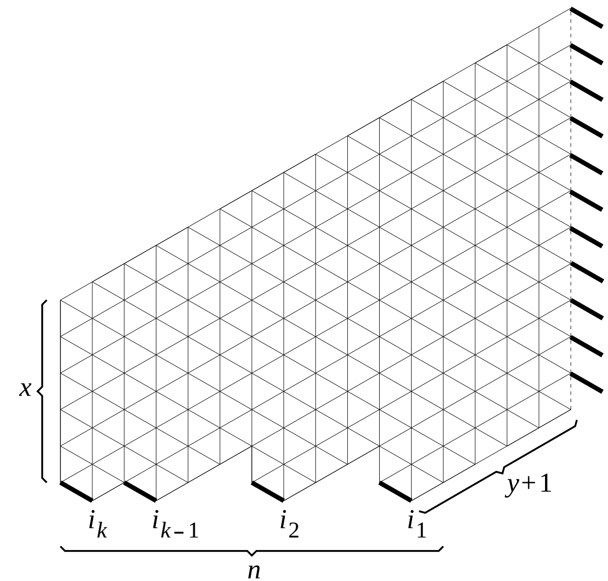
It turns out we can reduce to enumerating tilings of such regions.

Great strike of luck: *They are given by "round" formulas!*

Proposition. *For any integers* $n, x \geq 0$ *and* $y \geq -1$*, and for any integers* $1 \leq i_1 < \cdots < i_k \leq n$*, we have*

$$\mathrm{M}_f(D_{n,x,y}^{i_1,\ldots,i_k}) = \prod_{a=1}^{k} \binom{x+y+n+i_a}{y+2i_a} \prod_{1 \leq a < b \leq k} \frac{i_b - i_a}{y + i_b + i_a}.$$



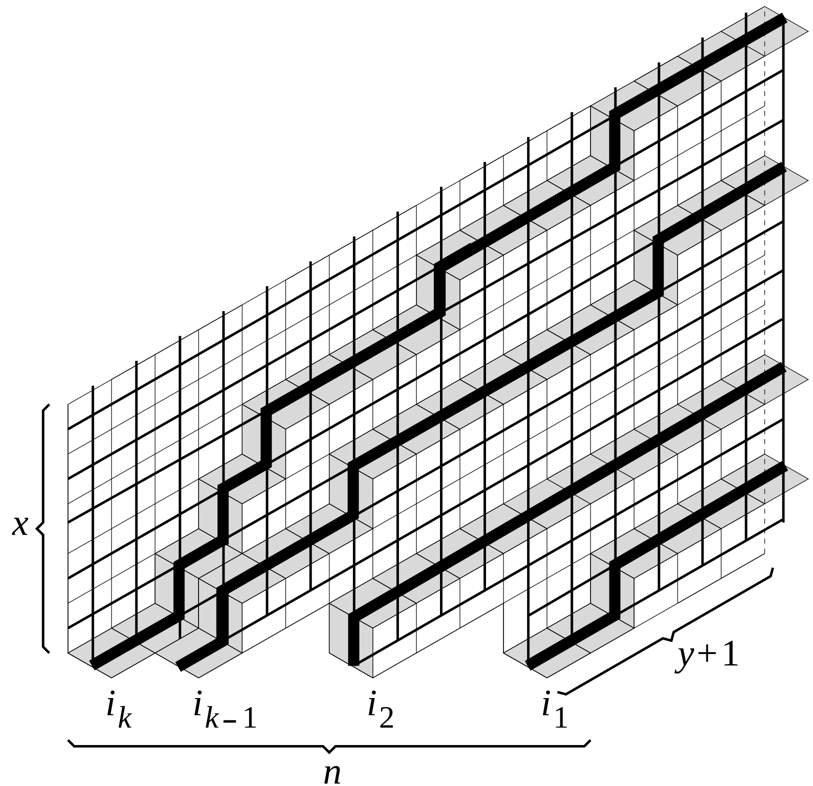


Tilings and paths

Starting and ending segments

The tilings are in bijection with non-intersecting families of paths of rhombi:

- starting points: fixed
- ending points: can vary among a specified set



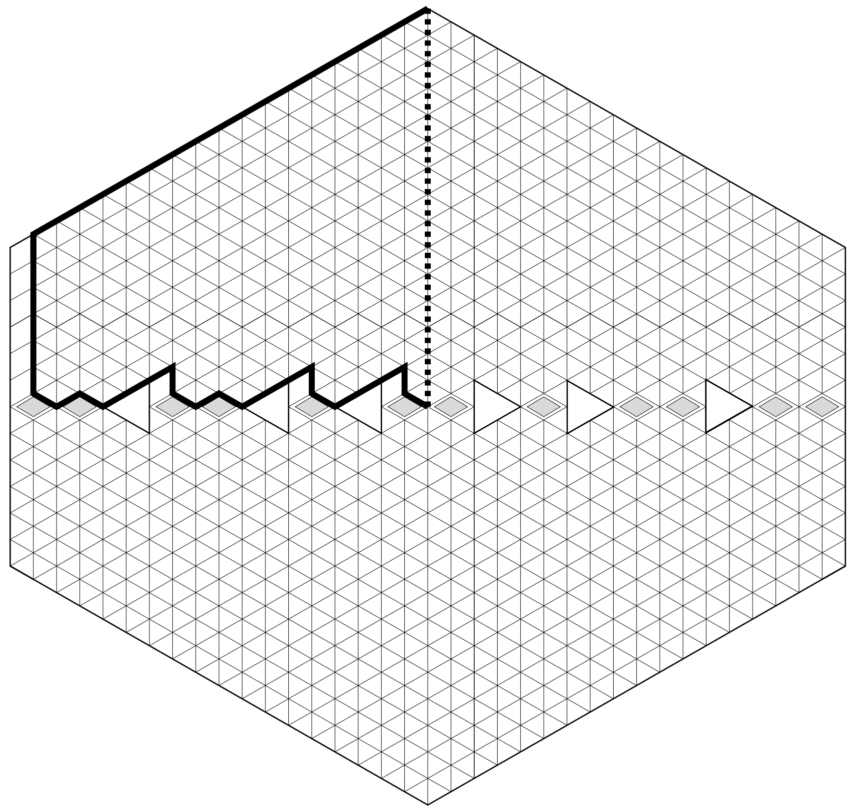
Regarding the paths of lozenges as lattice paths in $\mathbb{Z}^2$

A result of Stembridge expresses this as a Pfaffian.

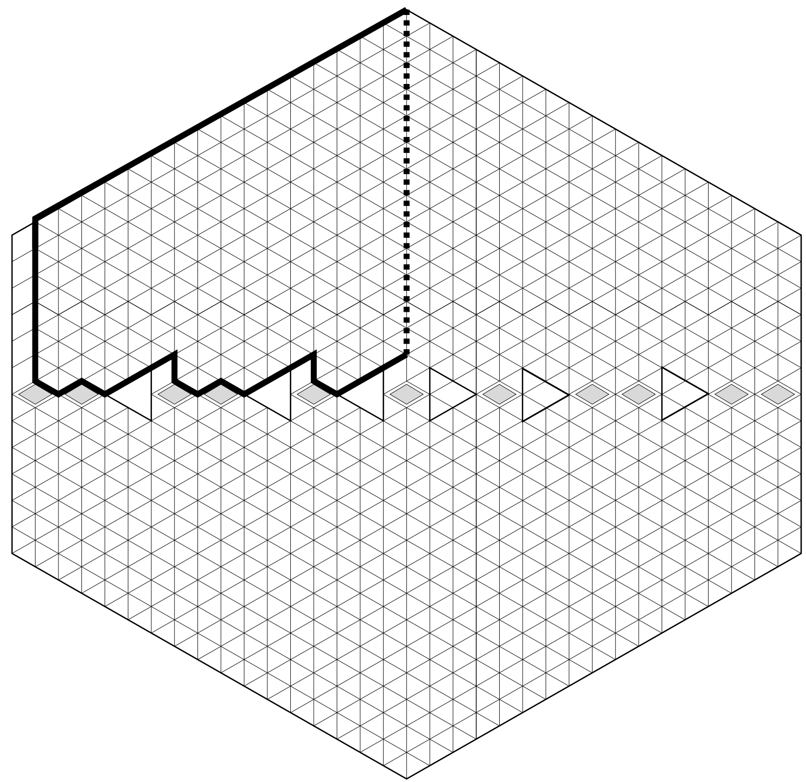
After using some combinatorial identities, this Pfaffian can be evaluated explicitly using Schur's Pfaffian Identity:

THEOREM (SCHUR'S PFAFFIAN IDENTITY). *Let n be even, and let $x_1, \ldots, x_n$ be indeterminates. Then we have*

$$\operatorname{Pf}\left[\frac{x_j - x_i}{x_j + x_i}\right]_{i,j=1}^{n} = \prod_{1 \leq i < j \leq n} \frac{x_j - x_i}{x_j + x_i}.$$



Generalization of SSC plane partitions, even by even by even case.



Generalization of SSC plane partitions, even by odd by odd case.

Corollary (Generalization of SSC plane partitions). *Let $n, x \geq 0$ and $1 \leq k_1 < \cdots < k_s \leq n$ be integers. If $k_1 > 1$ set $t = 0$, otherwise define t by requiring $k_i - i = 0$, $i = 1, \dots, t$, and $k_{t+1} - (t+1) > 0$. Let $\{1, \dots, n\} \setminus \{k_1, \dots, k_s\} = \{i_1, \dots, i_{n-s}\}$.*
Then we have:
(a).

$$\mathrm{M}_{-,|}(H_{2n,2n,2x}(k_1, \dots, k_s)) = \mathrm{M}_f(D_{n,x,2t-1}^{i_1,\dots,i_{n-s}})$$

$$= \prod_{a=1}^{n-s} \binom{x + 2t + n + i_a - 1}{2t + 2i_a - 1} \prod_{1 \leq a < b \leq n-s} \frac{i_b - i_a}{2t + i_a + i_b - 1}.$$

(b).

$$\mathrm{M}_{-,|}(H_{2n+1,2n+1,2x}(k_1, \dots, k_s)) = \mathrm{M}_f(D_{n,x,2t}^{i_1,\dots,i_{n-s}})$$

$$= \prod_{a=1}^{n-s} \binom{x + 2t + n + i_a}{2t + 2i_a} \prod_{1 \leq a < b \leq n-s} \frac{i_b - i_a}{2t + i_a + i_b}.$$

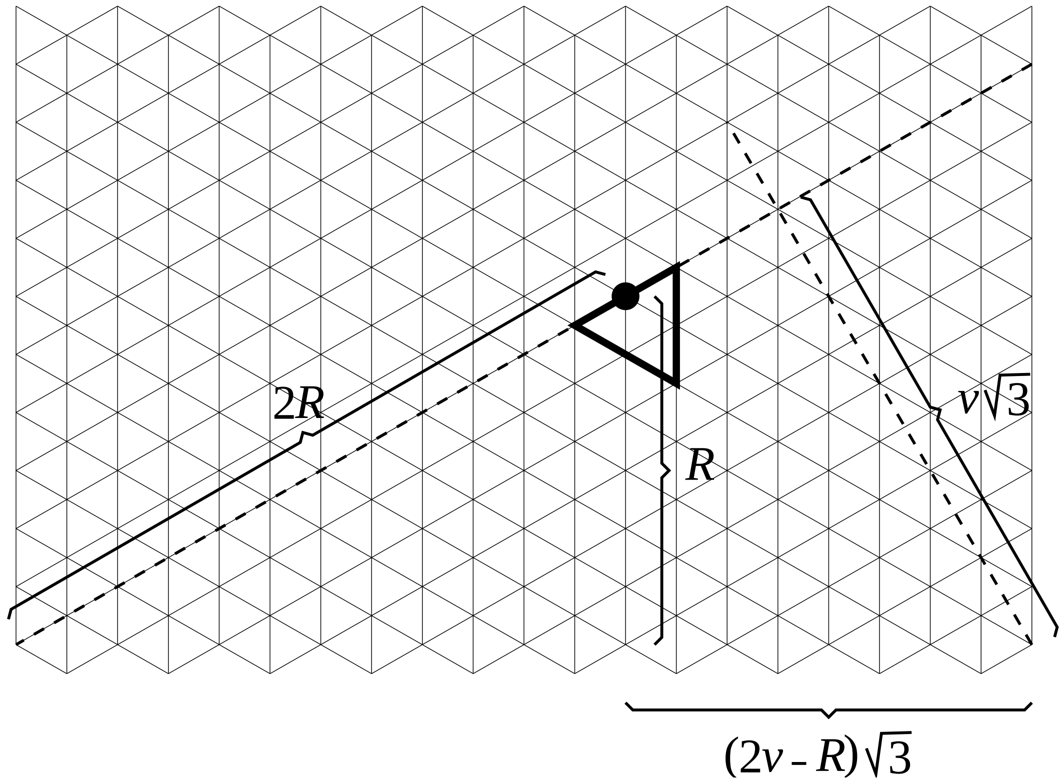
A limit formula for regions with two dents

PROPOSITION. *For any fixed integers* $1 \le i < j$, *we have*

$$\lim_{n\to\infty} \frac{\mathrm{M}_f\left(D_{n,n,0}^{[n]\setminus\{i,j\}}\right)}{\mathrm{M}_f\left(D_{n,n,0}^{[n]\setminus\{1,2\}}\right)} = 4\,\frac{j-i}{j+i}\,\frac{1}{2^{2i-2}}\binom{2i-1}{i-1}\frac{1}{2^{2j-2}}\binom{2j-1}{j-1}.$$

To finish the proof:

- a double sum formula
- its asymptotic analysis



Changing from (α, β) to (R, v)-coordinates.

