経路空間上のGibbs測度に関連した微分作用素の一意性問題とその応用

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2007年9月21日

§1. Introduction

▶ Model Case (State space $\cdots\cdots \mathbb{R}^n$)

- $\rho : \mathbb{R}^n \to \mathbb{R}$, strictly positive & locally Lipschitz continuous
- $\mathcal{D} \equiv C_0^\infty(\mathbb{R}^n)$ $(\subset L^2(\rho dx))$
- Pre-Dirichlet form $(\mathcal{E}, \mathcal{D})$

$$\mathcal{E}(u,v) = \frac{1}{2}\int_{\mathbb{R}^n} (\nabla u(x), \nabla v(x))_{\mathbb{R}^n} \rho(x) dx \quad \text{for } u, v \in \mathcal{D}$$

$$\updownarrow$$

$$\mathcal{E}(u,v) = (-\mathcal{L}_0 u, v)_{L^2(\rho dx)} \cdots (*)$$

$$\updownarrow$$

- Pre-Dirichlet operator $(\mathcal{L}_0, \mathcal{D})$

$$\mathcal{L}_0 u = \frac{1}{2}\Delta u + \frac{1}{2}\left(\frac{\nabla \rho}{\rho}, \nabla u\right)_{\mathbb{R}^n}, \quad u \in \mathcal{D}$$

$$\left(\begin{aligned} A_0 &= \sqrt{\rho}\,\mathcal{L}_0 \sqrt{\rho}^{-1} \\ &= \underline{\frac{1}{2}\Delta - \frac{1}{2}\frac{\Delta(\sqrt{\rho})}{\sqrt{\rho}}} \quad \text{in } L^2(dx) \end{aligned} \right)$$

Unitary Equivalence

Schrödinger operator

$\rightsquigarrow$ By (✱), $(\mathcal{L}_0, \mathcal{D})$ has a self-adjoint extension on $L^2(\rho dx)$

(Friedrichs extension $(\mathcal{L}_\mu, \mathrm{Dom}(\mathcal{L}_\mu))$)

▷ <u>Uniqueness Problem</u>

Is $(\mathcal{L}_0, \mathcal{D})$ essentially self-adjoint on $L^2(\rho dx)$?

($\Leftrightarrow$ Is $(\mathcal{L}_\mu, \mathrm{Dom}(\mathcal{L}_\mu))$ the unique self-adjoint extension of $(\mathcal{L}_0, \mathcal{D})$?)

$\rightsquigarrow$ **Yes!!** (Wielens ('85), ……; Eberle's book)

▷ $\{e^{t\mathcal{L}_\mu}\}_{t\geq 0}$ is the unique symmetric C_0-semigroup on $L^2(\rho dx)$ whose generator is a extension of $(\mathcal{L}_0, \mathcal{D})$.

$\rightsquigarrow$ Uniqueness of

(i) Cauchy Problem:

$$\begin{cases} \dfrac{\partial}{\partial t} u = \dfrac{1}{2}\Delta u + \dfrac{1}{2}\left(\dfrac{\nabla \rho}{\rho}, \nabla u\right)_{\mathbb{R}^n} \\ u_0 = f \in L^2(\rho dx) \end{cases} \rightsquigarrow u(t,\cdot) = e^{t\mathcal{L}_\mu} f$$

(ii) Markov Process:

