Almost principal fiber bundles

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The purpose of the talk

Let G be an algebraic group acting on $X = \operatorname{Spec} B$. A principal G-bundle is a very good quotient, but the map $X = \operatorname{Spec} B \to \operatorname{Spec} B^G = Y$ is rarely a principal fiber bundle. However, if we remove closed subsets of codimension two or more from both X and Y, the remaining part is often a principal G-bundle. Thus we can compare the reflexive sheaves, class gropus, and the canonical modules of X and Y in this case.

Modules over Krull rings

Let R be a Krull domain. An R module M is said to be torsionless if there exist some $n \geq 0$ and some injection $M \hookrightarrow R^n$. M is torsionless if and only if $\dim_{Q(R)} M \otimes_R Q(R) < \infty$ and M is a lattice in $M \otimes_R Q(R)$, where $Q(R)$ is the field of fractions of $Q(R)$. If M is torsionless and the canonical map $M \to M^{**}$ is an isomorphism, then we say that M is reflexive (or divisorial).

Locally Krull schemes

A scheme is said to be locally Krull if it has an open covering consisting of the prime spectra of Krull domains. Note that a locally Krull scheme is a (possibly infinite) disjoint union of integral locally Krull closed open subschemes.

Let Z be a locally Krull scheme, and $\mathcal{M}$ a quasi-coherent sheaf over Z. We say that $\mathcal{M}$ is torsionless (resp. reflexive) if for any $z \in Z$, there exists some affine open neighborhood $U = \operatorname{Spec} R$ of z such that R is a Krull domain and $\Gamma(U, \mathcal{M})$ is torsionless (resp. reflexive).

Fundamental settings

Throughout the talk, let S be a scheme, and G a flat, quasi-compact, quasi-separated S-group scheme.

Equivariant structure of the dual of a sheaf (1)

Let Z be a locally Krull G-scheme, and $\mathcal{M} \in \mathrm{Qch}(G, \mathcal{O}_Z)$. Assume that $\mathcal{M}/\mathcal{M}_{\mathrm{tor}}$ is torsionless as an $\mathcal{O}_Z$-module. Then $\mathcal{M}^* = \underline{\mathrm{Hom}}_{\mathcal{O}_Z}(\mathcal{M}, \mathcal{O}_Z)$ has a canonical structure of quasi-coherent $(G, \mathcal{O}_Z)$-module.

Equivariant structure of the dual of a sheaf (2)

Lemma 1

Let Z be as above, and $\mathcal{M} \in \mathrm{Qch}(G, \mathcal{O}_Z)$.

1. If $\mathcal{M}/\mathcal{M}_{\mathrm{tor}}$ is torsionless, then the canonical map $\mathcal{M} \to \mathcal{M}^{**}$ is $(G, \mathcal{O}_Z)$-linear.
2. If $\mathcal{M}$ is rank one reflexive, then the canonical map $(\mathcal{M}^* \otimes \mathcal{M})^{**} \to \mathcal{O}_Z$ is an isomorphism of $(G, \mathcal{O}_Z)$-modules.

Equivariant class group (1)

Let Z be a locally Krull G-scheme. Then we define $\mathrm{Cl}(G, Z)$ (resp. $\mathrm{Pic}(G, Z)$) to be the set of isomorphism classes of $(G, \mathcal{O}_Z)$-modules which are rank-one reflexive (invertible sheaves) as $\mathcal{O}_Z$-modules. $\mathrm{Cl}(G, Z)$ and $\mathrm{Pic}(G, Z)$ are called the equivariant class group (resp. Picard group) of Z.

Equivariant class group (2)

$\mathrm{Pic}(G, Z)$ is an additive group by the sum

$$[\mathcal{L}] + [\mathcal{L}'] = [\mathcal{L} \otimes \mathcal{L}'].$$

$\mathrm{Cl}(G, Z)$ is an additive group by the sum

$$[\mathcal{M}] + [\mathcal{N}] = [(\mathcal{M} \otimes \mathcal{N})^{**}].$$

Forgetful map

There is an obvious map

$$\alpha : \mathrm{Cl}(G, Z) \to \mathrm{Cl}(Z),$$

forgetting the action of G.

