Hook formulas for skew shapes

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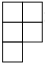
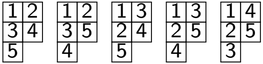
Standard Young Tableaux

Irreducible representations of S_n:

Specht modules $\mathbb{S}_\lambda$, for all $\lambda \vdash n$.

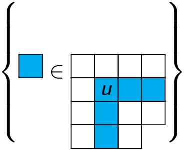
Basis for $\mathbb{S}_\lambda$: **Standard Young Tableaux** of shape λ:

$\lambda = (2, 2, 1)$:  :



Hook-length formula [Frame-Robinson-Thrall]:

$$\dim \mathbb{S}_\lambda = \#\{\text{SYTs of shape } \lambda\} = f^\lambda = \frac{|\lambda|!}{\prod_{u\in\lambda} h_u} = \frac{5!}{4*3*2*1*1}$$

Hook length of box $u = (i, j) \in \lambda$: $h_u = \lambda_i - j + \lambda'_j - i + 1 = \#$ 

Counting skew SYTs

Outer shape λ, inner – μ, e.g. for $\lambda = (5,4,4,2), \mu = (2,2,1)$:

		2	3	6
		7	8	
	1	5	10	
4	9			
11				

Counting skew SYTs

Outer shape λ, inner – μ, e.g. for $\lambda = (5,4,4,2), \mu = (2,2,1)$:

$$\begin{array}{|c|c|c|c|c|}\hline & & 2 & 3 & 6 \\ \hline & & 7 & 8 & \\ \hline & 1 & 5 & 10 & \\ \hline 4 & 9 & & & \\ \hline 11 & & & & \\ \hline\end{array}$$

Origins:
Representations of $GL_n(\mathbb{C})$:

Weyl modules V_λ, for all λ with $\ell(\lambda) \le n$.
Characters – **Schur functions** $s_\lambda(x_1, \ldots, x_n)$.

Tensor product: $V_\mu \otimes V_\nu = \oplus_\lambda V_\lambda^{c^\lambda_{\mu\nu}}$, where $c^\lambda_{\mu\nu}$ – **Littlewood-Richardson coefficients**

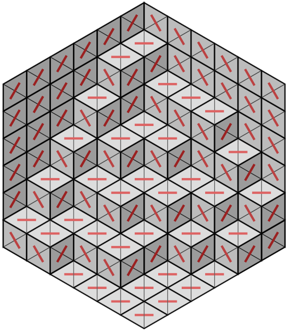
$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda \iff c^\lambda_{\mu\nu} = \langle s_\mu s_\nu, s_\lambda \rangle = \langle s_\nu, \underbrace{s_{\lambda/\mu}}_{\text{skew Schur}} \rangle$$

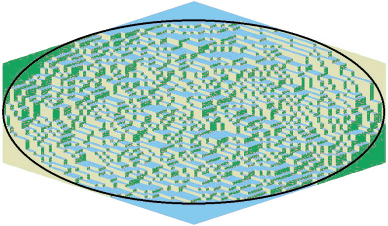
Skew Schur functions and **skew (semi)standard Young Tableaux (SSYTs)**:

$$s_{(3,2)/(1)}(x_1, x_2, x_3, x_4) = \underset{\begin{array}{|c|c|c|}\hline & 1 & 2 \\ \hline 3 & 4 & \\ \hline\end{array}}{x_1 x_2 x_3 x_4} + \underset{\begin{array}{|c|c|c|}\hline & 2 & 3 \\ \hline 1 & 4 & \\ \hline\end{array}}{x_1 x_2 x_3 x_4} + \cdots + \underset{\begin{array}{|c|c|c|}\hline & 1 & 2 \\ \hline 1 & 3 & \\ \hline\end{array}}{x_1^2 x_2 x_3} + \underset{\begin{array}{|c|c|c|}\hline & 1 & 1 \\ \hline 2 & 3 & \\ \hline\end{array}}{x_1^2 x_2 x_3} + \cdots$$

