Surjectivity of multiplication and F-regularity of multigraded rings

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1 Introduction

Let R be a noetherian $\mathbb{Z}^r$-graded integral domain. Then the subset $\Sigma(R) := \{\lambda \in \mathbb{Z}^r \mid R_\lambda \neq 0\}$ is a finitely generated subsemigroup of $\mathbb{Z}^r$. We say that R is *surjectively graded* if for any $\lambda, \mu \in \Sigma(R)$, the product $R_\lambda \otimes_{R_0} R_\mu \to R_{\lambda+\mu}$ is surjective. This is essentially a generalization of the degree-one generation property of $\mathbb{N}$-graded rings. The purpose of this paper is to study this property, mainly for normal domains. After that, we show that surjectively graded normal domains in positive characteristic behaves well with respect to strong F-regularity, utilizing the notion of global F-regularity defined and studied by Smith [17]. This approach gives yet another abstraction of beautiful ring theoretic properties of multicones over G/B, as in [9], [11].

In section 2, we review the definition and some basic properties of global F-regularity (with an obvious generalization). This section is essentially a proper subset of [17]. A lemma on multicones will be used later.

In section 3, we define and study the first property of surjectively graded rings. As is a multihomogeneous coordinate ring of a closed subscheme of a product of projective spaces, such a ring gives a projective variety. As combinations of the Segre and the Veronese embeddings give various embeddings of the same projective variety, we prove that lines in the 'middle' of graded part give homogeneous coordinate rings of the same projective variety.

We can prove more for a normal surjectively graded algebra, and we discuss this in section 4. Such rings are given as multicones over the projective schemes given in section 3.

As an application, we prove a criterion for strong F-regularity of surjectively graded algebras in section 5. Note that if the ring is $\mathbb{N}$-graded, then much more has long been known [20, (3.4)]. A multicone over G/B is a typical example. Finally, we prove that a normal semigroup scheme in characteristic zero which admits a dominating semigroup homomorphism from a reductive group has at most rational singularities.

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2 Global F-regularity

For a ring A, we denote by A° the subset $\bigcup_{P\in\mathrm{Min}(A)} P$, where $\mathrm{Min}(A)$ is the set of minimal primes of A.

Let p be a prime number, and k a perfect field of characteristic p. Let A be a commutative k-algebra. For $e \geq 0$, the Frobenius map $A \to A$ ($x \mapsto x^{p^e}$) is denoted by F_A^e. For $r \in \mathbb{Z}$, $A^{(r)}$ denotes the ring A with the k-algebra structure $u_A \circ F_k^{-r} : k \to A$, where $u_A : k \to A$ is the original k-algebra structure of A. For a k-algebra map $f : A \to B$, we define $f^{(r)} : A^{(r)} \to B^{(r)}$ to be f. Then $(?)^{(r)}$ is an autofunctor of the category of k-algebras, and we have $(?)^{(r')} \circ (?)^{(r)} \cong (?)^{(r+r')}$ and $(?)^{(0)} \cong \mathrm{Id}$. Note that $F^e : (?)^{(r+e)} \to (?)^{(r)}$ is a natural transformation for $e \geq 0$.

For an A-module (resp. A-algebra) M, we denote the same additive group (resp. ring) M viewed as an $A^{(r)}$-module (resp. $A^{(r)}$-algebra) by $M^{(r)}$. An element m of M, viewed as an element of $M^{(r)}$ is denoted by $m^{(r)}$. So if $a \in A$, then $F_A^e(a^{(e+r)}) = (a^{p^e})^{(r)}$. If $r = 0$, then the superscript $(?)^{(0)}$ may be omitted.

This convention is standard in representation theory of algebraic groups, see [8]. As we need to consider gradings (or actions of diagonalizable group schemes), it is convenient to use this convention here. The relationship with the standard notation in commutative ring theory is explained as follows. $A^{(r)}$ for $r \leq 0$ is denoted ${}^{|r|}A$. Sometimes $A^{(e)}$ for $e \geq 0$ is denoted A^{p^e} (usually p^e is denoted q for short (so q depends on e), and A^q is used).

A similar convention applies to schemes and quasi-coherent sheaves. For a k-scheme X, we always assume that $X^{(r)}$ is an $X^{(r+e)}$-scheme via the Frobenius map $F_X^e : X^{(r)} \to X^{(r+e)}$. If G is a k-group scheme, then $F_G^e : G \to G^{(e)}$ is a k-group homomorphism. For a G-action X, $X^{(e)}$ is a $G^{(e)}$-action. So $X^{(e)}$ is also a G-action via the group homomorphism $F_G^e : G \to G^{(e)}$, and $F_X^e : X \to X^{(e)}$ is a morphism of G-actions. In particular, considering the case where G is a split torus T over $\mathbb{F}_p$, $F_A^e : A^{(e)} \to A$ is a graded homomorphism of $X(T)$-graded rings, if A is $X(T)$-graded (if a has degree λ in A, then $a^{(e)}$ has degree $p^e\lambda$ in $A^{(e)}$).

We say that A is *F-finite* if A is a finite $A^{(1)}$-module.

(2.1) Let Σ be a finitely generated torsion free abelian group. Let A be a Σ-graded integral domain which is an $\mathbb{F}_p$-algebra. We say that A is quasi$F$-regular, if for any homogeneous nonzero element a of A of degree λ, there exists some $e > 0$ such that $aF_A^e : A^{(e)} \to A(\lambda)$ splits as a homogeneous $A^{(e)}$-linear map, where (λ)

denotes the shifting of degree. If A is non-graded, then we consider that A is trivially graded when we speak of quasi$F$-regularity of A. If A is noetherian and F-finite, then A is quasi$F$-regular if and only if A is strongly F-regular in the sense of Hochster-Huneke [5]. In particular, quasi$F$-regularity of a noetherian F-finite domain A of positive characteristic is independent of grading of A.

(2.2) Let A be a $\mathbb{Z}^r$-graded domain, and t a new variable which is homogeneous. If A is quasi$F$-regular, then the polynomial ring $A[t]$ is quasi$F$-regular.

Let $f(t) = a_0 + a_1 t + \cdots + a_r t^r$ ($a_r \neq 0$) be a nonzero homogeneous polynomial. Note that a_i is homogeneous for any i. Take $e > 0$ sufficiently large so that $a_r F_A^e$ has a homogeneous splitting $\pi : A(\lambda) \to A^{(e)}$, and that $r < q$, where λ is the degree of a_r, and $q = p^e$. Let $\Psi : A[t](r\mu) \to A[t^q] = A \otimes_{A^{(e)}} (A[t])^{(e)}$ be the homogeneous $A[t^q]$-linear map given by $t^i \mapsto 0$ for $0 \leq i < q$ and $i \neq r$, and $t^r \mapsto 1$, where μ is the degree of t. As

$$(\pi \otimes 1)\Psi f(t) F_{A[t]}^e(1) = 1,$$

$(\pi \otimes 1)\Psi$ is an $(A[t])^{(e)}$-linear homogeneous splitting of $f(t)F_{A[t]}^e : (A[t])^{(e)} \to A[t](\lambda + r\mu)$.

(2.3) Let $\Sigma \subset \mathbb{Z}^r$ be a subgroup of $\mathbb{Z}^r$, and A a $\mathbb{Z}^r$-graded domain of characteristic p. If A is quasi$F$-regular, then so is $A_\Sigma := \bigoplus_{\lambda \in \Sigma} A_\lambda$.

For $\lambda \in \Sigma$ and $a \in A_\lambda \setminus 0$, there exists some $e > 0$ and a homogeneous splitting $\pi : A(\lambda) \to A^{(e)}$ of aF_A^e. Let $i : A_\Sigma \to A$ be the inclusion, and $\eta : A \to A_\Sigma$ be the projection. Then

$$\eta^{(e)} \pi i a F_{A_\Sigma}^e = \eta^{(e)} \pi a F_A^e i^{(e)} = \eta^{(e)} i^{(e)} = \mathrm{id}.$$

As Σ is a subgroup, i and η are homogeneous and A_Σ-linear. So $\eta^{(e)}\pi i : A_\Sigma(\lambda) \to A_\Sigma^{(e)}$ is a homogeneous splitting of $aF_{A_\Sigma}^{(e)}$.

Lemma 2.4 *Let $A := \bigoplus_{\lambda \in \mathbb{Z}^r} A_\lambda$ be a $\mathbb{Z}^r$-graded commutative ring of characteristic p, and C a rational convex polyhedral cone* (*see* (3.1)) *in $\mathbb{R}^r$. If A is quasi$F$-regular* (*resp. strongly F-regular*), *then so is $A_C := \bigoplus_{\lambda \in C \cap \mathbb{Z}^r} A_\lambda$.*

Proof. As C is the intersection of some finitely many half-spaces whose boundaries are rational hyperplanes through the origin, we may assume that C is such a half-space. Let $H := C \cap (-C)$ be the boundary of C, and set $N := H \cap \mathbb{Z}^r$. Let λ be any element of $C \cap \mathbb{Z}^r$ such that λ and N together generate $\mathbb{Z}^r$. Let t be a new variable of degree $-\lambda$. Then $A[t]$ is quasi$F$-regular. Hence $A_C \cong A[t]_H$ is also quasi$F$-regular. Noetherian F-finite property is also inherited by $A[t]_H$, and we are done. $\square$

(2.5) Let X be a noetherian integral scheme. For invertible sheaves $\mathcal{L}_1, \ldots, \mathcal{L}_r$ on X, we define $R(X; \mathcal{L}_1, \ldots, \mathcal{L}_r)$ to be $\Gamma(V, \mathcal{O}_V)$, where $V = \underline{\mathrm{Spec}}\, \underline{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r)$. Then we have

$$R(X; \mathcal{L}_1, \ldots, \mathcal{L}_r) = \Gamma(X, \underline{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r)) \cong \bigoplus_{\lambda \in \mathbb{N}^r} \Gamma(X, \mathcal{L}_\lambda),$$

where $\mathcal{L}_\lambda := \mathcal{L}_1^{\otimes \lambda_1} \otimes \cdots \otimes \mathcal{L}_r^{\otimes \lambda_r}$ for $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r$. Set E to be the closed subset of V defined by the ideal sheaf of $\underline{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r)$ generated by $\mathcal{L}_{(1,1,\ldots,1)}$. Note that the open subscheme $V \setminus E$ is nothing but $\underline{\mathrm{Spec}}\, \mathcal{B}$, where

$$\mathcal{B} = \mathcal{B}(X; \mathcal{L}_1, \ldots, \mathcal{L}_r) := \bigoplus_{\lambda \in \mathbb{Z}^r} \mathcal{L}_\lambda.$$

We denote $\Gamma(V \setminus E, \mathcal{O}_{V \setminus E})$ by $B = B(X; \mathcal{L}_1, \ldots, \mathcal{L}_r)$.