Lemma 2

If $\Gamma(G \times Z, \mathcal{O}_{G \times Z})^\times \cong \mathrm{pr}_1^* \Gamma(G, \mathcal{O}_G)^\times$, then

$$\mathrm{Ker}\, \alpha \cong X(G) := \mathrm{Hom}_{\mathrm{grpsch}\,/S}(G, \mathbb{G}_m).$$

A corollary

Corollary 3

Let $S = \operatorname{Spec} R$, $G = \operatorname{Spec} H$, and $Z = \operatorname{Spec} B$ be all affine, and assume that $B = R[x_1, \ldots, x_n]$ is a polynomial ring. Then $\operatorname{Ker} \alpha \cong X(G)$.

The case of finite group

Lemma 1

If G is a finite group, then $\operatorname{Ker}\alpha \cong H^1(G, \Gamma(Z, \mathcal{O}_Z)^\times)$.

The group $\mathrm{Cl}'(G, B)$

Let $S = \operatorname{Spec} R$, G, and $Z = \operatorname{Spec} B$ be all affine. Let $\mathrm{Cl}'(G, Z)$ denote the subgroup of $\mathrm{Cl}(G, Z)$ generated by divisorial fractional ideals I of B such that I is a (G, B)-submodule of aB for some $a \in Q(B^G) \setminus \{0\}$ and $I^G \neq 0$.

Lemma 4

Let k be a field, $S = \operatorname{Spec} k$, G an affine connected algebraic group over k, and $Z = \operatorname{Spec} B$ be affine. If k is integrally closed in B, then $\mathrm{Cl}'(G, Z) \cap \operatorname{Ker} \alpha$ is a subquotient of $X(G)$.

Finite generation (1)

Lemma 5

Let $S = \operatorname{Spec} k$, and G an affine k-group scheme of finite type. Assume one of the following:

1. $\Gamma(G \times Z, \mathcal{O}_{G\times Z})^\times = \mathrm{pr}_1^* \Gamma(G, \mathcal{O}_G)^\times$;
2. G is connected smooth, $Z = \operatorname{Spec} B$ is affine, and k is integrally closed in B.

If $\mathrm{Cl}(Z)$ is a finitely generated abelian group, then $\mathrm{Cl}'(G, Z)$ is so.

Finite generation (2)

Proof.

This is because $0 \to \mathrm{Cl}'(G, Z) \cap \operatorname{Ker}\alpha \to \mathrm{Cl}'(G, Z) \xrightarrow{\alpha} \mathrm{Cl}(Z)$ is exact, $\mathrm{Cl}'(G, Z) \cap \operatorname{Ker}\alpha$ is a subquotient of $X(G)$, and both $X(G)$ and $\mathrm{Cl}(Z)$ are finitely generated. $\square$

Waterhouse type theorem (1)

Lemma 6 (Waterhouse)

Let B be a G-algebra which is a Krull domain. Then $\mathrm{Cl}(B^G)$ is a subquotient of $\mathrm{Cl}'(G, B)$.

Waterhouse type theorem (2)

Theorem 7

Let $S = \operatorname{Spec} k$, G an affine algebraic k-group scheme, and B a G-algebra which is a Krull domain. Assume that $\mathrm{Cl}(B)$ is finitely generated. Assume one of the following.

1. $(k[G] \otimes_k B)^\times = k[G]^\times$ (e.g., $B = k[x_1, \ldots, x_n]$);
2. (Waterhouse) G is a connected and smooth, and k is integrally closed in B.

Then $\mathrm{Cl}(B^G)$ is a finitely generated abelian group.

Principal fiber bundle

Let N be an S-flat closed normal subgroup scheme of G.

Definition 8

We say that $\pi : X \to Y$ is a G-equivariant principal N-bundle if

1. π is a G-morphism. That is, G acts on X and Y, and $\pi(gx) = g\pi(x)$.
2. N acts trivially on Y.
3. π is faithfully flat and quasi-compact.
4. $\Phi : N \times X \to X \times_Y X$ $(\Phi(g,x) = (gx, x))$ is an isomorphism.

