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Cohen-Macaulay property of graded rings associated to contracted ideals in dimension 2

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Abstract: We study the Cohen-Macaulay property of the associated graded ring of contracted homogeneous ideals in $K[x,y]$. Surprising, the problem is closely related to the description of the Gröbner fan of the ideal of the rational normal curve. We completely classify the contracted ideals with a Cohen-Macaulay associated graded ring in terms of the numerical invariants arising from Zariski's factorization. These results are contained in "Contracted ideals and the Gröbner fan of the rational normal curve" arXiv0705.3767, joint work with E.De Negri and M.E.Rossi which is going to appear in the first volume of the new journal "Algebra and Number Theory".

Let K be a field, $R = K[x,y]$ and I be a homogeneous ideal of R with $\sqrt{I} = \mathbf{m} = (x,y)$. Denote by $\mathrm{gr}_I(R)$ the associated graded ring of I, that is, $\mathrm{gr}_I(R) = \oplus_k I^k/I^{k+1}$. Denote by $\mu(I)$ the minimal number of generators of I and by $o(I)$ the order of I which is, by definition, the least degree of a non-zero element in I. By the Hilbert-Burch theorem we know that $\mu(I) \leq o(I)+1$. The ideal I is said to be contracted if $\mu(I) = o(I)+1$. Contracted ideals can be characterized also as the ideals which are contracted from a quadratic extension. Explicitly, for a linear form ℓ one considers a quadratic extension $R[x/\ell, y/\ell]$ of R. Then I is contracted if and only if $IR[x/\ell, y/\ell] \cap R = I$ for some ℓ. Contracted ideals have been introduced by Zariski in his studies on the factorization property of integrally closed ideals, see [ZS, App.5]. Every integrally closed ideal I is contracted and has a Cohen-Macaulay associated graded ring $\mathrm{gr}_I(R)$, see [LT]. In general, however, the associated graded ring of a contracted ideal need not be Cohen-Macaulay. So we are led to consider the following:

Problem 0.1. Describe the contracted homogeneous ideals I of $K[x,y]$ such that $\mathrm{gr}_I(R)$ is Cohen-Macaulay.

Zariski proved a factorization theorem for contracted ideals asserting that every contracted ideal I can be written as $I = L_1 \cdots L_s$ where the L_i are themselves contracted but of a very special kind. In the homogeneous case and assuming K is algebraically closed, each L_i is a lex-segment monomial ideal in a specific system of coordinates depending on i.

Recall that a monomial ideal L in R is a lex-segment ideal (lex-ideal for short) if whenever $x^a y^b \in L$ with $b > 0$ then also $x^{a+1}y^{b-1} \in L$. Every lex-ideal L of order d can be written as

$$L = (x^d, x^{d-1}y^{a_1}, \dots, y^{a_d})$$

and hence can be encoded by the vector $a = (a_0, a_1, \dots, a_d)$ with increasing integral coordinates and $a_0 = 0$.

Therefore to every contracted ideal I with factorization $I = L_1 \cdots L_s$ we may associate sequences $a_1, \dots, a_s$, where $a_i = (a_{ij} : j = 0, \dots d_i) \in \mathbf{N}^{d_i+1}$ are increasing and $a_{i0} = 0$. For instance:

Example 0.2. Let

$$X = \begin{pmatrix} y^2 & 0 & 0 & 0 & 0 & 0 \\ -x-3y & y & 0 & 0 & 0 & 0 \\ -9y & -x+3y & y^3 & 0 & 0 & 0 \\ 0 & 0 & -x-y & y^3 & 0 & 0 \\ 0 & 0 & 0 & -x-y & y^2 & 0 \\ 0 & 0 & 0 & 0 & -x-y & y \\ 0 & 0 & 0 & 0 & 0 & -x \end{pmatrix}$$

and let I be the ideal of 6-minors of X. We have $\mu(I) = 7$ and $o(I) = 6$, so I is contracted. Zariski's factorization of I is

$$I = (x^3, x^2y^2, xy^3, y^9)(x_1^3, x_1^2y_1^4, x_1y_1^7, y_1^9)$$

where $x_1 = x + y$ and $y_1 = y$. Hence we associate to I the sequences $a_1 = (0, 2, 3, 9)$ and $a_2 = (0, 4, 7, 9)$.