The following was proved by Smith [17] (see also [20] and [16]) except that the condition **3** seems to be new here. Some trivial generalization is also done here. The whole proof is included for reader's convenience.

Theorem 2.6 *Let the notation be as above, and assume that X is a noetherian integral scheme of characteristic p with an ample invertible sheaf. Then the following are equivalent.*

1 *There exists some ample invertible sheaf $\mathcal{L}$ over X and some $r_0 \geq 0$ such that for any $r \geq r_0$ and $a \in \Gamma(X, \mathcal{L}^{\otimes r}) \setminus 0$, there exists some $e > 0$ such that the composite $\mathcal{O}_{X^{(e)}} \to F^e_* \mathcal{O}_X \xrightarrow{F^e_* a} F^e_* \mathcal{L}$ has an $\mathcal{O}_{X^{(e)}}$-linear splitting.*

2 *For any invertible sheaf $\mathcal{L}$ over X such that for any $a \in \Gamma(X, \mathcal{L}) \setminus 0$, there exists some $e > 0$ such that the composite $\mathcal{O}_{X^{(e)}} \to F^e_* \mathcal{O}_X \xrightarrow{F^e_* a} F^e_* \mathcal{L}$ has an $\mathcal{O}_{X^{(e)}}$-linear splitting.*

3 *For any $r \geq 1$ and any invertible sheaves $\mathcal{L}_1, \ldots, \mathcal{L}_r$, the $\mathbb{Z}^r$-graded ring $B(X; \mathcal{L}_1, \ldots, \mathcal{L}_r)$ is quasi$F$-regular.*

4 *For any ample invertible sheaf $\mathcal{L}$ over X, the $\mathbb{Z}$-graded ring $R(X; \mathcal{L})$ is quasi$F$-regular.*

5 *For some ample invertible sheaf $\mathcal{L}$ over X, $R(X; \mathcal{L})$ is quasi$F$-regular.*

Note that **4**$\Leftrightarrow$**5** is proved in [20, (3.4)] for more general graded rings.

Proof. **1**$\Rightarrow$**2** Let $\mathcal{L}$ be an invertible sheaf on X, and a its nonzero section. Take an ample invertible sheaf $\mathcal{A}$ which satisfies the condition in **1**. Take r sufficiently large so that $\mathcal{L}^{\otimes(-1)} \otimes \mathcal{A}^{\otimes r}$ has a nonzero section b. As $ba \in \Gamma(X, \mathcal{A}^{\otimes r})$ is nonzero (because B is a domain), there exists some $e > 0$ such that baF^e splits. Then aF^e splits, and we are done.

2⇒1 As X is assumed to have an ample invertible sheaf, this is obvious.

1,2⇒3 Set $\mathcal{B} := \mathcal{B}(X, \mathcal{L}_1, \cdots, \mathcal{L}_r)$. Note that $\mathcal{B}$ is a $\mathbb{Z}^r$-graded $\mathcal{O}_X$-algebra, whose degree λ component is $\mathcal{L}_\lambda := \mathcal{L}_1^{\otimes \lambda_1} \otimes \cdots \otimes \mathcal{L}_r^{\otimes \lambda_r}$ for $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r$. Thus $B := B(X; \mathcal{L}_1, \ldots, \mathcal{L}_r)$ is also $\mathbb{Z}^r$-graded whose degree λ component B_λ is $\Gamma(X, \mathcal{L}_\lambda)$.

Let $a \in B$ be a nonzero homogeneous element of degree λ. Take e sufficiently large so that $aF^e : \mathcal{O}_{X^{(e)}} \to F_*^e \mathcal{O}_X \xrightarrow{a} F_*^e \mathcal{L}_\lambda$ splits.

To verify that B is quasi$F$-regular, it suffices to show that $aF^e : B^{(e)} \to B(\lambda)$ homogeneously splits for this e. To verify this, it suffices to show that the corresponding $\mathcal{B}^{(e)}$-linear map $aF^e : \mathcal{B}^{(e)} \to F_*^e \mathcal{B}$ homogeneously splits. Let $\varphi : \mathbb{Z}^r \to \mathbb{Z}^r / p^e \mathbb{Z}^r$ be the natural map. As $\mathcal{B}$ is $\mathbb{Z}^r$-graded and $(F^e)^* \mathcal{B}^{(e)}$ is $p^e \mathbb{Z}^r$-graded, there is a direct sum decomposition

$$\mathcal{B} \cong \bigoplus_{\bar{\nu} \in \mathbb{Z}^r / p^e \mathbb{Z}^r} \mathcal{B}_{\bar{\nu}}$$

as a $\mathbb{Z}^r$-graded $(F^e)^* \mathcal{B}^{(e)}$-module, where $\mathcal{B}_{\bar{\nu}} := \bigoplus_{\mu \in \mathbb{Z}^r,\, \varphi(\mu) = \bar{\nu}} \mathcal{L}_\mu$. Let $\pi(\bar{\nu})$ denotes the projection $\mathcal{B} \to \mathcal{B}_{\bar{\nu}}$. Obviously, the product

$$(F^e)^* \mathcal{B}^{(e)} \otimes_{\mathcal{O}_X} \mathcal{L}_\mu \to \mathcal{B}_{\varphi(\mu)}$$

is a graded isomorphism for $\mu \in \mathbb{Z}^r$. By the projection formula, the product

$$\mathcal{B}^{(e)} \otimes_{\mathcal{O}_{X^{(e)}}} F_*^e \mathcal{L}_\mu \to F_*^e \mathcal{B}_{\varphi(\mu)}$$

is also a graded isomorphism.

It is easy to see that the composite

$$\mathcal{B}^{(e)} \xrightarrow{aF^e_{\mathcal{B}}} F_*^e \mathcal{B}(\lambda) \xrightarrow{\pi(\varphi(\lambda))} F_*^e \mathcal{B}_{\varphi(\lambda)}(\lambda)$$

is the composite

$$\mathcal{B}^{(e)} \xrightarrow{F^e_X} \mathcal{B}^{(e)} \otimes_{\mathcal{O}_{X^{(e)}}} F_*^e \mathcal{O}_X \xrightarrow{a} \mathcal{B}^{(e)} \otimes_{\mathcal{O}_{X^{(e)}}} F_*^e \mathcal{L}_\lambda(\lambda) \cong F_*^e \mathcal{B}_{\varphi(\lambda)}(\lambda).$$

As aF^e_X splits, $\pi(\varphi(\lambda)) aF^e_{\mathcal{B}}$ homogeneously splits, and hence $aF^e_{\mathcal{B}}$ also homogeneously splits.

3⇒4 Let $\mathcal{L}$ be ample. As $B(X; \mathcal{L})$ is assumed to be quasi$F$-regular, $R(X; \mathcal{L})$ is quasi$F$-regular by Lemma 2.4.

4⇒5 As we are assuming that there is an ample invertible sheaf, this is trivial.

5⇒1 Let $\mathcal{L}$ be an ample invertible sheaf such that $R = R(X, \mathcal{L})$ is quasi$F$-regular. Note that R is a domain. Let $r > 0$ be arbitrary, and let $a \in \Gamma(X, \mathcal{L}^{\otimes r}) \setminus 0$.

By assumption, there exists some $e > 0$ such that $aF^e : R^{(e)} \to R(r)$ has an $R^{(e)}$-linear homogeneous splitting $\alpha : R(r) \to R^{(e)}$.

Now passing to the associated morphisms on sheaves on $\operatorname{Proj} R^{(e)}$ and restricting it to the open subscheme $X^{(e)} \subset \operatorname{Proj} R^{(e)}$, we get the splitting $\alpha : F_*^e \mathcal{L}^{\otimes r} \to \mathcal{O}_{X^{(e)}}$ of aF^e, as desired. □

Definition 2.7 (Smith [17]) Let X be a noetherian integral $\mathbb{F}_p$-scheme with an ample invertible sheaf. We say that X is globally F-regular if the equivalent conditions in the proposition are satisfied.

Lemma 2.8 *Let X be a noetherian integral $\mathbb{F}_p$-scheme with an ample invertible sheaf. Then the following hold.*

1 *If X is globally F-regular, then $\mathcal{O}_{X,x}$ is quasi$F$-regular with respect to the trivial grading for $x \in X$. In particular, $\mathcal{O}_{X,x}$ is F-regular in the sense of Hochster-Huneke* [6] *(i.e., any ideal of any localization of $\mathcal{O}_{X,x}$ is tightly closed). In particular, X is normal. If moreover, X is locally excellent, then X is Cohen-Macaulay.*

2 *A globally F-regular scheme is a Frobenius split scheme in the sense of Mehta–Ramanathan* [11].