Counting skew SYTs

Other motivation: dimer models (lozenge tilings) in statistical mechanics





Counting skew SYTs

Outer shape λ, inner – μ,   e.g. for $\lambda = (5,4,4,2), \mu = (2,2,1)$:

		2	3	6
		7	8	
	1	5	10	
4	9			
11				

Jacobi-Trudi[Feit 1953]:

$$f^{\lambda/\mu} = |\lambda/\mu|! \cdot \det\left[\frac{1}{(\lambda_i - \mu_j - i + j)!}\right]_{i,j=1}^{\ell(\lambda)}.$$

Littlewood-Richardson:

$$f^{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu,\nu} f^{\nu}$$

No product formula, e.g. $\lambda/\mu = \delta_{n+2}/\delta_n$:

			5	6
		1	9	
	2	7		
3	4			
8				

$\leftrightarrow 8 > 3 < 4 > 2 < 7 > 1 < 9 > 5 < 6$ $f^{\delta_{n+2}/\delta_n} = E_{2n+1}$:

$$1 + E_1 x + E_2 \frac{x^2}{2!} + E_3 \frac{x^3}{3!} + E_4 \frac{x^4}{4!} + \ldots = \sec(x) + \tan(x).$$

Euler numbers: $2, 5, 16, 61 \ldots$

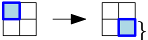
Hook-Length formula for skew shapes

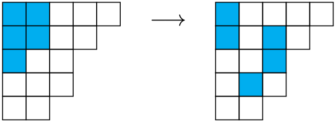
Theorem (Naruse, SLC, September 2014)

$$f^{\lambda/\mu} = |\lambda/\mu|! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{u \in [\lambda] \setminus D} \frac{1}{h(u)},$$

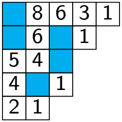
where $\mathcal{E}(\lambda/\mu)$ *is the set of* excited diagrams *of* λ/μ.

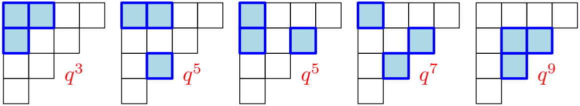
Excited diagrams:

$\mathcal{E}(\lambda/\mu) = \{D \subset \lambda :$ obtained from μ via  $\}$



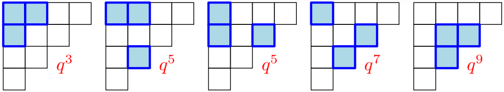
Hook lengths inside λ:





$$f^{(4321/21)} = 7!\left(\frac{1}{1^4 \cdot 3^3} + \frac{1}{1^3 \cdot 3^3 \cdot 5} + \frac{1}{1^3 \cdot 3^3 \cdot 5} + \frac{1}{1^2 \cdot 3^3 \cdot 5^2} + \frac{1}{1^2 \cdot 3^2 \cdot 5^2 \cdot 7}\right) = 61$$

Hook-Length formula for skew shapes



$$s_{\lambda/\mu}(1,q,q^2,\dots) = \sum_{T\in SSYT(4321/21)} q^{|T|} = \frac{q^3}{(1-q)^4(1-q^3)^3} + 2\times\frac{q^5}{(1-q)^3(1-q^3)^3(1-q^5)} + \cdots$$

Theorem (Morales-Pak-P)

For skew SSYTs, we have that

$$s_{\lambda/\mu}(1,q,q^2,\dots) = \sum_{T\in SSYT(\lambda/\mu)} q^{|T|} = \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in[\lambda]\setminus D} \left[\frac{q^{\lambda'_j - i}}{1-q^{h(i,j)}}\right].$$

Theorem (Morales-Pak-P)

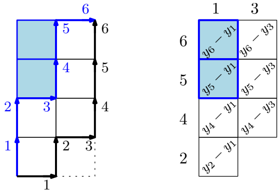
For (reverse) plane partitions of skew shape λ/μ we have that

$$\sum_{\pi\in RPP(\lambda/\mu)} q^{|\pi|} = \sum_{S\in PD(\lambda/\mu)} \prod_{u\in S} \left[\frac{q^{h(u)}}{1-q^{h(u)}}\right].$$

where $PD(\lambda/\mu) := \{S \subset [\lambda] : S \subset [\lambda]\setminus D,$ for some $D\in\mathcal{E}(\lambda/\mu)\}$ is the set of "pleasant diagrams".