With respect to the terminology introduced above, in [CDJR] it is shown that:

Theorem 0.3. *One has*

$$\operatorname{depth} \operatorname{gr}_I(R) = \min\{\operatorname{depth} \operatorname{gr}_{L_i}(R) : i = 1, \dots, s\}.$$

In particular, the Cohen-Macaulayness of $\operatorname{gr}_I(R)$ *is equivalent to the Cohen-Macaulayness of* $\operatorname{gr}_{L_i}(R)$ *for every* $i = 1, \dots, s$.

Therefore, to answer Problem 0.1, one has to characterize the lex-ideals L with Cohen-Macaulay associated graded ring.

Problem 0.4. For every d describe the sequences $a = (a_0, a_1, \dots, a_d) \in \mathbf{N}^{d+1}$ with increasing coordinates and $a_0 = 0$ such that $\operatorname{gr}_L(R)$ Cohen-Macaulay. Here L is the lex-ideal associated to a.

Since R is regular, $\operatorname{gr}_L(R)$ is Cohen-Macaulay iff $\operatorname{Rees}(L)$ is Cohen-Macaulay. As $\operatorname{Rees}(L)$ is an affine semigroup ring, the result of Trung and Hoa [TH] could be applied. But we have not been able to follow this line of investigation.

Denote by P the defining ideal of the Veronese embedding of $\mathbf{P}^1 \to \mathbf{P}^d$ in its standard coordinate system. It is well-known that P is the ideal of $K[t_0, \dots, t_d]$ generated by the 2-minors of the matrix

$$T_d = \begin{pmatrix} t_0 & t_1 & t_2 & \dots & \dots & t_{d-1} \\ t_1 & t_2 & \dots & \dots & t_{d-1} & t_d \end{pmatrix}$$

We show that:

Proposition 0.5. *Let L be a lex-ideal. Denote by a the sequence associated to L. Then*

$$\operatorname{depth} \operatorname{gr}_L(R) = \operatorname{depth} K[t_0, \dots, t_d]/\operatorname{in}_a(P).$$

Here $\operatorname{in}_a(P)$ *denotes the ideal of the initial forms of P with respect to the vector a. In particular,* $\operatorname{gr}_L(R)$ *is Cohen-Macaulay if and only if* $\operatorname{in}_a(P)$ *is perfect.*

Therefore Problem 0.4 becomes equivalent to:

Problem 0.6. For every d determine the vectors $a \in \mathbf{N}^{d+1}$ such that $\operatorname{in}_a(P)$ is perfect.

To answer 0.6 we first show:

Proposition 0.7. *The initial monomial ideals of P which are perfect are in bijective correspondence with the subsets of $\{1, 2, \dots, d-1\}$. So P has exactly 2^{d-1} perfect initial monomial ideals.*

The fact that P has exactly 2^{d-1} perfect monomial initial ideals can be derived also by combining results of Hosten and Thomas [HT] with results of O'Shea and Thomas [OT].

We explain this bijective correspondence with an example. Suppose $d = 6$ and take the sequence and the subset $\{3, 4\}$ of $\{1, 2, 3, 4, 5\}$, Set $\mathbf{i} = \{0, d\} \cup \{3, 4\} = \{0, 3, 4, 6\}$. The corresponding perfect initial ideal I of P is obtained by dividing the matrix T_6 in blocks (from column $i_v + 1$ to i_{v+1})

$$T_6 = \left(\begin{array}{ccc|c|cc} t_0 & t_1 & t_2 & t_3 & t_4 & t_5 \\ t_1 & t_2 & t_3 & t_4 & t_5 & t_6 \end{array} \right)$$

and then taking anti-diagonals of minors whose columns belong to the same block,

$$t_1^2, t_1t_2, t_2^2, t_5^2,$$

and main diagonals from minors whose columns belong to different blocks

$$t_0t_4, t_0t_5, t_0t_6, t_1t_4, t_1t_5, t_1t_6, t_2t_4, t_2t_5, t_2t_6, t_3t_5, t_3t_6.$$

