

**Riesz Transforms with
Stochastic Analysis
(A Survey Talk + ϵ ?)**

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§1. Introduction

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▷ (M, g) : complete $(m\text{-dim})$ Riemannian mfd

↳ $g = (g_{ij})$: Riemannian metric (Positive definite, Sym-matrix)

$$\Gamma \rightarrow G = \det(g), \quad g^{-1} = (g^{ij})$$

$$\int_M (dx) := \int \sqrt{G(x)} dx \quad (\text{Riemannian volume measure})$$

▷ ∇ : gradient operator (Geometric)

$$\nabla f(x) = g^{ij}(x) \frac{\partial}{\partial x_j} f(x) \left(\frac{\partial}{\partial x_i} \right)_x \quad (x \in T_x M)$$

▷ Δ_M : (non-positive) Laplace-Beltrami operator

$$\Delta_M f(x) = \frac{1}{\sqrt{G(x)}} \frac{\partial}{\partial x^i} \left(g^{ij}(x) \sqrt{G(x)} \frac{\partial f}{\partial x^j}(x) \right) \quad (\text{Analytic})$$

$$\int_M (\nabla f(x), \nabla g(x))_{T_x M} \nu_M(dx) = \int_M (-\Delta_M f(x)) g(x) \nu_M(dx)$$

Dirichlet form

for $\forall f, g \in C_0^\infty(M)$

... (*)

For $\alpha \geq 0$, we define Riesz transform $R_\alpha(\Delta_M)$ (2

by $R_\alpha(\Delta_M)f := \nabla(\alpha - \Delta_M)^{-1/2}f$, $f \in C_0^\infty(M)$

By (*),

$$\alpha \|f\|_{L^2(M)}^2 + \|\nabla f\|_{L^2(M)}^2$$

$$= \alpha \|f\|_{L^2(M)}^2 + \int_M (-\Delta_M f(x)) \cdot f(x) \, d\mu(x)$$

$$= (\alpha - \Delta_M)f, f)_{L^2(M)} = \|\sqrt{\alpha - \Delta_M} f\|_{L^2(M)}^2$$

$$\|\nabla f\|_{L^2(M)}^2 \leq \|\sqrt{\alpha - \Delta_M} f\|_{L^2(M)}^2$$

$$\|\nabla \sqrt{\alpha - \Delta_M} f\|_{L^2(M)} \leq \|f\|_{L^2(M)}$$

← boundedness of $R_\alpha(\Delta_M)$ on $L^2(M)$

Problem: How about boundedness of $R_\alpha(\Delta_M)$ on $L^p(M)$ ($p \geq 1$) ???

⋯ ⋯ ⋯ (X)

(~ Equivalence of Sobolev norms: $\|f\|_p + \|\nabla f\|_p \sim \|\sqrt{1 - \Delta_M} f\|_p$)

Some Examples & Applications

- $M = \mathbb{R}^m$, $R_\alpha(\Delta)f = (R_\alpha^0(\Delta)f, \dots, R_\alpha^{(m)}(\Delta)f)$

$$R_\alpha^0(\Delta)f(x) := \frac{2}{\alpha x^2} \sqrt{\alpha - \Delta} f(x) = \mathcal{F}^{-1} \left[\frac{\sqrt{1 - \xi^2}}{\sqrt{\alpha + |\xi|^2}} \mathcal{F}(f)(\xi) \right](x)$$

$$\text{where } \mathcal{F}(f)(\xi) := \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{-i \langle x, \xi \rangle} f(x) dx$$

Navier-Stokes Equation

$$\left(\frac{\partial}{\partial t} u(t, x) = \Delta u(t, x) - (u \cdot \nabla) u(t, x) + \nabla p(t, x), \nabla \cdot u(t, x) = 0 \right)$$