Other recent proof by [M. Konvalinka]

Algebraic proof for SSYTs:



$v = 245613$, $w = 361245$

[Ikeda-Naruse, Kreiman]:
Let $w \preceq v$ be Grassmannian permutations whose unique descent is at position d with corresponding partitions $\mu \subseteq \lambda \subseteq d \times (n-d)$. Then the Schubert class X_w for w at point v is:

$$[X_w]\big|_v = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in D} \left(y_{v(d+j)} - y_{v(d-i+1)}\right).$$

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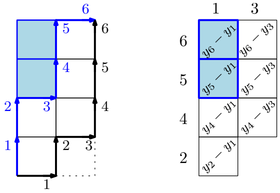

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Factorial Schur functions:

$$s_\mu^{(d)}(\mathbf{x}|\mathbf{a}) := \frac{\det\left[(x_j - a_1)\cdots(x_j - a_{\mu_i+d-i})\right]_{i,j=1}^{d}}{\prod_{1\le i<j\le d}(x_i - x_j)},$$

[Knutson-Tao, Lakshmibai–Raghavan–Sankaran] Schubert class at a point:

$$[X_w]\big|_v = (-1)^{\ell(w)} s_\mu^{(d)}\left(y_{v(1)},\ldots,y_{v(d)}|y_1,\ldots,y_{n-1}\right).$$

Algebraic proof for SSYTs:

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Evaluation at $y = 1, q, q^2, \ldots$, $v(d+1-i) = \lambda_i + d + 1 - i$, $x_i \to y_{v(i)} = q^{\lambda_i+d+1-i} \to$

$$y_{v(d+j)} - y_{v(d-i+1)} = y_{v(d+j)} - x_i = q^{d-\lambda'_j+j} - q^{\lambda_i+d+1-i} = q^{d-\lambda'_j+j}(1-q^{\overbrace{\lambda_i+\lambda'_j-i-j+1}^{h(i,j)}})$$

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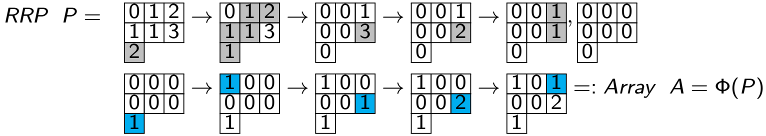
$$\sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in D} q^{d-\lambda'_j+j}(1-q^{h(i,j)}) = [X_w]|_v = s_\mu^{(d)}(q^{v(1)},\ldots|1,q,\ldots)$$

$$= \frac{\det[\prod_{r=1}^{\mu_j+d-j}(q^{\lambda_i+d+1-i} - q^r)]_{i,j=1}^d}{\prod_{i<j}(q^{\lambda+d+1-i} - q^{\lambda_j+d+1-j})} = \ldots[simplifications]\ldots$$

$$= (factor)\det[\underbrace{\frac{1}{(1-q)(1-q^2)\cdots(1-q^{\lambda_i-i-\mu_j+j})}}_{h_{\lambda_i-i-\mu_j+j}(1,q,\ldots)}] \underbrace{=}_{\text{Jacobi-Trudi}} s_{\lambda/\mu}(1,q,\ldots)$$

Combinatorial proofs:

Hillman-Grassl map Φ: Reverse Plane Partitions of shape λ to Arrays of shape λ:

