Combinatorics of the Selberg integral

Algebraic and Enumerative Combinatorics in Okayama

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Outline

① Motivation: Selberg integral

② Combinatorial interpretation: Selberg tableaux.

③ q-integrals over order polytopes

④ Applications

- Comb. inter. for q-Selberg
- reverse plane partitions.
- SYT, SSYT
- d-complete posets.

Motivation : Selberg integral

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} x_i^{\alpha-1}(1-x_i)^{\beta-1} \prod_{1\le i<j\le n} |x_i - x_j|^{2\gamma} dx_1 \cdots dx_n$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\alpha+(j-1)\gamma)\,\Gamma(\beta+(j-1)\gamma)\,\Gamma(1+j\gamma)}{\Gamma(\alpha+\beta+(n+j-2)\gamma)\,\Gamma(1+\gamma)}$$

- Pak found a combinatorial interpretation.
  Selberg integral ~ # linear extensions of poset.

- Motivation: Find a combinatorial proof.
  Find a q-analog.

Combinatorial interpretation

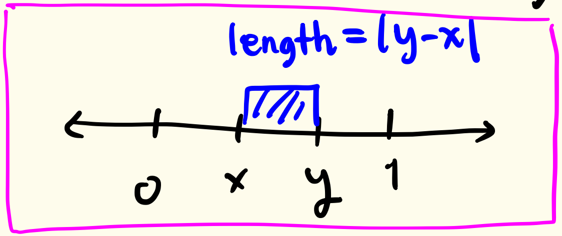
$$S_n(r,s,m)=\int_0^1\cdots\int_0^1 \prod_{i=1}^{n} x_i^r (1-x_i)^s \prod_{1\le i<j\le n} |x_j-x_i|^m \, dx_1\cdots dx_n .$$

$\Rightarrow S_n(r,s,m) = C \cdot$ # linear extensions of $P(n,r,s,m)$

Idea:

$$\int_0^1\int_0^1 |y-x| \, dx\, dy = \text{Prob}\left(\begin{array}{l} z \text{ lies between } x, y \\ \text{for random } x,y,z \in [0,1] \end{array} \right)$$

$$= \frac{1}{3}$$



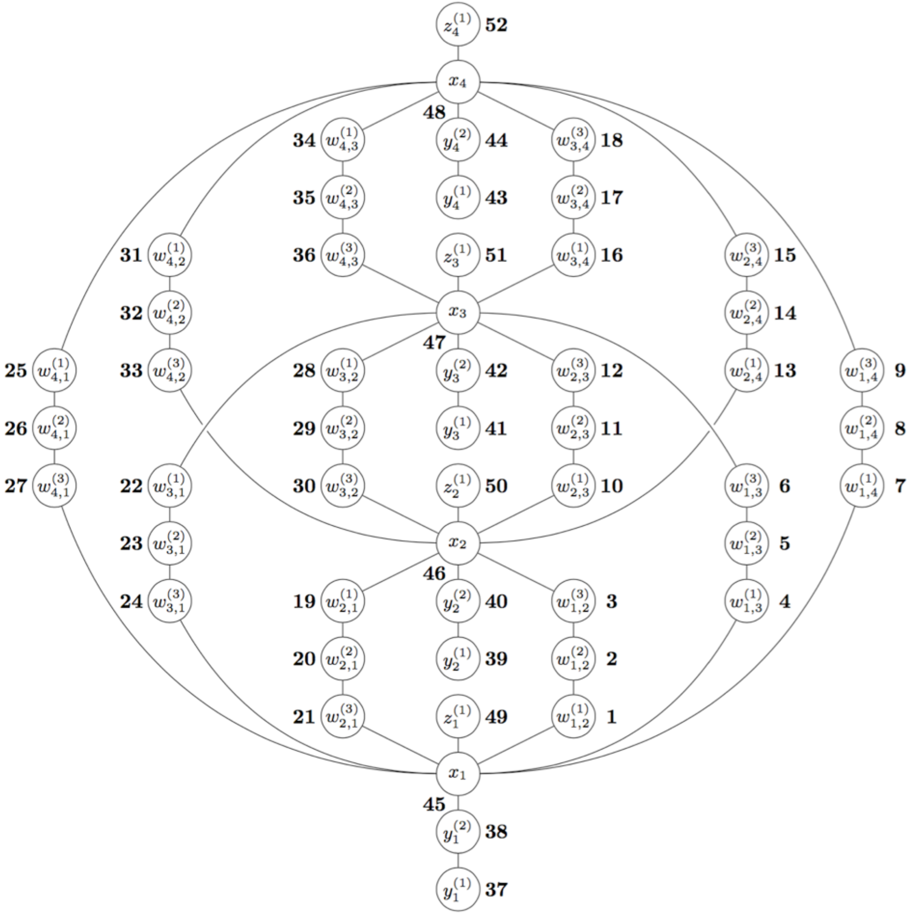
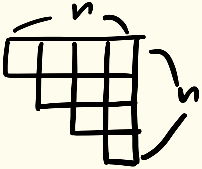


