Invariants of the unipotent radical of a Borel subgroup

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1 Introduction

Grassmannians and their Schubert subvarieties are fascinating objects and attract many mathematicians. The homogeneous coordinate ring of the Grassmann variety consisting of m-dimensional subspaces in an n-dimensional vector space over K is the subring of the polynomial ring over K generated by maximal minors of the $m \times n$ matrix of indeterminates. And the homogeneous coordinate ring of a Schubert subvariety is generated by the universal $m \times n$ matrix with the following property for some integers $b_1, b_2, \ldots, b_m$ with $1 \leq b_1 < b_2 < \cdots < b_m \leq n$.

(†) All the i-minors of first $b_i - 1$ columns are zero.

If a matrix M satisfies the property (†), then Mg also satisfies (†) for any upper triangular matrix g, so the Borel subgroup consisting of the upper triangular matrices of the general linear group and its subgroups act on the homogeneous coordinate ring of a Schubert subvariety of a Grassmannian and the algebra generated by the entries of the universal matrix with (†). We study the ring of invariants of the unipotent radical of this Borel subgroup in §3.

It is also known that there is an $m \times n$ universal matrix with conditions on minors related both to rows and columns. The direct product of Borel subgroups, consisting of lower triangular matrices and upper triangular matrices respectively, of the direct product of general linear groups, and its subgroups act on the algebra generated by the entries of the matrix with universal property. We also study the ring of invariants of the unipotent radical of this Borel subgroup.

2 Preliminaries

All rings and algebras in this note are commutative with identity element.

Let K be an infinite field of arbitrary characteristic. For an $s \times t$ matrix $M = (m_{ij})$ with entries in a K-algebra S, we denote by $K[M]$ the K-subalgebra of S generated by the entries of M, by $I_r(M)$ the ideal of S generated by all r-minors of M, by $M_{\leq j}$ the $s \times j$ matrix consisting of the first j columns of M, by $M^{\leq i}$ the $i \times t$ matrix consisting of the first i rows of M and by $\Gamma(M)$ the set of all maximal minors of M.

Let l be a positive integer. We set

$$H(l) := \{[a_1, a_2, \ldots, a_r] \mid 1 \leq a_1 < a_2 < \cdots < a_r \leq l, a_i \in \boldsymbol{Z}\}.$$

For $\alpha = [a_1, a_2, \ldots, a_r] \in H(l)$, we set $\mathrm{size}\alpha = r$. We define the order on $H(l)$ by

$$[a_1, \ldots, a_r] \leq [b_1, \ldots, b_s] \overset{\text{def}}{\Longleftrightarrow} r \geq s, a_i \leq b_i \text{ for } i = 1, 2, \ldots, s.$$

It is easy to verify that $H(l)$ is a distributive lattice.

For positive integers m and n, we set

$$\Delta(m \times n) := \{[\alpha|\beta] \mid \alpha \in H(m), \beta \in H(n), \mathrm{size}\alpha = \mathrm{size}\beta\}$$

and define the order on $\Delta(m \times n)$ by

$$[\alpha|\beta] \leq [\alpha'|\beta'] \overset{\text{def}}{\Longleftrightarrow} \alpha \leq \alpha' \text{ in } H(m) \text{ and } \beta \leq \beta' \text{ in } H(n).$$

For $\delta = [a_1, \ldots, a_r|b_1, \ldots, b_r] \in \Delta(m \times n)$ and an $m \times n$ matrix $M = (m_{ij})$, we set $\delta_M := \det(m_{a_i, b_j})_{i,j}$. We also set $\Delta(m \times n; \delta) := \{\gamma \in \Delta(m \times n) \mid \gamma \geq \delta\}$.

Now we fix integers m and n with $1 \leq m \leq n$. Let X be an $m \times n$ matrix of indeterminates, that is, $X = (X_{ij})$ and $\{X_{ij}\}_{1 \leq i \leq m, 1 \leq j \leq n}$ are independent indeterminates. Then

Fact 2.1 ([DEP1]) *$K[X]$ is an algebra with straightening law (ASL for short) over K generated by $\Delta(m \times n)$ with structure map $\delta \mapsto \delta_X$.*

Next we fix $\delta = [a_1, a_2, \ldots, a_r|b_1, b_2, \ldots, b_r] \in \Delta(m \times n)$. Since $\Delta(m \times n) \setminus \Delta(m \times n; \delta)$ is a poset ideal of $\Delta(m \times n)$, we see by [DEP2, Proposition 1.2],

Corollary 2.2

$$R(X;\delta) := K[X]/(\Delta(m\times n)\setminus\Delta(m\times n;\delta))K[X]$$

is an ASL over K generated by $\Delta(m\times n;\delta)$.