A remark

Remark 9

A principal N-bundle is locally trivial in the fpqc topology, and the converse is also true.

We set $H = G/N$

Let $q : G \to H$ be a homomorphism of S-group scheme, and assume that q is a G-equivariant principal N-bundle.

Remark 10

- We have $N = \operatorname{Ker} q$.
- Roughly speaking, $H = G/N$.

Important properties of principal bundles

Lemma 11

Let $\pi : X \to Y$ be a G-equivariant principal N-bundle. Then

1. π is quasi-separated.
2. If G is of finite presentation (resp. separated, affine, finite), then so is π.
3. $\pi^* : \mathrm{Qch}(H, Y) \to \mathrm{Qch}(G, X)$ is an equivalence, and $(\pi_* ?)^N$ is its quasi-inverse.

Affine quotients are rarely prinicipal fiber bundles

So principal fiber bundles are very good quotients. However, If $X = \operatorname{Spec} B$ is a spectrum of a G-algebra and $Y = \operatorname{Spec} B^N$, the canonical map $\pi : X \to Y$ is rarely a principal N-bundle.

Rational almost principal fiber bundles

Definition 12

We say that a diagram of S-schemes

$$X \xleftarrow{i} V \xrightarrow{\rho} U \xhookrightarrow{j} Y$$

is a G-equivariant rational almost principal N-bundle if

1. G acts on X and Y, and N acts trvially on Y.
2. V is a G-stable open subset of X, and $\operatorname{codim}_X(X \setminus V) \geq 2$.
3. U is an H-stable open subset of Y, and $\operatorname{codim}_Y(Y \setminus U) \geq 2$.
4. $\rho : V \to U$ is a G-equivariant principal N-bundle.

Almost principal fiber bundles

Definition 13

We say that $\pi : X \to Y$ is a G-equivariant almost principal N-bundle if

1. $\pi : X \to Y$ is a G-morphism.
2. There exist some open subsets V of X and U of Y such that

$$X \xleftarrow{\ i\ } V \xrightarrow{\ \rho\ } U \xrightarrow{\ j\ } Y$$

is a G-equivariant rational almost principal N-bundle.

Notation

From now on, we assume that G is of finite presentation.

Let Z be a locally Krull G-scheme. We denote the category of quasi-coherent $(G, \mathcal{O}_Z)$-modules which are reflexive as $\mathcal{O}_Z$-modules by $\operatorname{Ref}(G, Z)$.

Main theorem (1)

Theorem 14

Let

$$X \xleftarrow{i} V \xrightarrow{\rho} U \xhookrightarrow{j} Y$$

be a G-equivariant rational almost principal N-bundle such that X and Y are locally Krull. Then

1. $\mathcal{N} \mapsto i_*\rho^*j^*\mathcal{N} : \mathrm{Ref}(H, Y) \to \mathrm{Ref}(G, X)$ is an equivalence, and $\mathcal{M} \mapsto (j_*\rho_*i^*\mathcal{M})^N$ is its quasi-inverse.
2. The equivalence above induces an isomorphism $\mathrm{Cl}(H, Y) \cong \mathrm{Cl}(G, X)$.

Settings for discussing canonical modules

When we discuss canonical modules, we assume the following.

Assumption (#)

S is Noetherian, and has a fixed dualizing complex $\mathbb{I}_S$. X and Y are connected normal S-schemes separated of finite type over S.

Main theorem (2)

Theorem 15

Assume that Assumption (#) is satisfied.

1. Let G be smooth of relative dimension d. Set $\Theta = \bigwedge^d \operatorname{Lie} G$. Then there are a $(G, \mathcal{O}_X)$-isomorphism $\omega_X \cong i_* \rho^* j^* \omega_Y \otimes_{\mathcal{O}_X} (f^*\Theta)^*$ and an $(H, \mathcal{O}_Y)$-isomorphism $\omega_Y \cong (j_* \rho_* i^*(\omega_X \otimes_{\mathcal{O}_X} f^*(\Theta)))^N$, where $f : X \to S$ is the structure map.
2. Let $S = \operatorname{Spec} k$, and N be a finite linearly reductive group scheme. Then there are a $(G, \mathcal{O}_X)$-isomorphism $\omega_X \cong i_* \rho^* j^* \omega_Y$ and an $(H, \mathcal{O}_Y)$-isomorphism $\omega_Y \cong (j_* \rho_* i^* \omega_X)^N$.