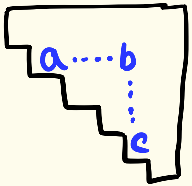
FIGURE 1. The Selberg poset $P(n, r, s, m)$ for $n = 4, r = 2, s = 1, m = 3$ with labeling.

Selberg tableaux

Def) **Selberg tableau of size n** is a filling of



with $1, 2, \ldots, \binom{n+1}{2}$ s.t.



$a < b < c$

$ST(n)$: set of Selberg tableaux of size n.

$ST(n; d_1, \ldots, d_{n-1})$: set of $T \in ST(n)$ with diagonal entries $a_1, \ldots, a_n$

$(d_i = a_{i+1} - a_i - 1, \quad a_1 = 1)$

ex) $ST(4;\ 3, 2, 1\ )$

1	2	3	4
3	5	6	7
2		8	9
1			10

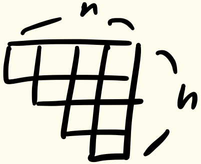
1	4	2	3
	5	7	6
		8	9
			10

1	3	4	2
	5	7	6
		8	9
			10

$\cdots$

$$\int_0^1 \cdots \int_0^1 \prod_{1 \le i < j \le n} |x_i - x_j| \, dx_1 \cdots dx_n \sim |ST(n)|.$$

$\Rightarrow$ $|ST(n)| =$ nice prod

Def) $YT(n)$ = set of SYT of shape 

$YT(n; d_1, \ldots, d_{n-1})$ = set of $T \in YT(n)$ with diag entries $a_1, \ldots, a_n$

$(d_i = a_{i+1} - a_i - 1)$

By hook-length formula,

$|YT(n)| =$ nice prod.

$\Rightarrow$ $|ST(n)| = |YT(n)| \, 1! \, 2! \cdots (n-1)!$

Thm (K., Oh)

$$|ST(n)| = |YT(n)|\, 1!\, 2! \cdots (n-1)!$$

$$|ST(n; d_1, \dots, d_{n-1})| = |YT(n; d_1, \dots, d_{n-1})|\, 1!\, 2! \cdots (n-1)!$$

ex). $n=4$, $(a_1, a_2, a_3, a_4) = (1, 5, 8, 10)$ or $(d_1, d_2, d_3) = (3, 2, 1)$

$|YT| = 1$

1	2	3	4
	5	6	7
		8	9
			10

$|ST| = 3!\,2!\,1!$

1				$\leftarrow$ 2, 3, 4
	5			$\leftarrow$ 6, 7
		8		$\leftarrow$ 9
			10	

ex) $n=4$, $(a_1, a_2, a_3, a_4) = (1, 3, 8, 10)$ or $(d_1, d_2, d_3) = (1, 4, 1)$

$|YT| = 2$

1	2	4		$\leftarrow$ 5, 6
	3		7	
		8	9	
			10	

$|ST| = 4! = 2 \cdot 1!\,2!\,3!$

1	2			$\leftarrow$ 4, 5, 6, 7
	3			
		8	9	
			10	

<u>Thm</u> (K., Oh)

$$|ST(n)| = |YT(n)|\, 1!\, 2! \cdots (n-1)!$$

$$|ST(n; d_1, \dots, d_{n-1})| = |YT(n; d_1, \dots, d_{n-1})|\, 1!\, 2! \cdots (n-1)!$$

Pf)
$$\sum_{d_1, \dots, d_{n-1} \geq 0} |ST(n; d_1, \dots, d_{n-1})| \frac{t_1^{d_1} \cdots t_{n-1}^{d_{n-1}}}{d_1! \cdots d_{n-1}!} = \prod_{1 \leq i < j \leq n} (t_i + t_{i+1} + \cdots + t_{j-1})^m$$

