Cubature formulas on Euclidean space and Wiener space

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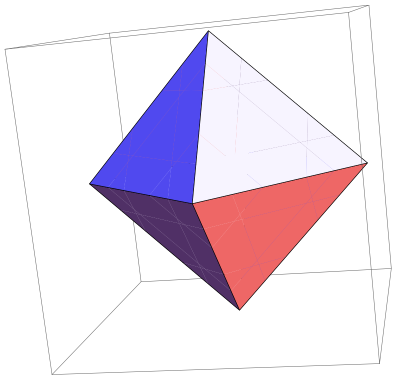
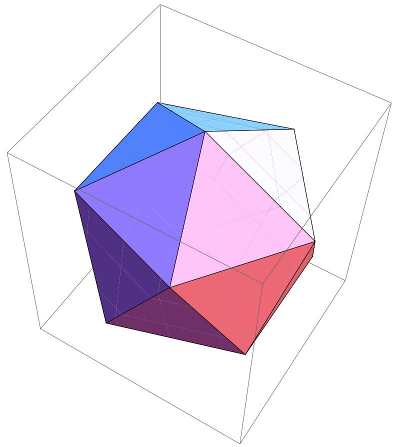
Cubature formulas on Euclidean space

$\Omega \subset \mathbb{R}^n$, μ : a probability meas. on Ω,
$X \subset \Omega$: a finite subset, i.e., $|X| < \infty$, $w : X \to \mathbb{R}_{>0}$.

Then, a pair (X, w) forms a cubature formula of degree t, iff

$$\int_\Omega f(x) d\mu(x) = \sum_{x \in X} w(x) f(x), \quad \forall f \in \mathcal{P}_t(\mathbb{R}^n).$$