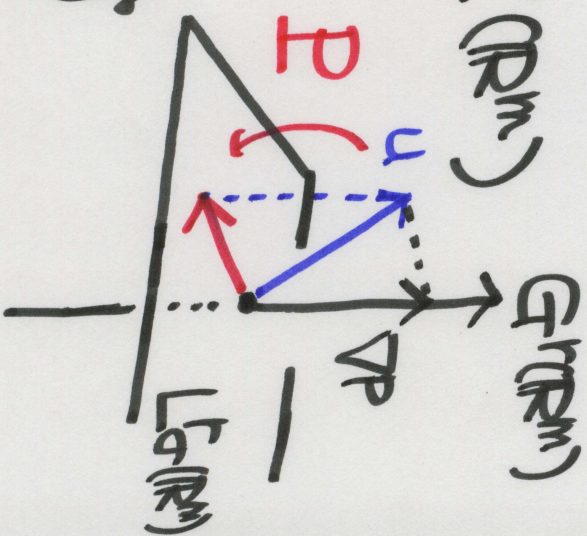
To vanish the pressure term, we use

Helmholtz decomposition in $L^r(\mathbb{R}^m)$

$$\underline{L^r(\mathbb{R}^m)} = \underline{L^r_0(\mathbb{R}^m)} \oplus G^r(\mathbb{R}^m) \quad \text{where}$$

$$L^r_0(\mathbb{R}^m) := \{ u \in L^r(\mathbb{R}^m; \mathbb{R}^m) \mid \nabla \cdot u = 0 \}$$

$$G^r(\mathbb{R}^m) := \{ \nabla p \in L^r(\mathbb{R}^m; \mathbb{R}^m) \mid p \in L^r_{loc}(\mathbb{R}^m) \}$$



$P: L^2(\mathbb{R}^m) \rightarrow L^2(\mathbb{R}^m)$ Projection

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Then

$$(Pu)^i = \sum_{j=1}^m (\delta_{ij} + R_i R_j) u^j \text{ where} \\ R_j = \frac{\partial}{\partial x_j} (-\Delta)^{-1/2} \quad (j=1, 2, \dots, m)$$

✦ History: (i) Analytic Approach (Stein, Coulhon, etc..)

$$P_\alpha(\Delta_M) = \int_0^\infty e^{-\alpha t} t^{-1/2} \nabla e^{t\Delta_M} dt$$

~ Analysis of **gradient bounds of heat kernel**
& Calderon-Zygmund theory (Singular integral operator)

(ii) **Stochastic Approach**

(Littlewood-Paley inequality $\rightarrow$ Riesz transform)

• P.A. Meyer ('76, 82) $M = \mathbb{R}^m$, Wiener space

• Bakry ('87) ... Bakry-Emery's Γ_2 -Calculus

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M: Complete Riemannian mfd with $\text{Ric}_M \geq -R$

Contrary to Analytic Approach, Stochastic Approach still works well when we replace M by ∞ -dim state spaces (singular mfd etc..)

[Advantage]

↳ • Shigekawa-Toshida ('92) ... L^p -inequality on a general metric space

↳ • Aida ('92) ... Sobolev norm equivalence on loop groups

↳ • Toshida ('94) ... $M^{\mathbb{Z}^d}$ with Gibbs measure

↳ • Shigekawa ('02) ... a generalization ($\leadsto$ Stochastic Ising model)

↳ • K-Miyokawa ('07), K ('08) etc ...

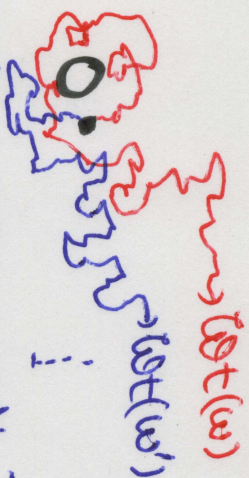


Relationship between Stochastic Analysis & Other Research Fields (via a simple example)

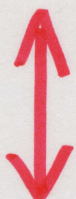
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Stochastic Analysis

BM on $\mathbb{R}^d$: $W_t = (W_t^1, \dots, W_t^d)$



(collection of sample paths)



Analysis (PDE)

Heat Equation on $\mathbb{R}^d$

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) = \frac{1}{2} \Delta u(t, x) \\ u(0, x) = f(x) \end{cases}$$

$$E[f(x + W_t(\omega))] = u(t, x) = e^{t(\frac{\Delta}{2})} f(x)$$

(Heat Semigroup P_t)

$$\left(\int_{\mathbb{R}^d} f(y) \frac{1}{\sqrt{2\pi t}} e^{-|x-y|^2/2t} dy \right)$$

Transition Probability

$$P(t, x, dy) = P(t, x, y) dy$$

(Heat kernel)

$t \geq 0$

$t \geq 0$

$$\lim_{t \rightarrow 0} 2t \log P(t, x, y) = -|x-y|^2$$

Analytic Geometric

Geometry: distance, curvature, etc ...