A remark

Remark 16

If $S = \operatorname{Spec} k$ with k a field of characteristic zero, then Theorem 15 is due to Knop.
The idea of the theorem is based on his result.

A corollary

Corollary 17

If Assumption ($\#$) is satisfied and $\Theta \cong \mathcal{O}_S$, then the following are equivalent.

1. $\omega_Y \cong \mathcal{O}_Y$ in $\mathrm{Ref}(H, Y)$;
2. $\omega_X \cong \mathcal{O}_X$ in $\mathrm{Ref}(G, X)$.

When is Θ trivial?

Remark 18

If $S = \operatorname{Spec} k$ and G an affine algebraic group over k, then the following hold.

1. If G is connected, then $\Theta \cong k$.
2. If G is finite, then $\Theta \cong k$.
3. (Knop) In general, Θ may not be trivial.

The case of almost principal fiber bundles (1)

Corollary 19

Let $\pi : X \to Y$ be a G-equivariant almost principal N-bundle such that X and Y are locally Krull.

1. $\mathcal{N} \mapsto (\pi^*\mathcal{N})^{**} : \mathrm{Ref}(H, Y) \to \mathrm{Ref}(G, X)$ is an equivalence, and $\mathcal{M} \mapsto (\pi_*\mathcal{M})^N$ is its quasi-inverse.
2. The equivalence induces $\mathrm{Cl}(H, Y) \cong \mathrm{Cl}(G, X)$.

The case of almost principal fiber bundles (2)

Corollary 20

Let $\pi : X \to Y$ be a G-equivariant almost principal N-bundle. Assume that $(\#)$ is satisfied. Then

1. if G is smooth of relative dimension d, there are a $(G, \mathcal{O}_X)$-isomorphism $\omega_X \cong (\pi^*\omega_Y)^{**} \otimes_{\mathcal{O}_X} (f^*\Theta)^*$ and an $(H, \mathcal{O}_Y)$-isomorphism $\omega_Y \cong (\pi_*(\omega_X \otimes_{\mathcal{O}_X} f^*(\Theta)))^N$.
2. If $S = \operatorname{Spec} k$ and N is finite linearly reductive, then there are a $(G, \mathcal{O}_X)$-isomorphism $\omega_X \cong (\pi^*\omega_Y)^{**}$ and an $(H, \mathcal{O}_Y)$-isomorphism $\omega_Y \cong (\pi_*\omega_X)^N$.

Example of finite groups (1)

Let k be an algebraically closed field, $B = k[x_1, \ldots, x_n]$, $V = \bigoplus_i kx_i$, and $G \subset \mathrm{GL}(V)$ a finite subgroup. Set $N = G$ and $H = \{e\}$. Let $A = B^G$, and $\pi : X = \operatorname{Spec} B \to \operatorname{Spec} A = Y$ be the canonical map.

Definition 21

We say that $g \in \mathrm{GL}(V)$ is a pseudo-reflection if $\operatorname{codim}_V \{v \in V \mid gv = v\} = 1$.

Lemma 22

$\pi : X \to Y$ is an almost principal G-bundle if and only if G does not have a pseudo-reflection.

Example of finite groups (2)

Lemma 23

Assume that G does not have a pseudo-reflection. Then

1. $\mathrm{Cl}(Y) \cong \mathrm{Cl}(G, X) \cong X(G)$.
2. $\omega_B \cong (B \otimes_A \omega_A)^{**}$ and $\omega_A \cong \omega_B^G$.
3. $(?)^G : \mathrm{Ref}(G, B) \to \mathrm{Ref}(A)$ is an equivalence.

Example of finite groups (3)

Corollary 24

The following are equivalent (by Watanabe if $\#G \neq 0$ in k).