$$e^{t\mathcal{L}_\mu} f = \mathbb{E}[f(X_t)]$$

$$dX_t = d\beta_t + \frac{1}{2}\left(\frac{\nabla \rho}{\rho}\right)(X_t)\,dt$$

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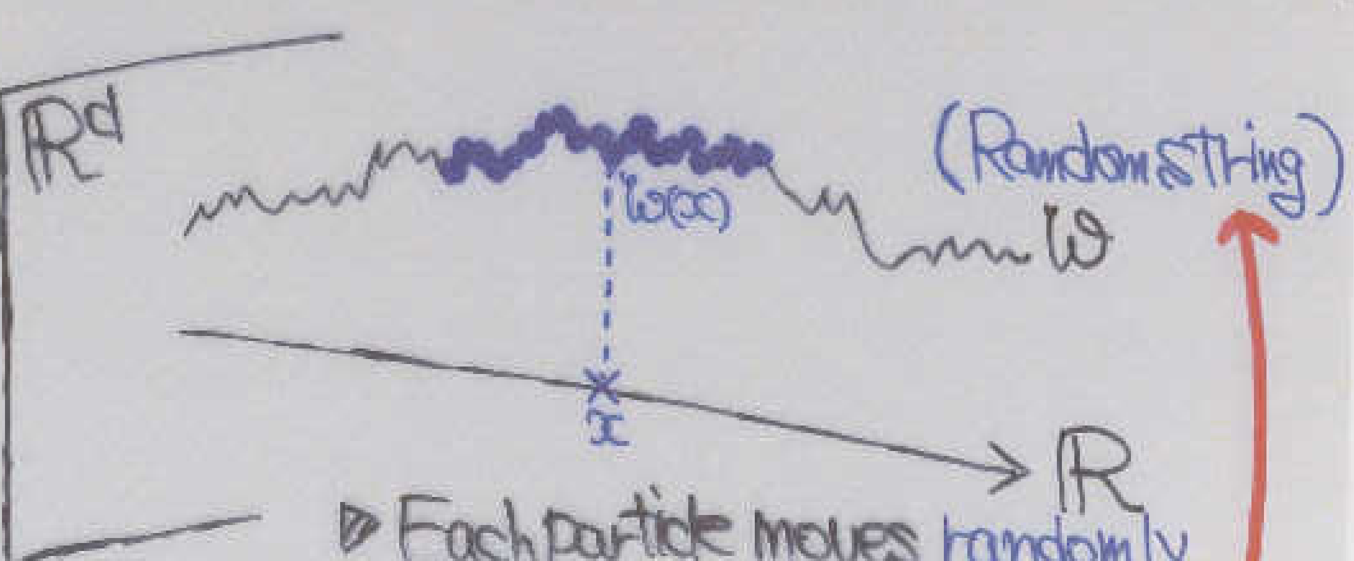
- **Our Concern** … infinite dimensional state space case ?
- **Our Model** … $P(\Phi)_1$-quantum field (QFT) interface model

- State space … $C(\mathbb{R}, \mathbb{R}^d)$
  (infinite volume path space)
- tangent space … $H = L^2(\mathbb{R}, \mathbb{R}^d)$
- Underlying measure ….
  Gibbs measure μ associated with the (formal) Hamiltonian

$$\mathcal{H}(w) \equiv \frac{1}{2}\int_{\mathbb{R}} |\dot{w}(x)|^2_{\mathbb{R}^d}\,dx + \int_{\mathbb{R}} U(w(x))\,dx$$

where $U : \mathbb{R}^d \longrightarrow \mathbb{R}$ (self-interaction potential)



- Each particle moves randomly
- interactions among particles.

Heuristically, μ is given by

$$\mu(dw) \overset{\text{``}}{=}{}_{\text{''}} \underbrace{Z^{-1}}_{\text{normalization}} \exp(-\mathcal{H}(w)) \underbrace{\prod_{x \in \mathbb{R}} dw(x)}_{\text{“flat measure”}}$$

- Consider a (Pre-) Dirichlet form

$$\mathcal{E}(F,G) = \frac{1}{2}\int (D_H F(w), D_H G(w))_H \mu(dw)$$

for $F, G \in \underbrace{\mathcal{F}C_b^\infty}_{\text{Smooth cylinder functions}}$

$\rightsquigarrow$ We can consider a (Pre-) Dirichlet operator $(\mathcal{L}_0, \mathcal{F}C_b^\infty)$ through

$$\mathcal{E}(F,G) = (-\mathcal{L}_0 F, G)_{L^2(\mu)}.$$

Our Problem

Is $(\mathcal{L}_0, \mathcal{F}C_b^\infty)$ essentially self-adjoint on $L^2(\mu)$?

The main purpose of this talk is to give an answer to this problem and discuss some applications !!

▶ Related works for infinite-dimensional settings

(I) • Takeda ('85), • Röckner-Zhang ('92)
• Shigekawa ('95) etc ...

$\Longrightarrow$ Functional analytic approach (e.g. Malliavin Calculus) under

$$\mu(dw) = \rho(w) W(dw)$$

(II) Finite dimensional approximation approach with

$\Longrightarrow$ Stochastic analysis

- Albeverio-Kondratiev-Röckner ('95~)
- Park-Yoo ('97)
- Eberle's book etc ...

$\Longrightarrow$ PDE

- Liskevich-Semenov ('92)
- Liskevich-Röckner ('98) etc ...