A double sum formula

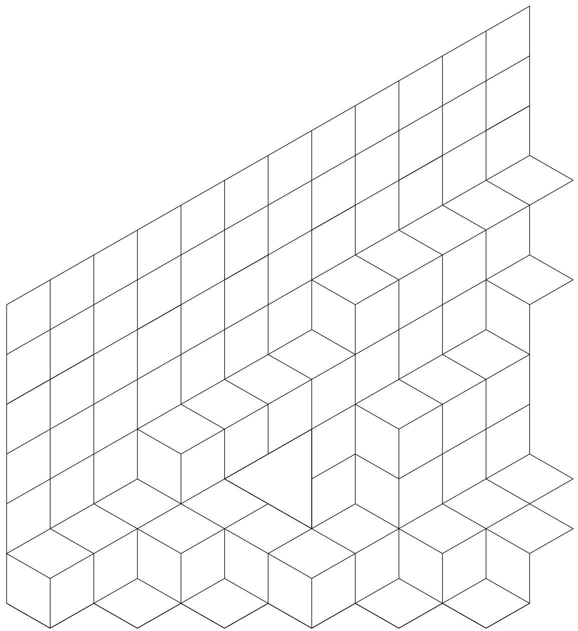
LEMMA. *Write* $\alpha = 2v - R$, $\beta = R$, *with* R *and* v *non-negative integers. Then we have*

$$\omega_c(\alpha, \beta) = \omega_c(2v - R, R)$$

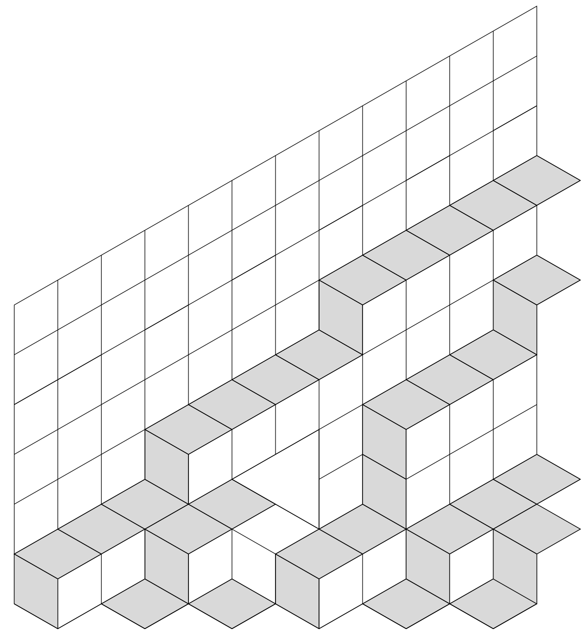
$$= 4R \left| \sum_{a=0}^{R} \sum_{b=0}^{R} (-1)^{a+b} \frac{(R+a-1)!\,(R+b-1)!}{(2a)!\,(R-a)!\,(2b)!\,(R-b)!} \right.$$

$$\left. \times \frac{(2v'+2a+1)!\,(2v'+2b+1)!}{2^{2(2v'+a+b)}(v'+a)!\,(v'+a+1)!\,(v'+b)!\,(v'+b+1)!} \quad \frac{(b-a)^2}{2v'+a+b+2} \right|,$$

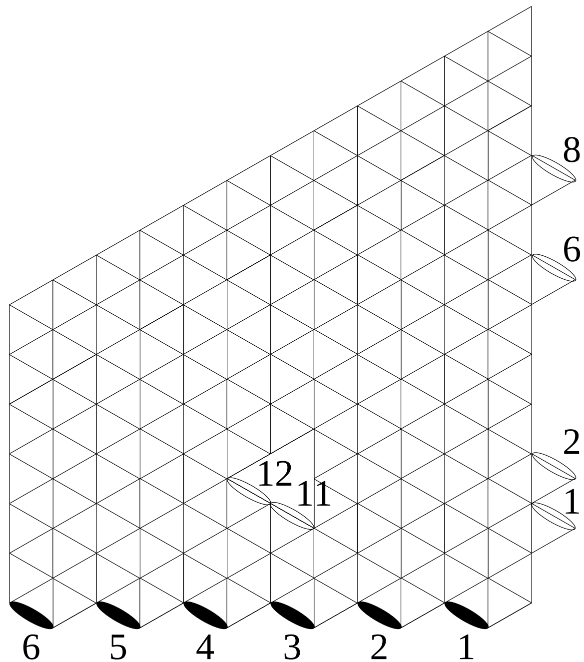
where $v' = 2v - R - 1$.



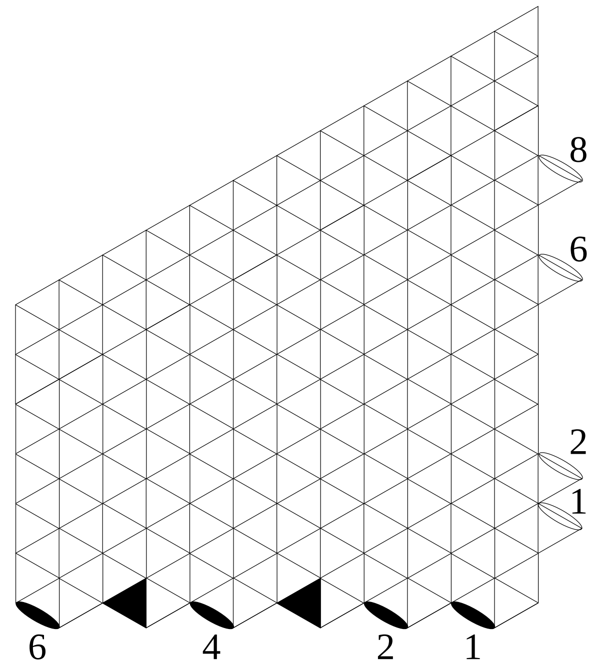
$D_{6,6,0}(3,3;\{1,2,6,8\})$



Paths of lozenges



Labeling starting and ending points



$D^{1,2,4,6}_{6,6,0}(\{1,2,6,8\})$

Outline of proof of double sum formula

- Free boundary is sum over constrained boundaries:

$$\mathrm{M}_f(D_{n,n,0}(\alpha,\beta)) = \sum_{\substack{S\subset T\\ |S|=n-2}} \mathrm{M}(D_{n,n,0}(\alpha,\beta;S))$$

- Use Pfaffian formula for lattice paths and Laplace expansion to get

$$\mathrm{M}(D_{n,n,0}(\alpha,\beta;S)) =$$
$$\left| \sum_{0\le a<b\le R} (-1)^{a+b} \frac{(b-a)(R+a-1)!\,(R+b-1)!}{(2a)!\,(R-a)!\,(2b)!\,(R-b)!}\,\mathrm{M}(D_{n,n,0}^{[n]\setminus\{2v-R+a,2v-R+b\}}(S)) \right|$$

- Sum over boundaries to get

$$\mathrm{M}_f(D_{n,n,0}(\alpha,\beta)) =$$
$$2R \left| \sum_{0\le a<b\le R} (-1)^{a+b} \frac{(b-a)(R+a-1)!\,(R+b-1)!}{(2a)!\,(R-a)!\,(2b)!\,(R-b)!} \mathrm{M}_f(D_{n,n,0}^{[n]\setminus\{2v-R+a,2v-R+b\}}) \right|$$

- divide by $\mathrm{M}_f(D_{n,n,0}(1,1))$, let $n \to \infty$, and use 2-dent limit formula

Reduction of the double sum to simple sums

- The double sum separates if we write

$$\frac{1}{2v' + a + b + 2} = \int_0^1 x^{2v'+a+b+1}\, dx$$

- Moment sums ($k \in \mathbb{Z}$, $x \in [0, 1]$):

$$T^{(k)}(R, v; x) := \frac{1}{R} \sum_{a=0}^{R} \frac{(-R)_a (R)_a (3/2)_{v+a}}{(1)_a (1/2)_a (2)_{v+a}} \left(\frac{x}{4}\right)^a a^k$$

LEMMA. *We have that*

$$\omega_c(2v-R,R)=$$

$$8R\left|\int_0^1 T^{(2)}(R,v';x)T^{(0)}(R,v';x)x^{2v'+1}dx-\int_0^1\left(T^{(1)}(R,v';x)\right)^2 x^{2v'+1}dx\right|,$$

where $v'=2v-R-1$.

The asymptotics of the integrals in the lemma

It follows from results in [C, Mem. AMS, 2005] that:

$$\int_0^1 T^{(2)}(R,v';x)T^{(0)}(R,v';x)x^{2v'+1}dx$$

$$\sim \frac{2}{\pi R}\int_0^1 x^{2qR}\frac{1}{(4-x)\sqrt{q^2+\frac{x}{4-x}}}\cos\left[2R\arccos\left(1-\frac{x}{2}\right)-\arctan\frac{1}{q}\sqrt{\frac{x}{4-x}}+\pi\right]dx$$

and

$$\int_0^1\left(T^{(1)}(R,v';x)\right)^2dx\sim$$

$$\frac{2}{\pi R}\int_0^1 x^{2qR}\frac{1}{(4-x)\sqrt{q^2+\frac{x}{4-x}}}\left\{1+\cos\left[2R\arccos\left(1-\frac{x}{2}\right)-\arctan\frac{1}{q}\sqrt{\frac{x}{4-x}}+\pi\right]\right\}dx$$

Lemma then implies

$$\omega_c(2v - R, R) \sim \frac{16}{\pi} \left| \int_0^1 x^{2qR} \frac{1}{(4-x)\sqrt{q^2 + \frac{x}{4-x}}} dx \right|$$

as R and v approach infinity so that $2v - R = qR$.

We have

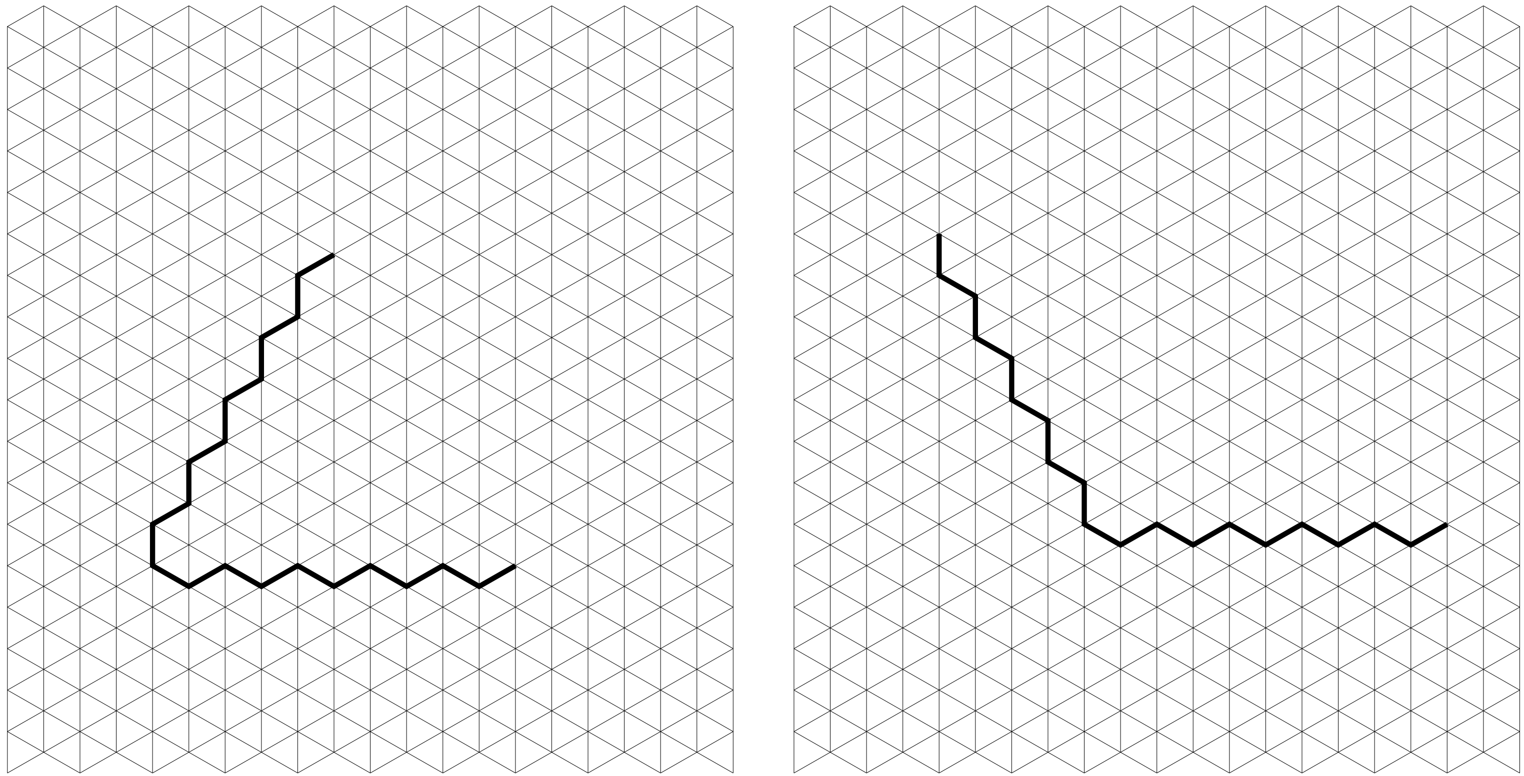
$$\int_0^1 x^{2qR} \frac{1}{(4-x)\sqrt{q^2+\frac{x}{4-x}}} dx \sim \frac{1}{3q\sqrt{q^2+\frac{1}{3}}} \frac{1}{R}, \qquad R \to \infty$$