The image $\overline{X}$ of X in $R(X;\delta)$ is the universal matrix which satisfies the condition

$$I_i(\overline{X}^{\le a_i-1}) = I_i(\overline{X}_{\le b_i-1}) = (0) \quad \text{for } i=1,2,\ldots,r+1,$$

where we set $a_{r+1}=m+1$ and $b_{r+1}=n+1$. That is, if M is an $m\times n$ matrix with entries in a K-algebra S and

$$(*) \qquad I_i(M^{\le a_i-1}) = I_i(M_{\le b_i-1}) = (0) \quad \text{for } i=1,2,\ldots,r+1,$$

then there is a unique K-algebra homomorphism $R(X;\delta)\to S$ mapping $\overline{X}$ to M.

3 Invariants of the unipotent radical of a Borel subgroup of $\mathrm{GL}(n,K)$

Now let $G=\mathrm{GL}(m,K)\times\mathrm{GL}(n,K)$, B^- the Borel subgroup of $\mathrm{GL}(m,K)$ consisting of lower triangular matrices, B^+ the Borel subgroup of $\mathrm{GL}(n,K)$ consisting of upper triangular matrices and U^- (resp. U^+) the set of all unipotent matrices in B^- (resp. B^+). If $g_1\in U^-$ and $g_2\in U^+$, then $g_1^{-1}\overline{X}g_2$ satisfies $(*)$. So there is an automorphism of $R(X;\delta)$ sending $\overline{X}$ to $g_1^{-1}\overline{X}g_2$. Therefore, $U^-\times U^+$ acts on $R(X;\delta)$. We may also consider the action of U^+ on $R(X;\delta)$.

We set

$$Y_\delta := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \\ Y_{a_1 1} & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ Y_{a_2 1} & Y_{a_2 2} & & 0 \\ \vdots & \vdots & & \vdots \\ Y_{a_r 1} & Y_{a_r 2} & \cdots & Y_{a_r r} \\ \vdots & \vdots & & \vdots \\ Y_{m1} & Y_{m2} & \cdots & Y_{mr} \end{bmatrix}$$

and

$$Z_\delta := \begin{bmatrix} 0 & \cdots & 0 & Z_{1b_1} & \cdots & Z_{1b_2} & \cdots & Z_{1b_r} & \cdots & Z_{1n} \\ 0 & \cdots & 0 & 0 & \cdots & Z_{2b_2} & \cdots & Z_{2b_r} & \cdots & Z_{2n} \\ & \cdots & & & \cdots & & \cdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & Z_{rb_r} & \cdots & Z_{rn} \end{bmatrix},$$

where Y_{ij} and Z_{ij} are independent indeterminates.

Lemma 3.1

$$\begin{array}{l} I_i((Y_\delta Z_\delta)^{\leq a_i - 1}) = (0) \\ I_i((Y_\delta Z_\delta)_{\leq b_i - 1}) = (0) \end{array} \quad \textit{for } i = 1, \ldots, r,\ r+1.$$

Therefore there is a unique K-algebra homomorphism $R(X;\delta) \to K[Y_\delta, Z_\delta]$ mapping $\overline{X}$ to $Y_\delta Z_\delta$.

We introduce the lexicographic monomial order on $K[Y_\delta, Z_\delta]$ induced by $Y_{a_1 1} > Y_{a_1+1,1} > \cdots > Y_{m1} > Y_{a_2 2} > \cdots > Y_{m2} > Y_{a_3 3} > \cdots > Y_{mr} > Z_{1b_1} > Z_{1b_1+1} > \cdots > Z_{1n} > Z_{2b_2} > \cdots > Z_{2n} > Z_{3b_3} > \cdots > Z_{rn}$.

Lemma 3.2 *If $\gamma = [c_1, \ldots, c_s | d_1, \ldots, d_s]$ is an element of $\Delta(m \times n; \delta)$, then*

$$\mathrm{lm}(\gamma_{Y_\delta Z_\delta}) = Y_{c_1 1} Y_{c_2 2} \cdots Y_{c_s s} Z_{1d_1} Z_{2d_2} \cdots Z_{sd_s}.$$

proof Since

$$\gamma_{Y_\delta Z_\delta} = \sum_{[e_1, \ldots, e_s] \in H(r)} [c_1, \ldots, c_s | e_1, \ldots, e_s]_{Y_\delta} [e_1, \ldots, e_s | d_1, \ldots, d_s]_{Z_\delta}$$

