The parabolic Harnack inequality and related topics on a path space with Gibbs measures

Hiroshi KAWABI

(Department of Mathematics, Faculty of Science

Okayama University)

Seminar at TU-Darmstadt

November 4, 2008

Main Object: Parabolic Harnack inequality (PHI)

- A comparison theorem for (non-negative) solutions of parabolic equations $\Longrightarrow$

  Heat kernel lower bound, Regularity, etc... (E.B. Davies' book)

♣ P. Li – S.T. Yau's PHI ('86)

◦ M: m-dim complete Riemannian manifold

◦ $\mathrm{Ric}_M \geq -K$, $P_t := e^{t\Delta_M}$

$\Longrightarrow$ For $f \geq 0$, $u(t,x) := P_t f(x)$ satisfies

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) = \Delta_M u(t,x), \quad x \in M, \ t > 0, \\ u(0,x) = f(x). \end{cases}$$

Moreover, the following type PHI holds for all $\alpha > 1$:

$$(0 \le)\ u(t,x) \ \le\ u(t+s,y) \cdot \left(\frac{s+t}{t}\right)^{\frac{m\alpha}{2}}$$

$$\times \exp\left(\frac{\alpha d_M(x,y)^2}{4s} + \frac{\alpha K m s}{4(\alpha-1)}\right).$$

- $\alpha = 1 + (s/d_M(x,y))\sqrt{mK}$ is optimal.

◇ How is an ∞-dim version PHI ?

(Of course, if we take $m \to \infty$ on the above PHI, we cannot get any meaningful inequality !)

♣ Feng-Yu Wang's PHI ('97): For all $\alpha > 1$,

$$\left|P_t f(x)\right|^{\alpha} \ \le\ P_t \left|f\right|^{\alpha}(y)$$

$$\times\ \exp\left(\frac{\alpha d_M(x,y)^2}{4(\alpha-1)} \cdot \frac{2K}{1-e^{-2Kt}}\right).$$

◇ Since Wang's inequality does not involve dimension m, we can generalize it to ∞-dim frameworks!

- Kusuoka ('92): Ornstein-Uhlenbeck semigroup on Wiener spaces
- Aida-K. ('01): Symmetric diffusion semigroups on Wiener spaces by using Malliavin calculus and Γ_2-calculus
- Röckner-Wang ('03): Generalized Mehler semigroups
- This Talk: K. ('05, POTA, '04, Bull.Sci.Math.) Diffusion semigroups on path space $C(\mathbb{R}, \mathbb{R}^d)$ with Gibbs measures (Coupling method for stochastic PDEs)

♣ Our Framework ($P(\phi)_1$-QFT):

- state space: infinite volume path space $C(\mathbb{R}, \mathbb{R}^d)$
- tangent space: $H := L^2(\mathbb{R}, \mathbb{R}^d)$
- underlying measure: Gibbs measure μ

associated with the (formal) Hamiltonian

$$\mathcal{H}(w) := \frac{1}{2}\int_{\mathbb{R}} |\dot{w}(x)|^2_{\mathbb{R}^d}\, dx + \int_{\mathbb{R}} U(w(x))\, dx,$$

where $U : \mathbb{R}^d \to \mathbb{R}$ is a self-interaction potential.

Heuristically, μ is given by

$$\mu(dw) = Z^{-1} e^{-\mathcal{H}(w)} \prod_{x\in\mathbb{R}} dw(x).$$

• This measure is constructed in terms of the ground state Ω of the Schrödinger operator

$$H_U := -\frac{1}{2}\Delta_z + U \quad \text{on} \quad L^2(\mathbb{R}^d, \mathbb{R}; dz).$$

♣ Conditions on the Potential Function U

(U1): $U \in C^1(\mathbb{R}^d, \mathbb{R})$ and $\exists K_1 \in \mathbb{R}$ s.t.

$$\left(\nabla U(z_1) - \nabla U(z_2), z_1 - z_2\right)_{\mathbb{R}^d} \geq -K_1 |z_1 - z_2|^2_{\mathbb{R}^d} \text{ for } z_1, z_2 \in \mathbb{R}^d.$$

(U2): $\exists K_2 > 0, \exists p > 0$ s.t.

$$|\nabla U(z)|_{\mathbb{R}^d} \leq K_2(1 + |z|^p_{\mathbb{R}^d}) \text{ for } z \in \mathbb{R}^d.$$

(U3): $\lim_{|z|_{\mathbb{R}^d} \to \infty} U(z) = \infty.$

Example: $U(z) = \sum_{j=0}^{2m} a_j |z|_{\mathbb{R}^d}^j, a_{2m} > 0, a_1 = 0.$

(Double-well potential functions

$U(z){=}a(|z|_{\mathbb{R}^d}^4 - |z|_{\mathbb{R}^d}^2), a > 0$ are included !)

