Hörmander theorem for Gaussian rough differential equations

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Sketch

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1. Introduction
   - SDEs driven by Gaussian processes
   - Malliavin derivatives
2. Hörmander theorem
3. Elements of proof

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Equation under consideration

Equation:

Standard differential equation driven by Gaussian process, $\mathbb{R}^n$-valued

$$dy_t = V_0(y_t)\, dt + V_j(y_t)\, dB_t^j, \tag{1}$$

with

- $t \in [0, 1]$.
- Vector fields $V_0, \ldots, V_j$ in $\mathcal{C}_b^\infty$.
- A d-dimensional Gaussian process B.
- Typical example: d-dimensional fBm B with $1/4 < H < 1/2$.

Brief summary of rough paths theory

Hypothesis: Consider a path x such that

- $x \in \mathcal{C}^{\gamma}(\mathbb{R}^d)$ with $\gamma > 1/4$
- x allows to define:
  - A Levy area $\mathbf{x^2} \in \mathcal{C}^{2\gamma}(\mathbb{R}^{d\times d}) \equiv \int dx \int dx$
  - Some volumes $\mathbf{x^3} \in \mathcal{C}^{3\gamma}(\mathbb{R}^{d\times d\times d}) \equiv \int dx \int dx \int dx$
- Vector fields $V_0, \ldots, V_j$ in $\mathcal{C}_b^{\infty}$.

Main rough paths theorem:

One can solve the equation $dy_t = V_0(y_t)\, dt + V_j(y_t)\, dx_t^j, \quad y_0 = a.$

Furthemore (Lyons-Qian, Friz-Victoir, Gubinelli)

$$F : \mathbb{R}^n \times \mathcal{C}^{\gamma}(\mathbb{R}^d) \times \mathcal{C}^{2\gamma}(\mathbb{R}^{d\times d}) \times \mathcal{C}^{3\gamma}(\mathbb{R}^{d\times d}) \longrightarrow \mathcal{C}^{\gamma}(\mathbb{R}^n)$$
$$(a, x, \mathbf{x^2}, \mathbf{x^3}) \longmapsto y$$

is a continuous map.

Canonical example: fractional Brownian motion

- $B = (B^1, \ldots, B^d)$
- B^j centered Gaussian process, independence of coordinates
- Variance of the increments:

$$\mathbf{E}[|B_t^j - B_s^j|^2] = |t - s|^{2H}$$

- $H^- \equiv$ Hölder-continuity exponent of B
- If $H = 1/2$, $B =$ Brownian motion
- If $H \neq 1/2$, most natural generalization of BM

Motivations: Engineering, Finance, Biophysics

Iterated integrals and fBm

Nice situation: $H > 1/4$

$\hookrightarrow$ 2 possible constructions for geometric iterated integrals of B.

- Malliavin calculus tools (Ferreiro-Utzet)
- Regularization or linearization of the fBm path (Coutin-Qian)

Conclusion: for $H > 1/4$, one can solve equation

$$dy_t = V_0(y_t)\, dt + V_j(y_t)\, dB_t^j,$$

in the rough paths sense.

Remark: Recent extensions to $H \leq 1/4$ (Unterberger, Nualart-T).

Iterated integrals and Gaussian processes

General setting: $B = (B^1, \ldots, B^d)$ with

- B^is independent copies of B^1
- $\mathbf{E}[B_s^1 B_t^1] \equiv R(s,t)$ covariance function

2-d variations: for $\rho \geq 1$ and $f : [0,1]^2 \to \mathbb{R}$, set

$$V_\rho(f; [0,1]^2) \equiv \sup_{\pi, \tilde{\pi}} \left(\sum_{t_i \in \pi, \tilde{t}_j \in \tilde{\pi}} \left| \Delta_{[t_i, t_{i+1}] \times [\tilde{t}_j, \tilde{t}_{j+1}]} f \right|^\rho \right)^{1/\rho}.$$

Basic assumption: $V_\rho(R; [0,1]^2) < \infty$ for $\rho < 2$.

Result: Under the basic assumption for B

- Iterated integrals of order 2 and 3 exist.
- One can solve equation (1).

Sketch

Malliavin derivative

Result 1, fBm case:
One can differentiate equation (1) in the Malliavin calculus sense.