1. $\omega_B \cong B$;
2. $G \subset \mathrm{SL}(V)$;
3. $\omega_A \cong A$;
4. A is quasi-Gorenstein (i.e., ω_A is projective).

If this is the case, the Cohen–Macaulay locus of A agrees with the Gorenstein locus (Braun).

Example of finite groups (4)

If $n = \dim B = 2$, then the equivalence $(?)^G : \mathrm{Ref}(G, B) \to \mathrm{Ref}(A)$ has the following interpretation.

$$\mathrm{Ref}(G, B) = \mathrm{Proj}(G, B) = \{M \in \mathrm{Mod}(G, B) \mid M \text{ is a finite projective } B\text{-module}\}$$

and $\mathrm{Ref}(A) = \mathrm{MCM}(A)$. If, moreover, $\#G \neq 0$ in k, then indecomposable objects of $\mathrm{Proj}(G, B)$ and irreducible representations of G are in one-to-one correspondence, and hence A is of finite representation type (well-known).

Example of Veronese subring (1)

Set $S = \operatorname{Spec} k$, $G = \mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}]$, $N = \mu_m = \operatorname{Spec} k[t]/(t^m - 1) \hookrightarrow G$ $(m > 1)$. $H = \operatorname{Spec} k[t^m, t^{-m}]$. A G-algebra is a $\mathbb{Z}$-graded k-alebra. For a G-algebra B, a (G, B)-module is nothing but a graded B-module. For a (G, B)-module M, M^N is nothing but the Veronese submodule $M^{(m\mathbb{Z})} = \bigoplus_{i \in m\mathbb{Z}} M_i$.

Let B be a Noetherian normal $\mathbb{Z}$-graded algebra such that $B_0 = k$ and $B = k[B_1]$. Assume that $B \neq k$ and $B \neq k[x]$. Or equivalently, $\dim B \geq 2$. B^N is the Veronese subring $B^{(m\mathbb{Z})} = \bigoplus_{i \in m\mathbb{Z}} B_i$.

Example of Veronese subring (2)

Lemma 25

Under the assumptions above,

1. $\pi : X = \operatorname{Spec} B \to \operatorname{Spec} B^N = Y$ is a G-equivariant almost principal N-bundle.
2. $\omega_{B^N} \cong \omega_B^N$ and $\omega_B \cong (B \otimes_{B^N} \omega_{B^N})^{**}$.
3. $\omega_B \cong B(rm) \Leftrightarrow \omega_{B^N} \cong B^N(rm)$. In particular, B^N is quasi-Gorenstein if and only if B is quasi-Gorenstein and $a(B)$ is divisible by m. A similar result (B is Cohen–Macaulay but may not be normal) is by Goto–Watanabe.
4. $\mathrm{Cl}(Y) \cong \mathrm{Cl}(N, X)$. If $B = k[x_1, \ldots, x_n]$, then $\mathrm{Cl}(Y) \cong X(N) \cong \mathbb{Z}/m\mathbb{Z}$.

Example of Veronese subring (3)

Consider the case that $G = N = \mu_m$, $H = \{e\}$, and $B = k[[x, y]]$. Then

$$\mathrm{MCM}(B^N) = \mathrm{Ref}(B^N) \cong \mathrm{Ref}(N, B).$$

The only indecomposables of $\mathrm{Ref}(N, B)$ are $B, B(-1), \ldots, B(-m+1)$. Hence B^N is of finite representation type.

Example of multi-section ring (1)

Let Y be a separated connected Noetherian normal scheme, and $D_1, \ldots, D_r \in \mathrm{Div}(Y)$. Assume that $\sum_{i=1}^r \mathbb{Z}D_i$ contains an ample Cartier divisor. Set $U = Y_{\mathrm{reg}}$. Let

$$V := \underline{\mathrm{Spec}} \bigoplus_{\lambda \in \mathbb{Z}^r} \mathcal{O}_U(\lambda_1 D_1' + \cdots + \lambda_r D_r') \xrightarrow{\rho} U$$

be the canonical map, where $D_i' := D_i|_U$. Let

$$R := \bigoplus_{\lambda \in \mathbb{Z}^r} \Gamma(Y, \mathcal{O}_Y(\lambda_1 D_1 + \cdots + \lambda_r D_r))$$

and set $X = \mathrm{Spec}\, R$. Set $N = G = \mathbb{G}_m^r$.