Then we get

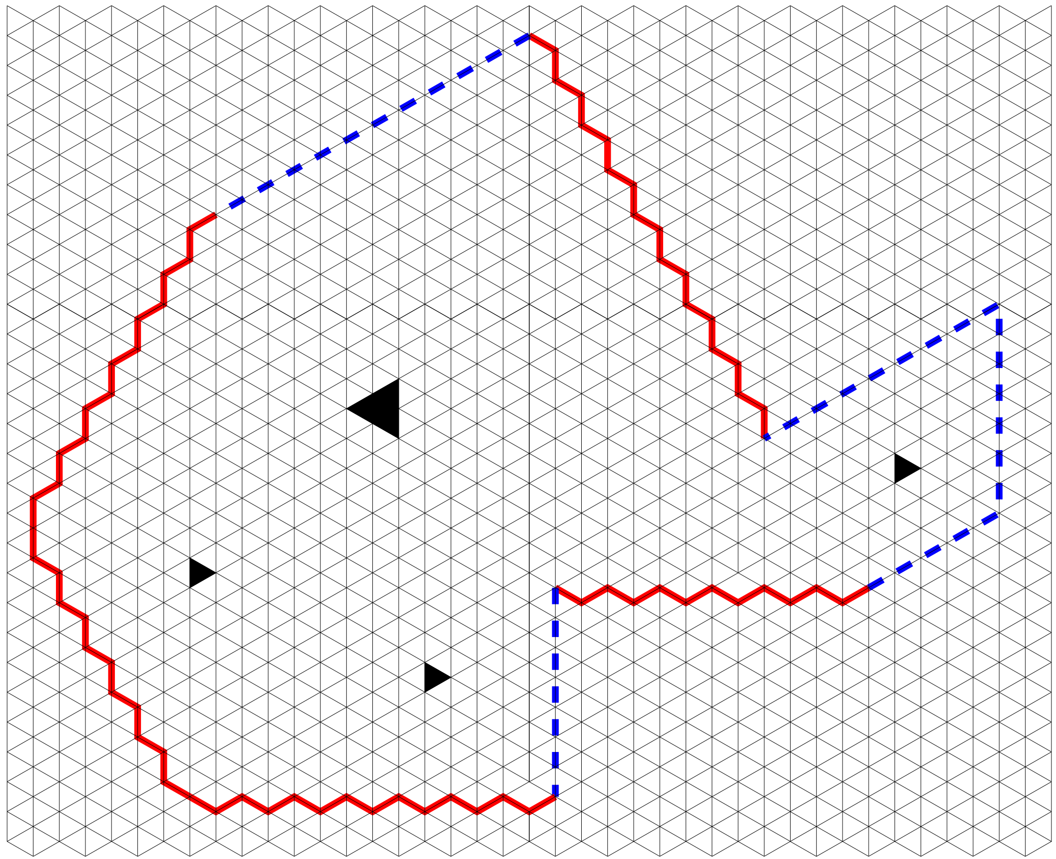
$$\omega_c(2v-R,R) \sim \frac{16}{3\pi q\sqrt{q^2+\frac{1}{3}}} \frac{1}{R},$$

which proves the Theorem.

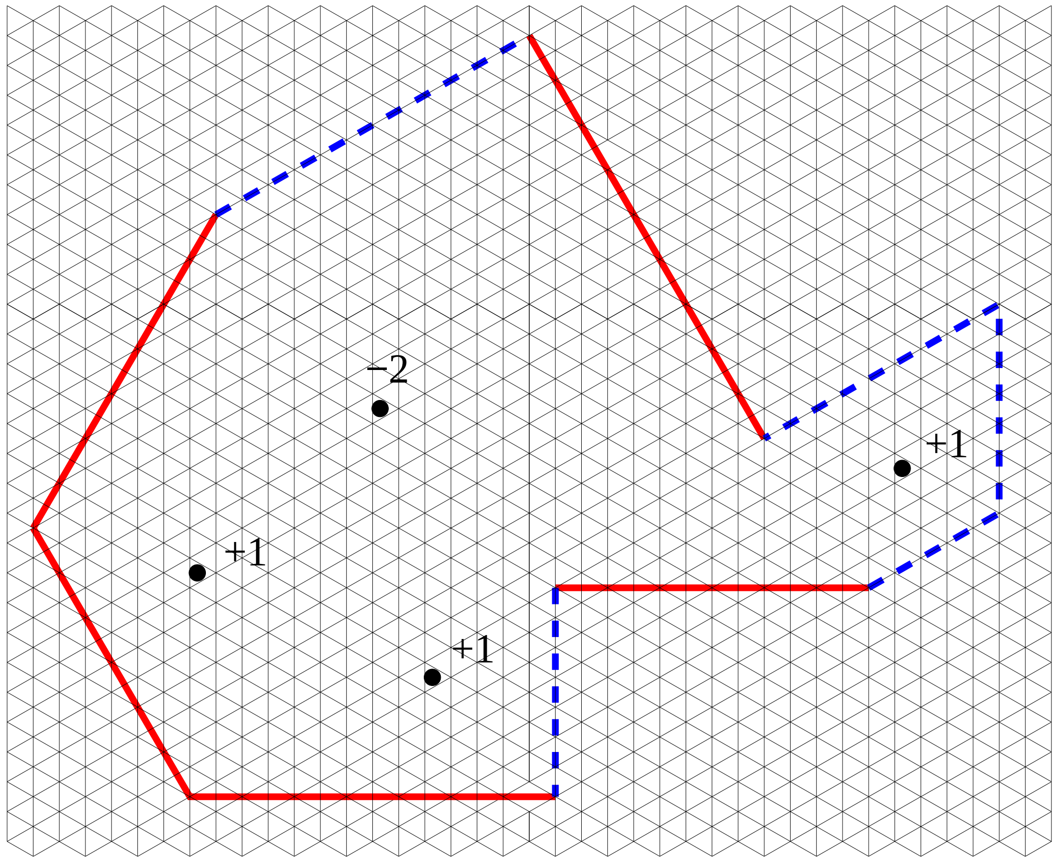
A general conjecture for regions Ω_n on the triangular lattice



The two types of zig-zag corners in Ω_n



An example of Ω_n



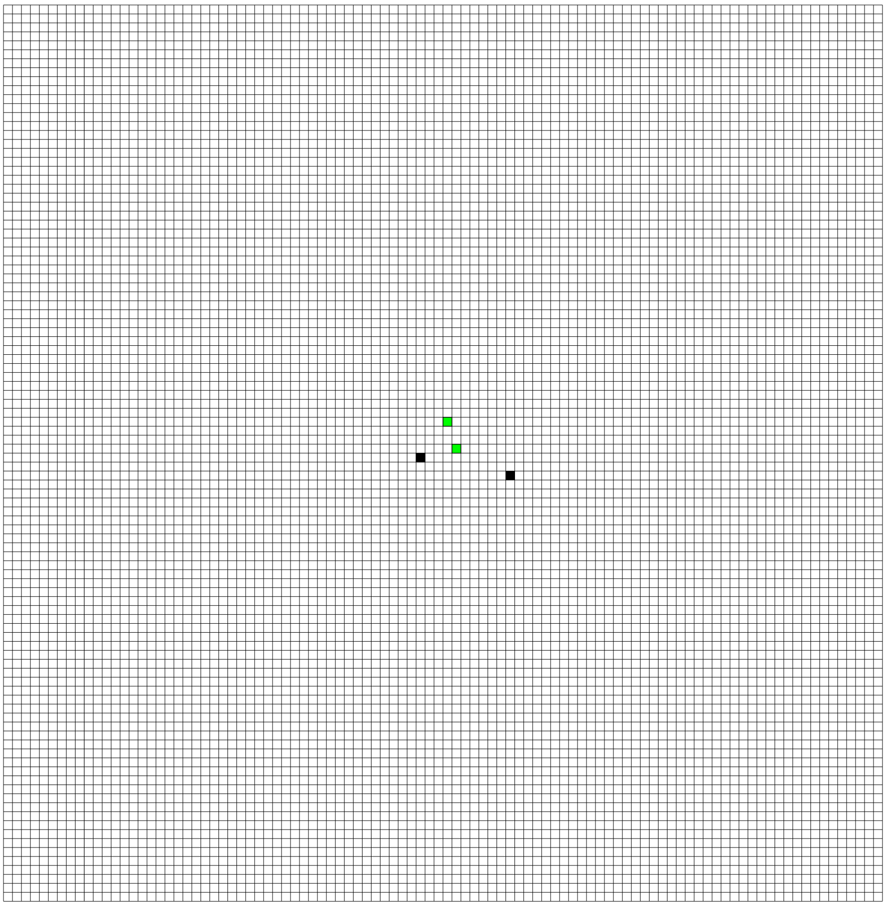
The corresponding steady state heat flow problem

- $O_1^{(n)}, \ldots, O_k^{(n)}$: finite unions of unit triangles from the interior of Ω_n (the gaps)
- for fixed i, $O_i^{(n)}$'s are translates of one another for all $n \geq 1$
- $O_i^{(n)}$ shrinks to point $a_i \in \Omega$ in scaling limit, $i = 1, \ldots, k$
- $\Omega_n \to \Omega$, $n \to \infty$
- E: heat energy when sources/sinks are at positions $a_1, \ldots, a_k$

Conjecture. *Let $O_i'^{(n)}$'s be translations of the $O_i^{(n)}$'s that shrink to distinct points $a_1', \ldots, a_k' \in \Omega$ in the scaling limit as $n \to \infty$. Then*

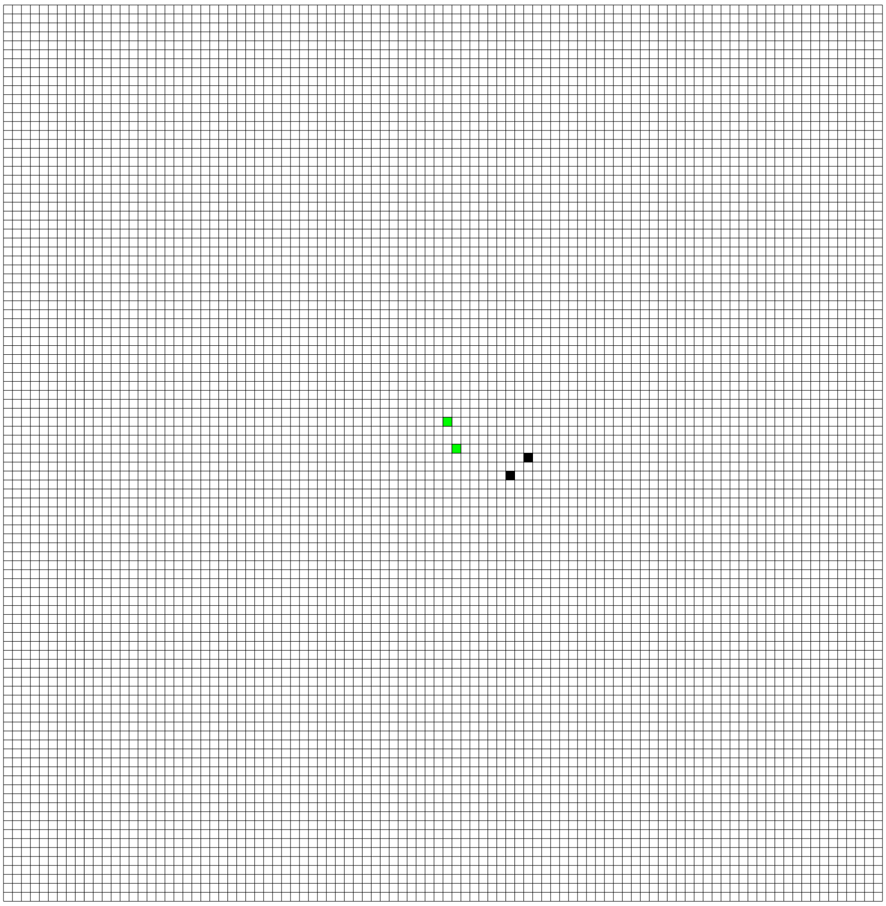
$$\frac{\mathrm{M}_f(\Omega_n \setminus O_1^{(n)} \cup \cdots \cup O_k^{(n)})}{\mathrm{M}_f(\Omega_n \setminus O_1'^{(n)} \cup \cdots \cup O_k'^{(n)})} \to \frac{\exp(-E)}{\exp(-E')},$$

where E' is the heat energy of the system obtained from S by moving the point heat sources to positions $a_1', \ldots, a_k'$.



A different direction: What if there are some defects in the tiling?