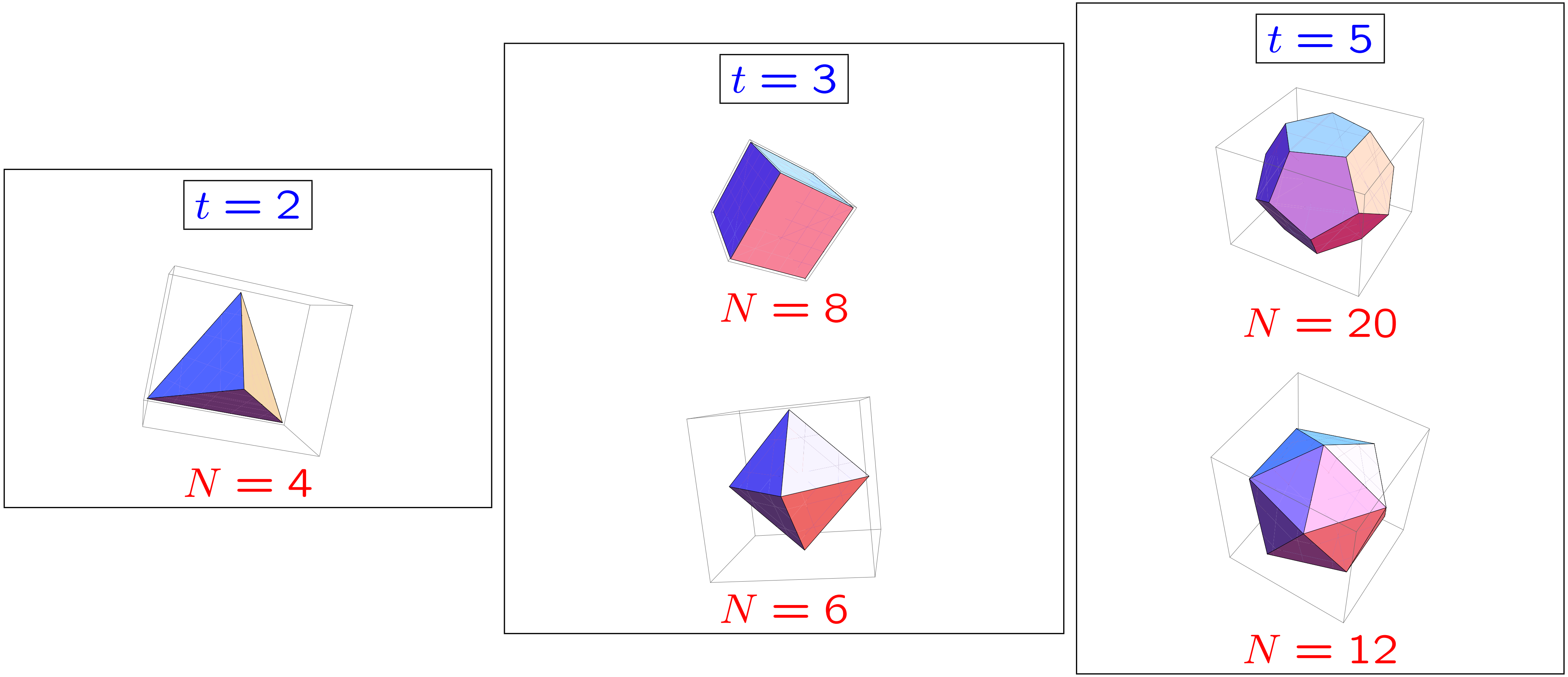


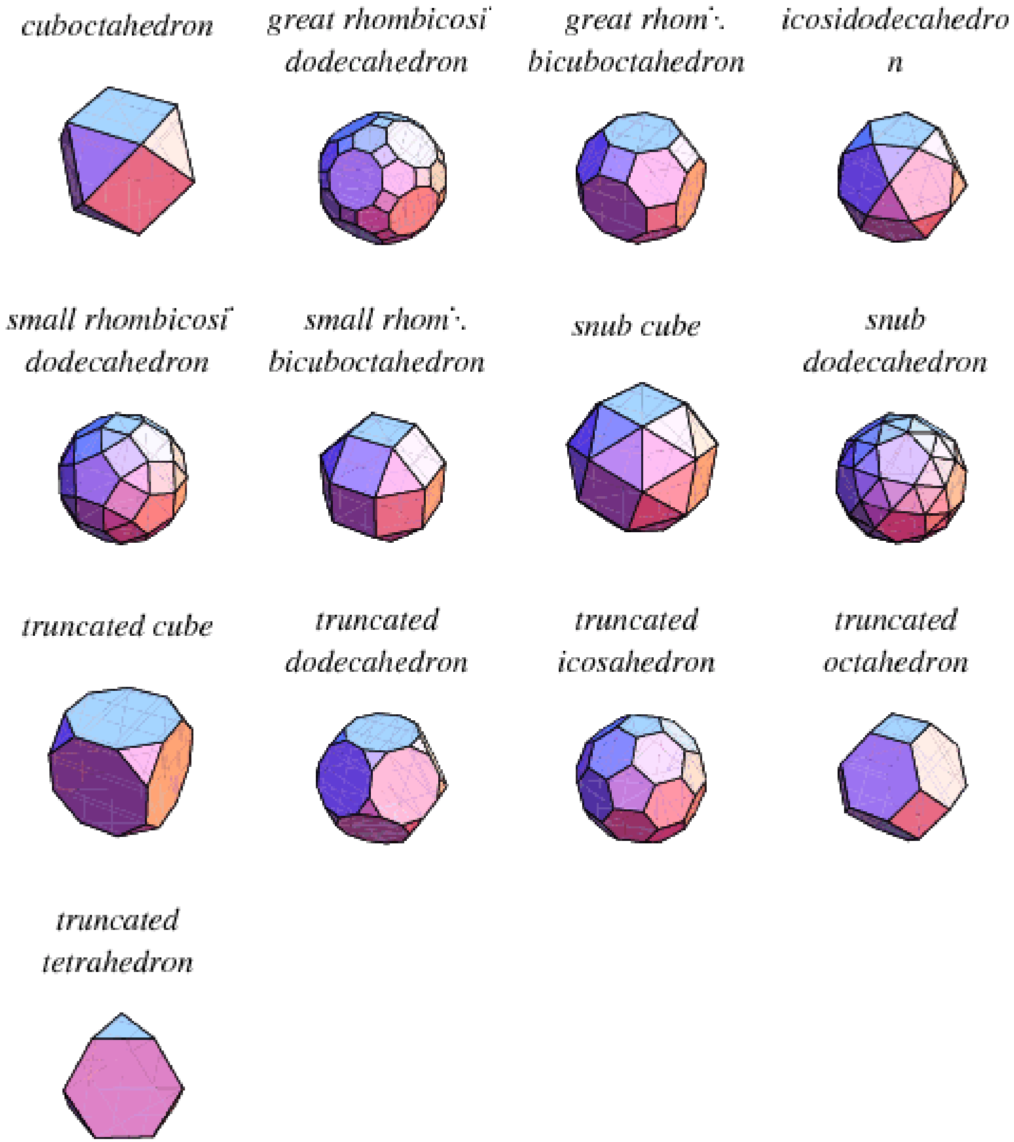


- Vertices of icosahedron form a cf for the integral on the sphere.

$$\Omega = S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}, \quad \sigma : \text{the surface meas. on } S^2,$$

$$\frac{1}{|S^2|} \int_{S^2} f(x) d\sigma(x) = \frac{1}{N} \sum_{i=1}^{N} f(x_i), \quad \forall f \in \mathcal{P}_t(\mathbb{R}^3).$$





Problem. Do these solids form cubature formulas?

Note. Cubature problem is expressed in terms of

a moment generating fun.

Y : a random variable, m_i : the ith moment

$$E[e^{\lambda Y}] = 1 + \lambda m_1 + \frac{\lambda^2 m_2}{2!} + \frac{\lambda^3 m_3}{3!} + \cdots + \frac{\lambda^t m_t}{t!} + \frac{\lambda^{t+1} m_{t+1}}{(t+1)!} + \cdots$$

We want to find a discrete random variable Z satisfying

$$E[e^{\lambda Z}] = 1 + \lambda m_1 + \frac{\lambda^2 m_2}{2!} + \frac{\lambda^3 m_3}{3!} + \cdots + \frac{\lambda^t m_t}{t!} + ?\lambda^{t+1} + ?\lambda^{t+2} + \cdots .$$

Why we find cubature formulas?

Observatory

Cubature formulas on the sphere are needed in statistics. For example, to estimate parameters of a regression model on the sphere, cubature formula are an important tool.

$S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$: the $(n-1)$-unit sphere,

$f_1(x), \ldots, f_N(x)$: a basis of $\mathcal{P}_e(S^{n-1})$, $\left(N < \binom{n+e}{e} = \dim(\mathcal{P}_e(\mathbb{R}^n)) \right)$

$\theta_1, \ldots, \theta_N$: unknown parameters.

$Y(x)$: a observation on a point $x \in S^{n-1}$, i.e., $Y(x)$ is of the form

$$Y(x) = \theta f^T(x) + \epsilon(x)$$

where $f = (f_1, \ldots, f_N)$, $\theta = (\theta_1, \ldots, \theta_N)$ and $\epsilon(x)$ is a noise;

$$E[\epsilon(x)] = 0, \quad E[\epsilon(x)\epsilon(y)] = \begin{cases} \sigma^2 & x = y, \\ 0 & x \neq y. \end{cases}$$

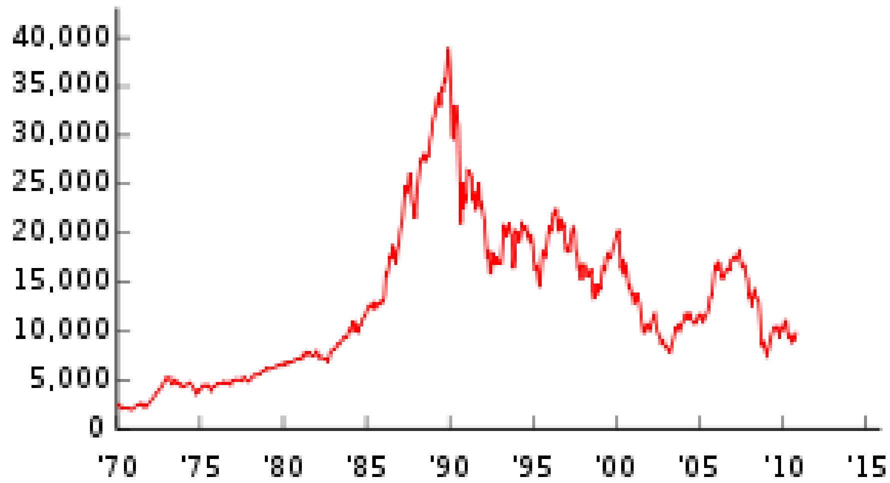
We want to find a "**good**" estimation θ by using

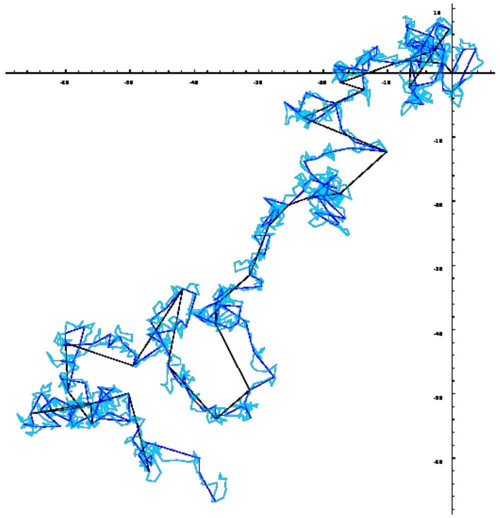
m observation on points $x_1, \ldots, x_m \in S^{n-1}$.

In the latter of this talk, we mainly focus on cubature formulas for the Gaussian integral on $\mathbb{R}^n$:

$$\frac{1}{\pi^{n/2}}\int_{\mathbb{R}^n} f(x)e^{-\|x\|^2}dx = \sum_{x\in X} w(x)f(x), \quad \forall f \in \mathcal{P}_t(\mathbb{R}^n).$$

- N. Victoir, Asymmetric cubature formulae with few points in high dimension for symmetric measures. *SIAM J. Numer. Analysis*, 2004.





Cubature formulas for the Gaussian integral are needed in stochastic analysis, mathematical finance and physics.