- Under **(U1)** and **(U3)**, H_U has purely discrete spectrum and a complete set of eigenfunctions.

$\Rightarrow$ · $\lambda_0(> \min U)$: the lowest eigenvalue of H_U,

· Ω : ground state of H_U with $\|\Omega\|_{L^2(\mu)} = 1$ and $\Omega > 0$.

i.e., $H_U\Omega = \lambda_0\Omega.$ $(e^{-tH_U}\Omega = e^{-t\lambda_0}\Omega)$

- $(H_U, L^2(dz)) \simeq (\hat{H}_U, L^2(\Omega(z)^2 dz))$, where

$\hat{H}_U f := \Omega^{-1} H_U(\Omega f) = -\frac{1}{2}\Delta_z - (\frac{\nabla\Omega}{\Omega}, \nabla)$

$\Longrightarrow$ Our Gibbs measure μ is the probability measure on $C(\mathbb{R}, \mathbb{R}^d)$ induced by

$$d\omega_t = d\beta_t + \frac{\nabla\Omega}{\Omega}(\omega_t)dt, \quad t \in \mathbb{R}, \ (\beta_t)_{t\in\mathbb{R}} : \text{BM}$$

$\big(\nu(dz) := \Omega(z)^2 dz$: reversible measure$\big)$

♣ Quasi-invariance: For every $k \in C_0^\infty(\mathbb{R}, \mathbb{R}^d)$, $\mu \sim \mu(k + \cdot)$ and $\mu(k + dw) = \Lambda(k, w)\mu(dw)$, where

$$\Lambda(k, w) = \exp\Big\{ \int_{\mathbb{R}} \Big(U(w(x)) - U(w(x) + k(x)) - \frac{1}{2}|k'(x)|^2 + (w(x), \Delta_x k(x))_{\mathbb{R}^d} \Big) dx \Big\}$$

and $\Delta_x := d^2/dx^2$.

- $\mathcal{F}\mathcal{C}_b^\infty$: smooth cylinder functions.

$$F(w) = f(\langle w, \varphi_1 \rangle, \cdots, \langle w, \varphi_n \rangle) (=: f((\langle w, \varphi. \rangle)),$$

where $f \in C_b^\infty(\mathbb{R}^n, \mathbb{R}), \{\varphi_i\}_{i=1}^n \subset C_0^\infty(\mathbb{R}, \mathbb{R}^d)$,

$$\langle w, \varphi_i \rangle := \int_{\mathbb{R}} (w(x), \varphi_i(x))_{\mathbb{R}^d} dx.$$

- H-Fréchet derivative $D_H F \in \mathcal{F}\mathcal{C}_b^\infty(H)$ is defined by

$$D_H F(w) := \sum_{i=1}^n \partial_i f((\langle w, \varphi. \rangle)) \varphi_i.$$

$\Longrightarrow$ We consider a (pre-)Dirichlet form on $\mathcal{F}\mathcal{C}_b^\infty$ by

$$\mathcal{E}(F, G) := \int (D_H F(w), D_H G(w))_H \mu(dw).$$

♣ Integration-by-Parts Formula [Iwata, Funaki]

$$\mathcal{E}(F,G) = -(\mathcal{L}_0 F, G)_{L^2(\mu)}, \quad F, G \in \mathcal{FC}_b^\infty,$$

where

$$\mathcal{L}_0 F(w) = \mathrm{Tr}(D_H^2 F(w)) + \Big\{ \langle w, \Delta_x D_H F(w(\cdot)) \rangle - \langle \nabla U(w(\cdot)), D_H F(w) \rangle \Big\}$$

$$= \sum_{i,j=1}^{n} \partial_i \partial_j f\big((\langle w, \varphi_\cdot \rangle)\big) \cdot \langle \varphi_i, \varphi_j \rangle$$

$$+ \sum_{i=1}^{n} \partial_i f\big((\langle w, \varphi_\cdot \rangle)\big) \cdot \{ \langle w, \Delta_x \varphi_i \rangle - \langle \nabla U(w(\cdot)), \varphi_i \rangle \}$$