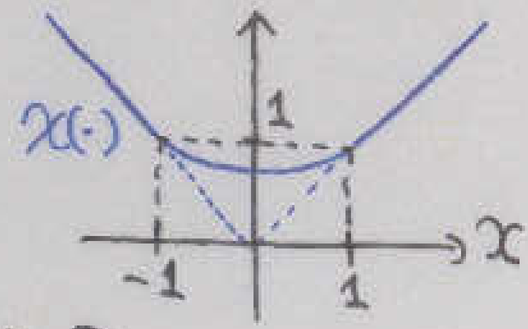
(III) Da Prato (etc)
SPDE approach
(This is based on the idea in (II))

§2. Framework & Results

At the beginning, we introduce some notations and objects we will working with.

- Weight function $\rho_r \in C^\infty(\mathbb{R},\mathbb{R})$, $r\in\mathbb{R}$

  $\rho_r(x) \equiv e^{r\chi(x)}$



- $E \equiv L^2(\mathbb{R},\mathbb{R}^d;\ \rho_{-2r}(x)dx)$

  ($r>0$ fixed with $k_1+2r^2>0$)

  $(X,Y)_E \equiv \int_{\mathbb{R}} (X(x),Y(x))_{\mathbb{R}^d}\, \rho_{-2r}(x)\,dx$

- $H \equiv L^2(\mathbb{R},\mathbb{R}^d)$

  ($E^* = L^2(\mathbb{R},\mathbb{R}^d;\rho_{2r}(x)dx) \subset H^* \equiv H \subset E$)

Before gluing a Gibbs measure μ, we impose some conditions on the potential function $U:\mathbb{R}^d \to \mathbb{R}$.

(U1): $U \in C(\mathbb{R}^d, \mathbb{R})$,

$\exists k_1 \in \mathbb{R}$, " $\nabla^2 U \geq -k_1$ " i.e.,

$$U(z) = -\frac{k_1}{2}|z|^2_{\mathbb{R}^d} + \underbrace{\tilde{U}(z)}_{\text{convex function}}$$

(In the case of $U \in C^1(\mathbb{R}^d, \mathbb{R})$

(U1) $\Longleftrightarrow$ **(U1)'**: one-sided Lipschitz)

(U2): $\exists k_2 > 0, \exists p > 0$ s.t

$$|\tilde{\nabla} U(z)|_{\mathbb{R}^d} \leq k_2 (1 + |z|^p_{\mathbb{R}^d}) \quad \text{a.e. } z \in \mathbb{R}^d$$

where $\tilde{\nabla} U(z) = -k_1 z + \underbrace{\partial_0 \tilde{U}}_{\text{minimal section of the subdifferential } \partial\tilde{U}}(z)$

(U3): $\lim_{|z| \to \infty} U(z) = \infty$

* <u>Example</u> $U(z) = \sum_{j=0}^{2m} a_j |z|^j$; $m \in \mathbb{N}$.

where $a_{2m} > 0$.

Especially, we are interested in

- $U(\underline{x}) = a|\underline{x}|^2_{\mathbb{R}^d}$ (square potential)
- $U(\underline{x}) = a(|\underline{x}|^4_{\mathbb{R}^d} - |\underline{x}|^2_{\mathbb{R}^d})$
  (double-well potential)

Under **(U1), (U3)**, we can construct a Gibbs measure μ in the following manner:

▷ We consider a Schrödinger operator

$$\underline{H_U \equiv -\frac{1}{2}\Delta + U} \text{ on } L^2(\mathbb{R}^d, \mathbb{R})$$

H_U has purely discrete spectrum and a complete set of eigenfunctions

$\rightsquigarrow$
- $\underline{\lambda_0 (> \min U)}$: the lowest eigenvalue of H_U
- $\underline{\Omega}$: ground state of H_U with $\underline{\|\Omega\|_{L^2(\mathbb{R}^d,\mathbb{R})} = 1}$, $\underline{\Omega > 0}$

i.e., $\underline{H_U \Omega = \lambda_0 \Omega}$ $\underline{(e^{-tH_U}\Omega = e^{-t\lambda_0}\Omega)}$