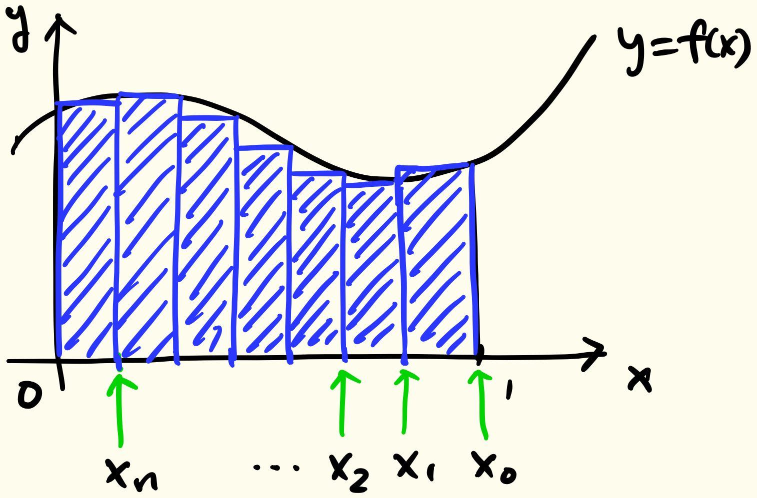
Postnikov's result on vol of Gelfand-Tsetlin polytope.

$$\sum_{d_1, \dots, d_{n-1} \geq 0} |YT(n; d_1, \dots, d_{n-1})| \frac{t_1^{d_1} \cdots t_{n-1}^{d_{n-1}}}{d_1! \cdots d_{n-1}!} = \prod_{1 \leq i < j \leq n} \frac{(t_i + t_{i+1} + \cdots + t_{j-1})^m}{j-i}$$

Open Problem: Find a bijective proof.

q-integrals

$$\int_0^1 f(x)\,dx = \text{area of}$$



$$= \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i)(x_i - x_{i+1})$$

q-integral : $x_i = q^i$ $\quad (0<q<1)$

$$\sum_{i=0}^{\infty} f(q^i)(q^i - q^{i+1}) = (1-q)\sum_{i=0}^{\infty} f(q^i)\,q^i$$

Def) $\int_0^1 f(x)\, d_q x = (1-q) \sum_{i=0}^{\infty} f(q^i)\, q^i$

Def) $\int_0^a f(x)\, d_q x = (1-q) \sum_{i=0}^{\infty} f(aq^i)\, aq^i$

$$\int_a^b f(x)\, d_q x = \int_0^b f(x)\, d_q x - \int_0^a f(x)\, d_q x$$

Note

$$\lim_{q \to 1} \int_a^b f(x)\, d_q x = \int_a^b f(x)\, dx$$

q-Selberg integral

Thm (Conjectured by Askey 1980, proved by Kadell, and Habsieger 1980)

$$\int_{0\le x_1\le\cdots\le x_n\le 1} \prod_{i=1}^{n} x_i^{\alpha-1}(qx_i)_{\beta-1} \prod_{1\le i<j\le n} x_j^{2m-1}\left(q^{1-m}x_i/x_j\right)_{2m-1} \Delta(x)\, d_qx_1\cdots d_qx_n$$

$$= q^{\alpha m\binom{n}{2}+2m^2\binom{n}{3}} \prod_{j=1}^{n} \frac{\Gamma_q(\alpha+(j-1)m)\,\Gamma_q(\beta+(j-1)m)\,\Gamma_q(jm)}{\Gamma_q(\alpha+\beta+(n+j-2)m)\,\Gamma_q(m)}$$

$$\Delta(x)=\prod_{1\le i<j\le 1}(x_j-x_i), \quad (a)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$$

* Volume of a simplex.

$$\int_{0 \le x_{\pi_1} \le \cdots \le x_{\pi_n} \le 1} dx_1 \cdots dx_n = \frac{1}{n!} \qquad (\pi \in S_n)$$

This integral is the same for all π.