For example, Lyons-Victoir ('04) proposed the concept of cubature formula on Wiener space: To compute the expectation of a solution of SDE by using bdd variation paths which is constructed by cf for the Gaussian integral.

$$dY_t^x = \sum_{j=1}^{n} V_j(Y_t^x) \circ dB_t^j, \quad Y_0^x = x \in \mathbb{R}^n, \qquad E[f(Y_1^x)] = ??$$

Cubature formula on Wiener space

$W^n = \{\omega : [0,1] \to \mathbb{R}^n, \text{conti. } \& \ \omega(0) = 0\}$, $(W^n, \mathbb{P})$, Wiener sp.,
$\tilde{W}^n = \{\omega \in W^n, \omega$ has bounded variations,
i.e., $\sup_\Delta \sum_{l=1}^k |\omega(t_{l+1}) - \omega(t_l)| < \infty\}$,

$B = \{(B^1(t), \ldots, B^n(t))\}$, n-dim. Brownian motion starting at 0.

Def. $\omega_1, \ldots, \omega_N \in \tilde{W}^n$, $\lambda_1, \ldots, \lambda_N > 0$

form a cf on Wiener sp. of degree t (at time 1) iff

$$E\Big[\int_{0<t_1<\ldots<t_k<1} \circ dB^{i_1}(t_1) \circ \cdots \circ dB^{i_k}(t_k)\Big]$$
$$= \sum_{j=1}^N \lambda_j \int_{0<t_1<\ldots<t_k<1} d\omega_j^{i_1}(t_1) \cdots d\omega_j^{i_k}(t_k), \ \ \forall (i_1, \ldots, i_k) \in \mathcal{A}_t,$$

where $\mathcal{A}_t = \{(i_1, \ldots, i_k) \in \{1, \ldots, n\}^k, \ k \leq t\}$.

- We construct a pair $(\{\omega_i\}_{1\leq i\leq N}, \{\lambda_i\}_{1\leq i\leq N})$ by using cf for the Gaussian integral of $\mathbb{R}^n$. $\leftarrow$ + "Rough Path ideas"

$$f(Y_1^x) = f(x) + \sum_{j=1}^{n} \int_0^1 V_j f(Y_s^x) \circ dB_s^j \qquad f : \text{ bounded smooth}$$

$$= f(x) + \sum_{j=1}^{n} \int_0^1 V_j \Big(f(x) + \sum_{i=1}^{n} \int_0^s V_i f(Y_u^x) \circ dB_u^i \Big) \circ dB_s^j$$

$$= f(x) + \sum_{j=1}^{n} V_j f(x) \int_0^1 \circ dB_s^j + \sum_{i,j=1}^{n} \int_0^1 \int_0^s V_j V_i f(Y_u^x) \circ dB_u^i \circ dB_s^j$$

$$= \cdots\cdots$$

$$= \sum_{(i_1,\cdots,i_k) \in \mathcal{A}_t} V_{i_1} \cdot \cdot V_{i_k} f(x) \int_{0<t_1<\cdots<t_k<1} \circ dB_{t_1}^{i_1} \circ \cdot \cdot \circ dB_{t_k}^{i_k} + \cdots .$$

(Note: $f(y) = f(a) \cdot 1 + f'(a)(y-a) + \frac{f''(a)}{2!}(y-a)^2 + \cdots .$)

Cf on Wiener sp. of degree 3

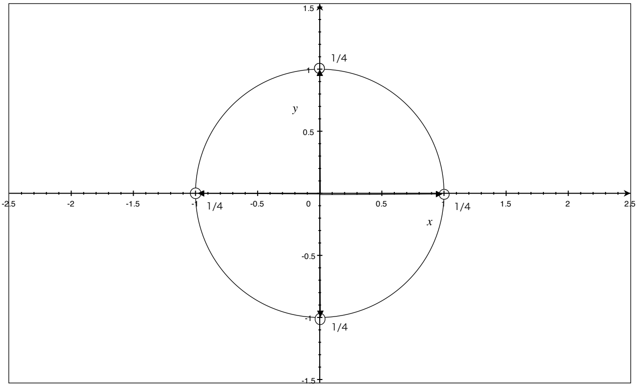
Let $x_1 = (\sqrt{2}, 0)$, $x_2 = (0, \sqrt{2})$, $x_3 = (-\sqrt{2}, 0)$, $x_4 = (0, -\sqrt{2}) \in \mathbb{R}^2$.
These points form a cf of deg. 3 for the Gaussian integral;

$$\frac{1}{2\pi}\int_{\mathbb{R}^n} f(x)e^{-\|x\|^2/2}dx = \frac{1}{4}\sum_{i=1}^{4} f(x_i), \quad \forall f \in \mathcal{P}_3(\mathbb{R}^2)$$

Then, $\boxed{\omega_i = x_i t,\ \lambda_i = \frac{1}{4}\ (i = 1, \ldots, 4)}$ form a cf on Wiener sp. of deg. 3;

$$E\left[\int_{0<t_1<\cdots<t_k<1} \circ dB_{t_1}^{i_1} \cdots \circ dB_{t_k}^{i_k}\right] = \frac{1}{4}\sum_{j=1}^{4}\int_{0<t_1<\cdots<t_k<1} d\omega_j^{i_1}(t_1)\cdots d\omega_j^{i_k}(t_k),$$

$$\forall (i_1, \ldots, i_k) \in \mathcal{A}_3 = \{(i_1, \ldots, i_k) \in \{1, 2\}^k, k \leq 3\}.$$



Cf on Wiener sp. of deg. 5

(X,λ) : a cf of deg. 5 for the Gaussian integral on $\mathbb{R}^n$ with $|X| = N$.

$$\mathcal{L}_{k,\pm 1} = x_k^1\epsilon_1 + \cdots + x_k^n\epsilon_n + \tfrac{1}{12}\textstyle\sum_{i<j}\Big(x_k^i(x_k^j)^2[[\epsilon_i,\epsilon_j],\epsilon_j] \pm 6x_k^i x_k^j \epsilon_i\otimes\epsilon_j + x_k^i(x_k^j)^2[[\epsilon_j,\epsilon_i],\epsilon_i]\Big).$$

Then we obtain

$$\sum_{k=1}^{N}\frac{\lambda_k}{2}\pi_5\Big(\exp(\mathcal{L}_{k,1})+\exp(\mathcal{L}_{k,-1})\Big) = \pi_5\Big(\exp\Big(\frac{1}{2}\sum_{j=1}^{n}\epsilon_j\otimes\epsilon_j\Big)\Big).$$

For $\omega \in \tilde{W}^n$ satisfies

$$\pi_5(\log(X_{0,1}(\omega))) = \epsilon_1 + \ldots + \epsilon_n + \tfrac{1}{12}\textstyle\sum_{i<j}\Big([[\epsilon_i,\epsilon_j],\epsilon_j] + 6\epsilon_i\otimes\epsilon_j + [[\epsilon_j,\epsilon_i],\epsilon_i]\Big),$$

- $w_i(t) = (x_i^1\omega^1(t),\ldots,x_i^n\omega^n(t))$ and $w_{N+i}(t) = (x_i^1\omega^n(t),\ldots,x_i^n\omega^1(t))$ form a cf of deg. 5 at time 1.

Upper and lower bounds for cf on $\mathbb{R}^n$

- We want to find cubature formulas for the Gaussian integral.