▷ A key connection between H_0 & a measure on path space

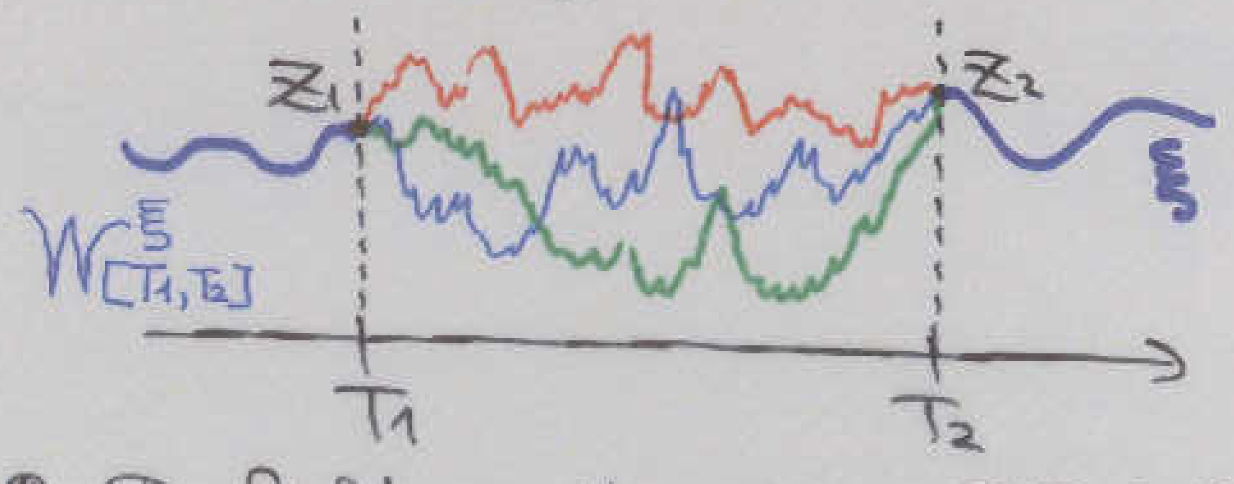
⇝ Feynman-Kac's formula

● $\mathcal{W}^{z_1,z_2}_{[T_1,T_2]}$ $(T_1 < T_2,\ z_1, z_2 \in \mathbb{R}^d)$

--- pinned BM measure on $C([T_1,T_2];\mathbb{R}^d)$

$$\mathcal{W}^{z_1,z_2}_{[T_1,T_2]}(C([T_1,T_2];\mathbb{R}^d)) = P(T_2-T_1, z_1, z_2)$$

$$\left(= \frac{1}{\sqrt{2\pi(T_2-T_1)}^{\,d}} \exp\left(-\frac{|z_2-z_1|^2_{\mathbb{R}^d}}{2(T_2-T_1)}\right)\right)$$



● σ-fields on the space $C(\mathbb{R};\mathbb{R}^d)$

$\mathcal{B} \equiv \sigma(\omega(x);\ x \in \mathbb{R})$

$\mathcal{B}_{[T_1,T_2]} \equiv \sigma(\{\omega(x) : T_1 \le x \le T_2\})$

$\mathcal{B}_{[T_1,T_2],c} \equiv \sigma(\{\omega(x) : x \le T_1,\ x \ge T_2\})$

▷ We sometimes regard $\mathcal{W}^{z_1,z_2}_{[T_1,T_2]}$ on $(C(\mathbb{R};\mathbb{R}^d), \mathcal{B})$ by $\xi(x) \equiv z_1\ (x \le T_1) \equiv z_2\ (x \ge T_2)$

$\rightsquigarrow$ We define a probability measure μ (U-Gibbs measure, $P(\Phi)_1$-measure) on $C(\mathbb{R}, \mathbb{R}^d)$ by

$$\mu(A) \equiv e^{(T_2-T_1)\lambda_0} \int_{\mathbb{R}^d} dz_1\, \Omega(z_1) \int_{\mathbb{R}^d} dz_2\, \Omega(z_2)\, \mathbb{E}^{W^{z_1,z_2}_{[T_1,T_2]}}\left[e^{-\int_{T_1}^{T_2} U(w(x))dx} ; A \right]$$

for $A \in \mathcal{B}_{[T_1,T_2]}$,

and by extending the above to a measure on $\mathcal{B}$.

▶ <u>Remark</u> $e^{-(T_2-T_1)H_0}(z_1,z_2)$ ← kernel function of e^{-tH_0}

$$= \mathbb{E}^{W^{z_1,z_2}_{[T_1,T_2]}}\left[e^{-\int_{T_1}^{T_2} U(w(x))dx} \right]$$

(Feynman-Kac's formula)

▶ <u>Properties of μ:</u>

(I) $\mu(\mathcal{C}) = 1$ where

$$\mathcal{C} \equiv \bigcap_{r>0} \{ w \in C(\mathbb{R}, \mathbb{R}^d) ; \|w\|_{r,\infty} < \infty \}$$

$$\|w\|_{r,\infty} := \sup_{x \in \mathbb{R}} |w(x)|_{\mathbb{R}^d} f_r(x)$$

($\rightsquigarrow$ Ω decays exponentially!)