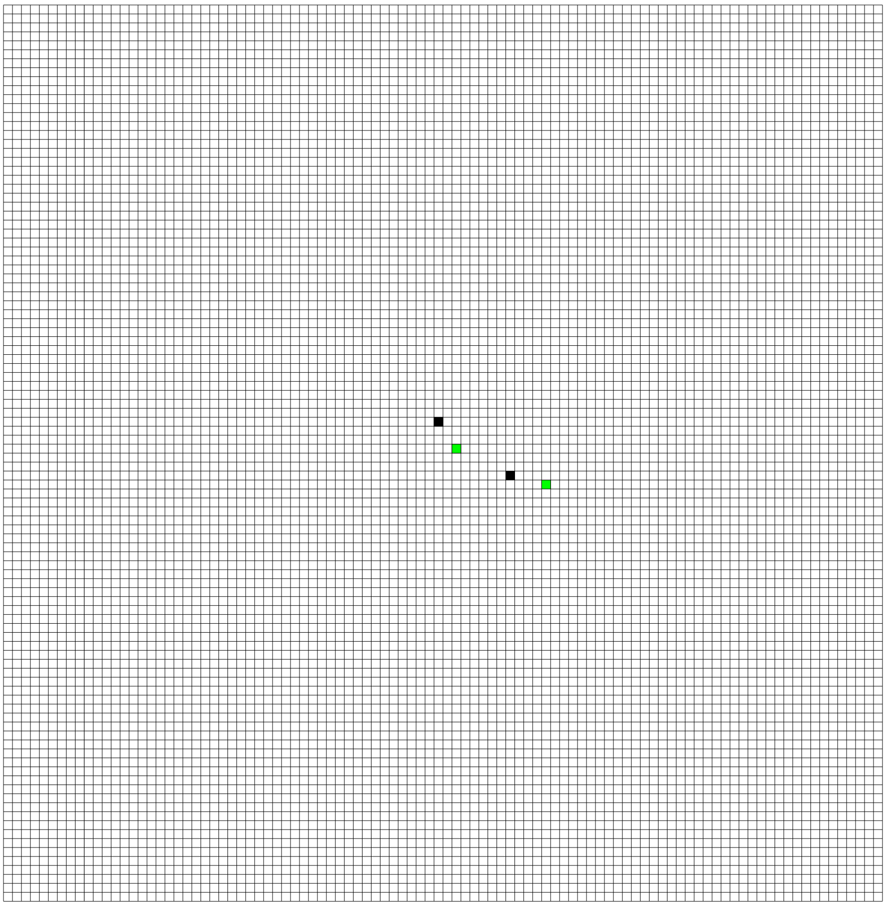
n_1=19950199996272315661984770563327145976410678066980942975683931965311961844841748608 3
894523528938454080130179249052794976495010594583960473633378239630712527177835 74
418948214177826916822854314194231955889171739907870172321370819692895251337426 39
981171648614524528019243829829571696546743169496377516093913926210047722886797 92
312792066862139707477334804155334433959899514513833374468070652065302156579613 63
056299106161865462090931894895242253477760646952434807070851296477273303570422 62
471232260654917105089700091813618745663999127453764792350777474589022836964454 73
180730777925391510598641117457547319998653742583779202474129891997820245928424 68
937002277850500852009559446559779474711160176031826435896998510098408151547481 35
637344467568167386030904920146550834713772785057482560956825400553003654070868 59
679279119228632974967511492436836926183243197013240033185472034719061994362646 65
438277561701005395075267947181216251536248670917247614664641917728407750249657 90
905223489047450915462148258169774729998114751936522243035521980598743458259782 14
949052955637474239241786949506912149365932144650473561487266626186834613559888 14
970887578122197145505105654378454095419217180040086740397224762806809233907947 03
47308994664374841278891412746224733990174274355 2 domino tilings



The holes moved to a different position

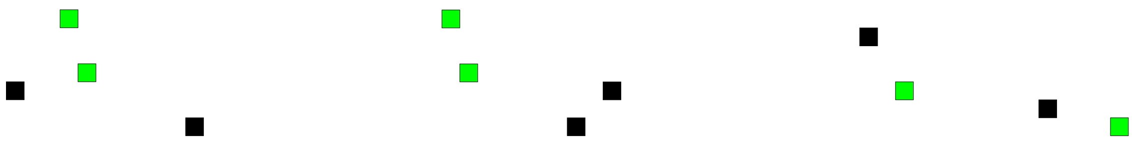
n_2=5205492024417980488854179994142348180886327072853128619326282956453535788126824625931875628414565383621414301924185275972851051263890553022346526859330240006786120213870397299173993363436759721564274097483402070880481993908828204061386307105435346309451288766338082787399904372806774917060878165779618062770859856294614766645993880305949647717709330793442745963550945742830210669480068857635439407442560275374395223493964493217637547049753318099315869405033700198794488936883061045742417066270748259896438660069959463639598577141838378030713697337762572338658629160881385134614636474144871405082783173246475238876058027087683789921293376636199119800552381575126815305675028715445223883053682439701436472239673319756088218325401486464284614156288580161330344628996801309757572660714967592626421619005005649414387568791362555468874376792306462409313857510629513710056706103355244844131389255384417384881758192273277596033437002136930550007100179029382176468566625529051569520281601847006266363352607769238235097446223109785991978994711296995720327587548197454916705362986734279273862748525944387643052732361487835288584233745443148176753511765104001716551222665889939180355892064762913891671449441534533272214224084563471399791997678084847727505949327736

domino tilings



A third position of the holes

n_3=358203230765959830199103574716365260175191383221678328992841789799425190001871904839
677813660657201378998511563122623973499482005079533523657403644823809420538959l6
7007390250535580374934513627032653670169785708999014514331911334980243463776861 7
969996564809446932208084847341633135132080509823292047455096612696356799381307 02
78019746981546228653379296465542450831987230523243236695713261677727398729711554
11158924797111834992719284069258569098893857042302895310152128758309321273297680
32746559409827558311094622516179613076341557302618259925415216499697080936418592
14232641344332079896830575529516640606985536581039260737502720933297649617852436
47637133925483392211694802644221978636100310774728662362030650488654342950071317
31797510275315697800016982244909540078183935567203334486630855687769131315205838
49884332623609638943028842478313738986824792997322114122583856182196514988167521
84053499545624097379396586199158686128049762799144185450318939186951940318372200
05244861351503878700025440512656803029292119466596099544923749805102445619425152
38843846735117761719460111692824048710120335344036607647816317713100574077940339
99749871363943126354908406155636699537379894930293305164271339204090476609636885
35268488135386029770958345159569617677367377 9200 domino tilings



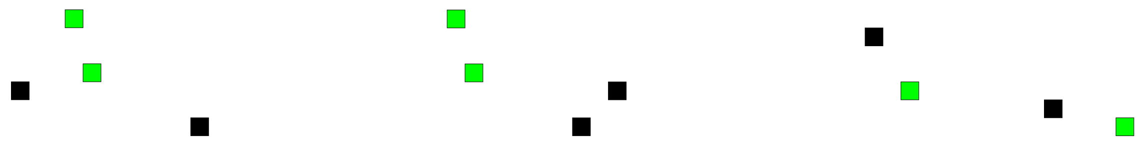
Relative sizes:

$$\frac{n_1}{n_2} = 3.832529164...$$

$$\frac{n_1}{n_3} = 0.556951983...$$

Can we understand *why* these ratios come out as they do?

Can we predict the ratios for other arrangements?



Positions of the green unit squares: $w_1, w_2 \in \mathbb{C}$

black unit squares: $b_1, b_2 \in \mathbb{C}$

Electrostatic Energy:

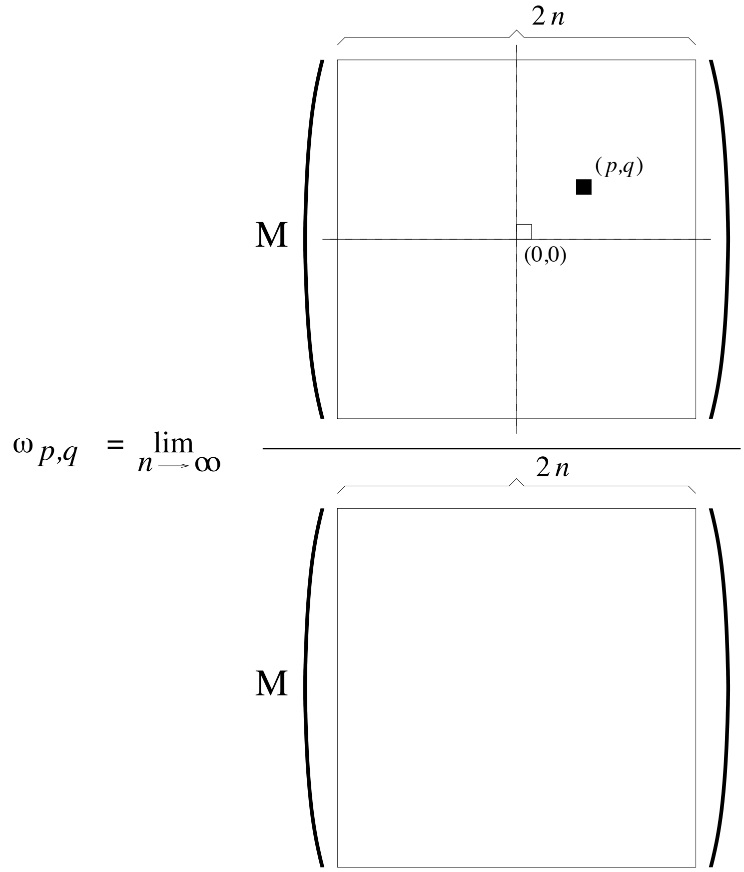
$$E(w_1, w_2, b_1, b_2) := \frac{\sqrt{|w_1 - w_2|}\sqrt{|b_1 - b_2|}}{\sqrt{|w_1 - b_1|}\sqrt{|w_1 - b_2|}\sqrt{|w_2 - b_1|}\sqrt{|w_2 - b_2|}}$$

Relative sizes of energies:

$$\frac{E_1}{E_2} = 3.726575104...$$

$$\frac{E_1}{E_3} = 0.570399674...$$

Fisher and Stephenson (1963): Correlation of monomers in a sea of dimers



Fisher and Stephenson conjectured (from exact data) that

$$\omega_{2p+1,0} \sim c\frac{1}{\sqrt{\mathrm{d}((2p+1,0),(0,0))}}$$

$$\omega_{p+1,p} \sim c\frac{1}{\sqrt{\mathrm{d}((p+1,p),(0,0))}}$$

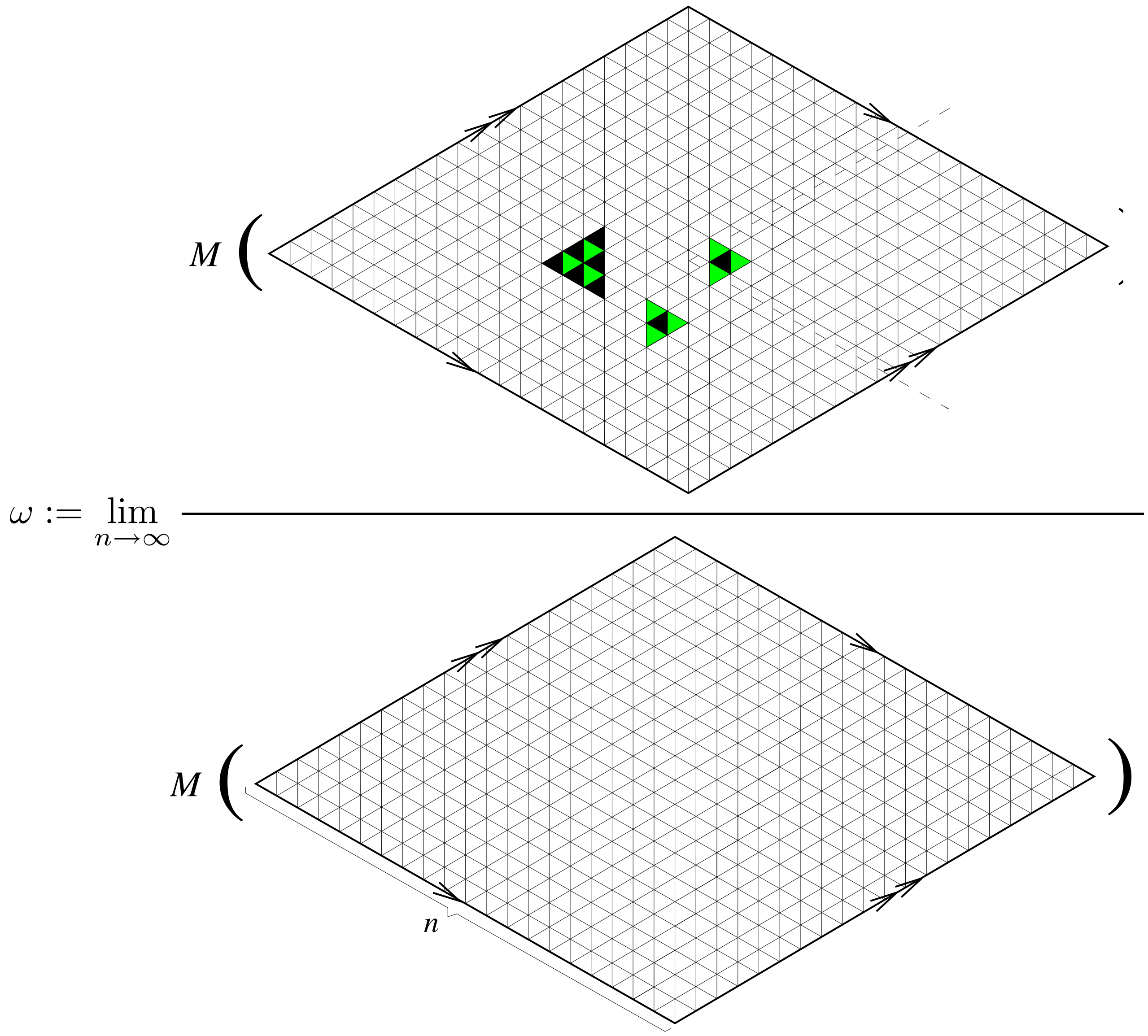
as $p \to \infty$, with *same* c.