Thm. (Tchakaloff, '57) We can find X with $|X| \leq \dim \mathcal{P}_t(\mathbb{R}^n) = \binom{n+t}{t}$, and positive weight function w, such that

$$\frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-\|x\|^2} dx = \sum_{x \in X} w(x) f(x), \quad \forall f \in \mathcal{P}_t(\mathbb{R}^n).$$

Thm. (Möller, '76) The smallest possible number $|X|$ in a cf of deg. t is bounded from below:

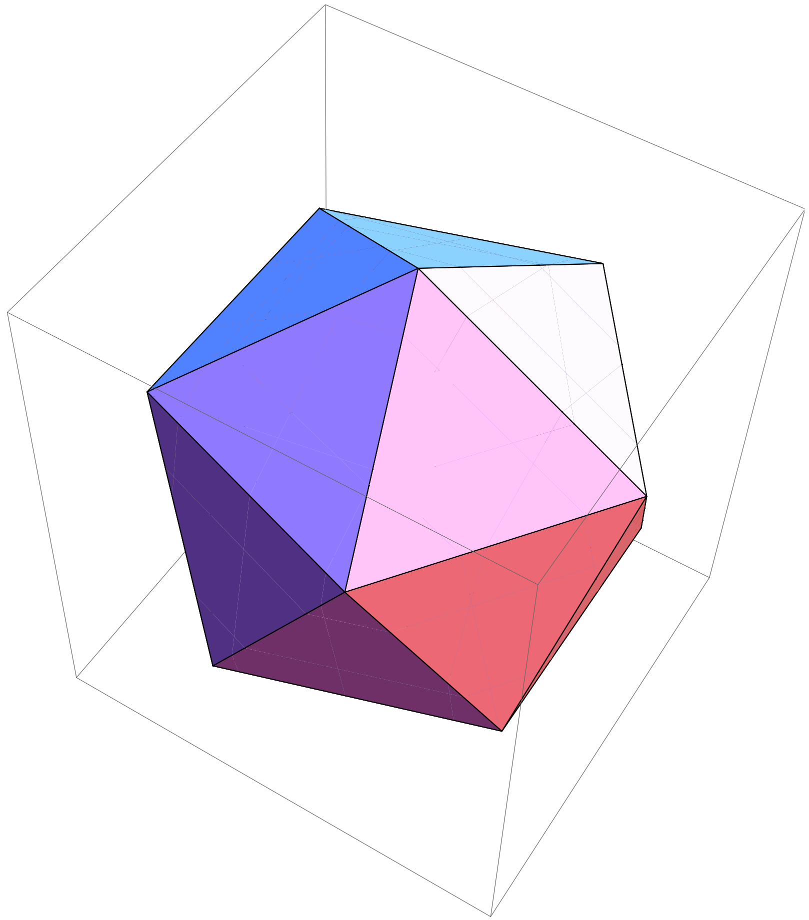
$$|X| \geq \begin{cases} \dim \mathcal{P}_e(\mathbb{R}^n) & t = 2e \\ 2\dim \mathcal{P}_e^*(\mathbb{R}^n) - 1 & t = 2e+1,\ e: \text{even } \&\ 0 \in X \\ 2\dim \mathcal{P}_e^*(\mathbb{R}^n) & \text{otherwise} \end{cases}$$

Here $\mathcal{P}_e^*(\mathbb{R}^n)$ is the subspace of $\mathcal{P}_e(\mathbb{R}^n)$ consisting all even or odd polynomials according to e being even or odd, respectively.

- $t = 4$: $|X| \geq \frac{1}{2}(n+1)(n+2)$, $\quad t = 5$: $|X| \geq n^2 + n + 1$.

- $t = 5, n = 3$:

$$\begin{aligned}|X| &\geq 3^2 + 3 + 1\\ &= 12 + 1\\ &= (\text{the vertices of icosahedron}) + (\text{the origin}).\end{aligned}$$



Our goal and outline of this talk

Our goal:

- The existence problem of minimum cf for the Gaussian integral.

However, minimum cf are very rare to exist. Thus, we need to find a cf with smaller number of points.

- Constructing and thinning methods of cf for the Gaussian integral.

Outline of this talk:

- Quadrature formula, Reproducing kernel, Euclidean design
- Our results (existence of min. cf & construction)
- Cubature on Wiener space (and its construction)
- Further problems

Quadrature formula

$I = (a, b) \subset \mathbb{R}$, μ : a probability measure on I,
$\{p_l\}_{l=0,1,\ldots}$: orthonormal poly. w.r.t. $\langle f, g \rangle := \int_I fgd\mu$,
$\{\lambda_1, \ldots, \lambda_{k+1}\}$: the set of zeros of $p_{k+1}(x)$, $\omega_j = (\sum_{i=0}^k p_i(\lambda_j)^2)^{-1}$.

Then, $(\{\lambda_1, \ldots, \lambda_{k+1}\}, \{w_1, \ldots, w_{k+1}\})$ forms a qf of degree $2k + 1$;

$$\int_I f(x) d\mu(x) = \sum_{j=1}^{k+1} w_j f(\lambda_j), \quad \forall f \in \mathcal{P}_{2k+1}(\mathbb{R}).$$

$\because$) For any polynomial $f(x)$ of deg. $2k+1$, there exit two polynomials $g(x), r(x)$ of deg. $\leq k$, such that $f(x) = g(x)p_{k+1}(x) + r(x)$. Then $r(\lambda_j) = f(\lambda_j)$ for each j and by orthogonality

$$\int_I f d\mu = \int_I g p_k d\mu + \int_I r d\mu = \int_I r d\mu = \sum_{j=1}^{k+1} w_j f(\lambda_j).$$

Reproducing kernel and minimum cf

$\Pi_k(\mathbb{R}^n)$: the vector sp. of all polynomial of deg. k,
$\{P_{k,i}\}_{1 \le i \le r_k^n}$: orthonormal poly. of deg. k w.r.t. $\langle f, g\rangle = \int_\Omega fg d\mu$,
$\mathbb{P} \equiv (P_{k,1}, \ldots, P_{k,r_k^n})$, $\quad r_k^n = \dim(\Pi_k(\mathbb{R}^n))$.

Then, the t-th rep. kernel is defined as follows:

$$\begin{cases} K_t(x,y) = \tilde{K}_t(x,y) + \tilde{K}_{t-1}(x,y), \\ \tilde{K}_t(x,y) = \sum_{0 \le l \le t,\ l \equiv t\ (2)} \mathbb{P}_l(x)\mathbb{P}_l^\top(y). \end{cases}$$

- For any $f \in \mathcal{P}_t(\mathbb{R}^n)$, we have $f(y) = \int_\Omega f(x) K_t(x,y) d\mu(x)$.

Thm. (Mysovskikh ('81)) There exists a minimum cf (X, w) of deg. $4k+1$ for a spherically sym. integral $\int_\Omega f \mathrm{d}\mu$, iff

(i) $\tilde{K}_{2k}(x,y) = 0$, $x, y \in X$, $x \ne y$,

(ii) $w(0) = \tilde{K}_{2k}(0,0)^{-1}$, $\quad w(x) = \tilde{K}_{2k}(x,x)^{-1}/2$, $x \in X \setminus \{0\}$.

Moreover, X is equal to the set of common zeros of $\{\mathcal{P}_{2k+1,i}\}_{1 \le i \le r^n_{2k+1}}$.