However, this is NOT true for q-integrals.

$$\int_{0 \le x_1 \le x_2 \le 1} d_q x_1 d_q x_2 = \frac{1}{1+q} \neq \int_{0 \le x_2 \le x_1 \le 1} d_q x_1 d_q x_2 = \frac{q}{1+q}$$

Thm (K., Stanton)

$$\int_{0 \le x_{\pi_1} \le \cdots \le x_{\pi_n} \le 1} d_q x_1 \cdots d_q x_n = \frac{q^{\mathrm{maj}(\pi)}}{[n]_q!}$$

$\pi = \pi_1 \pi_2 \cdots \pi_n$, $\quad i \in \mathrm{Des}(\pi) \iff \pi_i > \pi_i$

$$\mathrm{maj}(\pi) = \sum_{i \in \mathrm{Des}(\pi)} i$$

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$$

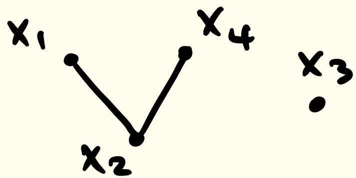
$$[n]_q! = [1]_q [2]_q \cdots [n]_q$$

ex)
$$\int_{0 \le x_3 \le x_4 \le x_1 \le x_5 \le x_2 \le 1} d_q x_1 \cdots d_q x_5 = \frac{q^{\mathrm{maj}(34152)}}{[5]_q!} = \frac{q^6}{[5]_q!}$$

* Order polytopes

P : poset on $\{x_1, \dots, x_n\}$.

$$O(P) = \{(x_1, \dots, x_n) \in [0,1]^n : x_i \le x_j \text{ if } x_i \le_P x_j\}$$

ex) $P =$ 

$$O(P) = \{(x_1, x_2, x_3, x_4) \in [0,1]^n : x_2 \le x_1,\ x_2 \le x_4\}$$

$$\int_{O(P)} d_q x_1 \cdots d_q x_4 = \int_0^1 \int_0^1 \int_0^{x_4} \int_{x_2}^1 d_q x_1\, d_q x_2\, d_q x_3\, d_q x_4.$$

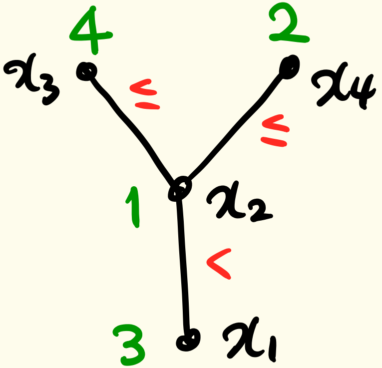
Def) P : poset , ω : labeling of P.

A (P, ω)-partition is an order-reversing map

$\sigma : P \to \mathbb{N}$ $\quad \left(x \le_P y \Rightarrow \sigma(x) \ge \sigma(y) \right)$

s.t. $x \le_P y$ & $\omega(x) > \omega(y) \Longrightarrow \sigma(x) > \sigma(y)$.

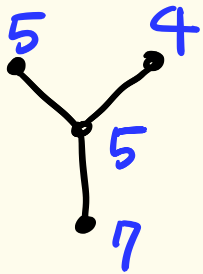
ex) $P =$



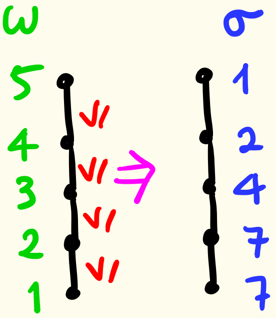
$\omega(x_1) = 3$, $\omega(x_2) = 1$
$\omega(x_3) = 4$, $\omega(x_4) = 2$.

$\sigma : P \to \mathbb{N}$

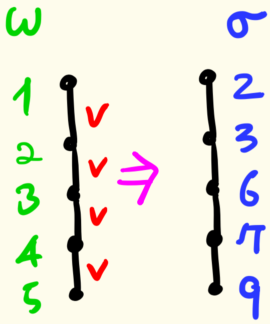
$\sigma(x_1) = 7$
$\sigma(x_2) = 5$
$\sigma(x_3) = 5$
$\sigma(x_4) = 4$.



ex)