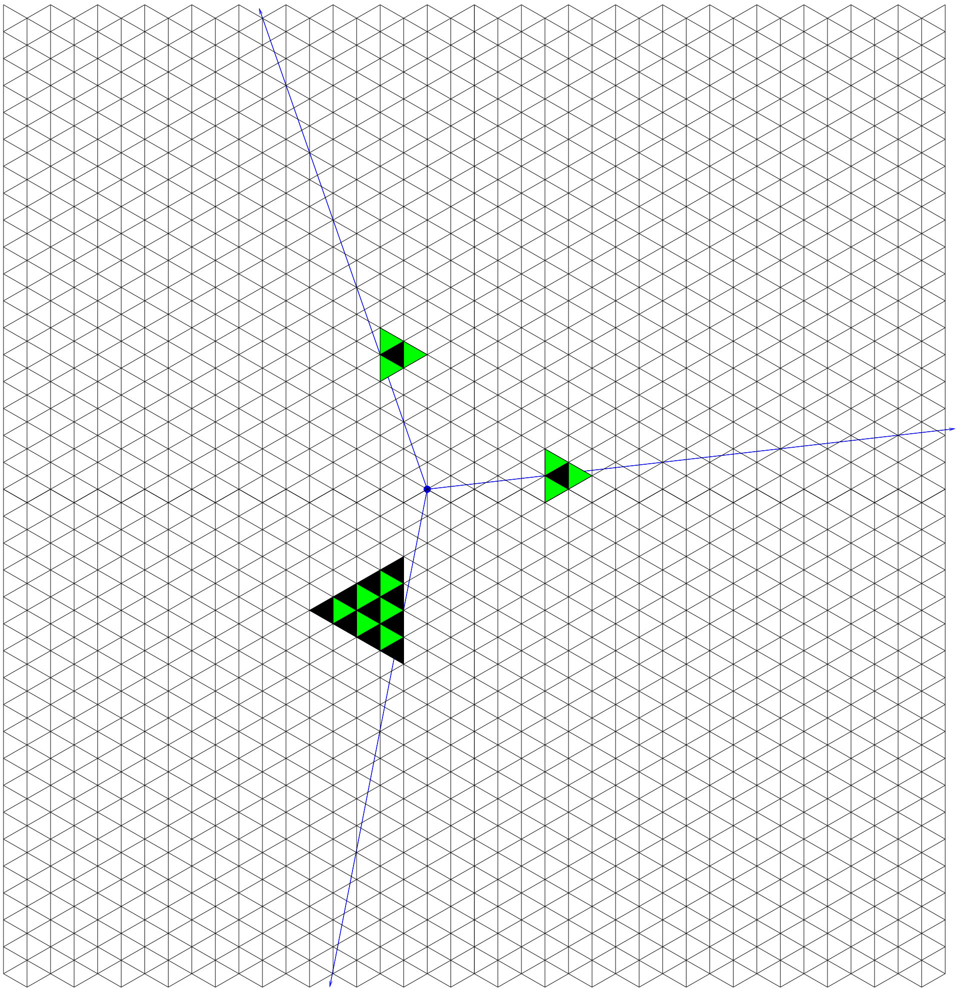
Based on this, they conjectured that $\omega_{p,q}$ is *rotationally invariant* for $p_k/q_k \to s$, $k \to \infty$, over all slopes s.

This still stands open.

Only proved direction is diagonal direction (Hartwig '66)

Correlation defined via tori

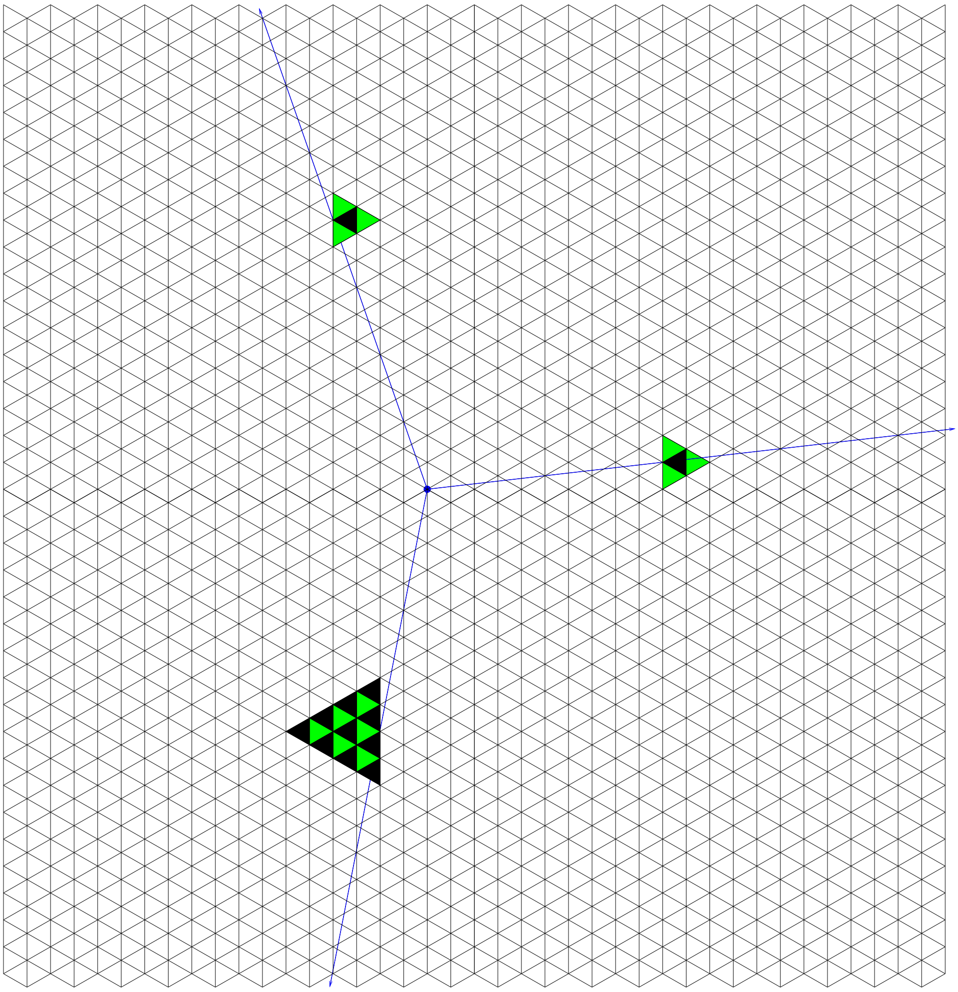




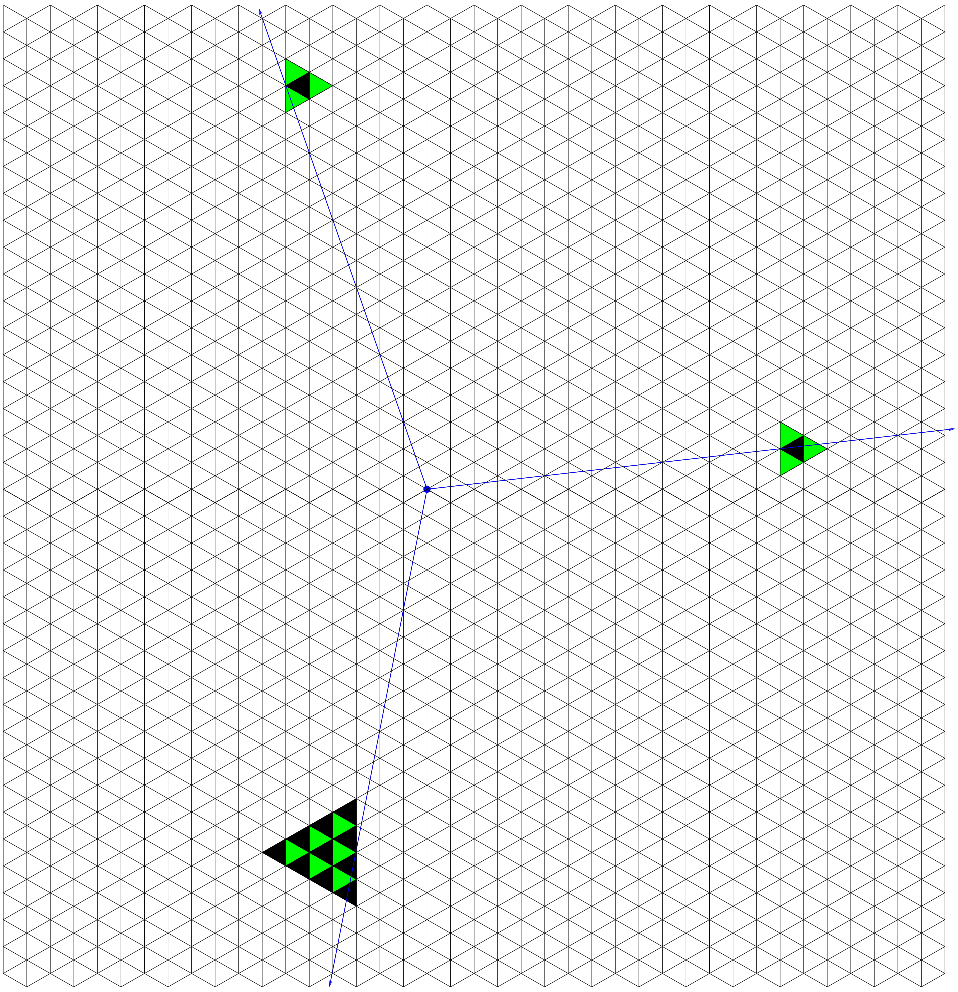
O_1, O_2, O_3

$\mathbf{r}$: position vector of O

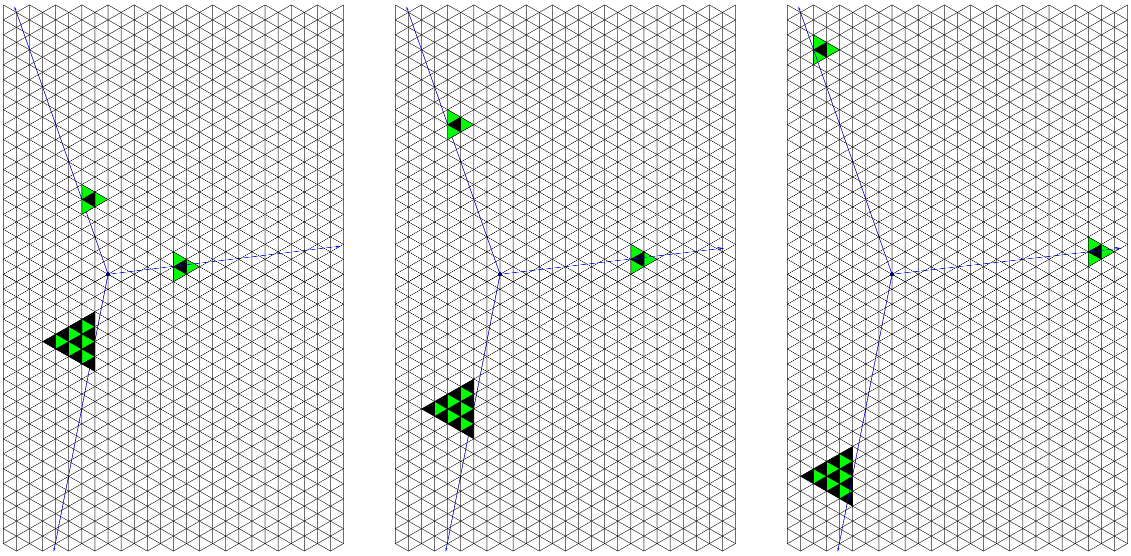
αO: translation of O whose position vector is $\alpha \mathbf{r}$



$2O_1$, $2O_2$, $2O_3$



$3O_1$, $3O_2$, $3O_3$



Charge of a hole:

$q(O) := \#$ (green monomers in O)$-\#$ (black monomers in O)

Electrostatic Energy:

$$E(O_1, \ldots, O_n) := \prod_{1 \leq i < j \leq n} \mathrm{d}(O_i, O_j)^{\frac{1}{2}\, \mathrm{q}(O_i)\, \mathrm{q}(O_j)}$$

Theorem (C., 2009). *Suppose O_i is either of type $\triangleright_{k_i}$ or of type $\triangleleft_{k_i}$, with k_i even, for $i = 1, \ldots, n$. Then*