3 *If X is globally F-regular, then $\Gamma(X, \mathcal{O}_X)$ is quasi$F$-regular with respect to the trivial grading. The converse is true if X is affine.*

Proof. **1** Let $\mathcal{A}$ be a very ample invertible sheaf over X, and $R := R(X, \mathcal{A})$ so that X is an open subscheme of $\operatorname{Proj} R$. Replacing X by $\operatorname{Proj} R$ and $\mathcal{A}$ by $\mathcal{O}(1)$, we may assume that $X = \operatorname{Proj} R$. By assumption, R is quasi$F$-regular with respect to the canonical $\mathbb{Z}$-grading. Note that any homogeneous localization of R is also quasi$F$-regular. Applying (2.3) to the subgroup $\{0\}$ of $\mathbb{Z}$, the degree zero component of any homogeneous localization of R is also quasi$F$-regular with respect to the trivial grading. So X is F-regular. Hence X is normal [7, (3.4)]. The last assertion also follows from the F-regularity of X, [7, (4.2)], and [19, Proposition 0.10].

2 This is trivial.

3 As in **1**, apply (2.3) to the ring $R = R(X, \mathcal{A})$ and the subgroup $\{0\}$ in $\mathbb{Z}$. The last assertion is trivial. $\square$

Lemma 2.9 *Let X be a noetherian normal connected $\mathbb{F}_p$-scheme with an ample invertible sheaf, and E a closed subset of X whose codimension is at least two. Then X is globally F-regular if and only if $X \setminus E$ is globally F-regular.*

Proof. Let $\mathcal{L}$ be an ample invertible sheaf of X. Then $\mathcal{L}|_{X \setminus E}$ is ample, and $R(X; \mathcal{L}) = R(X \setminus E; \mathcal{L}|_{X \setminus E})$. $\square$

Lemma 2.10 *Assume that X is globally F-regular. Let $D_1, \ldots, D_r$ be Weil divisors on X, and set $B := \bigoplus_{\lambda \in \mathbb{Z}^r} \mathcal{O}_X(\sum_i \lambda_i D_i) t^\lambda \subset Q(X)[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$, where $Q(X)$ denotes the rational function field of X, $t_1, \ldots, t_r$ are new variables, and $t^\lambda := t_1^{\lambda_1} \cdots t_r^{\lambda_r}$ for $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r$. Then B is quasi$F$-regular.*

Proof. Take the locally free locus U of $\bigoplus \mathcal{O}_X(D_i)$. As $X \setminus U$ has at least codimension two, the assertion follows immediately by Lemma 2.9. $\square$

Similarly, the following is also trivial.

Lemma 2.11 *Let X be a noetherian connected normal $\mathbb{F}_p$-scheme with an ample invertible sheaf. Then X is globally F-regular if and only if for any effective Weil divisor D of X, there exists some $e > 0$ such that*

$$\mathcal{O}_{X^{(e)}} \xrightarrow{F^e} F_*^e \mathcal{O}_X \to F_*^e \mathcal{O}_X(D)$$

splits.

Lemma 2.12 *An open subscheme U of a globally F-regular scheme X is globally F-regular.*

Proof. Let D be an effective Weil divisor of U. Then D is extended to an effective Weil divisor $\bar{D}$ of X. There exists some $e > 0$ such that $\mathcal{O}_{X^{(e)}} \to F_*^e \mathcal{O}_X \to F_*^e \mathcal{O}(\bar{D})$ splits. By restriction, $\mathcal{O}_{U^{(e)}} \to F_*^e \mathcal{O}_U \to F_*^e \mathcal{O}(D)$ splits. $\square$

Proposition 2.13 ([17, Corollary 4.3]) *Let A be a noetherian commutative ring of characteristic p, and X a globally F-regular projective A-scheme. Let $\mathcal{L}$ be an invertible sheaf on X such that there exists some $n_0 \geq 1$ such that for any $n \geq n_0$ and any ample invertible sheaf $\mathcal{A}$, the tensor product $\mathcal{A} \otimes \mathcal{L}^{\otimes n}$ is ample. Then $H^i(X, \mathcal{L}) = 0$ for $i > 0$. In particular, we have $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$.*

Proof. See [17].

3 Surjectively graded rings

(3.1) Let $r \geq 1$. By definition, a rational convex polyhedral cone in $\mathbb{R}^r$ is a subset of $\mathbb{R}^r$ of the form $\mathbb{R}^r \cap \bigcap_{i=1}^u Q_i$, where u is a nonnegative integer, and each Q_i is a half space in $\mathbb{R}^r$ of the form

$$\{(x_1, \ldots, x_r) \in \mathbb{R}^r \mid a_1^{(i)} x_1 + \cdots + a_r^{(i)} x_r \geq 0\},$$

with $(a_1^{(i)}, \ldots, a_r^{(i)}) \in \mathbb{Q}^r \setminus \{0\}$. Let C be a rational convex polyhedral cone in $\mathbb{R}^r$. Then $C \cap (-C)$ is a linear subspace of $\mathbb{R}^r$ defined over $\mathbb{Q}$. We say that H is a supporting hyperplane of C if there is a half space U in $\mathbb{R}^r$ such that 0 is on its boundary, $U \supset C$, and $H = U \cap (-U)$. A face of C is either C itself, or a subset of C of the form $C \cap H$ with H a supporting hyperplane of C. We denote the set of faces of C by $\mathcal{F}(C)$. Note that $\mathcal{F}(C)$ is a finite ordered set with respect to the incidence relation. Clearly, C is the maximum element, while $C \cap (-C)$ is the minimum element of $\mathcal{F}(C)$. For a face $\sigma \in \mathcal{F}(C)$, the relative interior $\sigma \setminus \left(\bigcup_{\mathcal{F}(C) \ni \rho \subsetneq \sigma} \rho \right)$ of σ is denoted by σ°.

(3.2) Let k be a field, and $R = \bigoplus_{n \in \mathbb{Z}} R_n$ a $\mathbb{Z}$-graded k-algebra. We say that R is a positively graded k-algebra, if $R_n = 0$ for $n < 0$, and $k \to R_0$ is an isomorphism. We say that R is a standard graded k-algebra, if R is positively graded, noetherian, and R is generated by R_1 as a k-algebra.

(3.3) Let $R = \bigoplus_{\lambda \in \mathbb{Z}^r} R_\lambda$ be a $\mathbb{Z}^r$-graded noetherian integral domain. We denote by $\Sigma(R)$ the subset $\{\lambda \in \mathbb{Z}^r \mid R_\lambda \neq 0\}$ of $\mathbb{Z}^r$. As R is generated by finitely many homogeneous elements over R_0, $\Sigma(R)$ is a finitely generated subsemigroup of $\mathbb{Z}^r$.

We denote the rational convex polyhedral cone $C(R) := \mathbb{R}_{\geq 0}\Sigma(R)$ in $\mathbb{R}^r$ by $C(R)$. For $x \in C(R)$, we denote by $\sigma(x)$ the smallest face of $C(R)$ on which x lies, which equals the intersection of the all faces on which x lies.

(3.4) For any subset S of $\mathbb{R}^r$, and any $\mathbb{Z}^r$-graded abelian group M, we set $M_S := \bigoplus_{\lambda \in S \cap \mathbb{Z}^r} M_\lambda$. If S is an additive subsemigroup of $\mathbb{R}^r$, then R_S is a graded R_0-subalgebra of R. This notation will be used also for a graded sheaves over a scheme.

(3.5) Let R be as above. We say that R is *surjectively graded* if the product $R_\lambda \otimes_{R_0} R_\mu \to R_{\lambda+\mu}$ is surjective for $\lambda, \mu \in \Sigma(R)$. It is easy to see that a homogeneous localization of a surjectively graded domain is again a surjectively graded domain. Any $\mathbb{N}$-graded noetherian domain generated by degree zero and one is surjectively graded. If R is surjectively graded and Ω is a subsemigroup of $\mathbb{Z}^r$, then R_Ω is an R_0-subalgebra of R which is surjectively graded.

(3.6) Let R be as above. Set $M(R) = M$ to be the additive group generated by $\Sigma(R)$. We denote $M_\mathbb{R}$ by $M \otimes_\mathbb{Z} \mathbb{R} \subset \mathbb{R}^r$. For a commutative ring A, set $T_A(R) = T_A := \operatorname{Spec} AM$, where AM denotes the group algebra of M over A. By a $T_A(R)$-action X we mean an A-scheme X with a left action $T_A(R) \times_{\operatorname{Spec} A} X \to X$ of the A-group scheme $T_A(R)$. Note that $T(R) := T_{R_0}(R)$ acts on $\operatorname{Spec} R$ in a natural way.

Let A be a commutative ring, Λ a finitely generated abelian group, and $\lambda \in \Lambda$. Set D to be the diagonalizable A-group scheme $\operatorname{Spec} A\Lambda$. The rank one A-free (D, A)-module A with the coaction $\omega : A \to A\Lambda$ such that $\omega(1) = \lambda$ is denoted by $A(\lambda)$. The corresponding $(D, \operatorname{Spec} A)$-module is denoted by $\tilde{A}(\lambda)$. For a D-action $f : X \to \operatorname{Spec} A$ and a quasi-coherent $(D, \mathcal{O}_X)$-module $\mathcal{M}$, we denote the $(D, \mathcal{O}_X)$-module $\mathcal{M} \otimes_{\mathcal{O}_X} f^*(\tilde{A}(\lambda))$ by $\mathcal{M}(\lambda)$.

(3.7) Let R be a $\mathbb{Z}^r$-graded integral domain. Then the subset $\{\lambda \in \Sigma(R) \mid R^\times \cap R_\lambda \neq \emptyset\}$ is denoted by $\chi(R)$, where $R^\times$ denotes the set of units of R. Clearly, $\chi(R)$ is a subgroup of $\Sigma(R)$. If R is surjectively graded and $\lambda \in \chi(R)$, then R_λ is a rank one free R_0-module. In particular, $R_{\chi(R)}$ is isomorphic to the Laurent polynomial ring $R_0\chi(R)$.