The ideal I is the initial ideal of P with respect to every term order τ refining the weight $a = (0, 3, 5, 6, 10, 16, 21)$ obtained from the "permutation" vector $\sigma = (3, 2, 1|4|6, 5) \in S_6$ by setting $a_0 = 0$ and $a_i = \sum_{j=1}^{i} \sigma_j$. With respect to this term order the 2-minors of T_6 are a Gröbner basis of P but not the reduced Gröbner basis. The corresponding reduced Gröbner basis is

$$\begin{array}{lllll} \underline{t_1^2} - t_0t_2, & \underline{t_1t_2} - t_0t_3, & \underline{t_2^2} - t_1t_3, & \underline{t_0t_4} - t_1t_3 & \underline{t_0t_5} - t_2t_3, \\ \underline{t_1t_4} - t_2t_3, & \underline{t_0t_6} - t_3^2, & \underline{t_1t_5} - t_3^2, & \underline{t_2t_4} - t_3^2, & \underline{t_1t_6} - t_3t_4, \\ \underline{t_2t_5} - t_3t_4, & \underline{t_2t_6} - t_4^2, & \underline{t_3t_5} - t_4^2, & \underline{t_3t_6} - t_4t_5, & \underline{t_5^2} - t_4t_6. \end{array}$$

So for every vector $a = (a_0, a_1, \dots, a_6) \in \mathbf{Q}^7_{\geq 0}$ satisfying the following system of linear inequalities

$$\begin{array}{llll} 2a_1 > a_0 + a_2* & a_1 + a_2 > a_0 + a_3 & 2a_2 > a_1 + a_3* & a_0 + a_4 > a_1 + a_3* \\ a_0 + a_5 > a_2 + a_3 & a_1 + a_4 > a_2 + a_3 & a_0 + a_6 > 2a_3 & a_1 + a_5 > 2a_3 \\ a_2 + a_4 > 2a_3 & a_1 + a_6 > a_3 + a_4 & a_2 + a_5 > a_3 + a_4 & a_2 + a_6 > 2a_4 \\ a_3 + a_5 > 2a_4 & a_3 + a_6 > a_4 + a_5* & 2a_5 > a_4 + a_6* & \end{array}$$

we have $\mathrm{in}_a(P) = I$. More precisely, if we set

$$C(\mathbf{i}) = \{a \in \mathbf{Q}^{d+1}_{\geq 0} : \mathrm{in}_a(P) = I\}$$

then $C(\mathbf{i})$ is the convex cone is defined by above system of inequalities. The $*$ indicates an essential inequalities. One has:

$$\overline{C(\mathbf{i})} = \{a \in \mathbf{Q}^{d+1}_{\geq 0} : I \text{ is an initial ideal of } \mathrm{in}_a(P)\}$$

For a given d we set:

$$CM_d = \{a \in \mathbf{Q}^{d+1}_{\geq 0} : \mathrm{in}_a(P) \text{ is perfect}\}$$

the "Cohen-Macaulay region" of the Gröbner fan of P. Our main theorem is the following:

Theorem 0.8.

$$CM_d = \cup_{\mathbf{i}} \overline{C(\mathbf{i})}$$

where the union is extended to the set of the 2^{d-1} sequences $\mathbf{i} = (0 = i_0 < i_1 < \cdots < i_k = d)$.

Combining these results we obtain an explicit characterization, in terms of the numerical invariants arising from the Zariski factorization, of the Cohen-Macaulay property of the associated graded ring to a contracted homogeneous ideal in $K[x,y]$.

Theorem 0.9. *Let I be a contracted homogeneous ideal of $K[x,y]$ with Zariski factorization $I = L_1 \cdots L_s$. Denote by d_i the order of L_i and by $a_i \in \mathbf{N}^{d_i+1}$ the sequence associated to L_i. Then $\mathrm{gr}_I(R)$ is Cohen-Macaulay iff $a_i \in CM_{d_i}$ for all $i = 1,\ldots,s$.*

As the regions CM_{d_i} are the union of cones $\overline{C(\mathbf{i})}$ which are described explicitly in terms of linear inequalities, Theorem 0.9 answers 0.1.

Two of the cones of the Cohen-Macaulay region CM_d are special as they correspond to opposite extreme selections:

(1) (the lex-cone) If $\mathbf{i} = (0,1,2,\ldots,d)$, then the closed cone $\overline{C(\mathbf{i})}$ is described by the inequality system
$$a_i + a_j \geq a_u + a_v$$
with $u = \lfloor (i+j)/2 \rfloor,\ v = \lceil (i+j)/2 \rceil$ for every i,j. Setting
$$b_i = a_i - a_{i-1}$$
the cone $C(\mathbf{i})$ can be described by:
$$b_{i+1} \geq b_i$$
for every $i = 1,\ldots,d-1$. In this case the initial ideal of P is $(t_it_j : j-i > 1)$ and it can be realized by the lex-order with $t_0 < t_1 < \cdots < t_d$ or by the lex-order with $t_0 > t_1 > \cdots > t_d$. This is the only radical monomial initial ideal of P. The lex-ideals "belonging" to $\overline{C(\mathbf{i})}$ are the integrally closed. Indeed, they are the products of d complete intersections of type (x, y^u).