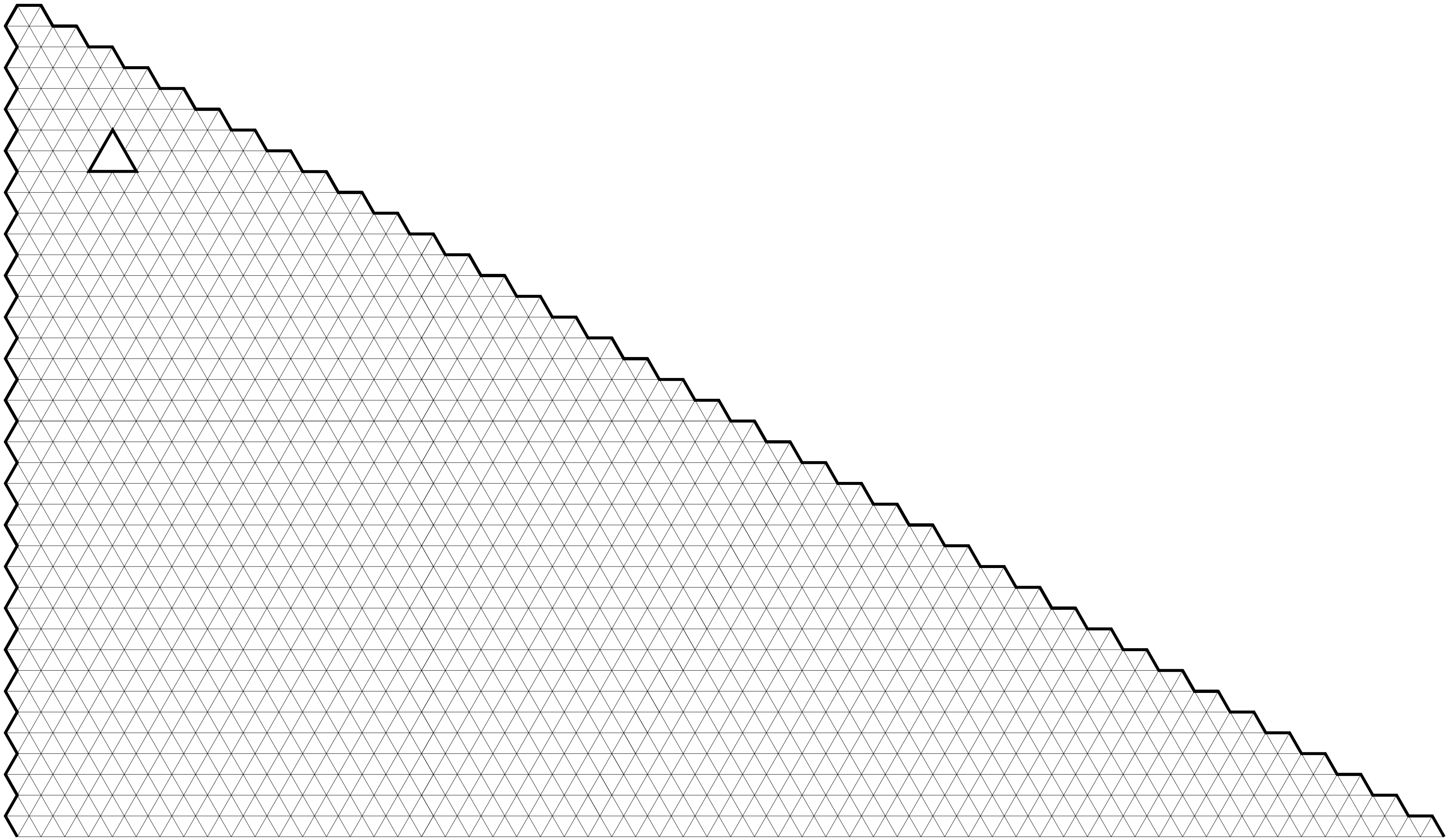
$$\omega(\alpha O_1 \ldots, \alpha O_n) \sim c E(\alpha O_1 \ldots, \alpha O_n), \qquad \alpha \to \infty.$$

Electrostatic Hypothesis

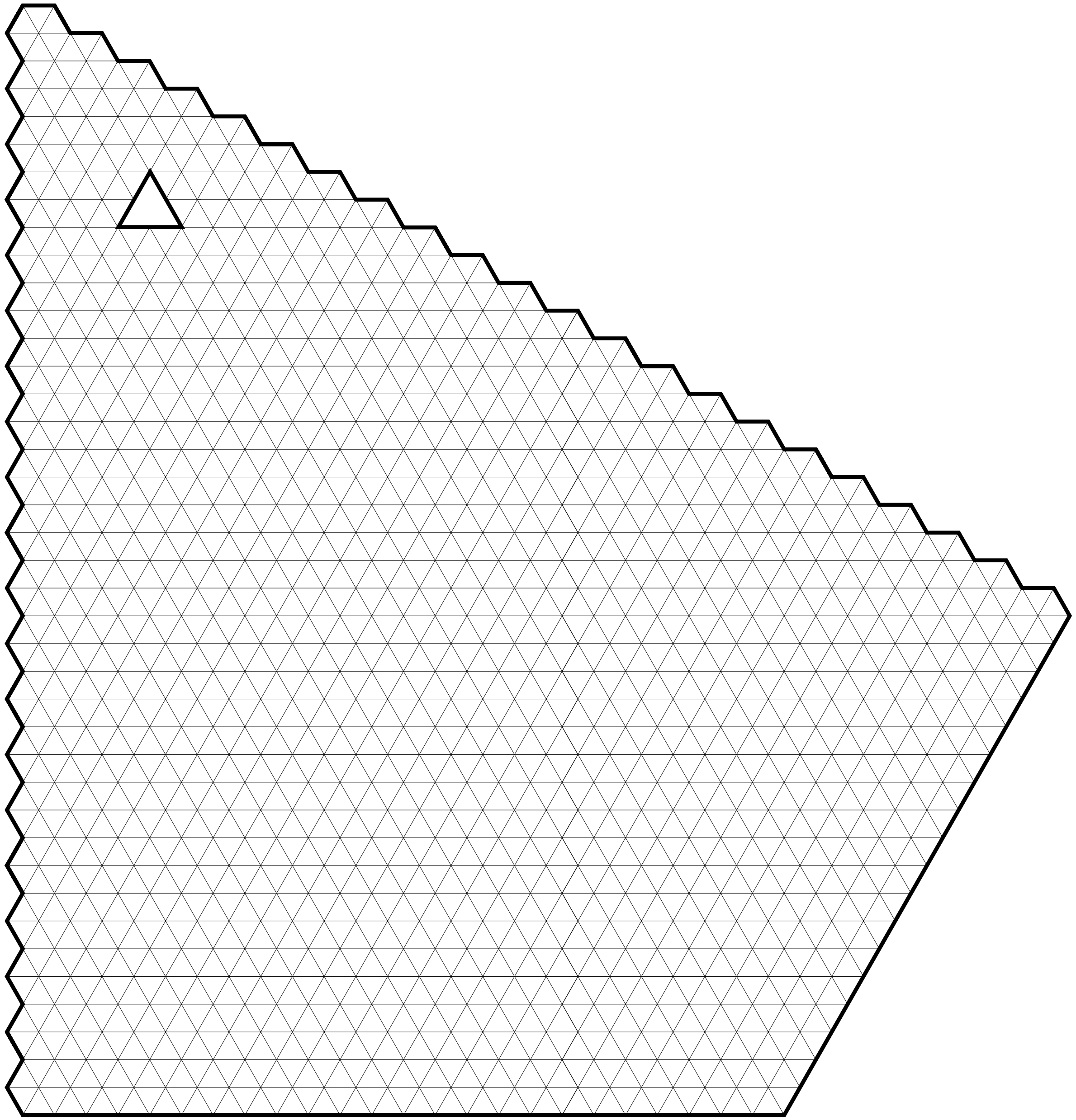
Conjecture. *For* **any** *holes* $O_1, \ldots, O_n$ *we have*

$$\omega(\alpha O_1 \ldots, \alpha O_n) \sim cE(\alpha O_1 \ldots, \alpha O_n), \qquad \alpha \to \infty.$$

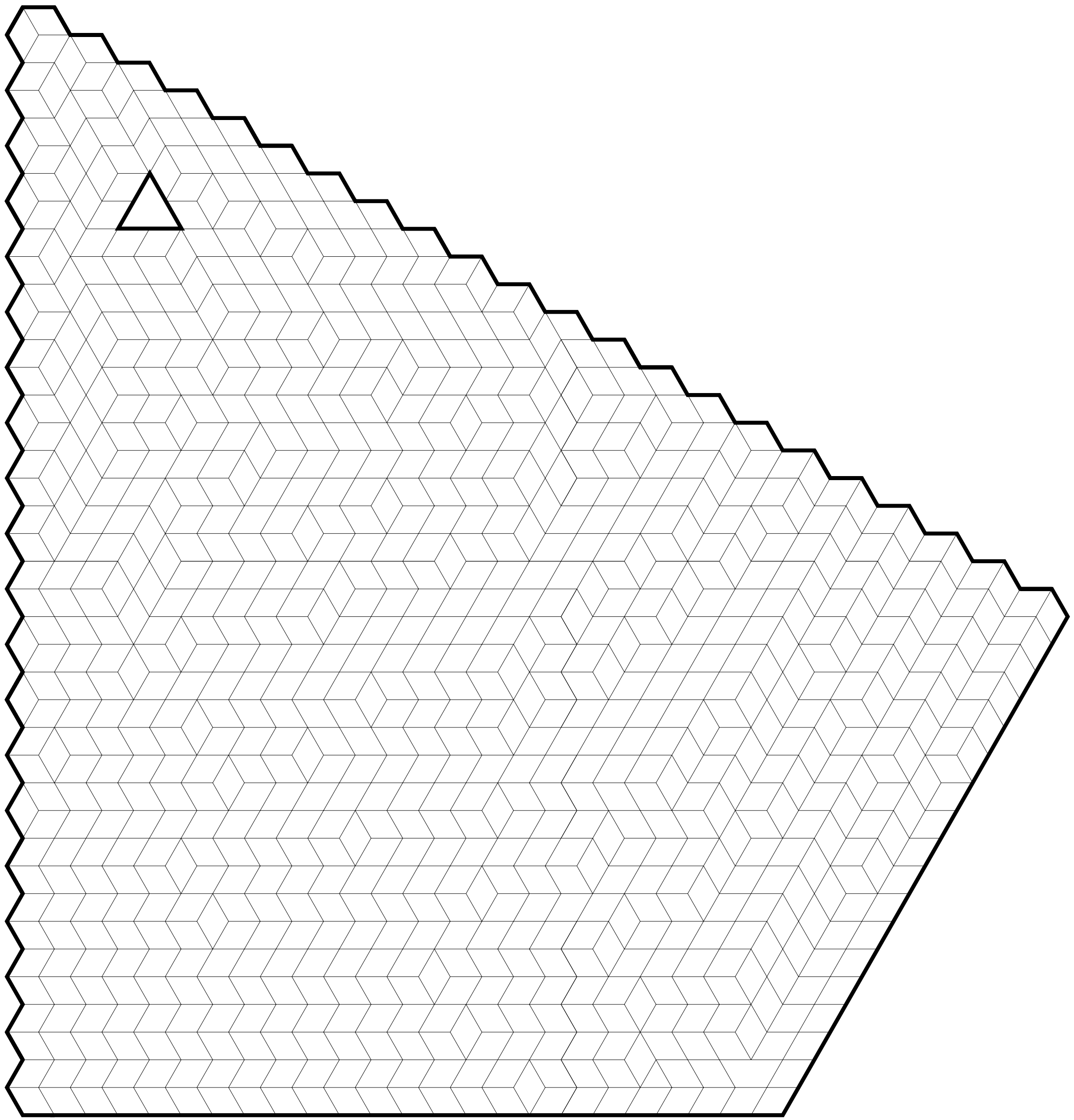
Remark. The square lattice analog of the above conjecture specializes in the case of a white and a black monomer to the original Fisher-Stephenson conjecture (the charges are $+1$ and -1).



