Riesz transforms associated with diffusion operators on a path space with Gibbs measures

Hiroshi KAWABI

(Department of Mathematics, Faculty of Science

Okayama University)

Oberseminar Stochastik, University of Bonn

October 23, 2008

Main Object: Riesz transforms

$$R_\alpha(\mathcal{L}) := D_H \sqrt{\alpha - \mathcal{L}}^{\,-1}, \quad \alpha > 0$$

$$\left(\quad (D_H F, D_H G)_{L^2(\mu;H)} = (-\mathcal{L}F, G)_{L^2(\mu)} \quad \right)$$

♣ L^p-boundedness of the Riesz transforms ?
(Meyer's equivalence of Sobolev norms)

$$\sqrt{\alpha}\|F\|_{L^p(\mu)} + \|D_H F\|_{L^p(\mu;H)} \sim \|\sqrt{\alpha - \mathcal{L}}F\|_{L^p(\mu)}, \quad 1 < p < \infty$$

- We are concerned with this problem on general metric spaces (especially ∞-dim state spaces).

♣ History: (i) Analytic Approach: Stein, Coulhon, etc...

$$R_\alpha(\Delta_M)=\int_0^\infty e^{-\alpha t}t^{-1/2}\nabla e^{t\Delta_M}\,dt$$

$\Longrightarrow$ Analysis of gradient bounds of the heat kernel !

(ii) Stochastic Approach: Meyer, Bakry, Shigekawa, etc...

- Meyer: Wiener space (Malliavin Calculus)
- Bakry: Complete Riemannian mfd with $\mathrm{Ric}_M \geq -R$

(Bakry-Emery's Γ_2-calculus

$\Longrightarrow$ Shigekawa–Yoshida (LPS on a general metric space))

- Yoshida: $M^{\mathbb{Z}^d}$ with Gibbs measures
- This Talk: Path space $C(\mathbb{R}, \mathbb{R}^d)$ with Gibbs measures

♣ Our Framework ($P(\phi)_1$-QFT):

- state space: infinite volume path space $C(\mathbb{R}, \mathbb{R}^d)$
- tangent space: $H := L^2(\mathbb{R}, \mathbb{R}^d)$
- underlying measure: Gibbs measure μ

associated with the (formal) Hamiltonian

$$\mathcal{H}(w) := \frac{1}{2}\int_{\mathbb{R}} |\dot{w}(x)|^2_{\mathbb{R}^d}\, dx + \int_{\mathbb{R}} U(w(x))\, dx,$$

where $U : \mathbb{R}^d \to \mathbb{R}$ is a self-interaction potential.

Heuristically, μ is given by

$$\mu(dw) = Z^{-1} e^{-\mathcal{H}(w)} \prod_{x\in\mathbb{R}} dw(x).$$

• This measure is constructed in terms of the ground state Ω of the Schrödinger operator

$$H_U := -\frac{1}{2}\Delta_z + U \quad \text{on} \quad L^2(\mathbb{R}^d, \mathbb{R}; dz).$$

Strictly speaking, it is the probability measure on $C(\mathbb{R}, \mathbb{R}^d)$ induced by

$$d\omega_t = d\beta_t - \frac{\nabla\Omega}{\Omega}(\omega_t)dt, \quad t \in \mathbb{R}, \ (\beta_t)_{t\in\mathbb{R}} : \text{BM}$$

♣ Conditions on the Potential Function U

(U1): $U \in C^2(\mathbb{R}^d, \mathbb{R})$ &

$\exists K_1 \in \mathbb{R}$ s.t. $\nabla^2 U \geq -K_1$.

(U2): $\exists K_2 > 0, \exists p > 0$ s.t.

$$|\nabla U(z)|_{\mathbb{R}^d} + |\nabla^2 U(z)|_{\mathbb{R}^d \otimes \mathbb{R}^d} \le K_2(1 + |z|_{\mathbb{R}^d}^p), \quad z \in \mathbb{R}^d.$$

(U3): $\lim_{|z|_{\mathbb{R}^d} \to \infty} U(z) = \infty.$

Example: $U(z) = \sum_{j=0}^{2m} a_j |z|_{\mathbb{R}^d}^j, a_{2m} > 0, a_1 = 0.$

(Double-well potential functions

$U(z) = a(|z|_{\mathbb{R}^d}^4 - |z|_{\mathbb{R}^d}^2), a > 0$ are included !)