SZ. Framework

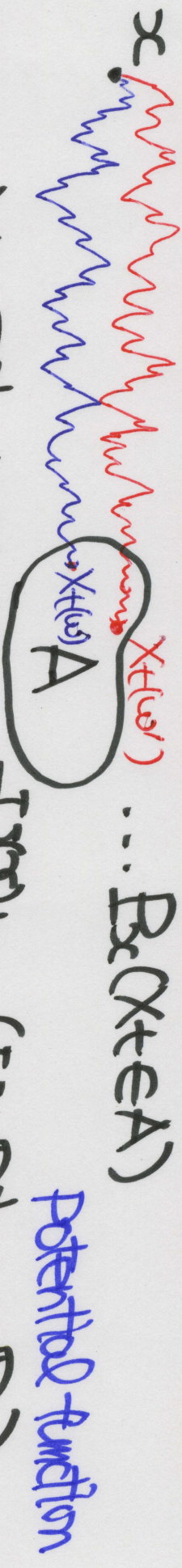
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- ▷ X : complete separable metric space
- ▷ μ : Borel Probability measure on X
- ▷ $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$: "good" Dirichlet form on $L^2(\mu)$ i.e.
 - i) $\mathcal{D}(\mathcal{E}) \subset L^2(\mu)$, ii) $\mathcal{E}: \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \rightarrow \mathbb{R}$
Symmetric, Markovian, Closed form
- ▷ $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$: generator of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ i.e.
 - i) $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$: non-positive definite self-adjoint op
on $L^2(\mu)$
 - ii) $\mathcal{E}(f, g) = (\sqrt{-\mathcal{A}}f, \sqrt{-\mathcal{A}}g)_{L^2(\mu)}$ for $\forall f, g \in \mathcal{D}(\mathcal{E})$
 $= (f, -\mathcal{A}g)_{L^2(\mu)}$ for $\forall f \in \mathcal{D}(\mathcal{E}), \underline{\forall g \in \mathcal{D}(\mathcal{A})}$

Remark: $\equiv M = (X_t, P_{X_t \in \cdot})$: diffusion Process on X (8)
 associated $(\mathbb{E}, P(\mathbb{E}))$
 (strong Markov Process
 continuous sample path)

i.e.
$$P_t f(x) (= e^{t\mathcal{L}} f(x))$$

$$= \int_X f(y) P_x(X_t \in dy) (= \mathbb{E}^x[f(X_t)])$$



Example: $X = \mathbb{R}^d$, $\mu(dx) = e^{-U(x)} dx$ ($U: \mathbb{R}^d \rightarrow \mathbb{R}$)

- $D(\mathbb{E}) = H^1_2(U) := \{u, |Du|_{\mathbb{R}^d} \in L^2(U)\}$ ($= H^1_2(U)$)
- $\mathcal{E}(f, g) = \int_{\mathbb{R}^d} (\nabla f(x), \nabla g(x))_{\mathbb{R}^d} \mu(dx)$ (*)

If g : nice function (like ω in $C_0(\mathbb{R}^d)$) &
 U has nice regularity?

$$(*) \Leftrightarrow \mathbb{E}(f, g) = - (f, \mathcal{I}g)_{L^2(\Omega)} \quad \text{where} \quad (9)$$

$$\mathcal{I}g(x) = \Delta g(x) - (\nabla U(x), \nabla g(x))_{\mathbb{R}^d}$$

Hence, we can see that the corresponding diffusion Process has the following SDE representation:

$$dX_t = \underbrace{\sqrt{2} dW_t}_{\text{(scaled-) BM part}} - \underbrace{\nabla U(X_t) dt}_{\text{drift part}}$$

Advantages of Dirichlet form technique

(1) Singular coefficient diffusions

(2) Diffusions with bdy conditions

$$\mathbb{E}(f, f) = \int_D (\nabla f, \nabla f) dx$$

$$\mathcal{D}(\mathcal{E}) = H^1_0(D) \dots \text{reflecting BM} \quad (\mathcal{I} = \Delta_N)$$

$$\mathcal{D}(\mathcal{E}) = H^1_0(D) \dots \text{distorted BM} \quad (\mathcal{I} = \Delta_D)$$

(3) ∞ -dim case, fractals (Riemannian Structure
... Kigami, Hilino etc...)