(2) (the revlex-cone) If $\mathbf{i} = (0,d)$ then the closed cone $\overline{C(\mathbf{i})}$ is described by inequality system
$$a_i + a_j \geq a_0 + a_{i+j}$$
if $i + j \leq d$ and
$$a_i + a_j \geq a_d + a_{i+j-d}$$
if $i + j \geq d$. It can be realized by the revlex-order with $t_0 < t_1 < \cdots < t_d$ or by the revlex-order with $t_0 > t_1 > \cdots > t_d$. The corresponding initial ideal of P is $(t_1,\ldots,t_{d-1})^2$. The lex-ideals L "belonging" to the cone are characterized by the fact that $L^2 = (x^d, y^{a_d})L$, that is, they are exactly the lex-ideals with a monomial minimal reduction and reduction number 1. It is not difficult to show that the simple homogeneous integrally closed ideals of $K[x,y]$ are exactly the ideals of the form $\overline{(x^d, y^c)}$ with $\mathrm{GCD}(d,c) = 1$. In other words, $\overline{C(\mathbf{i})}$ contains the exponent vectors of all the simple (i.e. not product of two proper ideals) integrally closed ideals of order d.

Example 0.10 (0.2 continued)**.** For the ideal I the corresponding sequences are $a_1 = (0,2,3,9)$ and $a_2 = (0,4,7,9)$. The region CM_3 is the union of 4 cones: $C_1 = \overline{C(0,3)}$ the revlex-cone, $C_2 = \overline{C(0,1,3)}, C_3 = \overline{C(0,2,3)}$ and $C_4 = \overline{C(0,1,2,3)}$ the lex-cone. The revlex-cone C_1 is described by the inequalities $b_1 \geq b_2 \geq b_3$. The union of the cones C_2, C_3, C_4 form what we call the big cone that is described by the inequality $b_1 \leq b_3$. So we see that

a_1 belongs to the big cone and $a_2 \in C_1$. Hence both a_1 and a_2 belong to CM_3. It follows that $\mathrm{gr}_I(R)$ is Cohen-Macaulay.

For a lex-segment L associated to a vector a there is a closed relationship between the Hilbert series of $\mathrm{gr}_L(R)$ and the multigraded Hilbert series of $\mathrm{in}_a(P)$.

Given a monomial initial ideal I of P (perfect or not) consider the associated closed maximal cone of the Gröbner fan:

$$C_I = \{a \in \mathbf{Q}^{d+1}_{\geq 0} : I \text{ is an initial ideal of } \mathrm{in}_a(P)\}.$$

The key observation is the following:

Lemma 0.11. *Let L be a lex-ideal with associated vector a belonging to C_I. For $k \in \mathbf{N}$ set $M_k(I) = \{\alpha \in \mathbf{N}^{d+1} : t^\alpha \notin I, |\alpha| = k\}$. Denote by $\sum M_k(I)$ the sum of the vectors in $M_k(I)$. By construction $\sum M_k(I) \in \mathbf{N}^{d+1}$ and*

$$\mathrm{length}(R/L^k) = a \cdot \sum M_k(I)$$

for all k.

In terms of Hilbert series Lemma 0.11 can be rewritten as in the following lemma.

Lemma 0.12. *Let L be a monomial ideal with associated sequence a belonging to C_I. Then*

$$H^1_L(z) = a \cdot \nabla H_{S/I}(\underline{t})_{t_i = z}$$

where $\nabla = (\partial/\partial t_0, \ldots, \partial/\partial t_d)$ is the gradient operator.

Where $H^1_L(z)$ is the Hilbert series $\sum \mathrm{length}(R/L^{k+1})z^k$ of L and $H_{S/I}(\underline{t})$ is the $\mathbf{Z}^{d+1}$-graded Hilbert series of S/I.