- By means of pathwise methods (rough paths)
- The derivative takes values in $\mathcal{H} = I_{0+}^{1/2-H}(L^2)$

Notation: $\eta^r = \{D_r y_t;\ t \geq r\}$, with

- $\eta_t^r \in \mathbb{R}^{n\times d}$
- $\eta_t^{r;ij} = D_r^j y_t^i$

Result 2: η^r is solution of the linear equation

$$\eta_t^{r;ij} = V_j^i(y_r) + \int_r^t \partial_k V_0^i(y_u)\, \eta_u^{r;kj}\, du + \int_r^t \partial_k V_l^i(y_u)\, \eta_u^{r;kj}\, dB_u^l. \qquad (2)$$

Malliavin derivatives and densities

Notation: Set $\gamma_t^{ij} = \langle Dy_t^i, Dy_t^j \rangle$.

Criterions: It is well known (see e.g. Nualart's book)

- $\|Dy_t\|_{\mathcal{H}} > 0$ almost surely $\Longrightarrow \mathcal{L}(y_t)$ admits a density.
- $\|\gamma_t^{-1}\| \in L^p \Longrightarrow$ smooth density.

Application of the criterion: In the elliptic case, one can show that γ_t^{-1} is governed by an equation of type (2)
$\hookrightarrow$ estimate $\|\gamma_t^{-1}\| \equiv$ estimate $\|\eta^r\|$.

(Lack of) moments for Malliavin derivative

Moment estimates for (2): on $[0, T]$ and for any $\gamma < H < 1/2$

$$\|\eta^r\|_\gamma \leq (1 + |a|) \exp\left(c\left(\|\mathbf{B^1}\|_\gamma^{1/\gamma} + \|\mathbf{B^2}\|_{2\gamma}^{1/(2\gamma)}\right)\right)$$

See Friz-Victoir, Besalú-Nualart.

Problem: **non integrable bound!**

Other occurences of equation (2):

- Derivatives of flows
- Convergence of numerical schemes
- Ergodic properties

Density results for RDEs

Existing results:

- Case $H > 1/2$:
  - Smooth density in the elliptic case: Hu-Nualart
  - Smooth density in the Hörmander case: Baudoin-Hairer.
  - Further estimates by Baudoin-Ouyang for $H > 1/2$.
- Case $H < 1/2$:
  - Existence of the density in elliptic and Hörmander cases: Cass-Friz, Hairer-Pillai.
  - Smoothness of density, nilpotent case: Hu-T.
  - Smoothness of density, skew-sym. case: Baudoin-Ouyang-T.
- Case $d = n = 1$:
  - Smooth density: Nourdin-Simon.

Recent results

Cass-Lyons-Litterer's breakthrough:

Moments for the Jacobian of RDEs driven by Gaussian processes

Another ingredient for Hörmander's theorem:

Norris type lemma (Hairer-Pillai; Hu-Tindel).

Aim of the talk:

Obtain smoothness of density under Hörmander's conditions for:

- Fractional Brownian motion with $1/4 < H < 1/2$.
- Gaussian process with $V_\rho(R; [0,1]^2) < \infty$ for $\rho < 2$.

Sketch

Hörmander's condition

Family of vector fields: set $\mathcal{V}_0(x) = \{V_i(x);\, i > 0\}$ and

$$\mathcal{V}_{k+1}(x) = \mathcal{V}_k(x) \bigcup \left\{[U, V_j](x);\, U \in \mathcal{V}_k,\, j \geq 0\right\}.$$

Ellipticity (weak form): for all $x \in \mathbb{R}^n$, we have $\mathrm{Span}(\mathcal{V}_0(x)) = \mathbb{R}^n$.

Hörmander's hypoellipticity: for all $x \in \mathbb{R}^n$
$\hookrightarrow$ there exists $p_0 \geq 0$ such that $\mathrm{Span}(\mathcal{V}_{p_0}(x)) = \mathbb{R}^n$.

Additional assumptions

Hypothesis: We assume that

(i) Regularity of vector fields: $V_0, \dots, V_d$ are C_b^∞.

(ii) Hörmander's condition: see previous slide.

(iii) Regularity of B: $V_\rho(R; [0,1]^2) < \infty$ for $\rho < 2$.

(iv) Non degeneracy of B: R satisfies

- Monotonicity for derivatives:
  $\partial_a R(a,b) > 0$ and $\partial^2_{ab} R(a,b) < 0$ for $0 \leq a < b \leq 1$.
- Strong ϕ-local nondeterminism:
  $\mathbf{Var}(\delta B_{st} | \mathcal{F}_{0s} \vee \mathcal{F}_{t1}) \geq \phi(t-s)$ for suitable ϕ.