Theorem 1 [K.–Röckner ('07. JFA)]

(i) The pre-Dirichlet operator $(\mathcal{L}_0, \mathcal{F}\mathcal{C}_b^\infty)$ is essentially self-adjoint in $L^2(\mu)$, i.e., $(\overline{\mathcal{L}}_0, \mathrm{Dom}(\overline{\mathcal{L}}_0))$: closure of $(\mathcal{L}_0, \mathcal{F}\mathcal{C}_b^\infty)$ in $L^2(\mu)$ is self-adjoint.

(ii)
$$e^{t\overline{\mathcal{L}}_0}F = P_t F, \quad F \in L^2(\mu),$$
where $\{P_t\}_{t\geq 0}$ is the transition semigroup corresponding to the parabolic SPDE

$$dX_t(x) = \{\Delta_x X_t(x) - (\nabla U)(X_t(x))\}dt$$
$$+\sqrt{2}dB_t(x), \quad x \in \mathbb{R}, \ t > 0, \cdots \text{(GL)}$$

where $\{B_t\}_{t\geq 0}$ is a H-cylindrical Brownian motion.

- By the Riesz-Thorin interpolation, $\{P_t\}_{t\geq 0}$ can be regarded as a strongly continuous contraction semigroup in $L^p(\mu), 1 \leq p < \infty$.

Theorem 2 [K. '05, POTA] Let $F \in L^\infty(\mu)$. Then for any $h \in C_0^\infty(\mathbb{R}, \mathbb{R}^d)$, $\alpha > 1$ and $t > 0$, the following PHI holds:

$$|P_t F(w)|^\alpha \leq P_t |F|^\alpha (w+h) \times \exp\left(\frac{\alpha |h|_H^2}{4(\alpha-1)} \cdot \frac{2K_1}{1-e^{-2K_1 t}}\right), \quad \mu\text{-a.e. } w.$$

♣ In the case $K_1 = 0$, we set $\dfrac{2K_1}{1-e^{-2K_1 t}} := \dfrac{1}{t}$.

♣ Outline of the Proof of Thm 2:

(1): A gradient bound for the diffusion semigroup:

$$|D_H P_t F|_H \le e^{K_1 t} P_t(|D_H F|_H), F \in \mathcal{D}(\mathcal{E}) \cdots (\dagger)$$

To show $(\dagger)$, we use the coupling method for SPDE (GL).

- $\|X_t^w - X_t^{w'}\|_r \le e^{(K_1+2r^2)t}\|w - w'\|_r, \quad P$-a.s.

where $\|w\|_r^2 := \|we^{r|x|}\|_H^2, \quad w, w' \in \mathcal{C} \subset C(\mathbb{R}, \mathbb{R}^d)$.

Sketch: $Y_t(x) := X_t^w(x) - X_t^{w'}(x)$ satisfies

$$\frac{\partial}{\partial t} Y_t(x) = \Delta_x Y_t(x)$$

$$-\{\nabla U(X_t^w(x)) - \nabla U(X_t^{w'}(x))\}, \quad x \in \mathbb{R}, \ t > 0.$$

Multiply both sides by $2Y_t(x)e^{-2r|x|}$, use the condition (U1), integrate over $(0, t) \times \mathbb{R}$, and apply integration by parts !!

(2): Introduce a "nice" interpolation function G $\Longrightarrow$ Differential inequality

For $F \in \mathcal{FC}_b^\infty(>0)$ and $h \in C_0^\infty(\mathbb{R}, \mathbb{R}^d)$, we set by

∘ $G(s) := P_s(P_{t-s}F)^\alpha(\cdot + v(s)), \quad 0 \le s \le t,$

where

$$v(s) := \frac{(\int_0^s e^{-2K_1\tau} d\tau)}{(\int_0^t e^{-2K_1\tau} d\tau)} \cdot h. \quad (v(0) = 0, v(t) = h)$$

$\Longrightarrow$ By differentiating on both sides and using the gradient bound (†), we obtain a certain differential inequality.