- $\mathcal{F}\mathcal{C}_b^\infty$: smooth cylinder functions.

$$F(w) = f(\langle w, \varphi_1 \rangle, \cdots, \langle w, \varphi_n \rangle)(=: f((\langle w, \varphi. \rangle)),$$

where $f \in C_b^\infty(\mathbb{R}^n, \mathbb{R}), \{\varphi_i\}_{i=1}^n \subset C_0^\infty(\mathbb{R}, \mathbb{R}^d),$

$$\langle w, \varphi_i \rangle := \int_{\mathbb{R}} (w(x), \varphi_i(x))_{\mathbb{R}^d} dx.$$

• $\mathcal{FC}_b^\infty(H)$: smooth H-valued cylinder functions.

$$\theta(w) = \sum_{k=1}^{m} F_k(w) e_k, \ \ F_k \in \mathcal{FC}_b^\infty, e_k \in C_0^\infty(\mathbb{R}, \mathbb{R}^d).$$

$$\left(\mathcal{FC}_b^\infty \hookrightarrow L^2(\mu), \mathcal{FC}_b^\infty(H) \hookrightarrow L^2(\mu; H) \right)$$

• H-Fréchet derivative $D_H F \in \mathcal{FC}_b^\infty(H)$ is defined by

$$D_H F(w) := \sum_{i=1}^{n} \partial_i f((\langle w, \varphi_\cdot \rangle))\varphi_i.$$

$\Rightarrow$ We consider a (pre-)Dirichlet form on $\mathcal{FC}_b^\infty$ by

$$\mathcal{E}(F, G) := \int \left(D_H F(w), D_H G(w)\right)_H \mu(dw).$$

♣ Integration-by-Parts Formula [Iwata, Funaki]

$$\mathcal{E}(F,G) = -(\mathcal{L}_0 F, G)_{L^2(\mu)}, \quad F, G \in \mathcal{FC}_b^\infty,$$

where

$$\begin{aligned}
\mathcal{L}_0 F(w) &= \mathrm{Tr}(D_H^2 F(w)) + \Big\{ \langle w, \Delta_x D_H F(w(\cdot)) \rangle \\
&\qquad - \langle \nabla U(w(\cdot)), D_H F(w) \rangle \Big\} \\
&= \sum_{i,j=1}^{n} \partial_i \partial_j f\big((\langle w, \varphi_\cdot \rangle)\big) \cdot \langle \varphi_i, \varphi_j \rangle \\
&\quad + \sum_{i=1}^{n} \partial_i f\big((\langle w, \varphi_\cdot \rangle)\big) \cdot \big\{ \langle w, \Delta_x \varphi_i \rangle \\
&\qquad - \langle \nabla U(w(\cdot)), \varphi_i \rangle \big\}
\end{aligned}$$

Theorem 1 [K–Röckner ('07. JFA)]

(i) The pre-Dirichlet operator $(\mathcal{L}_0, \mathcal{F}\mathcal{C}_b^\infty)$ is essentially self-adjoint in $L^2(\mu)$, i.e., $(\overline{\mathcal{L}}_0, \mathrm{Dom}(\overline{\mathcal{L}}_0))$: closure of $(\mathcal{L}_0, \mathcal{F}\mathcal{C}_b^\infty)$ in $L^2(\mu)$ is self-adjoint.

(ii)
$$e^{t\overline{\mathcal{L}}_0}F = P_tF, \quad F \in L^2(\mu),$$
where $\{P_t\}_{t\geq 0}$ is the transition semigroup corresponding to the parabolic SPDE

$$dX_t(x) = \{\Delta_x X_t(x) - (\nabla U)(X_t(x))\}dt$$
$$+\sqrt{2}dB_t(x), \ \ x \in \mathbb{R}, \ t > 0,$$

where $\{B_t\}_{t\geq 0}$ is a H-cylindrical Brownian motion.

- By the Riesz-Thorin interpolation, $\{P_t\}_{t\geq 0}$ can be regarded as a strongly continuous contraction semigroup in $L^p(\mu), 1 \leq p < \infty$.
- We denote by its generator $\mathcal{L} = \mathcal{L}_p$ in $L^p(\mu)$. (Note that $\overline{\mathcal{L}_0} = \mathcal{L}_2$.)

Theorem 2 (Boundedness of the Riesz transforms)
Under (U1), (U2) and (U3), $R_\alpha(\mathcal{L})$ is bounded in $L^p(\mu)$ for all $p > 1$ and $\alpha \geq K_1 \vee 0$, i.e.,

$$\|R_\alpha(\mathcal{L})F\|_{L^p(\mu)} \leq C_p \|F\|_{L^p(\mu)}, \quad F \in \mathcal{FC}_b^\infty.$$

♣ Outline of the Proof:

(1): Littlewood-Paley-Stein Inequality under a gradient bound condition:

$$\Gamma(P_tF, P_tF) \leq Ke^{2Rt} P_t(\Gamma(F,F)) \quad \cdots (\dagger)$$

[K–Miyokawa, '07, J.Math.Sci.Univ.Tokyo]