Combining Lemma 0.11 with Lemma 0.12 we have that Hilbert coefficients, the h-polynomials of L are linear functions in the a_i's whose coefficients just depend on I. The explicit expressions can be computed in terms of the multigraded Betti numbers or in terms of Stanley decompositions of S/I. For example:

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I:=Ideal(
t[2]^2, t[0]t[6], t[1]t[4], t[0]t[4], t[0]t[3], t[0]t[2],t[3]t[5]^2,
t[0]^2t[5], t[4]t[5]^3, t[5]^5, t[0]t[5]^4, t[4]t[6], t[3]t[6],
t[4]^2, t[2]t[6], t[3]t[4], t[2]t[4], t[2]t[5], t[3]^2, t[2]t[3]
)

Inequalities describing C_I

a[0]+a[5]<a[1]+a[4]     a[1]+a[2]<a[0]+a[3]      2a[1]+a[3]<2a[0]+a[5]
a[1]+4a[6]<5a[5]          a[4]+a[5]<a[3]+a[6]      a[3]+a[5]<a[2]+a[6]

h-vector of L associated to every vector a of C_I

(0)  a[0] + a[1] + a[2] + a[3] + a[4] + a[5] + a[6]
(1)  a[0] + 4a[1] - 2a[2] - a[3] - 2a[4] + 4a[5] + a[6]
(2)  -a[0] + a[1] + a[2] - a[3] + a[4] - 2a[5] + a[6]
(3)  a[3] - a[4] - a[5] + a[6]
(4)  -a[0] + a[1] + 2a[4] - 3a[5] + a[6]
(5)  a[0] - a[1] - a[4] + a[5]
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We discuss also how the formulas for the Hilbert series and polynomials of $\mathrm{gr}_L(R)$ change by varying the corresponding cones of the Gröbner fan of P. There are two Hilbert polynomials in this setting, the one associated to $\mathrm{length}(L^k/L^{k+1})$ and the one associated to $\mathrm{length}(R/L^{k+1})$. To distinguish one from the other we use an asterisk to denote the second. We have:

Proposition 0.13. *Let I, J be monomial initial ideals of P. Then*

(1) *The formula for the multiplicity that is valid in the cone C_I equals that that is valid in C_J iff $\sqrt{I} = \sqrt{J}$.*
(2) *The formula for the Hilbert series that is valid in the cone C_I equals that that is valid in C_J iff $I = J$.*
(3) *The formula for the Hilbert polynomial* that is valid in the cone C_I equals that that is valid in C_J iff I and J have the same saturation.*

Furthermore there is a conjectural relation with the hypergeometric Gröbner fan introduced by Saito, Sturmfels and Takayama in [SST] and the equality between the formulas giving the Hilbert polynomials. Precisely, we conjecture that the formula for the Hilbert polynomial valid in the cone C_I equals that that is valid in C_J iff I and J have the same minimal components.

The ideals I, J below are non-Cohen-Macaulay initial ideals of P. We display the formulas for the h-vectors and Hilbert coefficients e_0, e_1, e_2 valid in the corresponding cones (computed via Stanley decompositions).

I	$(t_1t_3, t_1t_2, t_0t_2, t_3^3, t_1^2t_4, t_1^3, t_2t_4, t_2t_3, t_2^2)$
(h_0)	$a_0 + a_1 + a_2 + a_3 + a_4$
(h_1)	$2a_0 + a_1 - 3a_2 + a_3 + 2a_4$
(h_2)	$2a_0 - 4a_1 + 3a_2 - 2a_3 + a_4$
(h_3)	$-a_0 + 2a_1 - a_2$
(e_0)	$4a_0 + 4a_4$
(e_1)	$3a_0 - a_1 - 3a_3 + 4a_4$
(e_2)	$-a_0 + 2a_1 - 2a_3 + a_4$

J	$(t_1t_3, t_1t_2, t_1^2, t_3^3, t_2t_4, t_2t_3, t_2^2)$
(h_0)	$a_0 + a_1 + a_2 + a_3 + a_4$
(h_1)	$3a_0 - a_1 - 2a_2 + a_3 + 2a_4$
(h_2)	$a_2 - 2a_3 + a_4$
(h_3)	0
(e_0)	$4a_0 + 4a_4$
(e_1)	$3a_0 - a_1 - 3a_3 + 4a_4$
(e_2)	$a_2 - 2a_3 + a_4$

In this case $I^{top} = J^{top} = (t_1t_3, t_2, t_3^3, t_1^2)$ as conjectured and $J = J^{sat} \neq I^{sat} = (t_2, t_1t_3, t_1^2t_4, t_3^3, t_1^3)$ as we know by Proposition 0.13.

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