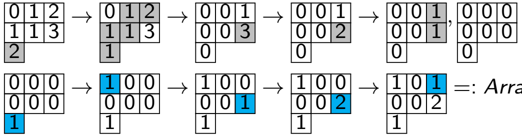


$Weight(P) = |P| = 0 + 1 + 2 + 1 + 1 + 3 + 2 = 10 =$
$= \sum_{i,j} A_{i,j} hook(i,j) = 1 * 5 + 1 * 2 + 2 * 1 + 1 * 1 =: weight(A)$

Combinatorial proofs:

Hillman-Grassl map Φ: Reverse Plane Partitions of shape λ to Arrays of shape λ:

$RRP \;\; P =$

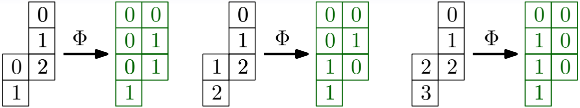


$=: Array \;\; A = \Phi(P)$

$Weight(P) = |P| = 0 + 1 + 2 + 1 + 1 + 3 + 2 = 10 =$
$= \sum_{i,j} A_{i,j} hook(i,j) = 1 * 5 + 1 * 2 + 2 * 1 + 1 * 1 =: weight(A)$

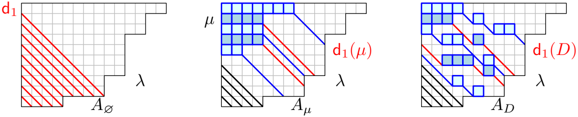
$$\sum_{P \in RPP(\lambda)} q^{|P|} = \sum_{A:Array(\lambda)} \prod_{(i,j)\in\lambda} q^{h(i,j)*A_{i,j}} = \prod_{(i,j)\in\lambda} \frac{1}{1 - q^{h(i,j)})}$$

Combinatorial proofs:



Theorem (Morales-Pak-P)

The restricted Hillman-Grassl map is a bijection from the SSYTs of shape λ/μ to the excited arrays (diagrams in $\mathcal{E}(\lambda/\mu)$ with nonzero entries on the broken diagonals) .



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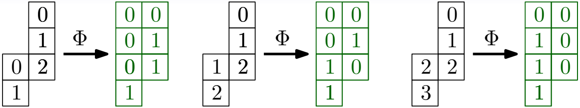

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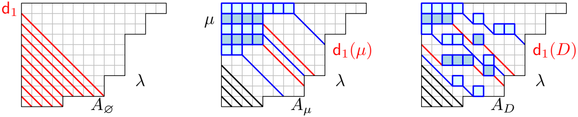


Proof sketch:

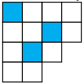
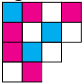
Issue: enforce 0s on μ and strict increase down columns on λ/μ.
Show $\Phi^{-1}(A)$ is column strict in λ/μ + support in λ/μ via properties of RSK
(Integer partition on kth diagonal
$(\ldots, P_{2,2+k}, P_{1,1+k}) = shape(RSK(A_k^T))$ is shape of RSK tableau on the corresponding subrectangle of A)
Thus, Φ^{-1} is injective: restricted arrays $\to$ SSYTs of shape λ/μ.
Bijective: use the algebraic identity.

Hillman-Grassl on skew RPPs

Weakly increasing rows:
Skew reverse plane partitions ⇔ arrays with support *"pleasant diagrams"*:

$$PD(\lambda/\mu) := \{S \subset [\lambda] : S \subset [\lambda] \setminus D, \text{ for some } D \in \mathcal{E}(\lambda/\mu)\}$$

– subsets of complements of the excited diagrams, identified by the "high peaks".