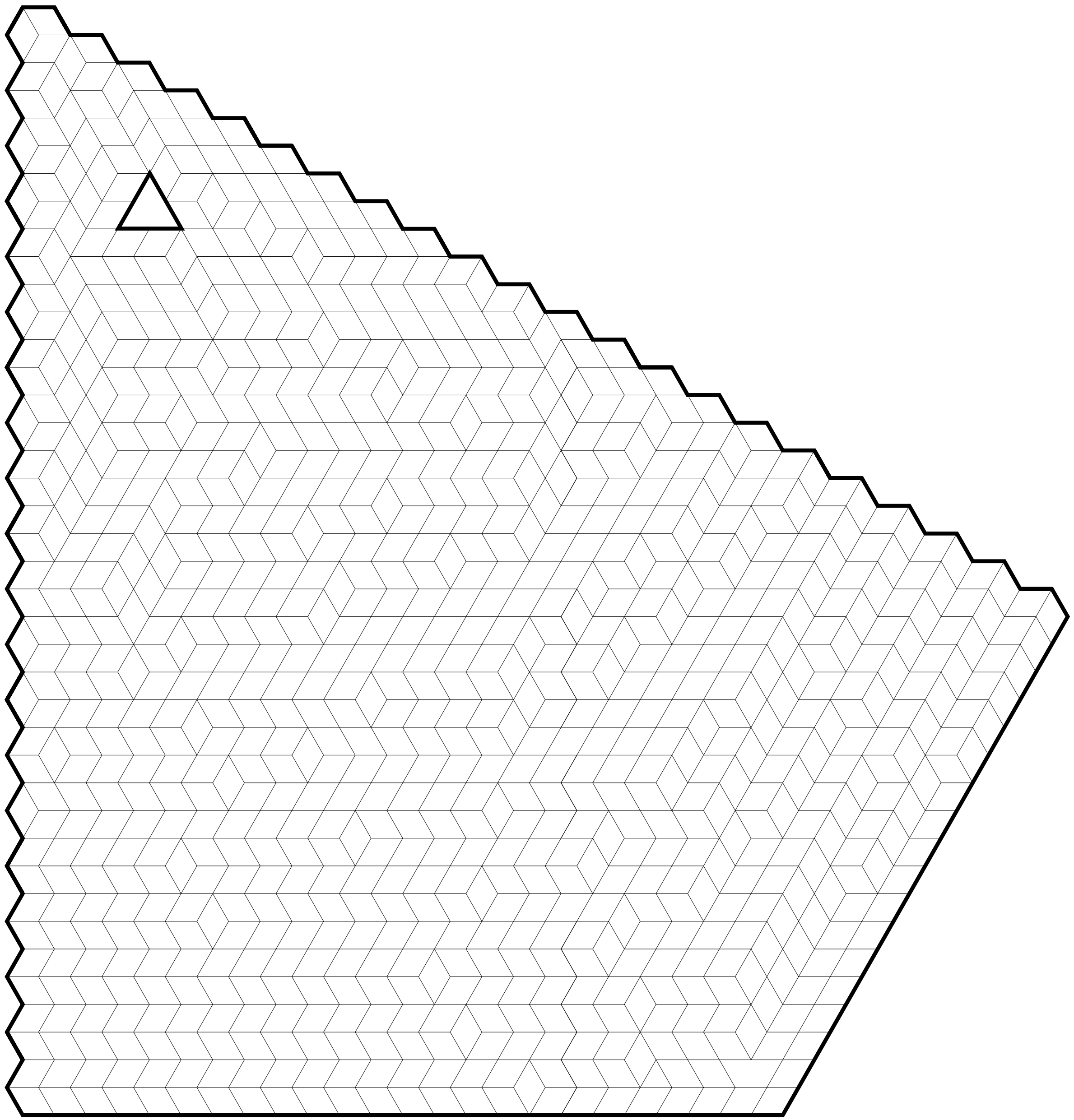
Want a natural interaction of gap with angle



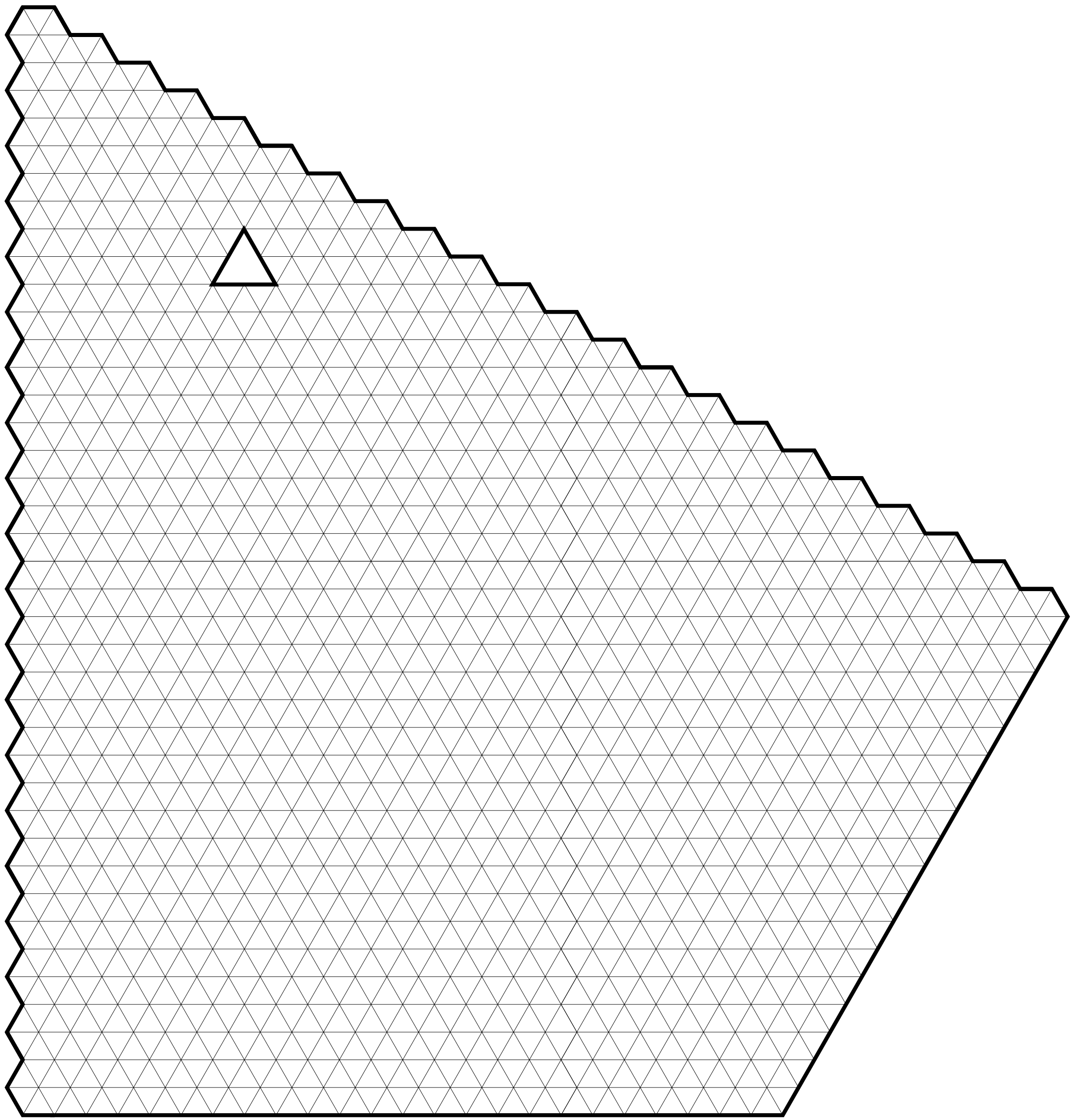
Enclose gap in a large finite region



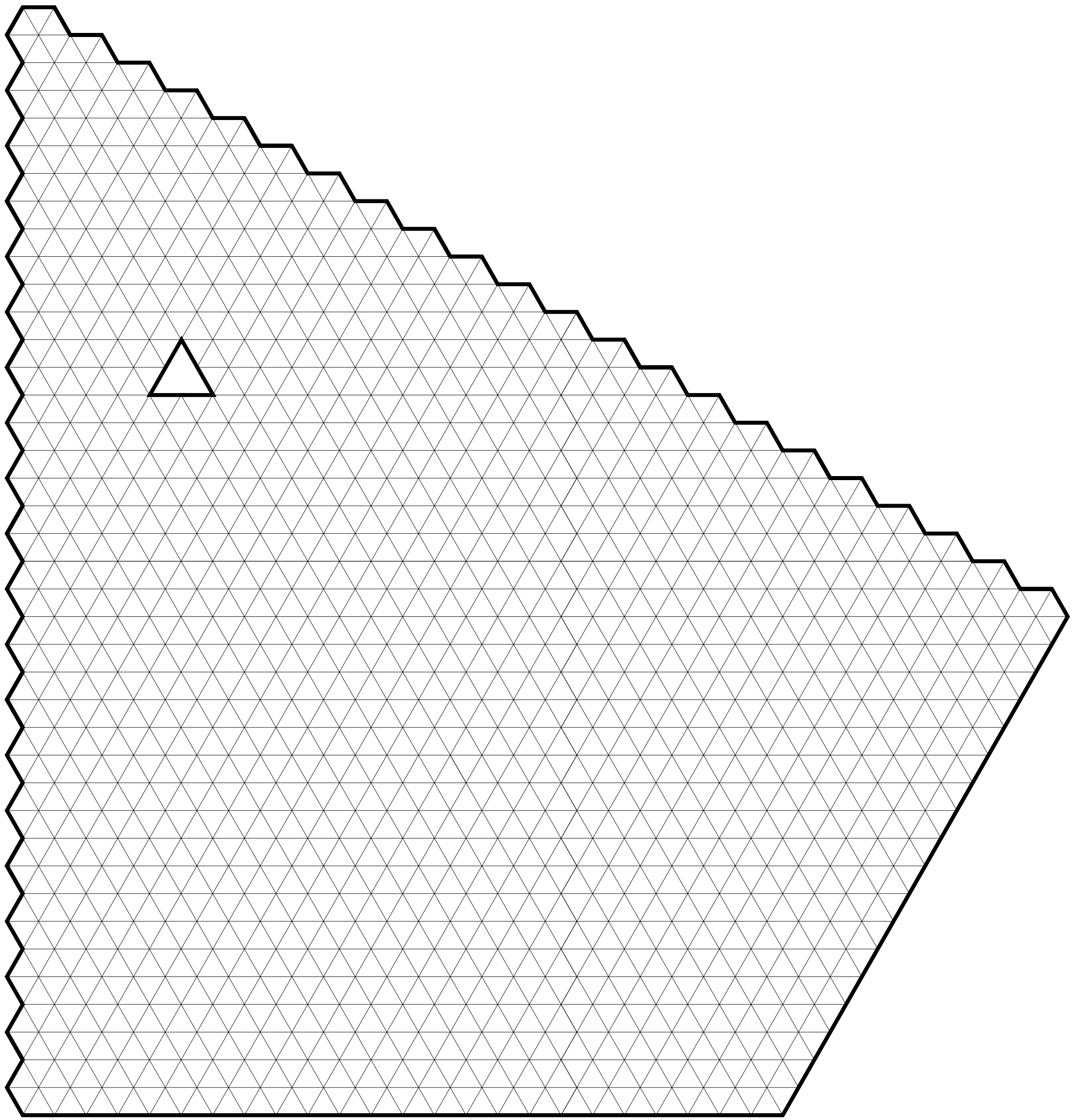
Count the number of tilings by unit rhombi (also called lozenges)



145709818682166414719489131047512214754142854128451883496255862310470677721813188722905560071136 0 such tilings



Change position of the gap: 6875618273315530597996800540627408141179940675855769850310123742702260907192122748128905284734912 tilings



A third position: 2471820827197543647149065107349125334045443362715444874 93039388218383238027875186785746769864166624 tilings

Denote the three numbers by n_1, n_2, n_3

Then

$$\frac{n_1}{n_2} = 0.2119224961...$$

$$\frac{n_1}{n_3} = 0.05894837404...$$

For regions with bottom edge of length 100 the numbers become:

166606268260400169290133111820249095991484330337426288027142302092192651906150724206212732367
561266713456739867535715032727350058341927620215077174496868482836145433479692200855384192517
336341101324822063323810094205548367585576256446939525974826649839900080605800038829866819206
116742643079630580558521596650969094826862353892972948675747721048946147888083683340960280835
574697374439213654058098633564929115153668125780133065207644493133683654740132061767803571636
857116813350251402836963067474728044881037738625114844844057023179245158626666756222839294695
888327002962115752797949759495181985522702350588968495508257114670184870444209840460205778875
966438741589752397264860577619199285752747476397295725885108270121020522970470974128777817358
792492013727632430250020461176003979722277482168852361790388532720412361938307473946635928021
301897124871073009116178578951886964935192586224691894223362445699171739502324097677002151486
886767278048880311378735853575800305898099108891992962127711717054562688427326681229192414725
296090522107257760629591563107348564421114578061783873750262156403584674429634506505932556721
797069985677149602936899175142358373528266342647037451105606483089989314830851767211759776934
420553061077683294662695886234645740353891024528521334450033104281478631156756597635248644715
775155863945296139774544389935842478494375839342055884352348543182146655415991593381502030841
637161577386480775345973023052764386421352852398658377791620735120644301573968914296715846238
428446083817181211616963037654393671176437967518514691561044755739467037640014480490736287112
651459870122688843293004540663585889901546270876876347824142404666401615686824622442909481100
543913544902212640031911897719782855813037280230432330089104613765263990360650918934212897645
294964681699586650348327865260057917026207949398768447714527650380772218650035007154714341443
420095584338699432977802162962173010829629623200522870165316325725757810132017314572729718924
770199068092810825546634638970345235652065160098339221618427675194908008926862602590106863595
78410368589024717508584391997347393390270341870718765718235037845757231104000000000