and

$$\begin{aligned} & \mathrm{lm}([c_1, \ldots, c_s | e_1, \ldots, e_s]_{Y_\delta} [e_1, \ldots, e_s | d_1, \ldots, d_s]_{Z_\delta}) \\ = \; & Y_{c_1 e_1} \cdots Y_{c_s e_s} Z_{e_1 d_1} \cdots Z_{e_s d_s}, \end{aligned}$$

the result follows form the definition of monomial order. ∎

If $\mu = \prod_{i=1}^{u} [c_{i1}, \ldots, c_{is(i)} | d_{i1}, \ldots, d_{is(i)}]$ is a standard monomial on $\Delta(m \times n; \delta)$ in the sense of ASL, then

$$\mathrm{lm}(\mu_{Y_\delta Z_\delta}) = \prod_{i=1}^{u} \prod_{j=1}^{s(i)} Y_{c_{ij} j} Z_{j d_{ij}}. \tag{3.1}$$

In particular, we can reconstruct μ form $\mathrm{lm}(\mu_{Y_\delta Z_\delta})$. So

Lemma 3.3 *If μ and μ' are different standard monomials on $\Delta(m \times n; \delta)$, then* $\mathrm{lm}(\mu_{Y_\delta Z_\delta}) \neq \mathrm{lm}(\mu'_{Y_\delta Z_\delta})$*. In particular, $\{\mu_{Y_\delta Z_\delta} \mid \mu$ is a standard monomial on $\Delta(m \times n; \delta)\}$ is linearly independent over K.*

Therefore

Proposition 3.4 *The K-algebra homomorphism in Lemma 3.1 is injective. In particular, $R(X; \delta) \simeq K[Y_\delta Z_\delta]$.*

For $g \in U^+$, we can define a K-algebra automorphism of $K[Z_\delta]$ which maps Z_δ to $Z_\delta g$. Therefore U^+ acts on $K[Z_\delta]$. As for this action we have

Lemma 3.5 $K[Z_\delta]^{U^+} = K[Z_{1b_1}, Z_{2b_2}, \ldots, Z_{rb_r}]$.

proof First we define the row degree on $K[Z_\delta]$ by $\deg Z_{ij} := \boldsymbol{e}_i \in \boldsymbol{N}^r$.

Since the action of U^+ fixes row degree, we may assume, by extending Z_δ, that $[b_1, b_2, \ldots, b_r] = [1, 2, \ldots, n]$, that is, Z_δ is the $n \times n$ upper triangular matrix of indeterminates.

Let f be an arbitrary element of $K[Z_\delta]^{U^+}$. Since the action of U^+ fixes the row degree, in order to prove that $f \in K[Z_{11}, Z_{22}, \ldots, Z_{nn}]$, we may assume that f is homogeneous of row degree $(d_1,\ d_2,\ \ldots,\ d_n)$. Write f as

$$\sum_{i_1=0}^{d_1} \cdots \sum_{i_n=0}^{d_n} f_{i_1 i_2 \cdots i_n}(Z_{12}, \ldots, Z_{1n}, Z_{23}, \ldots, Z_{n-1,n}) Z_{11}^{d_1-i_1} Z_{22}^{d_2-i_2} \cdots Z_{nn}^{d_n-i_n}$$

where $f_{i_1 i_2 \cdots i_n}$ is a homogeneous polynomial of Z_{12}, Z_{13}, $\ldots$, Z_{1n}, Z_{23}, Z_{24}, $\ldots$, Z_{2n}, Z_{34}, $\ldots$, $Z_{n-1,n}$ of row degree $(i_1,\ i_2,\ \ldots,\ i_n)$.

Let $g = (g_{ij})$ be an element of U^+. Since the image of Z_{ij} by the action of g is

$$\sum_{l=i}^{j} Z_{il} g_{lj} \tag{3.2}$$

for $i \leq j$, we see that the image of f is of the following form.

$$\sum_{i_1=0}^{d_1} \sum_{i_2=0}^{d_2} \cdots \sum_{i_n=0}^{d_n} f_{i_1 i_2 \cdots i_n}(g_{12}, \ldots, g_{1n}, g_{23}, \ldots, g_{n-1,n}) Z_{11}^{d_1} Z_{22}^{d_2} \cdots Z_{nn}^{d_n}$$
$$+ \quad \text{(terms of lower degree in } Z_{11},\ Z_{22},\ \ldots,\ Z_{nn})$$

Since $g(f) = f$ for any $g \in U^+$ and K is an infinite field, we see that

$$f_{i_1, i_2, \ldots, i_n} = 0 \quad \text{if } (i_1, i_2, \ldots, i_n) \neq (d_1, d_2, \ldots, d_n),$$

that is, $f \in K[Z_{11}, Z_{22}, \ldots, Z_{nn}]$.