$\rightsquigarrow$ Since $C \subset E$ is continuous, we can regard μ as a probability measure on E.

($\rightsquigarrow$ (E, H, μ) : rigged Hilbert space)

(II) $\mu \cdots$ Probability law on $C(\mathbb{R}, \mathbb{R}^d)$ induced by a Markov process $(\omega_t)_{t \in \mathbb{R}}$

with $\begin{cases} \text{generator} = \frac{1}{2}\Delta - \left(\frac{\nabla \Omega}{\Omega}, \nabla \cdot\right) \\ \text{Stationary measure} = \underline{\Omega(z)^2 dz} \end{cases}$

SDE $\cdots$ $d\omega_t = d\beta_t - \frac{\nabla \Omega}{\Omega}(\omega_t) dt$

$\rightsquigarrow$ $\int \left(\int_{\mathbb{R}} |w(x)|^m_{\mathbb{R}^d} \rho_{-2r}(x) dx \right) \mu(dw)$

$\leq \frac{1}{r} \int_{\mathbb{R}^d} |z|^m \underline{\Omega(z)^2 dz} < +\infty$

$m \in \mathbb{N}$, $r > 0$

(III) DLR-equation

$\mathbb{E}^{\mu}[\mathbb{1}_A | \mathcal{B}_{[T_1, T_2], c}](\xi)$

$= Z^{-1}_{[T_1, T_2]}(\xi) \, \mathbb{E}^{W^{\xi}_{[T_1, T_2]}}\left[\mathbb{1}_A e^{-\int_{T_1}^{T_2} U(w(x)) dx}\right]$

for μ-a.s. ξ, $\forall T_1 < T_2$, $\forall A \in \mathcal{B}$.

(IV) As a corollary of DLR-equation, we have the following quasi-invariance

i.e. $\forall h \in C_0^\infty(\mathbb{R}, \mathbb{R}^d)$,

$$\mu \sim \mu(h + \cdot) \quad \&$$

$$\mu(h + dw) = \underline{\Lambda(h, w)}\, \mu(dw)$$

where
$$\underline{\Lambda(h, w)} = \exp\Big\{ \int_{\mathbb{R}} \Big(U(w(x)) - U(w(x) + h(x)) - \frac{1}{2} |\dot{h}(x)|^2_{\mathbb{R}^d} + (w(x), \Delta_x h(x))_{\mathbb{R}^d} \Big) dx \Big\}$$

$\Delta_x = \frac{d^2}{dx^2}$

▶ the space of smooth cylinder functions:

$$\underline{\mathcal{F}C_b^\infty} \equiv \Big\{ F(w) = f(\langle w, \varphi_1 \rangle, \cdots, \langle w, \varphi_n \rangle);\ n \in \mathbb{N},\ f \in C_b^\infty(\mathbb{R}^n, \mathbb{R}),\ \varphi_1, \varphi_2, \cdots \varphi_n \in C_0^\infty(\mathbb{R}, \mathbb{R}^d) \Big\}$$

where $\langle w, \varphi_i \rangle \equiv \int_{\mathbb{R}} (w(x), \varphi_i(x))_{\mathbb{R}^d}\, dx$ for $w \in E$.

- $\mathcal{F}C_b^\infty \underset{\text{dense}}{\hookrightarrow} L^2(\mu)$.

▷ H-Fréchet derivative $D_H F : E \to H$

$$D_H F(w)(\cdot) \equiv \sum_{i=1}^{n} \partial_i f(\langle w, \varphi_1 \rangle, \cdots \langle w, \varphi_n \rangle)\, \varphi_i(\cdot)$$

for $F \in \mathcal{FC}_b^\infty$

▷ We define a (pre-) Dirichlet form $(\mathcal{E}, \mathcal{FC}_b^\infty)$ by

$$\mathcal{E}(F, G) \equiv \frac{1}{2} \int_E (D_H F(w), D_H G(w))_H \, \mu(dw)$$

for $F, G \in \mathcal{FC}_b^\infty$.