81600333301429433626634406379584633563523161010882656788698331361732417954941459465526504996
82335320109672815789858214303369096343099097378164999310017793225488702613798769196761276039
49550104843183844522225304911243808672821745780174274407548663902083943790610963682615709826
77155653090476642158976271911387538093835376910996828925593090060421830401868001781641032682
76856654441992565846830049797065553128410218174684791544183226382922735660151361589627188768
26381186538455306712484833744721638967559391629531954114499172434724482401191670370121382032
40383474286788724723506274305636657611216346847606652068685787093232299581132814659027743938
00604949100586469656328533103814006246757342192015139927592102858559709586722746298174729863
89863612478365047935480600341730556408438077072235098973616281146292478296920677471706755545
61875672998210746950491788477226981224742333430117428084944829311547686370145949727828373636
47925778023678467568654096296975747890867341343526289585578174576745628922993906146479209905
12523234517360180987721140294240570902687266997624163466451718126781744161553408473079833864
62388990458654119164759490990774485099442670039874661009136755862987014432799668204782539596
05011082430633603872899848598181993671154088010704486029512725548505705304027806276579521760
00245451153960969508276293067910689692197807535360783805055470665671636149027736271898595789
22950726214742990183594981041908073323293489526016299295888302428102778177081067036875393635
67385072658506103534619411748110290252539970230391731970256778754436787709469558571398418329
13351322820998181706817236812124573621412924142221183668455118105560224596123409427678405799
01969452770730229000306806112883201259692778990844131935851389229270438432367921520667476342
28683704061809708623617637529263393581988777486881053972472827312451828301276627259158060499
15206132134487885755543024886741261025294950492776343097616880282450097988164449473711828037
46226716591978172179012934809851206577757841482286375619603445083678297191116321819006860281
77464257925474824290022355517538128973471079153674263263434946936048844800000000000000,

and

2225245057849352377315972031406344101903920328218160404875091029152484485406173669020510224299
0597204677591374463562741248059653536344782056189298764705445629606730505723487916311159779411
8710339204787037937950900399616231299103355728810968764922187211801058875907894406124660133121
3022802342212475967457332891083245894681255968258877043662134800044112284964822082225224831711
2053558846334807714876700701696185330601151693094271452997796575842633407012911424675261685677
2466741852340639363325704144340042822905196070268524371776335376031479409659437320123990273955
6546489890907275991439842681075606148705411355551458092884538905949547890032358094087133943011
2934338523712270068243154122572102924512397312690291558398319614433061761288798032481225538155
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7512152845316018972376692046895653723291375320908085962101164166214606470640053090183736533922
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6362154669623849896644491666111042250014505786411595654957594199259373336339040006898539022399
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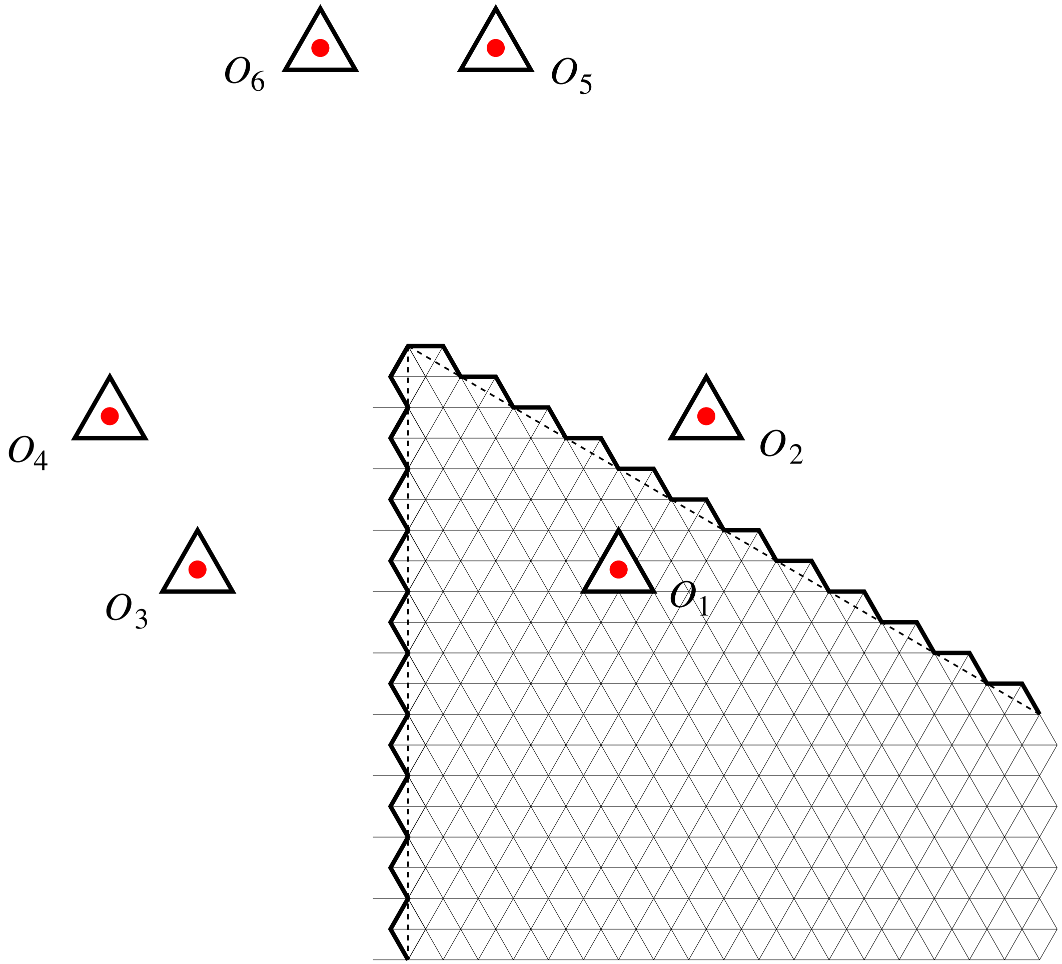
6170332510529411263701647640386969372816086748082360459392262519817779976449500023556240577844
5652492870521151321530454770262368055348686051757147573541985165130264248320000000000

Their ratios are

$$\frac{N_1}{N_2} = 0.2041735144...$$

$$\frac{N_1}{N_3} = 0.07487097552...$$

Main question: What are these ratios in the limit?



- O_1: original gap; $O_2, \ldots, O_6$ its mirror images
- $c_1, \ldots, c_6$: the centers
- $E := \mathrm{d}(c_1, c_2)\, \mathrm{d}(c_1, c_3)\, \mathrm{d}(c_1, c_4)\, \mathrm{d}(c_1, c_5)\, \mathrm{d}(c_1, c_6)$

Then

$$\frac{E_1}{E_2} = 0.2011066817...$$

$$\frac{E_1}{E_3} = 0.07969684211...$$

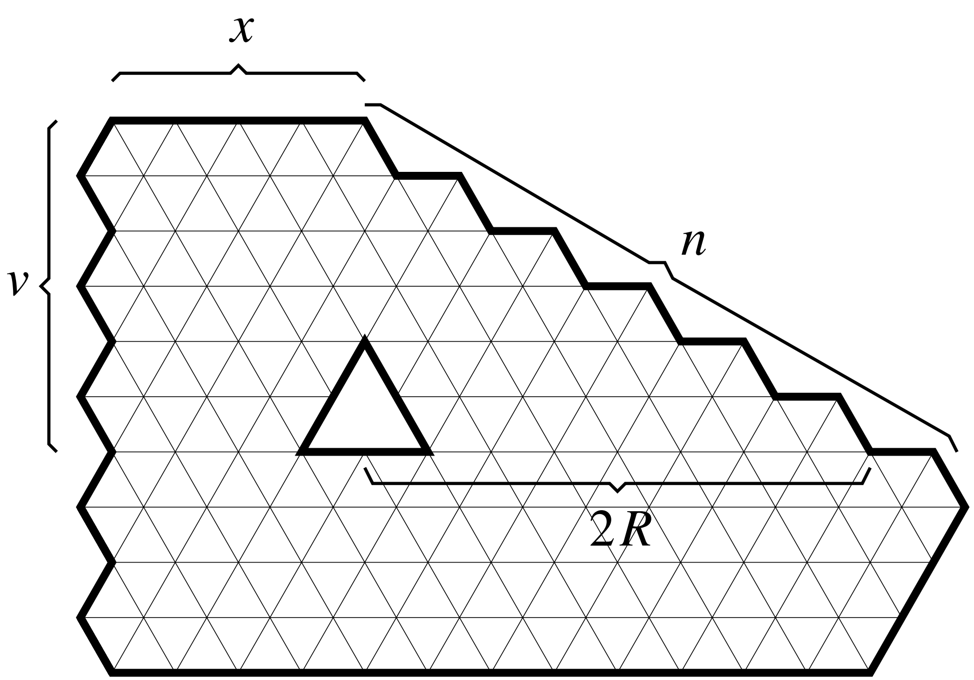
(Recall

$$\frac{N_1}{N_2} = 0.2041735144...$$

$$\frac{N_1}{N_3} = 0.07487097552...)$$

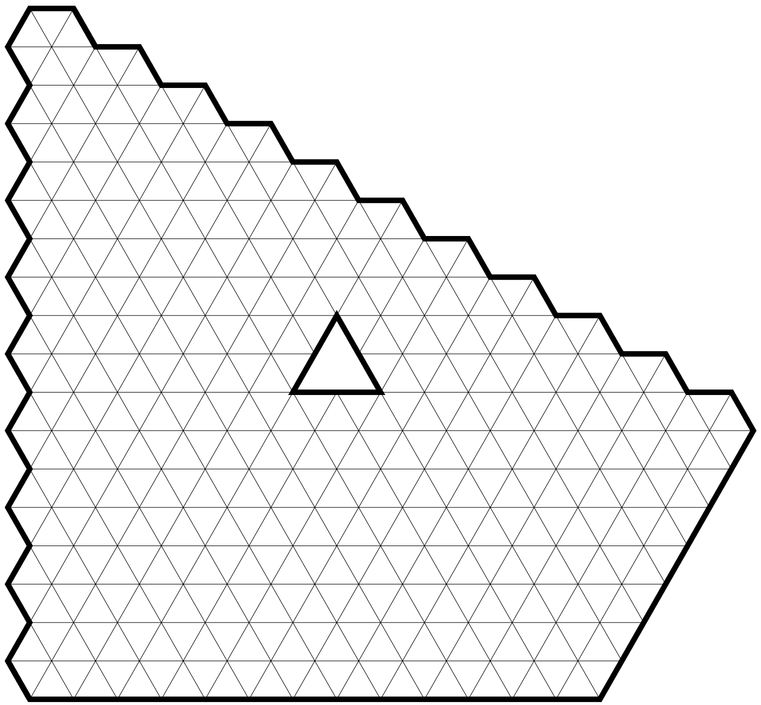
The correlation $\omega_c(R, v)$

Previous regions were of the following type:

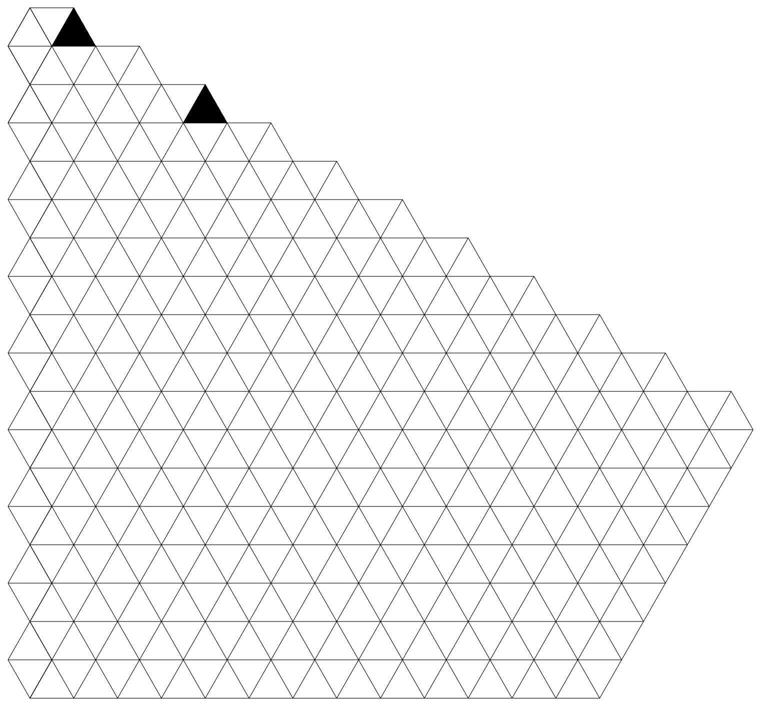


$D_{n,x}(R, v)$

(here $n = 7$, $x = 4$, $R = 4$, $v = 3$)



$D_{11,1}(4,5)$



$D^0_{11,1}$

- M(region): the number of tilings of that region
- Define the correlation of the gap with the corner of the angle by

$$\omega_c(R,v) := \lim_{n\to\infty} \frac{\mathrm{M}(D_{n,1}(R,v))}{\mathrm{M}(D^0_{n,1})}$$

This is "correlation in a sea of dimers."

THEOREM (C. AND FISCHER, 2012). *The correlation of the gap with the corner is*

$$\omega_c(R,v) =$$

$$\begin{cases} \frac{1}{81} R(3v-R)(3v-2R)(4R^2-12Rv+12v^2-8R+16v+3), & R = 0 \pmod 3 \\ \frac{1}{162} R(2R+1)(3v-R+1)(6v-4R+1)(2R^2-6Rv+6v^2-R-v), & R = 1 \pmod 3 \\ \frac{1}{162} R(2R-1)(6v-2R+1)(3v-2R+1)(2R^2-6Rv+6v^2+2R-v), & R = 2 \pmod 3 \end{cases}$$

$$\sim \frac{1}{1944}\,\mathrm{d}(O_1,O_2)\,\mathrm{d}(O_1,O_3)\,\mathrm{d}(O_1,O_4)\,\mathrm{d}(O_1,O_5)\,\mathrm{d}(O_1,O_6).$$

Note. Product of distances is $2(2R-1)(3v-2R)(6v-2R-1)(12R^2-36Rv+36v^2-6v+1)$.