Example of multi-section ring (2)

Lemma 26

Under the notation above,

1. R is a Krull domain.
2. The diagram
$$X \xleftarrow{\ i\ } V \xrightarrow{\ \rho\ } U \xhookrightarrow{\ j\ } Y$$
is a rational almost principal G-bundle.
3. The functor $\beta : \mathrm{Ref}(Y) \to \mathrm{Ref}(G, R)$ given by $\mathcal{M} \mapsto \bigoplus_{\lambda \in \mathbb{Z}^r} \Gamma(Y, (\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(\lambda_1 D_1 + \cdots + \lambda_r D_r))^{**})$ is an equivalence, and give an isomorphism $\beta' : \mathrm{Cl}(Y) \cong \mathrm{Cl}(G, R)$.

Example of multi-section ring (3)

Theorem 27

1. (Elizondo–Kurano–Watanabe) The sequence

$$\mathbb{Z}^r \xrightarrow{\gamma} \mathrm{Cl}(Y) \xrightarrow{\alpha\beta'} \mathrm{Cl}(R) \to 0$$

   is exact, where $\gamma(\lambda) = \sum_{i=1}^r \lambda_i D_i$ and $\alpha\beta'(D) = [\bigoplus_\lambda \Gamma(Y, \mathcal{O}_Y(D + \sum_{i=1}^r \lambda_i D_i))]$.

2. (Kurano–H) Assume (#). Then

$$\omega_R = \bigoplus_{\lambda \in \mathbb{Z}^r} \Gamma(Y, \mathcal{O}_Y(K_Y + \sum_{i=1}^r \lambda_i D_i)).$$

Example of multi-section ring (4)

Example 28 (well-known)

Consider the case that $Y = \mathbb{P}^1$, $r = 1$, and $D_1 = \{0\}$. Then

$$\mathrm{vb}(\mathbb{P}^1) = \mathrm{Ref}(\mathbb{P}^1) \to \mathrm{Ref}(\mathbb{G}_m, k[x, y])$$

is an equivalence. Any finitely generated graded free $k[x, y]$-module is a direct sum of rank-one free modules $k[x, y](m)$ $(m \in \mathbb{Z})$. Thus any vector bundle of $\mathbb{P}^1$ is a direct sum of $\mathcal{O}_{\mathbb{P}^1}(m)$ $(m \in \mathbb{Z})$.

Example of determinantal ring (1)

Let $S = \operatorname{Spec} k$, $m, n, t \in \mathbb{Z}$, and $m, n \geq t \geq 1$. Set $V = k^n$, $W = k^m$, and $E = k^{t-1}$. Define $X = \operatorname{Hom}(E, W) \times \operatorname{Hom}(V, E)$ and $Y = \{\varphi \in \operatorname{Hom}(V, W) \mid \operatorname{rank} \varphi < t\}$. Then $\pi : X \to Y$ is defined by $\pi(f, g) = f \circ g$.

Lemma 29

$\pi : X \to Y$ is a $\mathrm{GL}(V) \times \mathrm{GL}(E) \times \mathrm{GL}(W)$-equivariant almost principal $\mathrm{GL}(E)$-bundle.

Example of determinantal ring (2)

Corollary 30

1. (well-known) $\mathrm{Cl}(Y) \cong X(\mathrm{GL}(E)) \cong \mathbb{Z}$.
2. (Svanes) The following are equivalent.
    1. $m = n$.
    2. $\omega_X \cong \mathcal{O}_X$ as $(\mathrm{GL}(E), \mathcal{O}_X)$-modules.
    3. $\omega_Y \cong \mathcal{O}_Y$ as $\mathcal{O}_Y$-modules.
    4. Y is Gorenstein.

Thank you

This slide will soon be available at
http://www.math.nagoya-u.ac.jp/~hasimoto/