- $|D_H P_t F|_H \leq e^{K_1 t} P_t(|D_H F|_H)$ [K, '05, POTA]

♣ Gaveau's diffusion $(B_t^{(1)}, B_t^{(2)}, A_t)$ on the Heisenberg group (sub-Riemannian mfd): Quite recently, Driver–Melcher, H.Q. Li, etc, proved that, $(\dagger)$ also holds, i.e.,

$$|\nabla P_t f|^p \leq K_p P_t(|\nabla f|^p), \quad p \geq 1.$$

(2): Intertwining Property for Diffusion Semigroups

$$D_H P_t F = \vec{P}_t D_H F, \quad F \in \mathcal{D}(\mathcal{E}) \quad \cdots (\star)$$

How to show this identity?

Step 1:Generator version of $(\star)$ (rather easier part)

$$D_H \mathcal{L} F = \vec{\mathcal{L}} D_H F, \quad F \in \mathcal{F}\mathcal{C}_b^\infty \quad \cdots (\star)'$$

where $(\vec{\mathcal{L}}, \mathcal{F}\mathcal{C}_b^\infty(H))$ is given by

$$\vec{\mathcal{L}}\theta(w)(x) = \sum_{i,j=1}^{n} \sum_{k=1}^{m} \partial_i \partial_j f_k((\langle w, \varphi.\rangle)) \langle \varphi_i, \varphi_j \rangle e_k(x)$$

$$+ \sum_{i=1}^{n} \sum_{k=1}^{m} \partial_i f_k((\langle w, \varphi.\rangle)) \cdot \{ \langle w, \Delta_x \varphi_i \rangle$$

$$- \langle \nabla U(w(\cdot)), \varphi_i \rangle \} e_k(x)$$

$$+\sum_{k=1}^{m} f_k((\langle w, \varphi.\rangle))\{\Delta_x e_k(x) - \nabla^2 U(w(x))[e_k(x)]_{\mathbb{R}^d}\}$$

for $\theta(w) = \sum_{k=1}^{m} f_k((\langle w, \varphi.\rangle))e_k \in \mathcal{FC}_b^\infty(H)$.

Step 2: Construction of $\vec{P}_t$ Define a bi-linear form by

- $\vec{\mathcal{E}}(\theta, \eta) := (-\vec{\mathcal{L}}\theta, \eta)_{L^2(\mu;H)}, \quad \theta, \eta \in \mathcal{FC}_b^\infty(H)$

$$\underset{\text{(U1)}}{\Longrightarrow} \quad \vec{\mathcal{E}}(\theta, \theta) \geq -K_1 \|\theta\|^2_{L^2(\mu;H)}$$

$\Longrightarrow$

$\exists(\vec{\mathcal{L}}, \mathcal{D}(\vec{\mathcal{L}}))$: Friedrichs extension of $(\vec{\mathcal{L}}, \mathcal{FC}_b^\infty(H))$

($\leftrightarrow$ $(\vec{\mathcal{E}}, \mathcal{D}(\vec{\mathcal{E}}))$: minimal extension)

- $\vec{P}_t := e^{t\vec{\mathcal{L}}}$: symmetric strongly continuous semigroup on $L^2(\mu; H)$

Step 3: $(\star)' \underset{\text{Theorem 1}}{\Longrightarrow} (\star)$ [Shigekawa, '06, JFA]

♣ Application (Original Motivation ?):

- A Non-Symmetric Diffusion Process on $C(\mathbb{R}, \mathbb{R}^d)$:

We consider an SPDE (GL-B) given by

$$dY_t(x) = \{\Delta_x Y_t(x) - (\nabla U)(Y_t(x))\}dt + BY_t(x)dt + \sqrt{2}dB_t(x), \quad x \in \mathbb{R}, \ t > 0,$$

where $B \in \mathbb{R}^d \otimes \mathbb{R}^d$.

♣ Additional Conditions on U and B

(U4): U is radial symmetric, i.e.,

$$U = U(z) \text{ is a function of } |z|_{\mathbb{R}^d}.$$

(B): $B^* = -B$.

Example: In the case of $d = 2$, as an example of B, we give $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Since this matrix generates $e^{tB} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$, we can regard the solution of the SPDE (GL-B) as the diffusion process containing rotation.

• Under these conditions, our Gibbs measure still keeps invariance (Not Reversible !), i.e.,

$$\int P_t^{(B)} F(w) \mu(dw) := \int \mathbb{E}[F(Y_t^w)] \mu(dw) = \int F(w) \mu(dw)$$

(Key point: $(\nabla U(z), Bz)_{\mathbb{R}^d} = 0, \ \nabla \cdot (Bz) = 0$)

$\Longrightarrow \{P_t^{(B)}\}_{t \geq 0}$: strongly continuous contraction semigroup in $L^p(\mu), 1 \leq p < \infty$.