Excited:  Pleasant boxes ■: 

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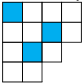
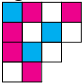
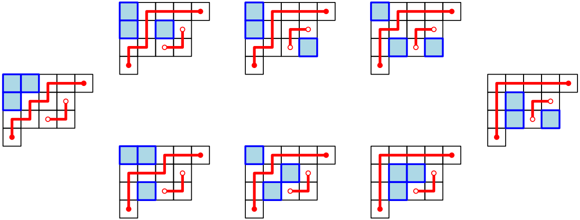
Excited:  Pleasant boxes: 

Excited diagrams $\leftrightarrow$ complements of lattice paths:



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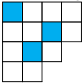
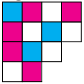
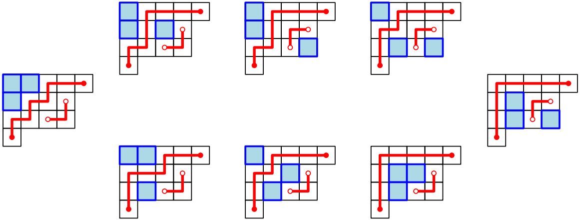
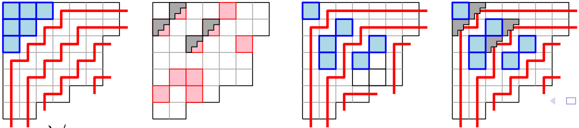
Excited:  Pleasant boxes: 

Excited diagrams ↔ complements of lattice paths:



Pleasant diagrams ↔ subsets of such complements:



λ/μ S $D^* = \varphi_*(S)$ $D = \varphi(S)$

Hillman-Grassl on skew RPPs

Weakly increasing rows:
Skew reverse plane partitions ⇔ arrays with support *"pleasant diagrams"*:

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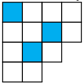
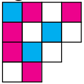
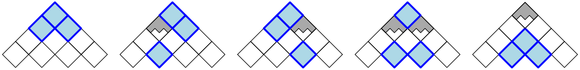
Excited:  Pleasant boxes ■: 

Theorem (MPP)

The HG map is a bijection between skew RPPs of shape λ/μ and arrays with certain nonzero entries (at the "high peaks"):

$$\sum_{\pi \in RPP(\lambda/\mu)} q^{|\pi|} = \sum_{S \in PD(\lambda/\mu)} \prod_{u \in S} \left[\frac{q^{h(u)}}{1 - q^{h(u)}} \right].$$



2^7 2^6 2^6 2^5 2^6

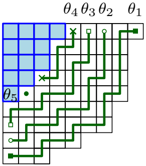
P-partitions/limit: combinatorial proof of original Naruse Hook-Length Formula for $f^{\lambda/\mu}$..

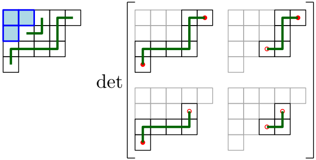
Non-intersecting lattice paths

Theorem[Lascoux-Pragacz, Hamel-Goulden] If $(\theta_1, \ldots, \theta_k)$ is a Lascoux–Pragacz decomposition (i.e. maximal outer border strip decomposition) of λ/μ, then

$$s_{\lambda/\mu} = \det \left[s_{\theta_i \# \theta_j} \right]_{i,j=1}^{k}.$$

where $s_{\varnothing} = 1$ and $s_{\theta_i \# \theta_j} = 0$ if the $\theta_i \# \theta_j$ is undefined.
θ_1 – border strip following the inner border of λ;
θ_i – inner border of $\lambda \setminus (\theta_1 \cup \cdots \cup \theta_{i-1})$ etc until μ is hit,
then – border strips from each connected part etc.
Ordering: corners.
Strip $\theta_i \# \theta_j :=$ shape of θ_1 between the diagonals of the endpoints of θ_i and θ_j.





