Riesz transforms on a path space with Gibbs measures

Hiroshi KAWABI

(Department of Mathematics, Faculty of Science

Okayama University)

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Main Object: Riesz transforms

$$R_\alpha(\mathcal{L}) := D_H \sqrt{\alpha - \mathcal{L}}^{\,-1}, \quad \alpha > 0$$

$$\left(\quad (D_H F, D_H G)_{L^2(\mu;H)} = (-\mathcal{L}F, G)_{L^2(\mu)} \quad \right)$$

♣ L^p-boundedness of the Riesz transforms ?
(Meyer's equivalence of Sobolev norms)

$$\sqrt{\alpha}\|F\|_{L^p(\mu)} + \|D_H F\|_{L^p(\mu;H)}$$
$$\sim \|\sqrt{\alpha - \mathcal{L}}F\|_{L^p(\mu)}, \quad 1 < p < \infty$$

- We are concerned with this problem on general metric spaces (especially ∞-dim state spaces).

♣ History: (i) Analytic Approach: Stein, Coulhon, etc...

$$R_\alpha(\Delta_M)=\int_0^\infty e^{-\alpha t}t^{-1/2}\nabla e^{t\Delta_M}\,dt$$

$\Longrightarrow$ Analysis of gradient bounds of the heat kernel !

(ii) Stochastic Approach: Meyer, Bakry, Shigekawa, etc...

- Meyer: Wiener space (Malliavin Calculus)
- Bakry: Complete Riemannian mfd with $\mathrm{Ric}_M \geq -R$

(Bakry-Emery's Γ_2-calculus

$\Longrightarrow$ Shigekawa–Yoshida (LPS on a general metric space))

- Yoshida: $M^{\mathbb{Z}^d}$ with Gibbs measures
- This Talk: Path space $C(\mathbb{R}, \mathbb{R}^d)$ with Gibbs measures

♣ Our Framework ($P(\phi)_1$-QFT):

- state space: infinite volume path space $C(\mathbb{R}, \mathbb{R}^d)$
- tangent space: $H := L^2(\mathbb{R}, \mathbb{R}^d)$
- underlying measure: Gibbs measure μ

associated with the (formal) Hamiltonian

$$\mathcal{H}(w) := \frac{1}{2}\int_{\mathbb{R}} |\dot{w}(x)|^2_{\mathbb{R}^d}\, dx + \int_{\mathbb{R}} U(w(x))\, dx,$$

where $U : \mathbb{R}^d \to \mathbb{R}$ is a self-interaction potential.

Heuristically, μ is given by

$$\mu(dw) = Z^{-1} e^{-\mathcal{H}(w)} \prod_{x\in\mathbb{R}} dw(x).$$

• This measure is constructed in terms of the ground state Ω of the Schrödinger operator

$$H_U := -\frac{1}{2}\Delta_z + U \quad \text{on} \quad L^2(\mathbb{R}^d, \mathbb{R}; dz).$$

Strictly speaking, it is the probability measure on $C(\mathbb{R}, \mathbb{R}^d)$ induced by

$$d\omega_t = d\beta_t - \frac{\nabla\Omega}{\Omega}(\omega_t)dt, \quad t \in \mathbb{R}, \ (\beta_t)_{t\in\mathbb{R}} : \text{BM}$$

♣ Conditions on the Potential Function U

(U1): $U \in C^2(\mathbb{R}^d, \mathbb{R})$ &

$\exists K_1 \in \mathbb{R}$ s.t. $\nabla^2 U \geq -K_1$.

(U2): $\exists K_2 > 0, \exists p > 0$ s.t.

$$|\nabla U(z)|_{\mathbb{R}^d} + |\nabla^2 U(z)|_{\mathbb{R}^d \otimes \mathbb{R}^d} \le K_2(1 + |z|_{\mathbb{R}^d}^p), \quad z \in \mathbb{R}^d.$$

(U3): $\lim_{|z|_{\mathbb{R}^d} \to \infty} U(z) = \infty.$

Example: $U(z) = \sum_{j=0}^{2m} a_j |z|_{\mathbb{R}^d}^j, a_{2m} > 0, a_1 = 0.$

(Double-well potential functions

$U(z) = a(|z|_{\mathbb{R}^d}^4 - |z|_{\mathbb{R}^d}^2), a > 0$ are included !)