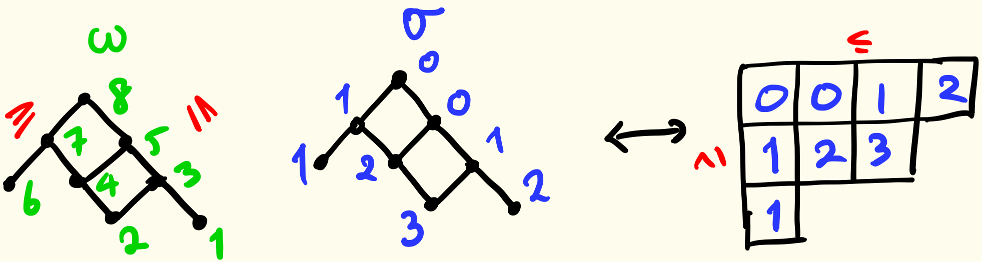


partition

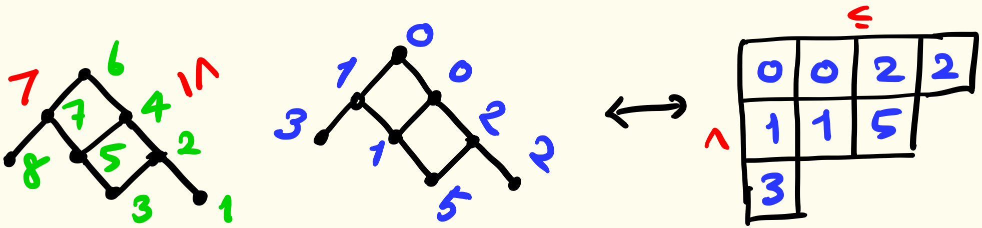


strict partition.

ex)



RPP



SSYT

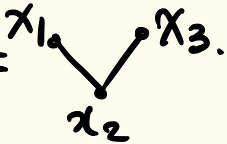
__Thm__ (K., Stanton)

P : poset on $\{x_1, \ldots, x_n\}$

ω : labeling of P given by $\omega(x_i) = i$

$$\int_{O(P)} d_q x_1 \cdots d_q x_n = (1-q)^n \sum_{\sigma : (P,\omega)\text{-partitions}} q^{|\sigma|}$$

$$= \frac{1}{[n]_q!} \sum_{\pi \in \mathcal{L}(P,\omega)} q^{\mathrm{maj}(\pi)}$$

ex) $P =$ 

$\mathcal{L}(P,\omega) = \{213, 231\}$.

$$\int_{O(P)} d_q x_1 d_q x_2 d_q x_3 = \frac{1}{[3]_q!}\left(q^{\mathrm{maj}(213)} + q^{\mathrm{maj}(231)}\right) = \frac{q+q^2}{[3]_q!}$$

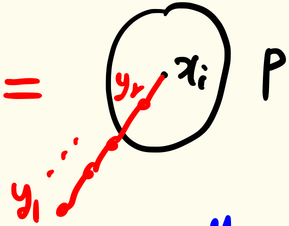
Back to q-Selberg integral.

Equivalent form:

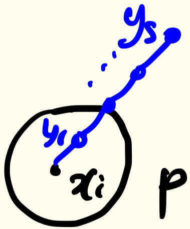
$$\int_{0\le x_1\le\cdots\le x_n\le 1} \prod_{i=1}^{n} x_i^r (qx_i)_s \prod_{1\le i<j\le n} x_i^m (x_j/x_i)_m \, x_j^m (x_i/x_j)_m \, d_qx_1\cdots d_qx_n$$

$\boxed{\text{Lem}}$ P poset on $\{x_1,\dots,x_n\}$

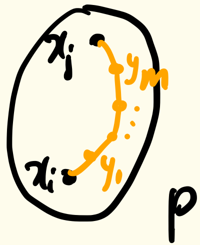
$$\int_{O(P)} x_i^r \, d_qx = [r]_q! \int_{O(Q)} d_qy \, d_qx$$

$Q =$ 

$$\int_{O(P)} (qx_i)_s \, d_qx = [s]_q! \int_{O(Q)} d_qx \, d_qy$$

$Q =$ 

$$\int_{O(P)} x_i^m (x_j/x_i)_m \, d_qx = [m]_q! \int_{O(Q)} d_qy \, d_qx$$

$Q =$ 

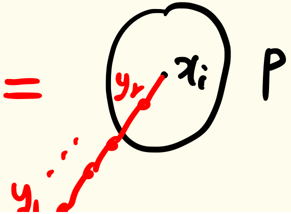
Back to q-Selberg integral.