⟿ By the quasi-invariance of μ, we have

☆ $\boxed{\mathcal{E}(F, G) = (-\mathcal{L}_0 F, G)_{L^2(\mu)}}$ where

$$\begin{aligned} \mathcal{L}_0 F(w) &= \frac{1}{2} \mathrm{Tr}(D^2 F(w)) \\ &\quad + \frac{1}{2} \left\{ \langle w, \Delta_x D_H F(w(\cdot)) \rangle - \langle \widetilde{\nabla} U(w(\cdot)), D_H F(w)(\cdot) \rangle \right\} \\ &= \frac{1}{2} \sum_{i,j=1}^{n} \partial_i \partial_j f(\cdots\cdots) \langle \varphi_i, \varphi_j \rangle \\ &\quad + \frac{1}{2} \sum_{i=1}^{n} \partial_i f(\cdots) \left\{ \langle w, \Delta_x \varphi_i \rangle - \langle \widetilde{\nabla} U(w(\cdot)), \varphi_i \rangle \right\} \end{aligned}$$

By ☆, $(\mathcal{L}_0, \mathcal{F}C_b^\infty)$ is dissipative on $L^2(\mu)$

i.e. $\underline{(\mathcal{L}_0 F, F)_{L^2(\mu)} \leq 0}$ $\forall F \in \mathcal{F}C_b^\infty$

$\rightsquigarrow$ $(\mathcal{L}_0, \mathcal{F}C_b^\infty)$: closable

$\rightsquigarrow$ $\exists (\overline{\mathcal{L}_0}, \mathrm{Dom}(\overline{\mathcal{L}_0}))$: closure of $(\mathcal{L}_0, \mathcal{F}C_b^\infty)$ w.r.t graph-norm

(dissipativity also holds)

<u>Main Theorem</u> [K-Röckner]

(i) The pre-Dirichlet operator $(\mathcal{L}_0, \mathcal{F}C_b^\infty)$ is essentially self-adjoint on $L^2(\mu)$ under (U1)~(U3).

$\Longleftrightarrow$ $(\overline{\mathcal{L}_0}, \mathrm{Dom}(\overline{\mathcal{L}_0}))$ is self-adjoint on $L^2(\mu)$

$\Longleftrightarrow$ $(\mathcal{L}_\mu, \mathrm{Dom}(\mathcal{L}_\mu)) = (\overline{\mathcal{L}_0}, \mathrm{Dom}(\overline{\mathcal{L}_0}))$

(ii) If we impose $\underline{U \in C^1(\mathbb{R}^d, \mathbb{R})}$,

$$e^{t\overline{\mathcal{L}_0}} F = P_t F \quad \mu\text{-a.s} \quad F \in L^2(\mu).$$

where $\{P_t\}_{t \geq 0}$ is the transition semi-group corresponding to the parabolic SPDE **(GL)**

$$dX_t(x) = \frac{1}{2}\{\Delta_x X_t(x) - \nabla U(X_t(x))\}dt + dB_t(x), \quad x \in \mathbb{R},\ t > 0$$

where $\{B_t\}_{t\geq 0}$: H-cylindrical BM.

Remark: Friedrichs extension $(\mathcal{L}_\mu, \mathrm{Dom}(\mathcal{L}_\mu))$

$\rightsquigarrow (\mathcal{E}, \mathcal{D}(\mathcal{E}))$: the closure of $(\mathcal{E}, \mathcal{F}C_b^\infty)$ w.r.t $\mathcal{E}_1^{1/2}$-norm

(Minimal Dirichlet form)

- How about the uniqueness of Dirichlet form? $\rightsquigarrow$ Markov uniqueness

▷ $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$: Dirichlet form in $L^2(\mu)$ is an extension of $(\mathcal{L}_0, \mathcal{F}C_b^\infty)$

$$\overset{\text{def}}{\Longleftrightarrow} \begin{cases} \cdot\ \mathcal{F}C_b^\infty \subset \mathrm{Dom}(\mathcal{E}) \\ \cdot\ \mathcal{E}(F,G) = (-\mathcal{L}_0 F, G)_{L^2(\mu)} \quad \text{for } \forall F \in \mathcal{F}C_b^\infty,\ \forall G \in \mathrm{Dom}(\mathcal{E}) \end{cases}$$

- Albeverio–Kusuoka (–Röckner)

$\longrightarrow$ Characterization of the maximal Dirichlet form $(\mathcal{E}^+, \mathcal{D}(\mathcal{E}^+))$

Corollary

$$(\mathcal{E}, \mathcal{D}(\mathcal{E})) = (\mathcal{E}^+, \mathcal{D}(\mathcal{E}^+))$$

§3. Sketch of the Proof

Aim $(\mathcal{L}_0, \mathrm{Dom}(\mathcal{L}_0))$: m-dissipative

i.e. $\exists \lambda > 0,\ \mathrm{Range}(\lambda - \mathcal{L}_0) = L^2(\mu)$

$\Uparrow$ Lumer–Phillips thm

It is sufficient to show

$\exists \lambda > 0,\ \mathcal{F}C_b^\infty \subset \mathrm{Range}(\lambda - \mathcal{L}_0)\ (\subset L^2(\mu))$