- $\mathcal{FC}_b^\infty$: smooth cylinder functions.

$$F(w) = f(\langle w, \varphi_1 \rangle, \cdots, \langle w, \varphi_n \rangle)(=: f((\langle w, \varphi . \rangle))),$$

where $f \in C_b^\infty(\mathbb{R}^n, \mathbb{R}), \{\varphi_i\}_{i=1}^n \subset C_0^\infty(\mathbb{R}, \mathbb{R}^d),$

$$\langle w, \varphi_i \rangle := \int_{\mathbb{R}} (w(x), \varphi_i(x))_{\mathbb{R}^d} dx.$$

• $\mathcal{FC}_b^\infty(H)$: smooth H-valued cylinder functions.

$$\theta(w)=\sum_{k=1}^{m} F_k(w)e_k,\ F_k \in \mathcal{FC}_b^\infty, e_k \in C_0^\infty(\mathbb{R},\mathbb{R}^d).$$

$$(\ \mathcal{FC}_b^\infty \hookrightarrow L^2(\mu), \mathcal{FC}_b^\infty(H) \hookrightarrow L^2(\mu;H)\ )$$

• H-Fréchet derivative $D_H F \in \mathcal{FC}_b^\infty(H)$ is defined by

$$D_H F(w) := \sum_{i=1}^{n} \partial_i f((\langle w, \varphi.\rangle))\varphi_i.$$

$\Rightarrow$ We consider a (pre-)Dirichlet form on $\mathcal{FC}_b^\infty$ by

$$\mathcal{E}(F,G) := \int (D_H F(w), D_H G(w))_H \mu(dw).$$

♣ Integration-by-Parts Formula [Iwata, Funaki]

$$\mathcal{E}(F,G) = -(\mathcal{L}_0 F, G)_{L^2(\mu)}, \quad F, G \in \mathcal{FC}_b^\infty,$$

where

$$\begin{aligned} \mathcal{L}_0 F(w) &= \mathrm{Tr}(D_H^2 F(w)) + \Big\{ \langle w, \Delta_x D_H F(w(\cdot)) \rangle \\ &\qquad - \langle \nabla U(w(\cdot)), D_H F(w) \rangle \Big\} \\ &= \sum_{i,j=1}^n \partial_i \partial_j f\big((\langle w, \varphi_\cdot \rangle)\big) \cdot \langle \varphi_i, \varphi_j \rangle \\ &\quad + \sum_{i=1}^n \partial_i f\big((\langle w, \varphi_\cdot \rangle)\big) \cdot \{ \langle w, \Delta_x \varphi_i \rangle \\ &\qquad - \langle \nabla U(w(\cdot)), \varphi_i \rangle \} \end{aligned}$$

Theorem 1 [K–Röckner ('07. JFA)]

(i) The pre-Dirichlet operator $(\mathcal{L}_0, \mathcal{F}\mathcal{C}_b^\infty)$ is essentially self-adjoint in $L^2(\mu)$, i.e., $(\overline{\mathcal{L}}_0, \mathrm{Dom}(\overline{\mathcal{L}}_0))$: closure of $(\mathcal{L}_0, \mathcal{F}\mathcal{C}_b^\infty)$ in $L^2(\mu)$ is self-adjoint.

(ii)

$$e^{t\overline{\mathcal{L}}_0} F = P_t F, \quad F \in L^2(\mu),$$

where $\{P_t\}_{t\geq 0}$ is the transition semigroup corresponding to the parabolic SPDE

$$dX_t(x) = \{\Delta_x X_t(x) - \nabla U(X_t(x))\}dt$$
$$+\sqrt{2}dB_t(x), \quad x \in \mathbb{R}, \ t > 0,$$

where $\{B_t\}_{t\geq 0}$ is a H-cylindrical Brownian motion.

• By the Riesz-Thorin interpolation, $\{P_t\}_{t\geq 0}$ can be regarded as a strongly continuous contraction semigroup in $L^p(\mu), 1 \leq p < \infty$.

• We denote by its generator $\mathcal{L} = \mathcal{L}_p$ in $L^p(\mu)$. (Note that $\overline{\mathcal{L}_0} = \mathcal{L}_2$.)