Here, we impose the following conditions:

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(I) conservativeness $P_t \mathbb{1} = \mathbb{1}$ i.e. lifetime of $M = \infty$

(II) Existence of Carre du Champ Γ (Square Root op) ("D" $\Gamma(f, g) = (Df, Dg)_H$)

$$\exists \Gamma: \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \longrightarrow L^1(\mu)$$

s.t. Symmetric conti, bi-linear form

$$\Gamma(f, g) = -\frac{1}{2} \{ \mathcal{L}(fg) - f \mathcal{L}g - \mathcal{L}f \cdot g \}$$

$$\mathcal{E}(f, g) = \int_X \Gamma(f, g)(x) \mu(dx)$$

(III) Pointwise gradient bound condition

$$\equiv k > 0, \exists R \in \mathbb{R} \text{ s.t.}$$

$$\Gamma(P_t f, P_t f) \leq k e^{2Rt} P_t(\Gamma(f, f))$$

(Intuitively, $\|D(P_t f)\|_{L^2}^2 \leq k e^{2Rt} P_t(\|Df\|_{L^2}^2)$)

Littlewood-Paley (-Stein) inequality

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To state this inequality, we need to define the Sub-ordination (Semigroup) $e^{-\sqrt{\Delta}t}$

Example $\mathcal{L} = \Delta$ on $\mathbb{R}^d$.

$$e^{-t\sqrt{\Delta}} f(x) = C(y) \int_{\mathbb{R}^d} \frac{t f(y)}{(t^2 + |x-y|^2)^{d+1/2}} dy \quad \text{where}$$

$$(C(y) = \int_{\mathbb{R}^d} (1 + |y|^2)^{-(d+1)/2} dy)$$

$$\begin{aligned} \circ \lambda_t(ds) &:= \frac{t}{2\sqrt{\pi}} e^{-t^2/4s} s^{-3/2} ds \quad (t \geq 0) \quad (\text{Prob mass on } (0, \infty)) \\ & \quad \left(\int_0^\infty e^{-ts} \lambda_t(ds) = e^{-\sqrt{t}t} \right) \dots (*) \end{aligned}$$

$\{O_t\}_{t \geq 0}$: Sub-ordination of $\{P_t\}_{t \geq 0}$ ($\alpha > 0$)

$$O_t f := \int_0^\infty e^{-\alpha s} P_s f \lambda_t(ds) \dots \text{Strongly continuous, contraction semigroup on } L^p(\mu)$$

$$\left(O_t f = \int_0^\infty e^{-\alpha s} (e^{s\mathcal{L}} f) \lambda_t(ds) = \int_0^\infty e^{-(\alpha^2 + s)} s^\alpha \lambda_t(ds) = e^{-\sqrt{\alpha^2 + t}t} \right) (*)$$

Littlewood-Paley's G-functions

(12)

$$g_f^{\rightarrow}(x,t) := \left| \frac{\partial}{\partial t} (O_t^{\theta} f)(x) \right| \quad (\text{Horizontal})$$

$$g_f^{\uparrow}(x,t) := \Gamma(O_t^{\theta} f, O_t^{\theta} f)(x)^{1/2} \quad (\text{Vertical})$$

$$g_f = \sqrt{(g_f^{\rightarrow})^2 + (g_f^{\uparrow})^2}$$

$$(\equiv \|D(O_t^{\theta} f)(x)\|_{H^*})$$

$$G_f(x) := \left(\int_0^{\infty} t g_f(x,t)^2 dt \right)^{1/2}$$

Theorem 1 (Bakry, Shigekawa-Yoshida, K-Miyokawa ('07))

Under the conditions (I) ~ (III), for $1 < p < \infty$, $\forall \alpha > R_{\text{VO}}$,

$$C(p, \bar{\kappa}) \|f\|_{L^p(\mu)} \leq \|G_f\|_{L^p(\mu)} \leq C(p, \kappa) \|f\|_{L^p(\mu)}$$

for $\forall f \in L^2 \cap L^p(\mu)$

(Remark) If $t \neq 0$, $g_f^{\rightarrow}(x,t) = |(-\sqrt{\alpha - \frac{1}{t}}) O_t^{\theta} f(x)| \doteq |\sqrt{\alpha - \frac{1}{t}} f(x)|$

$$g_f^{\uparrow}(x,t) = |D(O_t^{\theta} f)(x)| \doteq \|Df(x)\|_{H^*}$$

Hence, we can regard Thm 1 as an average version of norm equivalence

Remark Geometric interpretation of Condition (III)

(13)