♣ To expand $\frac{d}{ds} P_s(P_{t-s}F)^\alpha(\cdot + v(s))$, we must remark that $(P_{t-s}F)^\alpha \notin \mathrm{Dom}(\mathcal{L}_2)$ generally. Here we adopt a stochastic approach (Itô's formula) to overcome this difficulty.

- $H(r_1, r_2, r_3) := P_{r_1}(P_{t-r_2}F)^{\alpha}(\cdot + v(r_3))$
  $(0 < r_1, r_2, r_3 < t)$
- $M_{r_1} := (P_{t-r_2}F)(X_{r_1}) - (P_{t-r_2}F)(X_0)$
  $- \int_0^{r_1} \mathcal{L}_2(P_{t-r_2}F)(X_\tau)d\tau$

By using Itô's formula, we have

$$
\begin{aligned}
(P_{t-r_2}F)^{\alpha}(X_{r_1}) = (P_{t-r_2}F)^{\alpha}(X_0) \\
&+\alpha \int_0^{r_1} (P_{t-r_2}F)^{\alpha-1}(X_\tau) \cdot \mathcal{L}_2(P_{t-r_2}F)(X_\tau)d\tau \\
&+\alpha \int_0^{r_1} (P_{t-r_2}F)^{\alpha-1}(X_\tau)dM_\tau \\
&+\frac{\alpha(\alpha-1)}{2} \int_0^{r_1} (P_{t-r_2}F)^{\alpha-2}(X_\tau)d\langle M\rangle_\tau .
\end{aligned}
$$

Here, we recall

- $\langle M\rangle_t = 2\int_0^t |D(P_{t-r_2}F)(X_\tau)|_H^2 d\tau.$

Then we obtain

$$\begin{aligned} H(r_1,r_2,r_3) &= \mathbb{E}\big[(P_{t-r_2}F)^\alpha(X_{r_1}^{\cdot+v(r_3)})\big] \\ &= (P_{t-r_2}F)^\alpha(\cdot+v(r_3)) \end{aligned}$$

$$+\alpha\int_0^{r_1} P_\tau\Big\{(P_{t-r_2}F)^{\alpha-1}\mathcal{L}_2(P_{t-r_2}F)\Big\}(\cdot+v(r_3))d\tau$$

$$+\frac{\alpha(\alpha-1)}{2}\int_0^{r_1} P_\tau\Big\{(P_{t-r_2}F)^{\alpha-2}$$

$$\cdot\big(2|D(P_{t-r_2}F)|_H^2\big)\Big\}(\cdot+v(r_3))d\tau.$$

Hence we can proceed as

$$
\begin{aligned}
&\frac{d}{ds} P_s (P_{t-s} F)^{\alpha} (\cdot + v(s)) \\
&\qquad = \sum_{i=1}^{3} \frac{\partial}{\partial r_i} \Big|_{r_1 = r_2 = r_3 = s} H(r_1, r_2, r_3) \\
&\qquad = \alpha(\alpha - 1) P_s \Big\{ (P_{t-r_2} F)^{\alpha - 2} \\
&\qquad\qquad \cdot |D(P_{t-r_2} F)|_H^2 \Big\} (\cdot + v(r_3)) \\
&\qquad\quad + \Big(D P_s (P_{t-s} F)^{\alpha} (\cdot + v(s)), \dot{v}(s) \Big)_H .
\end{aligned}
$$

♣ Application and Further Topics:

(1) Varadhan type short time asymptotics

As an application of Thm 2, we can obtain a certain lower bound of

$$p_t(A, B) := \int_A P_t 1_B(w)\mu(dw), \quad \mu(A), \mu(B) > 0.$$

(Since this bound is very complicated, we omit in this talk.)
By combining this bound with Lyons-Zheng's martingale decomposition thm ($\Rightarrow$ upper bound), we have

$$\lim_{t \searrow 0} 4t \log p_t(A, B) = -d_H(A, B)^2$$

under A or B is H-open.

(2) • Non-symmetric case (K. '04, Bull.Sci.Math)

• Log-Sobolev inequality (K. '06, IDAQP)

• Littlewood-Paley inequality, Riesz transforms, etc.