Equivalent form :

$$\int_{0\le x_1\le\cdots\le x_n\le 1} \prod_{i=1}^{n} x_i^r (qx_i)_s \prod_{1\le i<j\le n} x_i^m (x_j/x_i)_m \, x_j^m (x_i/x_j)_m \, d_q x$$

$\boxed{\text{Lem}}$ P poset on $\{x_1, \ldots, x_n\}$

$$\int_{O(P)} x_i^r \, d_q x = [r]_q! \int_{O(Q)} d_q y \, d_q x$$

$Q =$ 

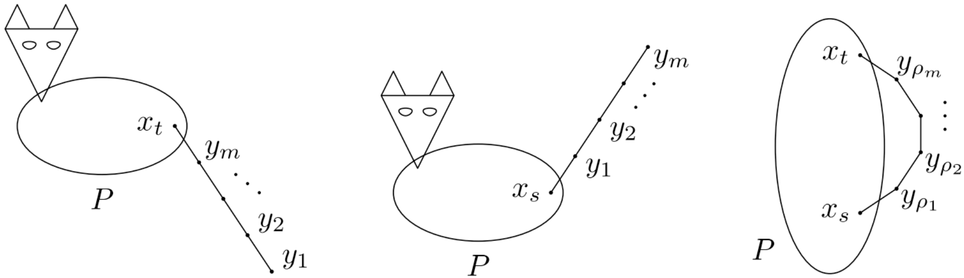


Fig. 1. Illustrations of the scaredy cat lemma (left), the happy cat lemma (middle), and the attaching chain lemma (right).

Combinatorial interpretation for q-Selberg

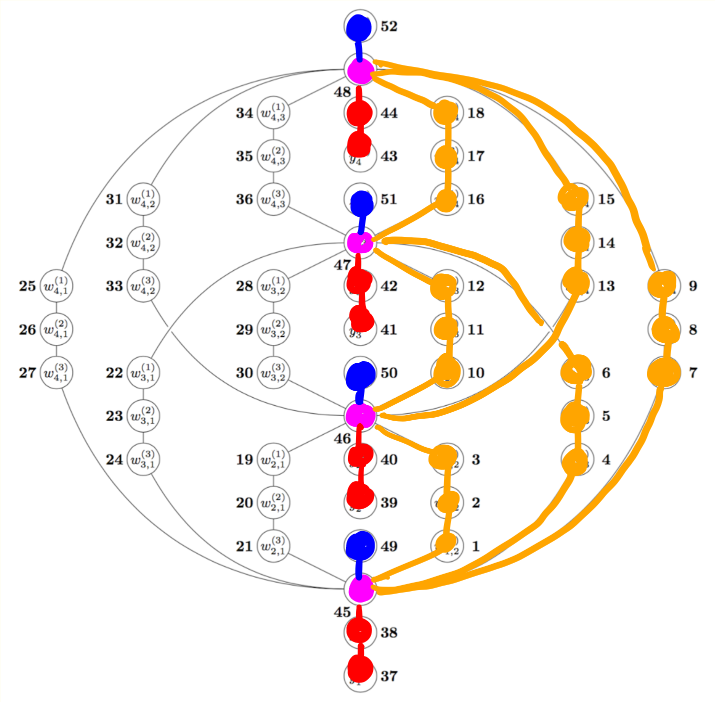
Thm (K., Stanton)

$$\int_{0\le x_1\le\cdots\le x_n\le 1} \prod_{i=1}^{n} x_i^r (qx_i)_s \prod_{1\le i<j\le n} x_i^m (x_j/x_i)_m \, x_j^m (x_i/x_j)_m \, d_qx$$

$$= \frac{1}{[n(r+s+1)+m\binom{n}{2}]_q!} \sum_{\pi\in\mathcal{L}(P(n,r,s,m),\omega)} q^{\mathrm{maj}(\pi)}$$

ex) $P(n, r, s, m)$

$n = 4$, $r = 2$, $s = 1$, $m = 3$



Application to reverse plane partitions.

$$\pi = \begin{array}{|c|c|c|c|} \hline d_n & & & \\ \hline & \ddots & & \\ \hline & & d_2 & \\ \hline & & & d_1 \\ \hline \end{array}$$

$$d_i(\pi) = d_i$$

$$tr(\pi) = d_1 + \cdots + d_n$$

Let $$wt_{a,b}(\pi) = q^{|\pi|}(q^a)^{tr(\pi)} \prod_{i=1}^{n} \left(q^{d_i(\pi)+i}\right)_b .$$