Under some technical assumptions (if we work on ∞ -dim state spaces, these are serious !!, However...), Condition (III) is equivalent to.

$$\Delta (G) \quad \Gamma_2(f, f) \geq -R \Gamma(f, f) \quad (\text{Bakry-Emery's } \Gamma_2\text{-condition})$$

where $\Gamma_2(f, g) := \frac{1}{2} \{ 2\Gamma(f, g) - \Gamma(2f, g) - \Gamma(f, 2g) \}$

o Geometric view point ... *Bakker formula*

$$\text{Ric}: T_{\mathbb{R}M} \times T_{\mathbb{R}M} \rightarrow \mathbb{R} \quad (0,2)\text{-tensor.}$$

$$2\Gamma_2(f, f)$$

$$2 \text{Ric}(\nabla f, \nabla f) = \Delta_M(|\nabla f|^2) - 2(\nabla f, \nabla \Delta_M f)_{T_{\mathbb{R}M}} - 2|\text{Hess}(f)|_{T_{\mathbb{R}M} \times T_{\mathbb{R}M}}^2$$

Hence

(III)

(G)

$$\text{Ric}(\nabla f, \nabla f) \geq -R \Gamma(f, f)$$

$$\Leftrightarrow \text{Ric}(\nabla f, \nabla f) \geq -R |\nabla f|^2 \Leftrightarrow \text{Ric} \geq -R$$

with technical ass.

Sketch ... Stochastic Approach (Itô's formula) (14)

$$f(X_t) - f(X_0) = M_t^f + \int_0^t \sigma f(X_s) ds \text{ with } \langle M^f \rangle_t = 2 \int_0^t \Gamma(f, f)(X_s) ds$$

martingale
term

quadratic variation

$$(f(W_t) - f(W_0)) = \int_0^t (\nabla f(W_s) dW_s) + \frac{1}{2} \int_0^t \Delta f(W_s) ds, \quad \langle M^f \rangle_t = \int_0^t |\nabla f(W_s)|^2 ds$$

$$\Rightarrow d(f^\alpha)(X_t) \stackrel{If}{=} \alpha f^{\alpha-1}(X_t) df(X_t) + \frac{\alpha(\alpha-1)}{2} f^{\alpha-2}(X_t) df(X_t) df(X_t)$$

$$= \alpha f^{\alpha-1}(X_t) dM_t^f + \alpha f^{\alpha-1}(X_t) \sigma f(X_t) dt$$

Vanilla!!

$$+ \frac{\alpha(\alpha-1)}{2} f^{\alpha-2}(X_t) d\langle M^f \rangle_t = 2 \Gamma(f, f)(X_t) dt$$

$$\Rightarrow E^x[f^\alpha(X_t)] - E^x[f^\alpha(X_0)]$$

$$= \alpha \int_0^t E^x[f^{\alpha-1} \sigma f(X_s)] ds + \alpha(\alpha-1) \int_0^t E^x[f^{\alpha-2} \Gamma(f, f)(X_s)] ds$$

$$\Leftrightarrow E(f^\alpha)(x) - P_0(f^\alpha)(x) = \alpha \int_0^t P_s(f^{\alpha-1} \sigma f)(x) ds + \alpha(\alpha-1) \int_0^t P_s(f^{\alpha-2} \Gamma(f, f)(x)) ds$$

$$\Rightarrow f(f^\alpha)(x) = \alpha f^{\alpha-1}(x) \sigma f(x) + \alpha(\alpha-1) f^{\alpha-2}(x) \Gamma(f, f)(x)$$

3.3. Riesz Transform

Here, we impose the existence of

We should check these
for each concrete
problem! (15)

▷ differential structure ... D (tangent sp H ,
cotangent sp H^*)

▷ Semigroup $\{\vec{P}_t\}$ on $L^p(\mu; H^*)$

Set $D\vec{P}_t f = \vec{P}_t Df$ (Intertwining Property)

(Example $X=M$ (cpt Riemannian mfd). $D=d$
 $\vec{P}_t = e^{-t\Delta}$ on $L^p(M, T^*M)$ $\square = d\vec{P}_t + \vec{P}_t d \dots$ Hodge-Kobayashi)

As a consequence of L^p -inequality & Intertwining Property,

Theorem 2

For $1 < p < \infty$, $\forall \alpha > RND$, $\|R_\alpha(f)\|_{L^p(\mu)} \leq C(k, p) \|f\|_{L^p(\mu)}$