Lemma 3.8 *Let the notation be as above.*

0 *For each $\sigma \in C(R)$, the sum $P(\sigma) := R_{\Sigma(R)\setminus\sigma} = \bigoplus_{\lambda\in\Sigma(R)\setminus\sigma} R_\lambda$ is a graded prime ideal of R, and $R/P(\sigma) \cong R_\sigma$.*

1 *The closed subset of non-semi-stable points of* $\operatorname{Spec} R$ *with respect to the $T(R)$-linearlized invertible sheaf $\mathcal{O}(\lambda)$ is defined by $J(\lambda) := R \cdot (\bigoplus_{n\geq 1} R_{n\lambda})$. If R is surjective, then $J(\lambda)$ is generated by R_λ.*

2 *If R is surjective, then $\sqrt{J(\lambda)}$ agrees with $J := \bigoplus_\mu R_\mu$, where the sum is taken over all $\mu \in \Sigma(R)$ such that $\sigma(\mu) \supset \sigma(\lambda)$, or equivalently, $\lambda \in \sigma(\mu)$.*

3 *Assume that R is surjective. Let $\lambda \in \Sigma(R)$ such that $\sigma(\lambda) = C(R)$. Set $Y(R) := \operatorname{Spec} R \setminus V(J(\lambda))$, and let $\pi(R) : Y(R) \to X(R)$ be the categorical quotient. Then $\pi(R)$ is a principal $T(R)$-fiber bundle.*

Proof. The assertion **0** is trivial. **1** is also trivial by the definition of semi-stability [12, Definition 1.7] and the fact $\Gamma(\operatorname{Spec} R, \mathcal{O}(n\lambda))^T = R_{n\lambda}$.

2 Let μ be an element of $\Sigma(R)$ such that $\sigma(\mu) \supset \sigma(\lambda)$. Let N be the subgroup of M spanned by $\Sigma(R) \cap \sigma(\mu)$, and set $A := (\mathbb{C}\Sigma(R))_{\sigma(\mu)}$ and $\bar{A} := (\mathbb{C}N)_{\sigma(\mu)}$. Note that $\bar{A}$ is the normalization of A. As $\bar{A}$ is A-finite, $\mathfrak{c} := [A :_A \bar{A}]$ is a nonzero homogeneous ideal of A. Take $\gamma \in \Sigma(R) \cap \sigma(\mu)$ such that $\mathfrak{c}_\gamma \neq 0$. By the definition of the conductor ideal $\mathfrak{c}$ and the surjectivity, we have $\gamma + (\Sigma(R) \cap \sigma(\mu)) = \gamma + (N \cap \sigma(\mu))$.

Since $\mu \in \sigma(\mu)^\circ$, there exists some $n \geq 1$ such that $n\mu - \gamma - \lambda \in N \cap \sigma(\mu)$. By the choice of γ, we have $n\mu \in \lambda + \Sigma(R)$. By surjectivity, $a^n \in J(\lambda)$ for $a \in R_\mu$. In particular, $R_\mu \subset \sqrt{J(\lambda)}$. So $J \subset \sqrt{J(\lambda)}$.

On the other hand, $J \supset J(\lambda)$ is trivial. So it remains to show that R/J is reduced. Let Ω be the set of maximal elements of the subset

$$\{\sigma \in \mathcal{F}(C(R)) \mid \sigma \not\supset \sigma(\lambda)\}$$

of $\mathcal{F}(C(R))$, where $\mathcal{F}(C(R))$ is ordered by the incidence relation. For $\sigma \in \Omega$, we have that $P(\sigma) \supset J$, and there is a canonical map $R/J \to \prod_{\sigma\in\Omega} R/P(\sigma)$. By the definition of Ω, this map is injective. As a subring of a finite direct product of integral domains is reduced, R/J is reduced. This completes the proof of **2**.

3 Note that the categorical quotient $\pi(R) : Y(R) \to X(R)$ exists by **1** and GIT [12, Theorem 1.10]. It is independent of $\lambda \in C(R)^\circ \cap \Sigma(R)$ by **2**.

Let $\lambda_1, \ldots, \lambda_n$ generate $\Sigma(R)$. As λ is a point in the relative interior of $C(R)$, there exist a sequence of positive integers $c_1, \ldots, c_n$ and a positive integer m such that $m\lambda = c_1\lambda_1 + \cdots + c_n\lambda_n$. By the surjectivity assumption, it suffices to show that for any nonzero element of the form $y = y_1^{c_1} \cdots y_n^{c_n}$ with $y_i \in R_{\lambda_i}$, the morphism $\operatorname{Spec} R[1/y] \to \operatorname{Spec} R[1/y]_0$ is a trivial bundle. This is obvious, since $\chi(R[1/y]) = M$. □

Let R be surjectively graded. By the lemma, for $\sigma \in \mathcal{F}(C(R))$ and $\lambda \in \sigma^\circ \cap \Sigma(R)$, we may write $J(\sigma)$ or $J_R(\sigma)$ instead of $\sqrt{J(\lambda)}$, since $\sqrt{J(\lambda)}$ depends only on $\sigma(\lambda)$ in this case. We set $Y(R) := \operatorname{Spec} R \setminus V(J(C(R)))$, and denote the quotient under the action of $T(R)$ by $\pi(R) : Y(R) \to X(R)$, as in the lemma. We denote $(\pi(R)_* \mathcal{O}_{Y(R)}(\lambda))^{T(R)}$ by $\mathcal{L}[\lambda]$. It is easy to show the following.

Lemma 3.9 *Let the notation be as above.*

1 $\mathcal{L}[\lambda]$ *is an invertible sheaf for each* $\lambda \in M$.

2 *The canonical map* $\mathcal{O}_{X(R)} \to \mathcal{L}[0]$ *is an isomorphism.*

3 *For* $\lambda, \mu \in M$, *we have that the canonical map* $\mathcal{L}[\lambda] \otimes_{\mathcal{O}_{X(R)}} \mathcal{L}[\mu] \to \mathcal{L}[\lambda + \mu]$ *is an isomorphism.*

The proof is left to the reader.

Lemma 3.10 *Let* R *be surjectively graded as above. Let* M *be the subgroup of* $\mathbb{Z}^r$ *generated by* $\Sigma(R)$. *Set* $\operatorname{rank} M = s$, *and let* $\lambda^{(1)}, \dots, \lambda^{(s)}$ *be a* $\mathbb{Z}$*-basis of* M. *Let* L *be a line in* $\mathbb{R}^r$ *through the origin such that* $\#(L \cap C(R)^\circ \cap M) \geq 2$. *Then*

1 $X(R_L) \cong X(R)$ *in a natural way, and is independent of choice of* L.

2 *If* $\lambda \in C(R)^\circ \cap \Sigma(R)$, *then* $\mathcal{L}[\lambda]$ *is very ample relative to* $\operatorname{Spec} R_0$. *Moreover,* $X(R)$ *is projective over* $\operatorname{Spec} R_0$.

Proof. As $\#M \geq 2$, we have that $s > 0$. Replacing $\mathbb{Z}^r$ by M, we may assume that $M = \mathbb{Z}^r$ and $s = r > 0$. It is easy to see that the assertion is independent of choice of $\mathbb{Z}$-basis of M, and so we may assume that $\lambda^{(j)} = \varepsilon_j$ for $1 \leq j \leq r$, where ε_j is the jth unit vector $(0, 0, \dots, 0, 1, 0, \dots, 0)$.

Let λ be any point on $L \cap \Sigma(R) \cap C(R)^\circ$, which exists by assumption. Then simply $\lambda \in \Sigma(R_L) \cap C(R_L)^\circ$. As $J(\lambda)$ is generated by $R_\lambda \subset R_L$, there is a canonical morphism $\eta(R, L) : Y(R) \to Y(R_L)$ induced by the inclusion $R_L \hookrightarrow R$. Let y be an element as in the proof of Lemma 3.8, **3**. Then it is easy to see that $R_L[1/y]_0 \to R[1/y]_0$ is an isomorphism. Namely, $X(R) \to X(R_L)$ is an isomorphism. This completes the proof of **1**.

We prove **2**. Assume that $\lambda \in \Sigma(R) \cap C(R)^\circ$. There are two cases. If $(-\mathbb{N}\lambda) \cap \Sigma(R) \neq \{0\}$, then $C(R) = \mathbb{R}^r$. In this case, we have $X(R) = \operatorname{Spec} R_0$, and the assertion is trivial.

Next, consider the case $(-\mathbb{N}\lambda) \cap \Sigma(R) = \{0\}$. Then the cohomology ring $R(X(R); \mathcal{L}[\lambda]) = \bigoplus_{n \geq 0} R_{n\lambda}$ is generated by R_λ as an R_0-algebra by surjectivity, and $\operatorname{Proj} R(X(R); \mathcal{L}[\lambda]) = X(R)$ by the argument above. So $\mathcal{L}[\lambda]$ is very ample in this case, too.

In both cases, the projectivity of $X(R)$ is obvious. $\square$

(3.11) Let A be a commutative ring, T a group scheme of the form $\operatorname{Spec} AG$, where G is a finitely generated abelian group (and the coalgebra structure is given so that each element of G is group-like). Let X be an A-scheme on which T acts trivially. Then a quasi-coherent $(T, \mathcal{O}_X)$-module is nothing but a G-graded quasi-coherent $\mathcal{O}_X$-module.

Lemma 3.12 *Let T be as above, and $f : X \to Y$ a quasi-compact separated morphism of A-schemes on which T acts trivially. Let $\mathcal{M}$ be a quasi-coherent $(T, \mathcal{O}_X)$-module. Then the canonical map $f_*\mathcal{M}^T \to (f_*\mathcal{M})^T$ is an isomorphism.*

Proof. A $(T, \mathcal{O}_X)$-module is nothing but a G-graded $\mathcal{O}_X$-module, and the T-invariance is nothing but the degree zero component. As f_* preserves the grading and compatible with direct sums, the assertion is trivial. $\square$

Lemma 3.13 *Let $r \geq 1$, and R a noetherian $\mathbb{Z}^r$-graded integral domain. Assume that R is surjectively graded. Let σ be a face of $C(R)$, and $\xi(R,\sigma) : X(R) \to X(R_\sigma)$ be the canonical morphism. Then for $\lambda \in M(R_\sigma)$, we have that $\xi(R,\sigma)^*\mathcal{L}_{R_\sigma}[\lambda] \cong \mathcal{L}_R[\lambda]$.*

Proof. By Lemma 3.9, we may assume that $\lambda \in \Sigma(R_\sigma)$. Note that $J_{R_\sigma}(C(R_\sigma))R = J_R(\sigma)$ and $J_R(\sigma) \supset J(C(R))$. This shows that the inclusion $R_\sigma \hookrightarrow R$ induces the canonical morphism $\eta(R,\sigma) : Y(R) \to Y(R_\sigma)$. Passing to the quotient by $T(R)$, this induces $\xi(R,\sigma)$.