Since the t-th modified rep. kernel is a polynomial of $\|x\|^2, \|y\|^2$ and $\langle x, y\rangle$, we can calculate, e.g., the 4-th modified rep. kernel:

$$\begin{aligned}\tilde{K}_4(x,y) &= a_1 + a_2\langle x,y\rangle^2 + a_3\langle x,y\rangle^4 + a_4(\|x\|^2+\|y\|^2) \\ &+ a_5\|x\|^2\|y\|^2 + a_6(\|x\|^2+\|y\|^2)\langle x,y\rangle^2 + a_7\|x\|^2\|y\|^2\langle x,y\rangle^2 \\ &+ a_8(\|x\|^4+\|y\|^4) + a_9\|x\|^2\|y\|^2(\|x\|^2+\|y\|^2) + a_{10}\|x\|^4\|y\|^4\end{aligned}$$

By Mysovskikh's thm, we can determine the radii and weights.

$\rightarrow$ However, it is not easy to determine the position of points on each spheres.

$\rightarrow$ We focus on **algebraic structure** associated with cubature points.

Xu's compact formula ('98): $\int_{B^n} f(x) w_\mu (1-\|x\|^2)^{\mu-1/2} dx$

$$\tilde{K}_t(x,y) = c_\mu \int_{-1}^{1} C_t^{(\mu+\frac{n+1}{2})} \left(\langle x,y\rangle + \sqrt{1-\|x\|^2}\sqrt{1-\|y\|^2}\, t \right) (1-t^2)^{\mu-1} dt,$$

where $C_t^{(\lambda)}$ is the Gegenbauer polynomial of degree t.

• We want to find a compact formula for a spherically sym. integral!

Euclidean design

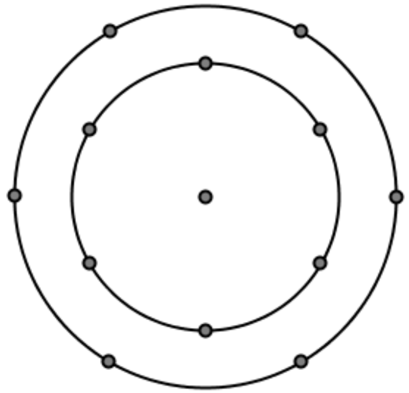
• To consider cf for spherically symmetric integrals, we prepare the concept of Euclidean design.

$X \subset \mathbb{R}^n$, $|X| < \infty$, $\quad w : X \to \mathbb{R}_{>0}$

$\{r_1, \ldots, r_p\} = \{\|x\| \mid x \in X\}$, $r_1 > \ldots > r_p$,

$S_i = \{x \in \mathbb{R}^n \mid \|x\|^2 = R_i\}$, $\quad S = \cup_{i=1}^{p} S_i$,

$X_i = X \cap S_i$, $\quad W_i = \sum_{x \in X_i} w(x)$.



Euclidean 7-design of $\mathbb{R}^2$

Then, a pair (X, λ) is a Euclidean t-design (on S), iff

$$\sum_{i=1}^{p} \frac{W_i}{|S_i|} \int_{S_i} f(x) d\sigma_i(x) = \sum_{x \in X} w(x) f(x), \quad \forall f \in \mathcal{P}_t(\mathbb{R}^n),$$

where σ_i is the surface measure on S_i.

In particular, X is a spherical t-des., iff $p = 1, r_1 = 1$ and $w(x) = \frac{1}{|X|}$.

Euclidean design and cubature formula

By the following, we know **a cf of deg. t is a Euclidean t-des.**

Thm. (Neumaier-Seidel, '88). (X, w) is a Euclidean t-design, iff

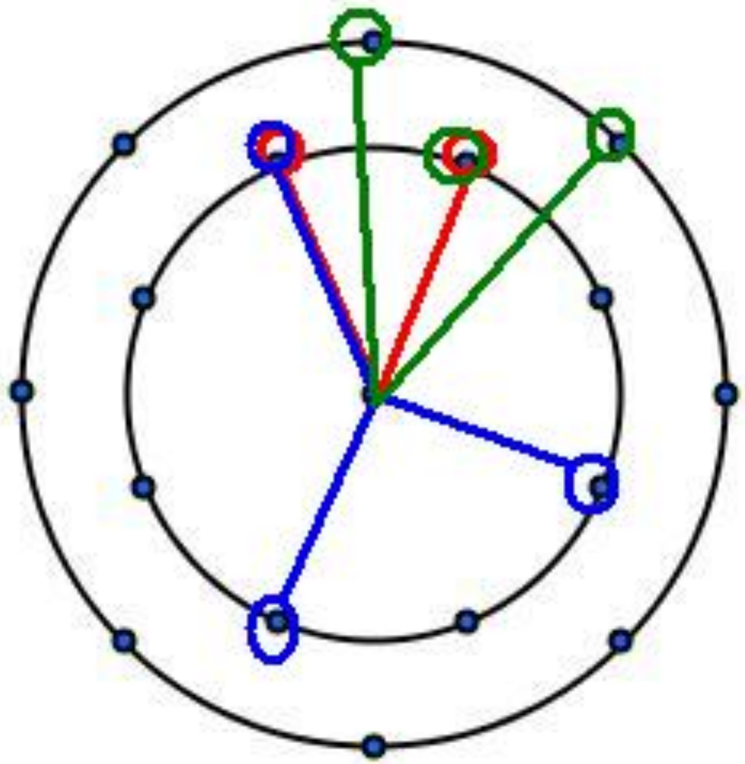
$$\sum_{x \in X} w(x) f(x) = 0, \quad \begin{cases} \forall f \in \|x\|^{2j} \mathrm{Harm}_l(\mathbb{R}^n), \\ 1 \le l \le t, \ 0 \le j \le [\frac{t-l}{2}]. \end{cases}$$

- Merit of considering Euclidean design.

1. Cf for spherically symmetric integral is a special class of Euclidean design.

2. We have many papers on minimum Euclidean t-des., in particular of those on one or two spheres (i.e., $p = 1, 2$).
For example, minimum Euclidean t-des. on 2 spheres were obtained for $t = 3, 4, 5, 6, 7$ (association scheme, coherent config.).

What is a coherent configuration?



Euclidean 9-design of $\mathbb{R}^2$

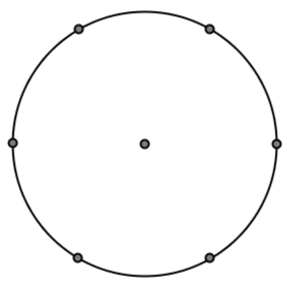
Existence of minimum cf

H.-Sawa ('09)

- **Structure of minimum cf of degree** $4k + 1$

Thm. Assume there exists a minimum cf (X, w) of degree $4k + 1$ for a spherically symmetric integral. Then the following hold:

1. X are distributed over k spheres and 0.
2. w take a constant on each sphere.
3. Each layer of $X \setminus \{0\}$ is similar to a spherical $(2k + 3)$-des.



degree 5

We showed nonexistence of minimum cf of degree 9 for Stroud's classical integrals. Recently, Bannai-Bannai completely showed nonexistence of minimum cf of degree 9 for a spherically sym. int.

- **Minimum cf of degree** 5

The points of minimum cf of deg. 5 for a spherically symmetric integral are supported by "the origin" and "a sphere".