Theorem 2 (Boundedness of the Riesz transforms)
Under (U1), (U2) and (U3), $R_\alpha(\mathcal{L})$ is bounded in $L^p(\mu)$ for all $p > 1$ and $\alpha > K_1 \vee 0$, i.e.,

$$\|R_\alpha(\mathcal{L})F\|_{L^p(\mu)} \leq C_p \|F\|_{L^p(\mu)}, \quad F \in \mathcal{FC}_b^\infty.$$

♣ Outline of the Proof:

(1): Littlewood-Paley-Stein Inequality under a gradient bound condition:

$$\Gamma(P_tF, P_tF) \leq Ke^{2Rt}P_t(\Gamma(F,F)) \quad \cdots (\dagger)$$

[K–Miyokawa, '07, J.Math.Sci.Univ.Tokyo]

- $|D_H P_t F|_H \leq e^{K_1 t} P_t(|D_H F|_H)$ [K, '05, POTA]

♣ Gaveau's diffusion $(B_t^{(1)}, B_t^{(2)}, A_t)$ on the Heisenberg group (sub-Riemannian mfd): Quite recently, Driver–Melcher, H.Q. Li, etc, proved that, surprisingly, $(\dagger)$ also holds, i.e.,

$$|\nabla P_t f|^p \leq K_p P_t(|\nabla f|^p), \quad p \geq 1.$$

(2): Intertwining Property for Diffusion Semigroups

$$D_H P_t F = \vec{P}_t D_H F, \quad F \in \mathcal{D}(\mathcal{E}) \ \cdots (\star)$$

How to show this identity?

Step 1:Generator version of $(\star)$ (rather easier part)

$$D_H \mathcal{L} F = \vec{\mathcal{L}} D_H F, \quad F \in \mathcal{F}\mathcal{C}_b^\infty \ \cdots (\star)'$$

where $(\vec{\mathcal{L}}, \mathcal{F}\mathcal{C}_b^\infty(H))$ is given by

$$\vec{\mathcal{L}}\theta(w)(x) = \sum_{i,j=1}^{n}\sum_{k=1}^{m} \partial_i\partial_j f_k((\langle w, \varphi.\rangle))\langle \varphi_i, \varphi_j\rangle e_k(x)$$

$$+\sum_{i=1}^{n}\sum_{k=1}^{m} \partial_i f_k((\langle w, \varphi.\rangle))\cdot\{\langle w, \Delta_x\varphi_i\rangle$$

$$-\langle \nabla U(w(\cdot)), \varphi_i\rangle\} e_k(x)$$

$$+\sum_{k=1}^{m} f_k((\langle w, \varphi.\rangle))\{\Delta_x e_k(x) - \nabla^2 U(w(x))[e_k(x)]_{\mathbb{R}^d}\}$$

for $\theta(w) = \sum_{k=1}^{m} f_k((\langle w, \varphi.\rangle)) e_k \in \mathcal{FC}_b^\infty(H)$.

Step 2: Construction of $\vec{P}_t$ Define a bi-linear form by

- $\vec{\mathcal{E}}(\theta, \eta) := (-\vec{\mathcal{L}}\theta, \eta)_{L^2(\mu;H)}, \quad \theta, \eta \in \mathcal{FC}_b^\infty(H)$

$\underset{\text{(U1)}}{\Longrightarrow}$ $\vec{\mathcal{E}}(\theta, \theta) \geq -K_1 \|\theta\|^2_{L^2(\mu;H)}$

$\Longrightarrow$

$\exists(\vec{\mathcal{L}}, \mathcal{D}(\vec{\mathcal{L}}))$: Friedrichs extension of $(\vec{\mathcal{L}}, \mathcal{FC}_b^\infty(H))$

($\leftrightarrow$ $(\vec{\mathcal{E}}, \mathcal{D}(\vec{\mathcal{E}}))$: minimal extension)

• $\vec{P}_t := e^{t\vec{\mathcal{L}}}$: symmetric strongly continuous semigroup on $L^2(\mu; H)$

Step 3: $(\star)' \underset{\text{Theorem 1}}{\Longrightarrow} (\star)$ [Shigekawa's machinary]

♣ Further Problems:

• Can we relax the condition (U1) ? (X.D. Li recently discusses under some gaugiability conditions on a complete Riemannian mfd.)

• Another ∞-dim framework (Wasserstein diffusion given by Sturm–von Renesse ?)