Let $T(R,\sigma)$ be the kernel of the natural map $T(R) \to T(R_\sigma)$ induced by the inclusion $M(R_\sigma) \hookrightarrow M(R)$. Note that $T(R,\sigma) \cong \operatorname{Spec} R_0(M(R)/M(R_\sigma))$, which is diagonalizable. Set $Z := Y(R)/T(R,\sigma)$, and $q : Y(R) \to Z$ to be the natural projection.

Since $T(R,\sigma)$ acts trivially on $Y(R_\sigma)$, there is a unique morphism $c : Z \to Y(R_\sigma)$ such that $cq = \eta(R,\sigma)$. Note that q, c and $\eta(R,\sigma)$ are T-stable. This means c is also $T(R_\sigma)$-stable.

Taking the quotient by $T(R_\sigma)$, we obtain a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{c} & Y(R_\sigma) \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle \pi(R_\sigma)} \\ X(R) & \xrightarrow{\xi(R,\sigma)} & X(R_\sigma), \end{array} \qquad (3.14)$$

where $q' : Z = Y(R)/T(R,\sigma) \to Z/T(R_\sigma) = Y(R)/T(R) = X(R)$ is the quotient map so that $q'q = \pi(R)$. As the square is $T(R_\sigma)$-equivariant and both q' and $\pi(R_\sigma)$ are $T(R_\sigma)$-principal fiber bundles, the square is a fiber square, as can be seen easily.

As $R \otimes_{R_\sigma} R_\sigma(\lambda) \cong R(\lambda)$ in a natural way, we have that $\eta(R,\sigma)^*(\mathcal{O}_{Y(R_\sigma)}(\lambda)) \cong \mathcal{O}_{Y(R)}(\lambda)$. Since $T(R,\sigma)$ acts trivially on $c^*(\mathcal{O}_{Y(R_\sigma)}(\lambda))$, we have

$$(q_*\mathcal{O}_{Y(R)}(\lambda))^{T(R,\sigma)} \cong (q_*(q^*(c^*\mathcal{O}_{Y(R_\sigma)}(\lambda)) \otimes_{\mathcal{O}_{Y(R)}} \mathcal{O}_{Y(R)}))^{T(R,\sigma)} \cong$$
$$(c^*(\mathcal{O}_{Y(R_\sigma)}(\lambda)) \otimes_{\mathcal{O}_Z} q_*\mathcal{O}_{Y(R)})^{T(R,\sigma)} \cong c^*(\mathcal{O}_{Y(R_\sigma)}(\lambda)) \otimes_{\mathcal{O}_Z} (q_*\mathcal{O}_{Y(R)})^{T(R,\sigma)}$$

by the projection formula. Thus we have $c^*(\mathcal{O}_{Y(R_\sigma)}(\lambda)) \cong (q_*\mathcal{O}_{Y(R)}(\lambda))^{T(R,\sigma)}$.

Since (3.14) is a fiber square, we have that

$$\xi(R,\sigma)^*\mathcal{L}_{R_\sigma}[\lambda] \cong (\xi(R,\sigma)^*\pi(R_\sigma)_*\mathcal{O}(\lambda))^{T(R_\sigma)} \cong (q'_*c^*\mathcal{O}(\lambda))^{T(R_\sigma)} \cong \mathcal{L}_R[\lambda],$$

as can be seen easily, utilizing Lemma 3.12. □

4 Normal surjectively graded rings

(4.1) Let $r \geq 1$, and R a surjectively graded noetherian $\mathbb{Z}^r$-graded *normal* domain.

Lemma 4.2 *Let σ be a face of $C(R)$ such that* $\operatorname{ht} P(\sigma) = 1$. *Then we have*

1 *σ is a maximal face (i.e., a maximal element of $\mathcal{F}(R) \setminus \{C(R)\}$).*

2 *The homogeneous localization $R_{(P(\sigma))}$ is isomorphic to the semigroup algebra $K(M(R_\sigma) + \mathbb{N}\lambda)$ for some field K and $\lambda \in M(R) \setminus \sigma$.*

3 *$M(R) \cap \mathbb{R}\sigma = M(R_\sigma)$.*

4 *The K in* **2** *agrees with the degree* 0 *component of the homogeneous localization $R_{((0))}$ at the ideal* (0).

Proof. If $\sigma \subsetneq \tau \subsetneq C(R)$, then $0 \subsetneq P(\tau) \subsetneq P(\sigma)$, and this contradicts to the assumption $\operatorname{ht} P(\sigma) = 1$. This proves **1**.

Set $N := M(R) \cap \mathbb{R}\sigma$. Then $M/N \cong \mathbb{Z}$, and we may assume that R is $\mathbb{N}$-graded (if $-\mathbb{N}$-graded, then we change the sign). We take the homogeneous localization of R at $P(\sigma)$ with respect to this $\mathbb{N}$-grading. It is nothing but the localization $R' := R \otimes_{R_\sigma} Q(R_\sigma)$. Obviously, $P(\sigma)R'$ is the irrelevant maximal ideal of the standard graded $Q(R_\sigma)$-algebra R', and its height is still one. So R' is a one-dimensional normal standard graded $Q(R_\sigma)$-algebra, and is isomorphic to $Q(R_\sigma)[t]$ for some nonzero homogeneous element t. The degree of t, say λ, is a generator of M/N. Note that for each $n \in \mathbb{N}$, the degree n component of R' is a one-dimensional $Q(R_\sigma)$-vector space.

Next, consider the homogeneous localization $R_{(P(\sigma))}$ with respect to the $\mathbb{Z}^r$-grading. Since this ring is M-graded, it is also $M/N \cong \mathbb{Z}$-graded, and each homogeneous component $(R_{(P(\sigma))})_n$ for $n \in \mathbb{Z}$ is a graded module over the N-graded ring $(R_{(P(\sigma))})_N$. Clearly, all non-zero homogeneous element whose degree is in N is a unit in $R_{(P(\sigma))}$, and hence $(R_{(P(\sigma))})_N \cong KM(R_\sigma)$, for some field K. Note that an N-graded module is $M(R_\sigma)$-graded, and any finitely generated $M(R_\sigma)$-graded $KM(R_\sigma)$-module is free. As each homogeneous component must be rank one by the first paragraph, we have that $R_{(P(\sigma))} \cong K(M(R_\sigma)+\mathbb{N}\lambda)$. This shows that $N = M(R_\sigma)$. So the proof of **2** and **3** are complete. The assertion **4** is obvious now. □

Remark 4.3 The normality assumption can be weakened to Serre's (R_1) condition in the lemma above. By the lemma, for a homogeneous element $a \neq 0$ of $R_{((0))}$, $a \in R_{(P(\sigma))}$ if and only if the degree of a lies in the unique half space H in $M_{\mathbb{R}}$ such that $H \cap (-H) = \mathbb{R} \cdot \sigma$ and $H \supset C(R)$. Note that the latter condition is only about the degree of a which is always satisfied by a degree in $C(R) \cap M(R)$.

Theorem 4.4 *Let $r \geq 1$, and R a surjectively graded $\mathbb{Z}^r$-graded noetherian normal domain. Then*

1 *$X := X(R)$ is connected normal.*

2 *For any $\mathbb{Z}$-basis $\nu_1, \ldots, \nu_s$ of M, there exists an isomorphism of $\Sigma(R)$-graded R_0-algebras*

$$R \cong \bigoplus_{\nu \in C(R)} \Gamma(X, \mathcal{L}[\nu]) = B(X; \mathcal{L}[\nu_1], \ldots, \mathcal{L}[\nu_s])_{C(R)}, \tag{4.5}$$

where $Ł[\nu]$ has degree ν.

Proof. Replacing $\mathbb{R}^r$ by $M_{\mathbb{R}}(R)$ and $\mathbb{Z}^r$ by $M(R)$, we may assume that $r = s$. The assertion is independent of choice of $\nu_1, \ldots, \nu_r$, so we may assume $\nu_i = \varepsilon_i$ for $i = 1, \ldots, r$, where $\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 at the ith place.

Set $U := Y(R) = \operatorname{Spec} R \backslash V(J(C(R)))$, and let $\pi : U \to X(R)$ be the quotient map. As U is connected normal and π is faithfully flat, $X(R)$ is connected normal. The part **1** was proved.

Since $\pi_* \mathcal{O}_U = \bigoplus_{\lambda \in \mathbb{Z}^r} (\pi_* \mathcal{O}_U(\lambda))^T$, we have

$$U = \underline{\operatorname{Spec}}\, \pi_* \mathcal{O}_U = \underline{\operatorname{Spec}} \left(\bigoplus_{\lambda \in \mathbb{Z}^r} \mathcal{L}[\lambda] \right).$$

In particular, $\Gamma(U, \mathcal{O}_U) = \bigoplus_{\lambda \in \mathbb{Z}^r} \Gamma(X, \mathcal{L}[\lambda])$.

As R is $\mathbb{Z}^r$-graded and normal, we have that $R = \bigcap_{\mathfrak{p}} R_{(\mathfrak{p})}$, where the intersection is taken in $R_{((0))}$, and $\mathfrak{p}$ runs over the all graded height one prime ideals of R.

For a height one graded prime $\mathfrak{p}$ of R, we have that either $U \cap V(\mathfrak{p}) \neq \emptyset$ or $\mathfrak{p} = P(\sigma)$ for some maximal face σ of $C(R)$. So it is easy to see that $R = \Gamma(U, \mathcal{O}_U) \cap (\bigcap_\sigma R_{(P_\sigma)})$, where the intersection is taken over the maximal faces σ of $C(R)$ such that $\operatorname{ht} P(\sigma) = 1$.