NHLF for border strips

Lemma (MPP)

For a border strip $\theta = \lambda/\mu$ with end points (a,b) and (c,d) we have

$$s_\theta(1,q,q^2,\ldots,) = \sum_{\substack{\gamma:(a,b)\to(c,d),\\ \gamma\subseteq\lambda}} \prod_{(i,j)\in\gamma} \frac{q^{\lambda'_j - i}}{1-q^{h(i,j)}}.$$

s$(1,q,q^2,\ldots) = \frac{q^3}{(1-q^2)(1-q^1)(1-q^3)(1-q^1)(1-q^2)}$ $\gamma=(3,1),(3,2),(2,2),(2,3),(1,3)$:  $+ \frac{q^4}{(1-q)(1-q^2)^2(1-q^3)(1-q^4)}$ 

$+ \frac{q^1}{(1-q)(1-q^2)^2(1-q^3)(1-q^4)}$  $+ \frac{q^7}{(1-q)^2(1-q^3)(1-q^4)^2}$  $+ \frac{q^6}{(1-q)^2(1-q^5)(1-q^4)^2}$ 

Proofs: induction on $|\lambda/\mu|$, or [multivariate] Chevalley formula for factorial Schurs.

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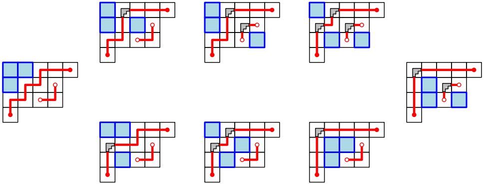
Excited diagrams for $\lambda/\mu \leftrightarrow$ Non-Intersecting Lattice Paths:



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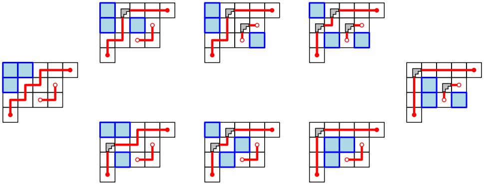
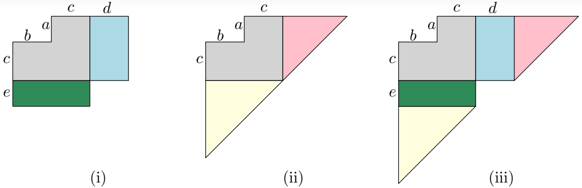


$$s_{\lambda/\mu} \overset{=}{\text{Lascoux-Pragacz}} \det\left[s_{\theta_i \# \theta_j} \right]_{i,j=1}^{k} \overset{=}{s_{\text{Border Strip}}} \det\left[\sum_{\gamma:(a_i,b_i)\to(c_j,d_j)} \prod_{u\in\gamma} \frac{q^{\cdot\cdot}}{1-q^{h_u}} \right]$$

$$\overset{=}{\mathit{Lindstrom-Gessel-Viennot}} \sum_{NILP:\gamma_1,\ldots} \prod_{u\in\gamma_1\cup\ldots} \frac{q^{\cdot\cdot}}{1-q^{h_u}} \overset{=}{\mathcal{E}(\lambda/\mu)=NILP} \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{u\in D} \frac{q^{\cdots}}{1-q^{h_u}}$$

Product formulas



(i) (ii) (iii)

$\Phi(n) := 1! \cdot 2! \cdots (n-1)!$, $\Psi(n) := 1!! \cdot 3!! \cdots (2n-3)!!$,
$\Psi(n;k) := (k+1)!! \cdot (k+3)!! \cdots (k+2n-3)!!$, $\Lambda(n) := (n-2)!(n-4)! \cdots$

Theorem (MPP)

For nonnegative integers a, b, c, d, e, let n be the size of the corresponding skew shape, then for the shapes in (i), (ii), (iii) we have the following product formulas for the number of skew SYTs:

$$f^{sh(i)} = n! \, \frac{\Phi(a)\Phi(b)\Phi(c)\Phi(d)\Phi(e)\Phi(a+b+c)\Phi(c+d+e)\Phi(a+b+c+e+d)}{\Phi(a+b)\Phi(e+d)\Phi(a+c+d)\Phi(b+c+e)\Phi(a+b+2c+e+d)},$$

$$f^{sh(ii)} = n! \, \frac{\Phi(a)\Phi(b)\Phi(c)\Phi(a+b+c)}{\Phi(a+b)\Phi(b+c)\Phi(a+c)} \, \frac{\Psi(c)\Psi(a+b+c)}{\Psi(a+c)\Psi(b+c)\Psi(a+b+2c)},$$