Remark: Assumptions satisfied when $B \equiv$ fBm with $1/4 < H < 1/2$.

Main result

Theorem

Consider the equation

$$dy_t = V_0(y_t)\, dt + V_j(y_t)\, dB_t^j.$$

Under the previous assumptions: for all $t \in (0,1]$
$\hookrightarrow$ *The random variable y_t admits a $\mathcal{C}^\infty$ density.*

Sketch

Global strategy

Stochastic analysis criterion: G admits a $\mathcal{C}^\infty$ density whenever

1. $G \in \cap_{k\geq 1} \cap_{p \geq 1} \mathbb{D}^{k,p}(\mathbb{R}^n)$ in the Malliavin calculus sense.
2. $\det(\gamma_t^{-1}) \in L^p$ for all $p \geq 1$, where $\gamma_t^{ij} = \langle DG^i, DG^j \rangle$.

Application: we can divide the proof in several steps

1. Integrability of the Malliavin derivatives.
2. Introduction of a process Z^F indexed by vector fields.
3. Lower bounds for $\| \cdot \|_{\mathcal{H}}$.
4. Norris type lemma.

Integrability of the Malliavin derivative

Jacobian of the equation: derivative w.r.t initial condition
$\hookrightarrow$ Denoted J_{st}, and $J_{st} = J_{0t} J_{0s}^{-1}$.

Relationship with Malliavin derivative: one can prove that

- $\mathcal{D}_s^j y_t = J_{st} \, V_j(y_s)$ for $0 \leq s \leq t$.
- Same kind of relation for higher order derivatives.

Consequence: We have

$$\mathbf{E}[\|J\|_\infty^p] < \infty \quad \Longrightarrow \quad X_t \in \cap_{k\geq 1} \cap_{p\geq 1} \mathbb{D}^{k,p}(\mathbb{R}^n)$$

Process Z^F

Definition: We consider

1. A vector field F on $\mathbb{R}^n$.
2. A deterministic vector $\eta \in \mathbb{R}^n$ with $|\eta| = 1$.
3. $Z_t^F \equiv \left\langle \eta,\, J_{0t}^{-1} F(y_t) \right\rangle_{\mathbb{R}^n}$.

Reduction of the non-degeneracy property: we have

$$\det(\gamma_t^{-1}) \in \cap_{p\geq 1} L^p \quad \Longleftarrow \quad \mathbf{P}\left(\|Z^{V_k}\|_{\mathcal{H}} < \varepsilon \right) \leq c_p \varepsilon^p$$

for at least one $k \in \{1, \ldots, m\}$ and for all $p \geq 1$.

Lower bound for $\|\cdot\|_{\mathcal{H}}$

Recall our aim: exhibit $k \in \{1, \ldots, m\}$ such that
$\hookrightarrow$ For all $p \geq 1$, $\mathbf{P}\left(\|Z^{V_k}\|_{\mathcal{H}} < \varepsilon\right) \leq c_p \varepsilon^p$.

Reduction 2: It is easier to prove

$$\mathbf{P}\left(\|Z^{V_k}\|_{\infty} < \varepsilon\right) \leq c_p \varepsilon^p.$$

Important ingredient: show, for all $f \in \mathcal{H} \cap L^{\infty}$:

$$\|f\|_{\mathcal{H}} \geq \|f\|_{\infty} \tag{3}$$

This is obtained by means of

- Non degeneracy conditions on R.
- Resolution of a quadratic programming problem.

Lie brackets showing up

Relation: Z^F is solution of the equation

$$Z_t^F = \langle \eta, F(x) \rangle_{\mathbb{R}^n} + \int_0^t Z_s^{[F,V_0]}\, ds + \sum_{i=1}^m \int_0^t Z_s^{[F,V_i]}\, dB_s^i.$$

In order to take advantage of Lie brackets: Norris type lemma
$\hookrightarrow$ For suitable (controlled) processes A and K, set

$$Z_t = z_0 + \int_0^t A_s\, ds + \int_0^t K_s^* dB_s.$$

Then there exists $r \in (0,1)$ such that

$$\{\|Z\|_\infty \leq \varepsilon\} \quad \Rightarrow_\varepsilon \quad \{\|A\|_\infty \leq \varepsilon^r\} \cap \{\|K\|_\infty \leq \varepsilon^r\}.$$