Plainly, $R = R_{\Sigma(R)} \subset \Gamma(U, \mathcal{O}_U)_{\Sigma(R)} \subset \Gamma(U, \mathcal{O}_U)_{C(R)}$. On the other hand, any nonzero homogeneous element of $\Gamma(U, \mathcal{O}_U)_{C(R)}$ is both in $\Gamma(U, \mathcal{O}_U)$ and $\bigcap_\sigma R_{(P_\sigma)}$ above by Remark 4.3. Namely, $\Gamma(U, \mathcal{O}_U)_{C(R)} \subset R$. Hence $R = \Gamma(U, \mathcal{O}_U)_{\Sigma(R)} = \Gamma(U, \mathcal{O}_U)_{C(R)}$. So **2** follows. $\square$

Proposition 4.6 *Let R be as above, and $\sigma \in \mathcal{F}(R)$. Then we have*

1 *The canonical morphism $\xi(R,\sigma) : X(R) \to X(R_\sigma)$ is projective and surjective.*

2 *We have $\xi(R,\sigma)_*\mathcal{O}_{X(R)} \cong \mathcal{O}_{X(R_\sigma)}$ as $(T, \mathcal{O}_{X(R_\sigma)})$-modules in a natural way.*

3 *We have $\xi(R,\sigma)_*\mathcal{L}_R[\lambda] = \mathcal{L}_{R_\sigma}[\lambda]$ for any $\lambda \in M(R_\sigma)$.*

Proof. **1** The projectivity is trivial by Lemma 3.10. As the composite of the dominating morphisms $\eta(R,\sigma) : Y(R) \to Y(R_\sigma)$ induced by $R_\sigma \hookrightarrow R$ and $\pi : Y(R_\sigma) \to X(R_\sigma)$ factors through $\xi(R,\sigma)$, $\xi(R,\sigma)$ must be dominating. As it is projective, it is surjective.

To prove **2** and **3**, using induction on the codimension of σ in $M_\mathbb{R}$, we may assume that the codimension of σ in $C(R)$ is one.

Let $N := M \cap \mathbb{R} \cdot \sigma$, $G := \operatorname{Spec} R_0(M/N) \cong \mathbb{G}_m \hookrightarrow T(R)$, and $Z = Y(R)/G$. Let $q : Y(R) \to Z$ be the projection, and $c : Z \to Y(R_\sigma)$ the induced morphism.

Let $\tilde{p} : \operatorname{Spec} R \to \operatorname{Spec} R_\sigma$ be the canonical morphism, and set $Y := \tilde{p}^{-1}(Y(R_\sigma))$ and $p := \tilde{p}|_Y$. Then p is affine, and $p_*\mathcal{O}_Y$ is $\mathbb{N}$-graded with its degree zero component $\mathcal{O}_{Y(R_\sigma)}$. Note that $Y \cap V(J(C(R)))$ is defined by the degree positive part $(p_*\mathcal{O}_Y)_{>0}$ of $p_*\mathcal{O}_Y$. This is because $(\bigoplus_{\lambda \notin \sigma} R_\lambda) J(R_\sigma)$ defines $V(C(R))$ in $\operatorname{Spec} R$.

So $c : Z \to Y(R_\sigma)$ is identified with the projection $Z = \underline{\operatorname{Proj}}\, p_*\mathcal{O}_Y \to Y(R_\sigma)$. The morphism $\xi(R,\sigma) : X(R) \to X(R_\sigma)$ is obtained by taking the quotient of c under the action of $\operatorname{Spec} R_0 N$.

Therefore, to prove **2**, it suffices to show that $c_*\mathcal{O}_Z = \mathcal{O}_{Y(R_\sigma)}$, because of Lemma 3.12. The question is local, so we may localize, and replace $Y(R_\sigma)$ by an open subscheme of the form $\operatorname{Spec} R_\sigma[1/y]$ with $y \neq 0$ homogeneous with its degree lies in $C(R_\sigma)^\circ$ and that $\chi(R_\sigma[1/y]) = M(R_\sigma)$.

Then, the assertion is reduced to the fact, if $A = \bigoplus_{n \in \mathbb{N}} A_n$ is a noetherian normal $\mathbb{N}$-graded domain generated by A_1 over A_0, then $A = \bigoplus_{n \in \mathbb{N}} \Gamma(\operatorname{Proj} A, \mathcal{O}(n))$, which is well-known. Hence the part **2** has been proved.

The assertion **3** is an immediate consequence of **2**, Lemma 3.13, and the projection formula. □

Corollary 4.7 *Let R be a surjectively graded $\mathbb{Z}^r$-graded noetherian normal domain. If $\lambda \in \Sigma(R)$, then $\mathcal{L}_R[\lambda]$ is generated by global sections.*

Proof. By Lemma 3.10, **2**, we have that $\mathcal{L}_{R_{\sigma(\lambda)}}[\lambda]$ is a very ample invertible sheaf on $X(R_{\sigma(\lambda)})$. So $\mathcal{L}_R[\lambda] \cong \xi(R,\sigma(\lambda))^*\mathcal{L}_{R_{\sigma(\lambda)}}[\lambda]$ is also generated by global sections, since $\Gamma(X(R), \mathcal{L}_R[\lambda]) = \Gamma(X(R_{\sigma(\lambda)}), \mathcal{L}_{R_{\sigma(\lambda)}}[\lambda])$ by **3** of the proposition. □

5 F-regularity of surjectively graded rings

Let p be a prime number.

Theorem 5.1 *Let $r \geq 1$, and R a surjectively graded noetherian normal $\mathbb{Z}^r$-graded F-finite domain of characteristic p. Let L be a line in $\mathbb{R}^r$ through the origin such that $\#(L \cap C(R)^\circ \cap M(R)) \geq 2$. Assume that R_L is strongly F-regular. Let $\sigma \in \mathcal{F}(R)$. Then we have*

1 *R_σ is F-finite noetherian, and strongly F-regular.*

2 *$X(R_\sigma)$ is globally F-regular.*

3 *For any $\lambda \in \Sigma(R)$, $R^i\xi(R,\sigma)_*\mathcal{L}_R[\lambda] = 0$ for $i > 0$.*

4 *For any $\lambda \in \Sigma(R)$, $H^i(X(R), \mathcal{L}_R[\lambda]) = 0$ for $i > 0$.*

Proof. As in Theorem 4.4, **2**, $R = B(X(R), \mathcal{L}[\nu_1], \ldots, \mathcal{L}[\nu_s])_{C(R)}$. The assumption on L simply says that L is defined over $\mathbb{Q}$ through the origin, and another $\mathbb{Q}$-rational point which lies in $C(R)^\circ \cap L$ exists. Thus $L \cap M(R) \cong \mathbb{Z}$. Let λ be a generator of the $\mathbb{Z}$-module $L \cap M(R)$ which lies in $C(R)^\circ$.

First consider the case that $C(R) = \mathbb{R} \cdot C(R)$. Then $\sigma = C(R)$, since $\mathcal{F}(R) = \{C(R)\}$. In this case, $X(R) = \operatorname{Spec} R_0$, and $R_L = \bigoplus_{n\in\mathbb{Z}} R_{n\lambda}$. As R_L is assumed to be strongly F-regular, so is R_0. As $X(R)$ is affine, $X(R)$ is globally F-regular, and hence $R = B(X(R), \mathcal{L}[\nu_1], \ldots, \mathcal{L}[\nu_s])$ is also strongly F-regular. Thus parts **1** and **2** are proved. Parts **3** and **4** are trivial this case. So we are done.

Next, consider the case that $C(R) \neq \mathbb{R} \cdot C(R)$, or equivalently, $C(R) \cap L$ is a half line. Then $X(R) \cong X(R_L) = \operatorname{Proj} \bigoplus_{n\geq 0} R_{n\lambda}$ is globally F-regular. Set $U := X(R) \setminus V(J(C(R)))$. Then $\Gamma(U, \mathcal{O}_U)$ is strongly F-regular by Theorem 2.6. By Lemma 2.4 and Theorem 4.4, **2**, R is strongly F-regular. By Lemma 2.4 again, **1** follows. Now take any $\lambda \in C(R_\sigma)^\circ \cap \Sigma(R)$, and consider $R_{\mathbb{Z}\lambda}$. It is a direct summand subring of R, and is strongly F-regular. Hence $R_{\mathbb{N}\lambda}$ is also strongly F-regular by Lemma 2.4. As $\mathcal{L}_{R_\sigma}[\lambda]$ is very ample on $X(R_\sigma)$, we have that $X(R_\sigma)$ is globally F-regular. To prove **3** for this case, as the question is local on $X(R_\sigma)$, we may and shall assume that σ is a linear subspace of $\mathbb{R}^r$, and $X(R_\sigma)$ is affine. Thus **3** is reduced to prove **4**. Since $\mathcal{L}[\lambda]$ is generated by global sections by Corollary 4.7, the assertion follows from Proposition 2.13. $\square$

As a corollary to the theorem, we have a special case (the case where both A and B are standard graded normal domains) of Theorem 5.2 by M. Hochster below. A proof is included here, as the author couldn't find a reference.

Theorem 5.2 *Let k be a perfect field of characteristic $p > 0$, and A and B be positively graded noetherian k-algebras such that $A_1 \otimes B_1 \neq 0$. Then, the following are equivalent.*

1 *The Segre product $A\#B := \bigoplus_{n\geq 0} A_n \otimes B_n$ is strongly F-regular.*

2 *Both A and B are strongly F-regular.*

3 $A \otimes B$ *is strongly F-regular.*

Proof. **1⇒2** We prove that A is strongly F-regular. The proof for B is the same. For this implication, k need not be perfect (but F-finite).