→ We construct explicitly for 1, 2, 3, 7, 23 dimensions.

Probem Let m be an integer with $m \geq 3$. Is there exists a minimum cf of deg. 5 for a spherically sym. int. in $\mathbb{R}^m$?

$$
\begin{array}{ccccccc}
(\pm\alpha, & \pm\alpha, & 0, & \pm\alpha, & 0, & 0, & 0) \\
(0, & \pm\alpha, & \pm\alpha, & 0, & \pm\alpha, & 0, & 0) \\
(0, & 0, & \pm\alpha, & \pm\alpha, & 0, & \pm\alpha, & 0) \\
(0, & 0, & 0, & \pm\alpha, & \pm\alpha, & 0, & \pm\alpha) \\
(\pm\alpha, & 0, & 0, & 0, & \pm\alpha, & \pm\alpha, & 0) \\
(0, & \pm\alpha, & 0, & 0, & 0, & \pm\alpha, & \pm\alpha) \\
(\pm\alpha, & 0, & \pm\alpha, & 0, & 0, & 0, & \pm\alpha) \\
(0, & 0, & 0, & 0, & 0, & 0, & 0)
\end{array}
$$

H.-Nozaki-Sawa-Vatchev (arXiv:1103.1111)

By applying a generalization of the Larman-Rogers-Seidel theorem (Nozaki, preprint), we obtain the following necessary condition:

Thm. Let k, n with $k \geq 2, n \geq 4k^2 - 2k + 1$. Assume there exists a minimum cf X of deg. $4k+1$ for a sherically symmetric integral. Then there exists X_l such that

$$\text{every } \alpha \in \left\{ \frac{\langle x,y \rangle}{r_l^2} \mid x, y \in X_l, x \neq y \right\} \text{ is a rational number.}$$

By using this theorem, we show that nonexistence of minimal formula of degrees $13, 17$ and 21 for a generalized Xu's integral.

Bannai-Bannai-H.-Sawa ('10)

By using the so-called Möller lower bound, we obtain **Fisher types of lower bounds of Euclidean designs.** And thereby we define minimum and almost minimum of the design.

By applying similar arguments by Verlinden-Cools ('92), we also give **classification of minimum Euclidean designs of** $\mathbb{R}^2$.

Thm. (X, λ) is a min.Euclidean $(4k+3)$-des. on $(k+1)$ spheres.

Then, letting $R_j = r_j^2$, it has the form

$$\sum_{j=1}^{k+1} \frac{W_j}{2k+4} \sum_{l=0}^{2k+3} f\left(\sqrt{R_j}\cos\left(\frac{j+2l}{2k+4}\pi\right), \sqrt{R_j}\sin\left(\frac{j+2l}{2k+4}\pi\right)\right), \text{ and}$$

$$W_i = \frac{R_1^{2k-p+3}}{R_i^{2k-p+3}} \frac{\prod_{j=2}^{i-1}(R_1 - R_j)\prod_{j=i+1}^{p}(R_1 - R_j)}{\prod_{j=2}^{i-1}(R_j - R_i)\prod_{j=i+1}^{p}(R_i - R_j)} W_1, \quad 2 \le i \le k+1.$$

H.-Sawa ('12?), Bannai-Bannai-H.-Sawa ('11)

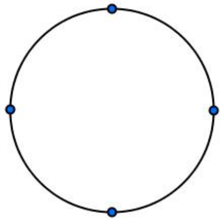
By combing BBHS('10) and orthogonal polynomial properties, we obtain **classification of minimum cf of the Gaussian int. of $\mathbb{R}^2$.**

Thm. Assume X is a minimum cf of deg. $4k+3$ with $\{\|x\|^2 \mid x \in X\} = \{R_1, \ldots, R_{k+1}\}$. Then the following holds:

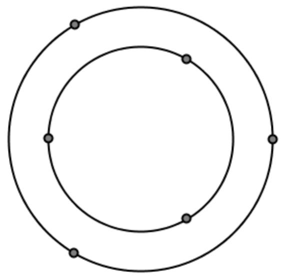
1. $\{R_1, \cdots, R_{k+1}\}$ is the set of zeros of $L_{k+1}(t)$.
2. $L_{k+1}(t)$ can be factored in the following two parts.

$$\begin{cases} L_a^{2a+2}(t)\Big(L_{a+1}^{2a+2}(t) + \gamma_1 L_a^{2a+2}(t) + \gamma_2 L_{a-1}^{2a+2}(t)\Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } k = 2a, \\ \\ \Big(L_{a+1}^{2a+3}(t) + \gamma_1 L_a^{2a+3}(t)\Big)\Big(L_{a+1}^{2a+3}(t) + \gamma_2 L_a^{2a+3}(t)\Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } k = 2a+1. \end{cases}$$

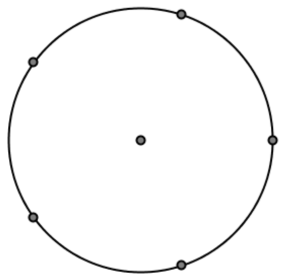
for some $\gamma_1, \gamma_2 \in \mathbb{R}$.



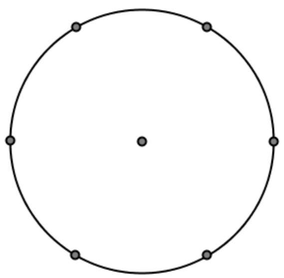
degree 3 (min)



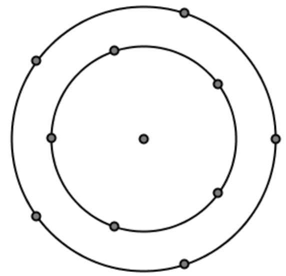
degree 4 (min)



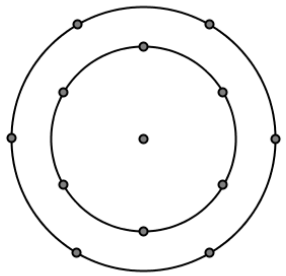
degree 4 (min)



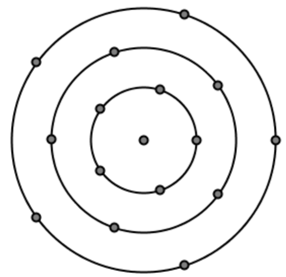
degree 5 (min)



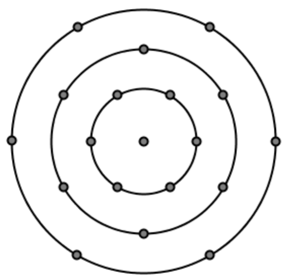
degree 6 (min+1)



degree 7 (min+1)



degree 8 (min+1)



degree 9 (min+2)

Constructing cf

H.-Sawa-Zhou ('10)

By using some orbits of a finite group, we construct **a minimum Euclidean** 5**-design on** 3 **concentric spheres**.

There seems to exist no minimum cf of higher degrees and dimensions for spherically symmetric integrals. So, we need to find some construction of cf with smaller number of points.

$\Rightarrow$ In H.-Jimbo-Sawa (preparation), we give a construction for cf with some statistical optimality. $\rightarrow$ cf for the Gaussian integral.