On the contrary, it is clear form (3.2) that $Z_{ii} \in K[Z_\delta]^{U^+}$ for $i = 1$, 2, ..., n. Therefore $K[Z_\delta]^{U^+} = K[Z_{11}, Z_{22}, \ldots, Z_{nn}]$. ∎

By symmetry, we see that U^- acts on $K[Y_\delta]$ and $K[Y_\delta]^{U^-} = K[Y_{a_1 1}, Y_{a_2 2}, \ldots, Y_{a_r r}]$.

Proposition 3.6 *$\{[c_1, \ldots, c_i | b_1, \ldots, b_i]_{Y_\delta Z_\delta} \mid [c_1, \ldots, c_i] \in H(m; [a_1, \ldots, a_r])\}$ is a sagbi basis of*

$$K[Y_\delta, Z_{1b_1}, Z_{2b_2}, \ldots, Z_{rb_r}] \cap K[Y_\delta Z_\delta].$$

In particular,

$$K[Y_\delta, Z_{1b_1}, Z_{2b_2}, \ldots, Z_{rb_r}] \cap K[Y_\delta Z_\delta] = K[\bigcup_{i=1}^{r} \Gamma((Y_\delta Z_\delta)_{b_1, b_2, \ldots, b_i})],$$

where $M_{b_1, b_2, \ldots, b_i}$ denotes the matrix consisting of b_1, b_2, $\ldots b_{i-1}$ and b_i-th columns of M.

proof It is clear that

$$\begin{aligned} & [c_1, \ldots, c_i | b_1, \ldots, b_i]_{Y_\delta Z_\delta} \\ = \; & [c_1, \ldots, c_i | 1, 2, \ldots, i]_{Y_\delta} [1, 2, \ldots, i | b_1, \ldots, b_i]_{Z_\delta} \\ \in \; & K[Y_\delta, Z_{1b_1}, Z_{2b_2}, \ldots, Z_{rb_r}] \cap K[Y_\delta Z_\delta]. \end{aligned}$$

Now suppose that $f \in K[Y_\delta, Z_{1b_1}, Z_{2b_2}, \ldots, Z_{rb_r}] \cap K[Y_\delta Z_\delta]$ and let

$$f = \sum_\mu r_\mu \mu$$

be the standard representation of f in the ASL $K[Y_\delta Z_\delta] \simeq R(X; \delta)$. Then by Lemma 3.3, we see that there is a unique standard monomial μ such that

$$\mathrm{lm}(f) = \mathrm{lm}(\mu_{Y_\delta Z_\delta}).$$

Since $\mathrm{lm}(\mu_{Y_\delta Z_\delta}) = \mathrm{lm}(f) \in K[Y_\delta, Z_{1b_1}, Z_{2b_2}, \ldots, Z_{rb_r}]$, we see, by (3.1), that μ is of the form $\prod_{i=1}^{u} [c_{i1}, \ldots, c_{is(i)} | b_{i1}, \ldots, b_{is(i)}]$. The result follows. ∎

The action of U^+ on $K[Z_\delta]$ induces an action of U^+ on $K[Y_\delta Z_\delta]$. Since

$$K[Y_\delta Z_\delta]^{U^+} = K[Z_\delta]^{U^+}[Y_\delta] \cap K[Y_\delta Z_\delta],$$

we see the following

Theorem 3.7

$$K[Y_\delta Z_\delta]^{U^+} = K[\bigcup_{i=1}^{r} \Gamma((Y_\delta Z_\delta)_{b_1,b_2,\ldots,b_i})].$$

And therefore,

$$R(X;\delta)^{U^+} = K[\bigcup_{i=1}^{r} \Gamma(\overline{X}_{b_1,b_2,\ldots,b_i})].$$

Note 3.8 *If* $[a_1,\ a_2,\ \ldots,\ a_r] = [1,\ 2,\ \ldots,\ m]$*, then* $K[\Gamma(Y_\delta Z_\delta)]$ *is the homogeneous coordinate ring of the Schubert subvariety.*