Note that $A\#B$ is normal. As it is positively graded, it must be a normal domain. By assumption, we can take $a \in A_1 \setminus \{0\}$ and $b \in B_1 \setminus \{0\}$. As $k[a]\#B \subset A\#B$, $k[a]\#B$ is an integral domain. It follows that a is transcendental over k, hence $B \cong k[a]\#B$, and B is also an integral domain. In particular, b is a nonzerodivisor of B, and is transcendental over k.

Take $\Omega_n \subset B_n$ for $n \geq 0$ so that $\Omega_0 = \{1\}$, and $\Omega_n \hookrightarrow B_n \to (B/bB)_n$ is injective and its image is a k-basis of $(B/bB)_n$. Then by [4, (2.3)], $\Omega = \bigcup_{n\geq 0} \Omega_n$ is a free basis of B as a $k[b]$-module, and we have a direct sum decomposition

$$A\#B = \bigoplus_{n\geq 0} \bigoplus_{c\in\Omega_n} \left[\bigoplus_{j\geq 0} A_{n+j} \otimes k b^j c \right]$$

of $A\#B$ as an $A\#k[b]$-module. So $A \cong A\#k[b]$ is a direct summand subring of $A\#B$, and hence A is strongly F-regular.

2⇒3 Take a homogeneous element $a \in A \setminus \{0\}$ such that $A[1/a]$ is regular, or k-smooth. Take $b \in B \setminus \{0\}$ similarly. Then $A[1/a] \otimes B[1/b]$ is k-smooth and hence is regular. Take e sufficiently large so that both aF_A^e and bF_B^e split. Then $(a \otimes b)F_{A\otimes B}^e$ splits, and this proves that $A \otimes B$ is strongly F-regular, see [5].

3⇒1 This is because $A\#B$ is a direct summand subring of $A \otimes B$. □

In view of Watanabe-Hara theorem [3], [2], which says that a $\mathbb{Q}$-Gorenstein normal domain of finite type over $\mathbb{C}$ has at most log-terminal singularities if and only if it is of strongly F-regular type, Theorem 5.2 is deeply related with the following characteristic zero result.

Remark 5.3 Tomari [18] kindly communicated the author with his result deeply related with the theorem above. Let A and B be positively graded finitely generated normal $\mathbb{C}$-algebras. If $\operatorname{Spec}(A\#B)$ has at most log-terminal singularities, then both $\operatorname{Spec} A$ and $\operatorname{Spec} B$ have at most log-terminal singularities. The converse is true, if $\operatorname{Spec}(A\#B)$ is $\mathbb{Q}$-Gorenstein. He also studied what conditions on A and B imply the $\mathbb{Q}$-Gorenstein property of $\operatorname{Spec}(A\#B)$.

Example 5.4 Let G be a connected reductive algebraic group over an algebraically closed field k, and T its maximal torus. Fix a base of the root system of G, and let B be the negative Borel subgroup of G defined accordingly. Then the multigraded algebra $R := \bigoplus_{\lambda\in X(T)} \Gamma(G/B, \mathcal{L}_\lambda)$ is a normal domain, surjectively graded, and is of finite type over k, where $\mathcal{L}_\lambda$ is the G-equivariant invertible sheaf corresponding to the one-dimensional B-module λ, see [8]. The surjectively graded property was first proved by Ramanan and Ramanathan [15]. For the history, see the introduction therein. This is also a consequence of Mathieu's tensor product theorem on modules with good filtrations [10].

A B-stable closed subvariety of G/B is called a Schubert variety in G/B. Ramanan and Ramanathan [15] also proved that the canonical map $\Gamma(G/B, \mathcal{L}_\lambda) \to \Gamma(\mathcal{X}, \mathcal{L}_\lambda|_{\mathcal{X}})$ is surjective for λ dominant and a Schubert subvariety $\mathcal{X}$ in G/B. It follows that $\bigoplus_\lambda \Gamma(\mathcal{X}, \mathcal{L}_\lambda|_{\mathcal{X}})$ is also surjectively graded.

The author does not know any proof of the assertion; any Schubert subvariety $\mathcal{X}$ in G/B is globally F-regular. The problem is reduced to the case where G is semisimple and simply connected. Mehta [14] pointed out that as there is an ample Cartier divisor D of $\mathcal{X}$ which eventually splits (see e.g., [13]), the question is local. That is, $\mathcal{X}$ is globally F-regular if and only if the all local rings of $\mathcal{X}$ are F-regular by [17, (3.10)].

Lemma 5.5 *Let Σ be a subsemigroup of $\mathbb{Z}^r$, k a field, and Λ a commutative Σ-graded k-algebra such that each homogeneous component Λ_σ is one-dimensional for $\sigma \in \Sigma$. If $\Lambda_\sigma \Lambda_\tau \neq 0$ for $\sigma, \tau \in \Sigma$, then Λ is isomorphic to the semigroup algebra $k\Sigma$ as a Σ-graded k-algebra.*

Proof. If Λ is not an integral domain, then Λ must have homogeneous elements $a, b \in \Lambda \setminus \{0\}$ such that $ab = 0$. So Λ is an integral domain by assumption. By assumption, $\Sigma = \Sigma(\Lambda)$.

Let A be the total homogeneous localization of Λ. Let $\Gamma := \Sigma - \Sigma$, $\gamma \in \Gamma$, and a/b and a'/b' be nonzero elements of A_γ, where a, a', b, b' are nonzero homogeneous elements of Λ of degree α, α', β, β', respectively. Then both ab' and ba' are nonzero homogeneous elements of $\Lambda_{\alpha+\beta'}$. So $ab' = cba'$ for some $c \in k^\times$. Thus $\Sigma(A) = \Gamma$, and A_γ is one-dimensional for $\gamma \in \Gamma$.

Let $\gamma_1, \ldots, \gamma_s$ be a $\mathbb{Z}$-basis of Γ, and t_i be a nonzero element of A_{γ_i} for $i = 1, \ldots, s$. Then it is easy to see that $A = k[t_1^{\pm 1}, \ldots, t_s^{\pm s}] \cong k\Gamma$ as Γ-graded k-algebras. We have an inclusion

$$\Lambda \hookrightarrow A_\Sigma \cong (k\Gamma)_\Sigma = k\Sigma,$$

but the inclusion must be surjective by dimension counting of homogeneous components. Hence $\Lambda \cong k\Sigma$. $\square$

Lemma 5.6 *Let k be an algebraically closed field, G a reductive k-group, Γ a finitely generated normal subsemigroup of the semigroup of dominant weights X^+ (we fix a maximal torus T of G and a base of the root system of G). Let $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ be a Γ-graded commutative G-algebra such that $A_\gamma \cong \nabla_G(\gamma)$ for $\gamma \in \Gamma$, and $A_\gamma \otimes A_{\gamma'} \to A_{\gamma+\gamma'}$ is nonzero for $\gamma, \gamma' \in \Gamma$, where $\nabla_G(\gamma)$ is the dual Weyl module of highest weight γ. Then A is F-regular if* $\operatorname{char}(k) > 0$*, and A is of F-regular type if* $\operatorname{char}(k) = 0$.

Proof. We may assume that $G = G' \times T'$, where G' is semisimple simply connected, and T' is a torus. For $\lambda \in X(T)$, let $\mathcal{L}_\lambda$ be the G-equivariant invertible sheaf over G/B corresponding to λ, where B is the negative Borel subgroup of G. Note that $H^0(G/B, \mathcal{L}_\lambda) \neq 0$ if and only if $\lambda \in X^+$, and $H^0(G/B, \mathcal{L}_\gamma) \cong \nabla_G(\gamma)$ if $\gamma \in X^+$.

Consider the cohomology ring

$$C = B(G/B; \mathcal{L}_{\lambda_1}, \ldots, \mathcal{L}_{\lambda_s}, \mathcal{L}_{\mu_1}, \ldots, \mathcal{L}_{\mu_t}),$$

where $s = \operatorname{rank} G'$, $t = \operatorname{rank} T'$, $\lambda_1, \ldots, \lambda_s$ are the fundamental dominant weights of G', and $\mu_1, \ldots, \mu_t$ are a $\mathbb{Z}$-basis of $X(T')$. Note that C is a $X(T)$-graded G-algebra defined over $\mathbb{Z}$. Moreover, when we set

$$C_{\mathbb{Z}} := B(G_{\mathbb{Z}}/B_{\mathbb{Z}}; \mathcal{L}_{\lambda_1,\mathbb{Z}}, \ldots, \mathcal{L}_{\lambda_s,\mathbb{Z}}, \mathcal{L}_{\mu_1,\mathbb{Z}}, \ldots, \mathcal{L}_{\mu_t,\mathbb{Z}}),$$

then $C_{\mathbb{Z}} \otimes_{\mathbb{Z}} K$ is the similar cohomology ring over K for any field K, by Kempf's vanishing [8, (II.4.5)] and the universal coefficient theorem [8, (I.4.18.b)]. By Example 5.4, Lemma 2.4 and (2.3), C_Γ is strongly F-regular if $\operatorname{char}(k) > 0$, and is of strongly F-regular type if $\operatorname{char}(k) = 0$.

So it suffices to show that $A \cong C_\Gamma$. Let $\gamma, \gamma' \in \Gamma$. By Mathieu's tensor product theorem [10], there is a short exact sequence of the form

$$0 \to V \to A_\gamma \otimes A_{\gamma'} \xrightarrow{\mu} \nabla_G(\gamma + \gamma') \to 0$$

such that V admits a filtration

$$0 = V_0 \subset V_1 \subset \cdots \subset V_u = V$$

of G-modules such that $V_i/V_{i-1} \cong \nabla_G(\lambda(i))$ for some $\lambda(i) < \gamma + \gamma'$. In particular, $\operatorname{Ext}^i_G(V, \nabla_G(\gamma + \gamma')) = 0$ for $i = 0, 1$. So

$$k \cong \operatorname{Hom}_G(\nabla_G(\gamma + \gamma'), \nabla_G(\gamma + \gamma')) \xrightarrow{\mu^*} \operatorname{Hom}_G(A_\gamma \otimes A_{\gamma'}, \nabla_G(\gamma + \gamma'))$$

are isomorphisms. In particular, any nonzero G-linear map from $A_\gamma \otimes A_{\gamma'}$ to $\nabla_G(\gamma + \gamma')$ is surjective. As we assume that the product $A_\gamma \otimes A_{\gamma'} \to \nabla_G(\gamma + \gamma')$ is nonzero, it must be surjective.