Note : $$wt_{a,0}(\pi) = q^{|\pi|}(q^a)^{tr(\pi)}$$

$$wt_{0,0}(\pi) = q^{|\pi|}$$

Thm (K., Stanton)

$$\sum_{\pi \in RPP(n^2)} wt_{a,b}(\pi) = \frac{1}{(1-q)^{n^2}} \prod_{j=1}^{n} \frac{[a+j-1]_q!\,[b+j-1]_q!}{[a+b+n+j-1]_q!\,[j-1]_q!}$$

Rmk If $b=0$, this reduces to the well known trace generating function for r.p.p.

$$\sum_{\pi \in RPP(n^2)} x^{tr(\pi)} q^{|\pi|} = \prod_{i,j=1}^{n} \frac{1}{1-xq^{i+j-1}}$$

Bounded SSYT

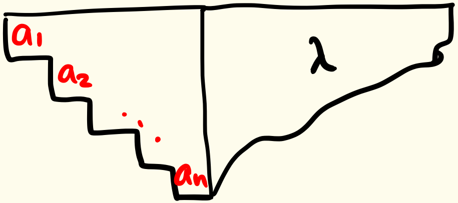
Thm (Conjectured by Askey, proved by Kadells and Evans)

$$\int_{[a,b]^n} \prod_{i=1}^{n} \left(\frac{qx_i}{a}\right)_{\alpha-1} \left(\frac{qx_i}{b}\right)_{\beta-1} \Delta(x) \prod_{\substack{1\le i<j\le n \\ 1-k\le l\le k-1}} (x_j - q^l x_i)\, d_q x$$

$$= (-1)^{k\binom{n}{2}} q^{k^2\binom{n}{3}-\binom{k}{2}\binom{n}{2}} \prod_{i=1}^{n-1} \frac{\Gamma_q(\alpha+ik)\Gamma_q(\beta+ik)\Gamma_q(k+ik)\left(\frac{a}{b}\right)_{\beta+ik}\left(\frac{b}{a}\right)_{\alpha+ik}(ab)^{1+ik}}{\Gamma_q(k)\Gamma_q(\alpha+\beta+(n+i-1)k)\,(a-b)}$$

Lem

$$a_{\lambda+\delta_n}(q^{a_1},\dots,q^{a_n}) = C\cdot \sum_{\substack{T\in SSYT(\lambda) \\ \mathrm{diag}(T)=(a_1,\dots,a_n)}} q^{|T|}$$

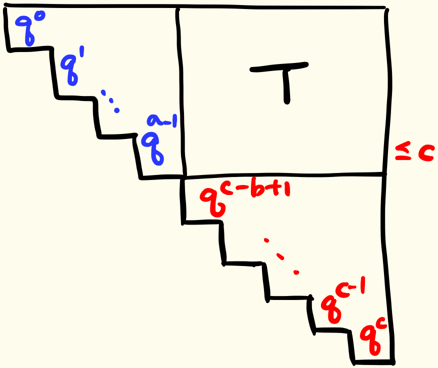


MacMahon Box Theorem

$$\sum_{\pi \subset B(a,b,c)} q^{|\pi|} = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}$$

$$\Leftrightarrow \sum_{T \in SSYT(a\,\overbrace{\boxed{\quad}}^{b},\, c)} q^{|T|} = \text{nice product.}$$

$\hookrightarrow$ SSYT with entries $\leq c$.



$$a_{\delta_{a+b}}(q^0, q^1, \dots, q^{a-1}, q^{c-b+1}, \dots, q^c)$$

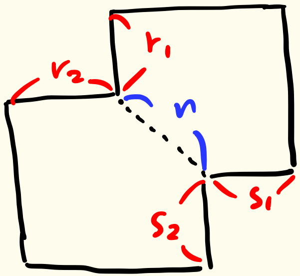
$$= D \cdot \sum_T q^{|T|}$$

$$\Rightarrow \sum_T q^{|T|} = \text{nice prod.}$$

Generalization of MBT

Thm (K., Yoo)