- $\epsilon_1, \ldots, \epsilon_n$: the standard basis of $\mathbb{R}^n$,
  G : the gr. generated by sine changes & permutat. of coordinates
  $\mathrm{Orb}_G(x)$: the orbit of x by G.

- When the case $n = 3m + 2$, let

$$\begin{cases} x_1 &= r_1 \epsilon_1, \\ x_2 &= \frac{r_1}{\sqrt{m+2}}(\epsilon_1 + \ldots + \epsilon_{m+2}), \\ \\ y_1 &= r_2 \epsilon_1, \\ y_2 &= \frac{r_2}{\sqrt{m+2}}(\epsilon_1 + \ldots + \epsilon_{m+2}). \end{cases}$$

where r_1^2, r_2^2 forms a qf of degree 2 for $\int \cdot s^{(n-2)/2} e^{-s} ds$.

Then, the set

$$X = \mathrm{Orb}_G(x_1) \cup \mathrm{Orb}_G(x_2) \cup \mathrm{Orb}_G(y_1) \cup \mathrm{Orb}_G(y_2)$$

form a cf of degree 5 for the Gaussian integral.

- For $n = 8$ $(m = 2)$,

$$|X| = 2 \cdot 2 \cdot 8 + 2 \cdot 2^4 \cdot \binom{8}{4} = 2272.$$

By using 3-(8,4,1) design (block size = 14), we can replace

$$
\begin{array}{llllllll}
(\pm\alpha, & \pm\alpha, & \pm\alpha, & 0, & \pm\alpha, & 0, & 0, & 0), \\
(\pm\alpha, & 0, & \pm\alpha, & \pm\alpha, & 0, & \pm\alpha, & 0, & 0), \\
(\pm\alpha, & 0, & 0, & \pm\alpha, & \pm\alpha, & 0, & \pm\alpha, & 0), \\
(\pm\alpha, & 0, & 0, & 0, & \pm\alpha, & \pm\alpha, & 0, & \pm\alpha), \\
(\pm\alpha, & \pm\alpha, & 0, & 0, & 0, & \pm\alpha, & \pm\alpha, & 0), \\
(\pm\alpha, & 0, & \pm\alpha, & 0, & 0, & 0, & \pm\alpha, & \pm\alpha), \\
(\pm\alpha, & \pm\alpha, & 0, & \pm\alpha, & 0, & 0, & 0, & \pm\alpha), \\
(0, & 0, & 0, & \pm\alpha, & 0, & \pm\alpha, & \pm\alpha, & \pm\alpha), \\
(0, & \pm\alpha, & 0, & 0, & \pm\alpha, & 0, & \pm\alpha, & \pm\alpha), \\
(0, & \pm\alpha, & \pm\alpha, & 0, & 0, & \pm\alpha, & 0, & \pm\alpha), \\
(0, & \pm\alpha, & \pm\alpha, & \pm\alpha, & 0, & 0, & \pm\alpha, & 0), \\
(0, & 0, & \pm\alpha, & \pm\alpha, & \pm\alpha, & 0, & 0, & \pm\alpha), \\
(0, & \pm\alpha, & 0, & \pm\alpha, & \pm\alpha, & \pm\alpha, & 0, & 0), \\
(0, & 0, & \pm\alpha, & 0, & \pm\alpha, & \pm\alpha, & \pm\alpha, & 0).
\end{array}
$$

$$|X'| = 2 \cdot 2 \cdot 8 + 2 \cdot 2^4 \cdot 14 = 480.$$

- This gives suggestion that we can reduce the points X by **combin. t-des. and OA.** (e.g., Kono('62), Victoir ('04))

Cubature on Wiener space

To compute the expectation of a solution of a SDE, Lyons-Victoir('04) introduced the concept of cubature formula on Wiener space.

$$dY_t^x = \sum_{j=1}^{n} V_j(Y_t^x) \circ dB_t^j, \quad Y_0^x = x \in \mathbb{R}^n,$$

$$E[f(Y_1^x)] = ?? \leftarrow \text{using cf on Wiener sp.}$$

To construct cubature formulas on Wiener space, we need to use cubature formulas for the Gaussian integral.

Before starting to introduce the def. of cubature on Wiener space, we prepare some notations and results.

Cubature formula on Wiener space

$(W^n, \mathbb{P})$, Wiener space,

$B = \{(B^1(t), \ldots, B^n(t)), t \geq 0\}$, n-dim. Brownian motion starting at 0

$$\tilde{W}^n = \{\omega \in W^n, \omega \text{ has bounded variations, i.e., } \sup_{\Delta} \sum_{l=1}^{k} |\omega(t_{l+1}) - \omega(t_l)| < \infty\},$$

Def. $\omega_1, \ldots, \omega_N \in \tilde{W}^n$, $\lambda_1, \ldots, \lambda_N > 0$

form a cf on Wiener sp. of degree t (at time 1) iff

$$E\Big[\int_{0<t_1<\ldots<t_k<1} \circ dB^{i_1}(t_1) \circ \cdots \circ dB^{i_k}(t_k)\Big] = \sum_{j=1}^{N} \lambda_j \int_{0<t_1<\ldots<t_k<1} d\omega_j^{i_1}(t_1) \cdots d\omega_j^{i_k}(t_k), \ \ \forall (i_1, \ldots, i_k) \in \mathcal{A}_t,$$

where $\mathcal{A}_t = \{(i_1, \ldots, i_k) \in \{1, \ldots, n\}^k, \ k \leq t\}$.

Constructing cubature formula on Wiener space

Prob. Find $\omega_1, \ldots, \omega_N \in \tilde{W}^n$, $\lambda_1, \ldots, \lambda_N > 0$ form a cubature formula on Wiener space of degree 3 at time 1, i.e.,

$$E\Big[\int_{0<t_1<\ldots<t_k<1} \circ dB^{i_1}(t_1) \circ \cdots \circ dB^{i_k}(t_k)\Big]$$
$$= \sum_{j=1}^{N} \lambda_j \int_{0<t_1<\ldots<t_k<1} d\omega_j^{i_1}(t_1) \circ \cdots d\omega_j^{i_k}(t_k), \quad \forall (i_1, \ldots, i_k) \in \mathcal{A}_3.$$

- We want to calculate

$$E[\int \circ dB^i], \quad E[\int \circ dB^i dB^j], \quad E[\int \circ dB^i \circ dB^j \circ dB^k].$$

$B = \{B^1(t)\epsilon_1 + \cdots + B^n(t)\epsilon_n,\ t \geq 0\}$: n-dim. BM starting at 0, where $\epsilon_1, \ldots, \epsilon_n$ are the standard basis of $\mathbb{R}^n$,

$$X^{(3)}_{s,t}(\circ B) = \sum_{l=0}^{3} \sum_{\substack{(i_1,\ldots,i_k) \\ \in \mathcal{A}_l \setminus \mathcal{A}_{l-1}}} \int_{0<t_1<\cdots<t_k<1} \circ dB^{i_1}(t_1) \cdots \circ dB^{i_k}(t_k) \epsilon_{i_1} \otimes \cdots \otimes \epsilon_{i_k}$$

$$\in \mathbb{R} \oplus \mathbb{R}^n \oplus (\mathbb{R}^n \otimes \mathbb{R}^n) \oplus (\mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n) = T^{(3)}(\mathbb{R}^n).$$