(We denote by its generator $\mathcal{L}_p^{(B)}$ in $L^p(\mu)$.)

- $P_t^{(B)}F = P_t Q_t F = Q_t P_t F$ for $F \in \mathcal{F}\mathcal{C}_b^\infty$, where $(Q_t F)(w) := F(e^{tB}[w(\cdot)]_{\mathbb{R}^d})$.

$$\frac{\partial}{\partial t}(e^{tB}X_t(x)) = (Be^{tB})X_t(x) + e^{tB}\frac{\partial}{\partial t}X_t(x)$$

$$= (Be^{tB})X_t(x)$$

$$+ e^{tB}\left(\Delta_x X_t(x) - \nabla U(X_t(x)) + \sqrt{2}\dot{B}_t(x)\right)$$

$$= \Delta_x(e^{tB}X_t(x)) - \nabla U(e^{tB}X_t(x))$$

$$+ B(e^{tB}X_t(x)) + \sqrt{2}(e^{tB}\dot{B}_t(x))$$

$\Longrightarrow e^{tB}(X_t(\cdot))$ and $Y_t(\cdot)$ have the same prob law !

♣<u>Problem:</u> For $p \geq 2$, $\mathrm{Dom}(\mathcal{L}_p^{(B)}) \subset \mathcal{D}(\mathcal{E})$?

(Of course, $\mathrm{Dom}(\mathcal{L}_p) \subset \mathcal{D}(\mathcal{E})$ holds.)

- Fukushima's decomposition: $u(Y_t) - u(Y_0) = M_t^{[u]} + N_t^{[u]}, \langle M^{[u]} \rangle_t = 2 \int_0^t |D_H u(Y_s)|_H^2 ds$

Thm 2 implies $\mathrm{Dom}(\sqrt{1-\mathcal{L}_p}) = W^{1,p}(\mu)$

$$(:= \{F \in L^p(\mu) \cap \mathcal{D}(\mathcal{E});\ |D_H F|_H \in L^p(\mu)\}).$$

Hence it is sufficient to show

$$\|\sqrt{1-\mathcal{L}_p}(1-\mathcal{L}_p^{(B)})^{-1} F\|_p \leq C\|F\|_p, F \in L^p(\mu).$$

$$(\Rightarrow \mathrm{Dom}(\mathcal{L}_p^{(B)}) \subset \mathrm{Dom}(\sqrt{1-\mathcal{L}_p}) = W^{1,p}(\mu))$$

$$
\begin{aligned}
&\|\sqrt{1-\mathcal{L}}(1-\mathcal{L}^{(B)})^{-1}F\|_p \\
&\leq \|\sqrt{1-\mathcal{L}}\int_0^\infty e^{-t}P_tQ_tFdt\|_p \\
&\leq \int_0^\infty \|\sqrt{1-\mathcal{L}}^{-1}(1-\mathcal{L})(e^{-t}P_t)(Q_tF)\|_p dt \\
&= \int_0^\infty dt \| \frac{1}{\Gamma(1/2)}\int_0^\infty ds\ s^{-1/2}e^{-s}e^{s\mathcal{L}} \\
&\qquad (1-\mathcal{L})(e^{-t}P_t)(Q_tF)\Big\|_p \\
&= \frac{1}{\Gamma(1/2)}\int_0^\infty dt \int_0^\infty ds \|\ s^{-1/2} \\
&\qquad (\mathcal{L}-1)e^{(s+t)(\mathcal{L}-1)}(Q_tF)\Big\|_p
\end{aligned}
$$

Here we recall $\|(\mathcal{L}-1)e^{t(\mathcal{L}-1)}\|_{p,p} \le C_p t^{-1} e^{-t}$.

Then we can continue as

$$
\begin{aligned}
&\le \frac{C_p}{\Gamma(1/2)} \int_0^\infty dt \int_0^\infty ds\ s^{-1/2} \\
&\qquad\qquad \times (s+t)^{-1} e^{-(s+t)} \|Q_t F\|_p \\
&= \frac{C_p\|F\|_p}{\Gamma(1/2)} \int_0^\infty dt \int_0^\infty ds\ s^{-1/2} (s+t)^{-1} e^{-(s+t)} \\
&\le \frac{C_p\|F\|_p}{\Gamma(1/2)} \Big(\int_0^\infty e^{-t} t^{-1/2} dt \Big) \qquad (s := \frac{t}{\tau}) \\
&\qquad \times \Big(\int_0^\infty \tau^{-1/2} (1+\tau)^{-1} d\tau \Big) \le C_p \|F\|_p.
\end{aligned}
$$