Hence it is sufficient to show

$\exists \lambda > 0,\ \forall F \in \mathcal{F}C_b^\infty,\ \exists \Phi \in \mathrm{Dom}(\mathcal{L}_0)$

s.t. $\lambda \Phi - \mathcal{L}_0 \Phi = F \cdots (\#)$

▷ Candidate:

$$\underline{\Phi = \int_0^\infty e^{-\lambda t} P_t F \, dt}, \quad \lambda > \frac{|k|}{2} + h^2$$

$$\Uparrow ?$$

▷ Facts on the SPDE (GL) (Iwata, Funaki, ⋯)

Under (U1)', (U2),

(I) SPDE (GL) has a unique (pathwise) solution $(X_t^w(\cdot))_{t \geq 0}$ living in $C([0,\infty), \mathcal{C})$ for an initial data $w \in \mathcal{C}$.

(II) For $F \in \mathcal{F}C_b^\infty$, we set by

$$P_t F(w) \equiv \mathbb{E}\left[F(X_t^w)\right], \quad w \in \mathcal{C}, \ t \geq 0.$$

Then $(P_t)_{t \geq 0}$ can be extended to a symmetric C_0-contraction semigroup on $L^2(\mu)$ (reversibility of μ)

(III) Its infinitesimal generator is an extension of $(\mathcal{L}_0, \mathcal{F}C_b^\infty)$.

(← An easy consequence of Itô's formula)

▷ Difficulty: It is difficult to show $\Phi \in Dom(\overline{\mathcal{L}_0})$ directly!!

⇝ How to show ?

We insert a tractable space which corresponds to the Ornstein-Uhlenbeck (OU-) operator L i.e.,

$$\overline{\mathcal{L}_0} = L + (b(\cdot), D\cdot)_E$$

$$\left(dY_t(x) = \frac{1}{2}\{\Delta_x Y_t(x) - {}^{\exists}K\, Y_t(x)\}dt + dB_t(x),\ x \in \mathbb{R},\ t > 0 \right)$$

▷ Then, it is sufficient to check

$$(\#)' \cdots \lambda\Phi - L\Phi - (b(\cdot), D\Phi(\cdot))_E = F$$

$$\left(\int_0^\infty e^{-\lambda t} DP_t F\, dt \right)$$

"Prop 3.6"

⇝ Representation of DP_t ($D_H P_t$) via coupling method & gradient estimate (Bismut, Elworthy-Li, Cerrai etc)

$$t>0 \begin{cases} \|DP_t F\|_E \le e^{(K+2\kappa)t/2} P_t(\|DF\|_E) \\ \|D_H P_t F\|_H \le e^{Kt/2} P_t(\|D_H F\|_H) \cdots ◎ \end{cases}$$

§4. Application (Riesz's transforms)

- Here we impose the following conditions instead of (U1), (U2).

(U1)'': $U \in C^2(\mathbb{R}^d, \mathbb{R})$, $\nabla^2 U \geq -{}^{\exists}K_1$

(U2)': ${}^{\exists}K_2 > 0$, ${}^{\exists}p > 0$ s.t.

$$|\nabla U(z)|_{\mathbb{R}^d} + |\nabla^2 U(z)|_{\mathbb{R}^d \otimes \mathbb{R}^d} \leq K_2 (1 + |z|^p_{\mathbb{R}^d})$$

- Moreover, we rewrite $(2\mathcal{L}_0, \mathcal{F}C_b^\infty)$ by $(\mathcal{L}_0, \mathcal{F}C_b^\infty)$. i.e.

$$(\mathcal{L}_0, \mathcal{F}C_b^\infty) \longleftrightarrow \mathcal{E}(F,G) = \int_E (D_H F(w), D_H G(w))\,\mu(dw)$$

$$\longleftrightarrow dX_t(x) = \{\Delta_x X_t(x) - \nabla U(X_t(x))\}dt + \sqrt{2}\, dB_t(x)$$

$\longleftrightarrow \{P_t\}_{t \geq 0}$
strongly continuous contraction semigroup on $L^p(\mu)$ $(1 \leq p < \infty)$
(Riesz-Thorin)

We denote by its generator
$$\mathcal{L} = \mathcal{L}_p \text{ on } L^p(\mu).$$
(Note that $\mathcal{L}_2 = \mathcal{L}_2 = \mathcal{L}_\mu$)

▷ Here we define <u>Riesz's transform</u> by

$$R_\alpha(\mathcal{L}) \equiv D_H(\alpha - \mathcal{L})^{-1/2}, \quad \alpha > 0.$$

▶ **Problem:** How about the boundedness of $R_\alpha(\mathcal{L})$ on $L^p(\mu)$ for $1 < {}^\forall p < \infty$?