Prob. Find $\omega_1, \ldots, \omega_N \in \tilde{W}^n$, $\lambda_1, \ldots, \lambda_N > 0$, which satisfy

$$E[X^{(3)}_{0,1}(\circ B)] = \sum_{j=1}^{N} \lambda_j X^{(3)}_{0,1}(\omega_j).$$

Moreover, Fawcett ('04) and Lyons-Victoir ('04) show that

$$E[X^{(3)}_{0,1}(\circ B)] = 1 + \frac{1}{2} \sum_{j=1}^{n} \epsilon_j \otimes \epsilon_j.$$

We would like to express the right hand side in terms of $\epsilon_1, \ldots, \epsilon_n$ by using Chen's theorem:

For $a, b \in T(\mathbb{R}^n)$, $\quad [a, b] := a \otimes b - b \otimes a$ (Lie bracket),

$\mathcal{U}^{(3)} = \mathbb{R}^n \oplus [\mathbb{R}^n, \mathbb{R}^n] \oplus [[\mathbb{R}^n, \mathbb{R}^n], \mathbb{R}^n]$.

(The element $\mathcal{U}^{(3)}$ is called a Lie polynomial of degree 3.)

Thm (Chen). $\forall \mathcal{L} \in \mathcal{U}^{(3)}$, $\exists \omega \in \tilde{W}^n$, s.t., $\mathcal{L} = \pi_3(\log(X_{s,t}(\omega)))$.

(For $a = 1 + c$, $c = (0, c_1, c_2, ..) \in T(\mathbb{R}^n)$, $\log(a) = \sum_{k \geq 1} (-1)^{k-1} k^{-1} c^{\otimes k}$.)

So, Prob. can be expressed in terms of Lie polynomials.

Prob. Find $\mathcal{L}_1, \ldots, \mathcal{L}_N \in \mathcal{U}^{(3)}$ and $\lambda_1, \ldots, \lambda_N > 0$, s.t.,

$$1 + \frac{1}{2} \sum_{i=1}^{n} \epsilon_i \otimes \epsilon_i = \sum_{j=1}^{N} \lambda_j \pi_3(\exp(\mathcal{L}_j)).$$

Hence our problem changes from finding bounded variation paths to finding Lie polynomials.

Cf on Wiener sp. of deg. 3

(X, λ): a cf of degree 3 for the Gaussian integral with $|X| = N$.

$\mathcal{L}_j = x_j^1 \epsilon_1 + \cdots + x_j^n \epsilon_n,\ j = 1, \ldots, N.$

$$\pi_3(\exp(\mathcal{L}_j)) = \sum_{k=0}^{3} \frac{(x_j^1 \epsilon_1 + \cdots + x_j^n \epsilon_n)^{\otimes k}}{k!}$$
$$= 1 + x_j^1 \epsilon_1 + \cdots + x_j^n \epsilon_n + \frac{1}{2}(x_j^1)^2 \epsilon_1 \otimes \epsilon_1 + \cdots + \frac{1}{2}(x_j^n)^2 \epsilon_n \otimes \epsilon_n$$
$$+ \frac{1}{6}(x_j^1)^3 \epsilon_1 \otimes \epsilon_1 \otimes \epsilon_1 + \cdots + \frac{1}{6}(x_j^n)^3 \epsilon_n \otimes \epsilon_n \otimes \epsilon_n.$$

Then we obtain

$$\pi_3\Big(\exp\Big(\frac{1}{2}\sum_{i=1}^{n} \epsilon_i \otimes \epsilon_i\Big)\Big) = 1 + \frac{1}{2}\sum_{i=1}^{n} \epsilon_i \otimes \epsilon_i = \sum_{j=1}^{N} \lambda_j \pi_3\Big(\exp(\mathcal{L}_j)\Big).$$

- $w_j : t \mapsto t(x_j^1, \ldots, x_j^n), j = 1, \ldots, N$, form a cf of deg. 3 at time 1.

Cf on Wiener sp. of deg. 5

(X,λ): a cf of degree 5 for the Gaussian integral with $|X|=N$.

$$\mathcal{L}_{k,\pm 1} = x_k^1\epsilon_1 + \cdots + x_k^n\epsilon_n$$
$$+\tfrac{1}{12}\textstyle\sum_{i<j}\Big(x_k^i(x_k^j)^2[[\epsilon_i,\epsilon_j],\epsilon_j] \pm 6x_k^ix_k^j\epsilon_i\otimes\epsilon_j + x_k^i(x_k^j)^2[[\epsilon_j,\epsilon_i],\epsilon_i]\Big).$$

Then we obtain

$$\sum_{k=1}^{N}\frac{\lambda_k}{2}\pi_5\Big(\exp(\mathcal{L}_{k,1})+\exp(\mathcal{L}_{k,-1})\Big) = \pi_5\Big(\exp\Big(\frac{1}{2}\sum_{j=1}^{n}\epsilon_j\otimes\epsilon_j\Big)\Big).$$

For $\omega\in\tilde{W}^n$ satisfies

$$\pi_5(\log(X_{0,1}(\omega))) = \epsilon_1 + \ldots + \epsilon_n$$
$$+\tfrac{1}{12}\textstyle\sum_{i<j}\Big([[\epsilon_i,\epsilon_j],\epsilon_j] + 6\epsilon_i\otimes\epsilon_j + [[\epsilon_j,\epsilon_i],\epsilon_i]\Big),$$

- $w_i(t) = (x_i^1\omega^1(t),\ldots,x_i^n\omega^n(t))$ and $w_{N+i}(t) = (x_i^1\omega^n(t),\ldots,x_i^n\omega^1(t))$ form a cf of degree 5 at time 1.

Error Estimates

For a bdd smooth fun. f,

$$f(Y_T^x) = \sum_{(i_1,\ldots,i_k)\in\mathcal{A}_m} V_{i_1}\cdots V_{i_k} f(x) \int_{0<t_1<\cdots<t_k<T} \circ dB_{t_1}^{i_1}\cdots\circ dB_{t_k}^{i_k} + R_m(T,x,f).$$

Let

$$E_{Q_T}[\int_{0<t_1<\cdots<t_k<1} \circ dB_{t_1}^{i_1}\cdots\circ dB_{i_k}^{i_k}] = \sum_{j=1}^{n} \lambda_j \int_{0<t_1<\cdots<t_k<T} dw_j^{i_1}(t_1)\cdots dw_j^{i_k}(t_k)$$

for all $(i_1,\ldots,i_k)\in\mathcal{A}_m$.

Then,

$$\sup_x E_{Q_T}[|R_m(T,x,f)|] \le C_{n,m,Q_1} T^{(m+1)/2} \sup_{(i_1,\ldots,i_k)\in\mathcal{A}_{m+2}\setminus\mathcal{A}_m} \|V_{i_1}\cdots V_{i_k} f\|_\infty.$$

Problems

• **For each k and n, find some methods of construction of cf of degree $2k+1$ for the n-dim.Gaussian integral.**

• For higher degree case, it doe not seem easy to construct Lie polynomials which form a cf on Wiener sp. I would like to find some explicit construction of Lie polynomials.

• Find the lower bound for the number of paths in a cf on Wiener sp. of deg. t!

• ... ...

Thank you for your attention!