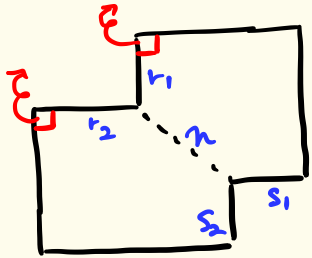
Let $\lambda =$



$$\sum_{T \in SSYT(\lambda, c)} q^{|T|} = \text{nice prod.}$$

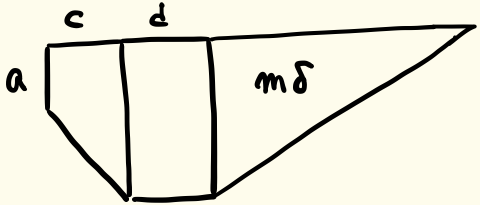
Conjectures (Morales, Pak, Panova)

Conj1) #SYT of shape



= nice.

conj2) #SYT of shape

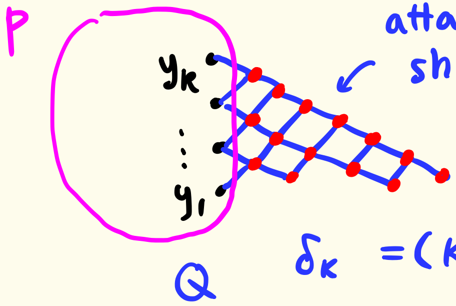


= nice

<u>Thm</u> (K., Yoo)

The above conjecture are true.

Attaching shifted shapes



attach shifted shape $\lambda+\delta_k$

$\delta_k = (k-1, \dots, 1, 0)$

Lem (K., Stanton)

P : poset on $\{u_1, \dots, u_n\}$

Q : poset on $\{v_1, \dots, v_N\}$

$y_1 < \dots < y_k$: chain in P

$$\int_{O(Q)} f(u_1, \dots, u_n)\, d_q v_1 \cdots d_q v_N$$

$$= C \cdot \int_{O(P)} f(u_1, \dots, u_n)\, a_{\lambda+\delta_k}(y_1, \dots, y_k)\, d_q u_1 \cdots d_q u_n$$

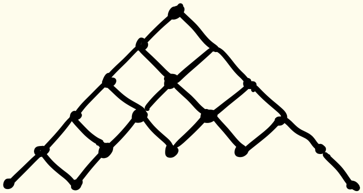
$$a_\lambda(x_1, \dots, x_n) = \det\left(x_i^{\lambda_j}\right)_{i,j=1}^{n}$$

q-hook length formula for SSYT.

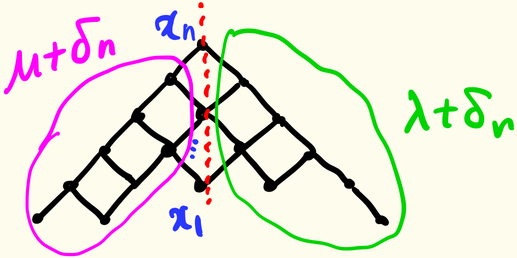
Thm (Stanley)

$$\sum_{T \in SSYT(\lambda)} q^{|T|} = q^{b(\lambda)} \frac{[n]_q!}{\prod_{x \in \lambda} [h_x]_q}$$

pf)



$\Rightarrow$

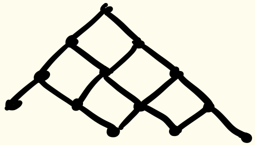


$$\text{LHS} \sim \int_{O(P)} d_q W \sim \int_{0 \le x_1 \le \cdots \le x_n \le 1} a_{\lambda+\delta_n} \cdot a_{\mu+\delta_n} \, d_q x_1 \cdots d_q x_n$$

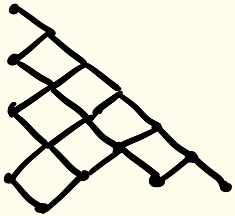
$\Rightarrow$ Selberg-type formula

Application to hook length formula for d-complete posets

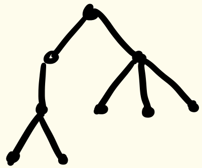
Proctor introduced d-complete posets generalizing shapes, shifted shapes, trees



(Stanley)



(Stanley)



(Björner & Wachs)

<u>Thm</u> (Proctor)

P : d-complete poset

$$\sum_{\sigma: P\to\mathbb{N}} q^{|\sigma|} = \prod_{x\in P} \frac{1}{1-q^{h(x)}}$$

Kim and Meesue Yoo gave a new proof using q-integrals.