4 Invariants of the unipotent radical of a Borel subgroup of $\mathrm{GL}(m, K) \times \mathrm{GL}(n, K)$

First we state the following

Proposition 4.1

$$\begin{aligned} & K[Y_\delta Z_\delta] \cap K[Y_\delta, Z_{1b_1}, Z_{2b_2}, \ldots, Z_{rb_r}] \cap K[Z_\delta, Y_{a_1 1}, Y_{a_2 2}, \ldots, Y_{a_r r}] \\ = \ & K[Y_{a_1 1} Y_{a_2 2} \cdots Y_{a_i i} Z_{1b_1} Z_{2b_2} \cdots Z_{ib_i} \mid i = 1, \ldots, r]. \end{aligned}$$

proof It is clear that $Y_{a_1 1} Y_{a_2 2} \cdots Y_{a_i i} Z_{1b_1} Z_{2b_2} \cdots Z_{ib_i} = [a_1,\ \ldots,\ a_i|b_1,\ \ldots,\ b_i]_{Y_\delta Z_\delta} \in K[Y_\delta Z_\delta] \cap K[Y_\delta,\ Z_{1b_1},\ Z_{2b_2},\ \ldots,\ Z_{rb_r}] \cap K[Z_\delta,\ Y_{a_1 1},\ Y_{a_2 2},\ \ldots,\ Y_{a_r r}]$ for $i = 1,\ 2,\ \ldots,\ r$.

Suppose that $f \in K[Y_\delta Z_\delta] \cap K[Y_\delta,\ Z_{1b_1},\ Z_{2b_2},\ \ldots,\ Z_{rb_r}] \cap K[Z_\delta,\ Y_{a_1 1},\ Y_{a_2 2},\ \ldots,\ Y_{a_r r}]$ and let

$$f = \sum_\mu r_\mu \mu$$

be the standard representation of f in the ASL $K[Y_\delta Z_\delta] \simeq R(X;\delta)$. Then there is unique standard monomial μ such that $\mathrm{lm}(f) = \mathrm{lm}(\mu_{Y_\delta Z_\delta})$.

Since $\mathrm{lm}(\mu_{Y_\delta Z_\delta}) = \mathrm{lm}(f) \in K[Y_\delta Z_\delta] \cap K[Y_\delta,\ Z_{1b_1},\ Z_{2b_2},\ \ldots,\ Z_{rb_r}] \cap K[Z_\delta,\ Y_{a_1 1},\ Y_{a_2 2},\ \ldots,\ Y_{a_r r}]$, we wee by (3.1) that μ is of the following form.

$$\mu = \prod_{t=1}^{u} [a_1, a_2, \ldots, a_{i(t)} | b_1, b_2, \ldots, b_{i(t)}]$$

So we see that

$$\begin{aligned} & \{Y_{a_1 1} Y_{a_2 2} \cdots Y_{a_i i} Z_{1b_1} Z_{2b_2} \cdots Z_{ib_i} \mid i = 1, \ldots, r\} \\ = \ & \{[a_1, \ldots, a_i | b_1, \ldots, b_i]_{Y_\delta Z_\delta} \mid i = 1, \ldots, r\} \end{aligned}$$

is a sagbi basis of $K[Y_\delta Z_\delta] \cap K[Y_\delta,\ Z_{1b_1},\ Z_{2b_2},\ \ldots,\ Z_{rb_r}] \cap K[Z_\delta,\ Y_{a_1 1},\ Y_{a_2 2},\ \ldots,\ Y_{a_r r}]$. The result follows. ∎

Since

$$\begin{aligned} & K[Y_\delta Z_\delta]^{U^- \times U^+} \\ = \ & K[Y_\delta Z_\delta]^{U^-} \cap K[Y_\delta Z_\delta]^{U^+} \\ = \ & K[Y_\delta, Z_{1b_1}, Z_{2b_2}, \ldots, Z_{rb_r}] \cap K[Y_\delta Z_\delta] \\ & \cap K[Z_\delta, Y_{a_1 1}, Y_{a_2 2}, \ldots, Y_{a_r r}] \cap K[Y_\delta Z_\delta], \end{aligned}$$

We see the following

Theorem 4.2

$$\begin{aligned} & K[Y_\delta Z_\delta]^{U^- \times U^+} \\ = \ & K[Y_{a_1 1} Y_{a_2 2} \cdots Y_{a_i i} Z_{1b_1} Z_{2b_2} \cdots Z_{ib_i} \mid i = 1, \ldots, r] \\ = \ & K[[a_1, \ldots, a_i | b_1, \ldots, b_i]_{Y_\delta Z_\delta} \mid i = 1, \ldots, r]. \end{aligned}$$

And therefore,

$$R(X;\delta)^{U^- \times U^+} = K[[a_1, a_2, \ldots, a_i | b_1, b_2, \ldots, b_i]_{\overline{X}} \mid i = 1, 2, \ldots, r].$$

In particular, it is isomorphic to the polynomial ring over K with r variables.

References

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