It follows that if a is a highest weight vector of A_γ and a' is a highest weight vector of $A_{\gamma'}$, then $aa' \neq 0$. By Lemma 5.5, $A^{U^+} \cong k\Gamma$, where U^+ is the unipotent radical of the positive Borel subgroup B^+ of G. Similarly, we have $C_\Gamma^{U^+} \cong k\Gamma$. Combining these, we have a Γ-graded k-algebra isomorphism $\varphi : A^{U^+} \to C_\Gamma^{U^+}$. For $\lambda \in \Gamma$,

$$k \to \operatorname{Hom}_G(\nabla_G(\lambda), \nabla_G(\lambda)) \to \operatorname{Hom}_k(\nabla_G(\lambda)^{U^+}, \nabla_G(\lambda)^{U^+}) \cong k$$

are isomorphisms. It follows that φ is uniquely extended to a graded G-linear isomorphism $\tilde{\varphi} : A \to C_\Gamma$.

It suffices to show that $\tilde{\varphi}$ is a k-algebra map. To prove this, it suffices to show that for $\gamma, \gamma' \in \Gamma$, we have $m_{C_\Gamma} \circ (\tilde{\varphi} \otimes \tilde{\varphi}) = \tilde{\varphi} \circ m_A$ as maps in $\mathrm{Hom}_G(A_\gamma \otimes A_{\gamma'}, A_{\gamma+\gamma'}) \cong k$, where m_{C_Γ} and m_A are the product maps. As the hom-group is one-dimensional and both hand sides are nonzero, there exists some $c \in k^\times$ such that $m \circ (\tilde{\varphi} \otimes \tilde{\varphi}) = c\tilde{\varphi} \circ m$. Let a be a highest weight vector of A_γ, and a' be a highest weight vector of $A_{\gamma'}$. Since φ is a k-algebra map,

$$m \circ (\tilde{\varphi} \otimes \tilde{\varphi})(a \otimes a') = \varphi(a)\varphi(a') = \varphi(aa') = c\tilde{\varphi}(aa').$$

As $\tilde{\varphi}(aa') = \varphi(aa') \neq 0$, we have that $c = 1$, as desired. □

Theorem 5.7 *Let G be a reductive group over $\mathbb{C}$, and H an affine semigroup scheme over $\mathbb{C}$ which is normal and of finite type over $\mathbb{C}$. If there is a dominating homomorphism of semigroup schemes $\varphi : G \to H$, then H has at most rational singularities.*

Proof. To prove the theorem, we may assume that G is a direct product of a simply connected semisimple group G' and a torus T'.

Let T be a maximal torus of G, and $X := X(T)$ the set of weights. We fix a base Δ of the set of roots Φ of G. Each $\alpha \in \Phi$ yields a one-parameter subgroup $\alpha^\vee$ of G. $\alpha^\vee$ gives an element of $\mathrm{Hom}_{\mathbb{R}}(X \otimes_{\mathbb{Z}} \mathbb{R}, \mathbb{R})$ determined by $\langle \lambda, \alpha^\vee \rangle = 0$ for $\lambda \in X(G)$, and $\langle \beta, \alpha^\vee \rangle = 2(\beta, \alpha)/(\alpha, \alpha)$ for $\beta \in \Phi$, where $X(G)$ is the set of isomorphism classes of one-dimensional representations of G, and $(\ , \ )$ is the Killing form of the semisimple Lie algebra $\mathrm{Lie}(G')$. We denote the set of dominant weights by X^+. The set of positive roots is denoted by Φ^+.

Via the left and right regular action, the coordinate ring $\mathbb{C}[G]$ is a $G \times G$-algebra, which is decomposed into a multiplicity free direct sum

$$\mathbb{C}[G] = \bigoplus_{\lambda \in X^+} \nabla_G(\lambda) \otimes \nabla_G(\lambda^*), \tag{5.8}$$

where $\nabla_G(\lambda) := \Gamma(G/B, \mathcal{L}_\lambda)$ is the dual Weyl module of highest weight λ, and $\lambda^* := -w_0\lambda$, where B is the negative Borel subgroup of G, and w_0 is the longest element of the Weyl group $W(G)$. Set $M(\lambda) := \nabla_G(\lambda) \otimes \nabla_G(\lambda^*)$.

Since φ is dominating, H is connected, and hence is integral. So the homomorphism $\mathbb{C}[H] \to \mathbb{C}[G]$ associated with φ is an injective $G \times G$-algebra map. In particular, $\mathbb{C}[H]$ is a partial sum of (5.8). Let Σ be the subset of X^+ such that $\mathbb{C}[H] = \bigoplus_{\lambda \in \Sigma} M(\lambda)$.

Let U^+ be the unipotent radical of the positive Borel subgroup B^+ of G. Then $\mathbb{C}[G]^{U^+ \times U^+}$ is isomorphic to $\mathbb{C}X^+$ by Lemma 5.5, and $\mathbb{C}[H]^{U^+ \times U^+}$ is identified with the subspace spanned by Σ. As $\mathbb{C}\Sigma$ is a subalgebra, Σ is a subsemigroup of X^+. By Grosshans theorem [1], $\mathbb{C}\Sigma$ must be finitely generated. Hence Σ is finitely generated. Moreover, $\mathbb{C}\Sigma$ must be integrally closed. Hence Σ is normal.

Let us consider $\rho^\vee := \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha^\vee$. Then $2\rho^\vee = \sum_{\alpha\in\Delta} c_\alpha^\vee \alpha^\vee$ with each (uniquely determined) $c_\alpha^\vee$ a positive integer. It follows that $\langle\lambda, 2\rho^\vee\rangle$ is a nonnegative integer for $\lambda \in X^+$. Note also that $\langle\alpha, 2\rho^\vee\rangle = 2$ for $\alpha \in \Delta$. In particular, $\langle\lambda - \mu, 2\rho^\vee\rangle > 0$, if $\lambda, \mu \in X(T)$ and $\lambda > \mu$.

For $\lambda, \mu \in X^+$, the tensor product $M(\lambda) \otimes M(\mu)$ has the highest weight $(\lambda + \mu, (\lambda + \mu)^*)$ as a $G \times G$-module, and any other weight of $M(\lambda) \otimes M(\mu)$ is smaller than $(\lambda+\mu, (\lambda+\mu)^*)$. It follows that the product $M(\lambda)M(\mu)$ is contained in $\bigoplus_{\gamma\leq\lambda+\mu} M(\gamma)$.

Let us define a $\mathbb{C}[t]$-algebra $\Lambda(G)$. We define $\Lambda = \mathbb{C}[t] \otimes_{\mathbb{C}} \mathbb{C}[G]$ as a $\mathbb{C}[t]$-module. For $\lambda, \mu \in X^+$, $a \in M(\lambda)$ and $b \in M(\mu)$, let $ab = \sum_\gamma c_\gamma$ be the original product in $\mathbb{C}[G]$, where $c_\gamma \in M(\gamma)$, and the sum is taken over $\gamma \in X^+$ such that $\gamma \leq \lambda + \mu$. We define the new product of Λ by

$$(t^u \otimes a)(t^v \otimes b) = \sum_\gamma t^{u+v+\langle\lambda+\mu-\gamma, 2\rho^\vee\rangle} \otimes c_\gamma.$$

It is easy to see that $\Lambda(G)$ is a commutative $\mathbb{C}[t]$-algebra. Note that $\Lambda(H) = \mathbb{C}[t] \otimes_{\mathbb{C}} \mathbb{C}[H] \subset \Lambda(G)$ is a $\mathbb{C}[t]$-subalgebra. Letting $G \times G$ acts on $\mathbb{C}[t]$ trivially, $\Lambda(G)$ is a $G \times G$-algebra, and $\Lambda(H)$ is its $G \times G$-subalgebra. It is easy to see that $\Lambda(H)[t^{-1}] \cong \mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[H]$, where the algebra structure of the right hand side is the real tensor product of algebras. In particular, it suffices to show that $\Lambda(H)$ has at most rational singularities.

The principal ideal generated by t is a $G \times G$-ideal of $\Lambda(H)$. Then $\Lambda(H)/t\Lambda(H) \cong \mathbb{C}[H]$ as G-modules, and the product of this ring is given by $ab = c_{\lambda+\mu}$ for $\lambda, \mu \in \Sigma$, $a \in M(\lambda)$ and $b \in M(\mu)$, where $ab = \sum_\gamma c_\gamma$ is the original product of $\mathbb{C}[H]$. It suffices to show that $\Lambda(H)/t\Lambda(H)$ has at most rational singularities. As $\Lambda(H)/t\Lambda(H)$ is identified with $(\Lambda(G)/t\Lambda(G))_\Sigma$ of the X^+-graded algebra $\Lambda(G)/t\Lambda(G)$ and Σ is a finitely generated normal subsemigroup, it suffices to show that $A = \Lambda(G)/t\Lambda(G)$ has at most rational singularities.

Note that $A = \bigoplus_{\lambda\in X^+} M(\lambda)$ is an X^+-graded $G \times G$-algebra. That is, $M(\lambda)M(\mu) \subset M(\lambda+\mu)$ for $\lambda, \mu \in X^+$, and the product $M(\lambda)\otimes M(\mu) \to M(\lambda+\mu)$ is $G\times G$-linear. The product is nonzero, since the product of highest weight vectors is nonzero. So the proof is completed by Lemma 5.6, applied to $G \times G$ and the subsemigroup $\Gamma := \{(\lambda, \lambda^*) \mid \lambda \in X^+(G)\} \subset X^+(G \times G)$. $\square$

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