(⇝ Equivalence of Sobolev norms:

$$\|F\|_{L^p(\mu)} + \|D_H F\|_{L^p(\mu;H)} \sim \|\sqrt{1-\mathcal{L}}\, F\|_{L^p(\mu)})$$

($\|R_\alpha(\mathcal{L})F\|_{L^2(\mu)} \le \|F\|_{L^2(\mu)}$ ⋯ Elementary spectral theory)

Theorem Under (U1)'', (U2)', (U3), for $1 < {}^\forall p < \infty$, ${}^\forall \alpha > k_1 \vee 0$,

$$\|R_\alpha(\mathcal{L})F\|_{L^p(\mu)} \le C_p \|F\|_{L^p(\mu)}, \quad F \in \mathcal{FC}_b^\infty$$

▷ Sketch of the Proof:

(I) Littlewood–Paley–Stein inequality [K–Miyokawa '07]

(gradient estimate ◎ plays a key role!)

(II) intertwining property for semigroups

(IT) .... $D_H P_t F = \vec{P}_t D_H F, \quad F \in \mathcal{D}(\mathcal{E})$

⇝ How to show this identity?

(i) generator version of (IT)

(Rather easier !!)

(IT)' .... $D_H \mathcal{L}_0 F = \vec{\mathcal{L}}_0 D_H F, \quad F \in \mathcal{F}C_b^\infty$

where $(\vec{\mathcal{L}}_0, (\mathcal{F}C_b^\infty)_H)$ is given by

vector-valued smooth cylinder fctions

$$\vec{\mathcal{L}}_0 \Theta(w) = \sum_{i,j=1}^{n} \sum_{k=1}^{m} \partial_i \partial_j f_k(\cdots) \langle \varphi_i, \varphi_j \rangle e_k(\cdot)$$

$$+ \sum_{i=1}^{n} \sum_{k=1}^{m} \partial_i f_k(\cdots) \{ \langle w, \Delta_x \varphi_i \rangle - \langle \nabla U(w(\cdot)), \varphi_i \rangle \} e_k(\cdot)$$

$$+ \underbrace{\sum_{k=1}^{m} f_k(\cdots) \{ \Delta_x e_k(\cdot) - \nabla^2 U(w(\cdot)) [e_k(\cdot)]_{\mathbb{R}^d} \}}_{\text{(New term)}}$$

for $\Theta(w) = \sum_{k=1}^{m} f_k(\langle w, \varphi_1 \rangle, \cdots \langle w, \varphi_n \rangle) e_k(\cdot)$

$\Theta \in (\mathcal{F}C_b^\infty)_H, \qquad e_k \in C_0^\infty(\mathbb{R}, \mathbb{R}^d) \subset H$

(ii) Construction of $\{\vec{P}_t\}$

Define a bi-linear form by

$$\vec{\mathcal{E}}(\theta,\eta) \equiv (-\vec{\mathcal{L}}_0\theta,\eta)_{L^2(\mu;H)}$$

for $\theta,\eta \in (\mathcal{F}C_b^\infty)_H$.

$\Longmapsto$ $\underline{\vec{\mathcal{E}}(\theta,\theta) \geq -K_1\|\theta\|^2_{L^2(\mu;H)}}$

(D1)"

$\Longmapsto$ $^\exists \underline{(\vec{\mathcal{L}}_\mu, \mathrm{Dom}(\vec{\mathcal{L}}_\mu))}$: Friedrichs extension of $(\vec{\mathcal{L}}_0, (\mathcal{F}C_b^\infty)_H)$ on $L^2(\mu;H)$

($\rightsquigarrow$ $(\vec{\mathcal{E}}, \mathcal{D}(\vec{\mathcal{E}}))$: minimal extension)

- $\underline{\vec{P}_t \equiv e^{t\vec{\mathcal{L}}_\mu}}$: Symmetric strongly conti-semigroup on $L^2(\mu;H)$

(iii) **(IT)'** $\Longrightarrow$ **(IT)**

By the Main Theorem (Essential self-adjointness of $(\mathcal{L}_0, \mathcal{F}C_b^\infty)$), we can use Shigekawa's result!!