$$f^{Sh(iii)} = \frac{n! \quad \Phi(a)\Phi(b)\Phi(c)\Phi(a+b+c)\,\Psi(c;d+e)\Psi(a+b+c;d+e)\,\Lambda(2a+2c)\Lambda(2b+2c)}{\Phi(a+b)\Phi(b+c)\Phi(a+c)\Psi(a+c)\Psi(b+c)\Psi(a+b+2c;d+e)\Lambda(2a+2c+d)\Lambda(2b+2c+e)}$$

Multivariate identities I

Set $z_{\lambda_i+d-i+1}(\lambda) = x_i$ and $z_{\lambda'_j+n-d-j+1}(\lambda) = y_j$.

Theorem (Ikeda-Naruse)

$$\sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in D} (x_i - y_j) = s_\mu^{(d)}(\mathbf{x} \mid z(\lambda))$$

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Let $\lambda/\mu \subset d \times (n-d)$ with $\lambda_d \geq \mu_1 + d - 1$. Then:

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In particular, the LHS is symmetric in $(x_1,\ldots,x_d)$.

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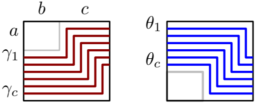

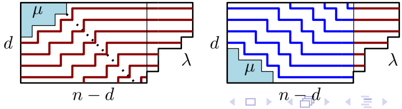

Multivariate identities II

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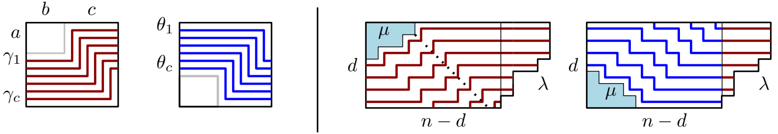

If $x_i = \lambda_i - i$ and $y_j = -\lambda_j + j - 1$, then $h_\lambda(i,j) = x_i - y_j$.
If λ is "nice", then any path θ : NW corner A $\to$ SE corner B has the same multiset of hooks $(h(\theta(1)), h(\theta(2)), \ldots)$

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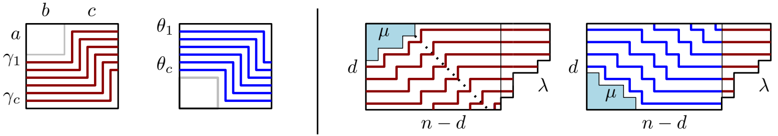

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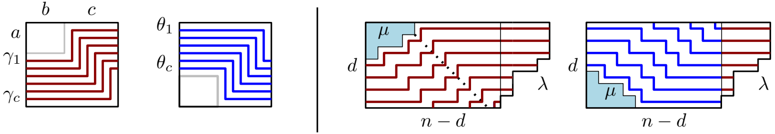

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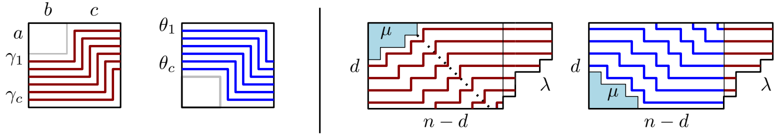

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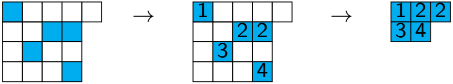
Excited diagrams $\leftrightarrow$ flagged tableaux of shape μ:



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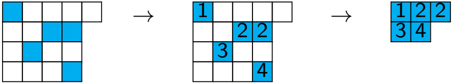
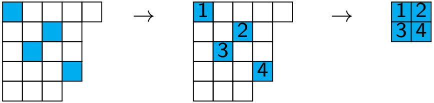


When $\mu = (b^a)$, then SSYTs with max entry $\leq \max\{k : \lambda_k \geq k + b - a\}$:



The end of day 1