Equivariant class group. III. Almost principal fiber bundles

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Dedicated to Professor Yuji Yoshino on the occasion of his 60th birthday

Abstract

As a formulation of 'codimension-two arguments' in invariant theory, we define a (rational) almost principal bundle. It is a principal bundle off closed subsets of codimension two or more. We discuss the behavior of the category of reflexive modules over locally Krull schemes, the category of the coherent sheaves which satisfy Serre's condition (S_2') over Noetherian (S_2) schemes with dualizing complexes, the class group, the canonical module, the Frobenius pushforwards, and global F-regularity, of a rational almost principal bundle. We give examples of finite group schemes, multisection rings, surjectively graded rings, and determinantal rings, and give unified treatment and new proofs to known results in invariant theory, algebraic geometry, and commutative algebra, and generalize some of them. In particular, we generalize the result on the canonical module of the multisection ring of a sequence of divisors by Kurano and the author. We also give a new proof of a generalization of Thomsen's result on the Frobenius pushforwards of the structure sheaf of a toric variety.

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Contents

0. Introduction

(0.1) This paper is a continuation of [Has9] and [Has11].

(0.2) Let k be a field, and G an algebraic group scheme over k. In geometric invariant theory, categorical quotients, geometric quotients, and principal fiber bundles play important roles. Among them, principal fiber bundles behave well with respect to quasi-coherent sheaves, and they are interesting from the viewpoint of algebraic invariant theory. Grothendieck's descent theorem tells us that if $\psi : X \to Y$ is a principal G-bundle, then $\psi^* : \mathrm{Qch}(Y) \to \mathrm{Qch}(G, X)$ is an equivalence of categories, see [Has9, (3.13)]. This fact has played central role in our treatment of equivariant class groups and Picard groups in [Has9] and [Has11].

(0.3) A G-invariant morphism $\varphi : X \to Y$ is said to be an algebraic quotient (or an affine quotient) by G if it is an affine morphism, and the canonical map $\mathcal{O}_Y \to (\varphi_* \mathcal{O}_X)^G$ is an isomorphism. If B is a G-algebra (a k-algebra on which G-acts), then the canonical map $\varphi : X = \operatorname{Spec} B \to \operatorname{Spec} B^G = Y$ is an algebraic quotient. It is the central object in algebraic invariant theory. This is not even a categorical quotient in general (see Example 10.14), and it seems that imposing geometric conditions should yield a good class of algebraic quotients. However, the algebraic quotient φ is rarely a principal G-bundle. For example, if $B = k[x_1, \ldots, x_n]$ is a polynomial ring and G acts on B linearly, then the origin is the fixed point, and hence φ is not a principal G-bundle unless G is the trivial group.

(0.4) However, when we remove closed subsets of codimension two or more both from X and Y, sometimes the remaining part is a principal bundle, and this is sometimes useful enough in invariant theory. We define that the diagram of G-schemes

$$X \xleftarrow{i} U \xrightarrow{\rho} V \xrightarrow{j} Y \tag{1}$$

is a *rational almost principal G-bundle* if G acts on V and Y trivially, i and j are open immersions, $\operatorname{codim}_Y(Y \setminus j(V)) \geq 2$, $\operatorname{codim}_X(X \setminus i(U)) \geq 2$, and $\rho : U \to V$ is a principal G-bundle (cf. Definition 10.2). The name 'rational' comes from the fact that $X \dashrightarrow Y$ is a rational map. A G-invariant morphism $\varphi : X \to Y$ is called an *almost principal G-bundle* if there exist some $U \subset X$ and $V \subset Y$ such that (1) is a rational almost principal G-bundle, where i and j are inclusions, and ρ is the restriction of φ (cf. Definition 10.3).

(0.5) If X and Y are locally Noetherian and normal, then the categories of reflexive modules $\mathrm{Ref}(V)$ and $\mathrm{Ref}(Y)$ are equivalent under the equivalences j_* and j^*. Similarly, the categories of G-equivariant reflexive modules $\mathrm{Ref}(G,U)$ and $\mathrm{Ref}(G,X)$ are equivalent under the functors i_* and i^*. Finally, by Grothendieck's descent theorem, $\mathrm{Ref}(V)$ and $\mathrm{Ref}(G,U)$ are equivalent under the equivalences ρ^* and $(?)^G \circ \rho_*$. Combining them, we have that $\mathrm{Ref}(Y)$ and $\mathrm{Ref}(G,X)$ are equivalent under the equivalences $i_*\rho^*j^*$ and $(?)^G j_*\rho_*i^*$ (note that $(?)^G j_*$ and $j_*(?)^G$ are equivalent) (cf. Theorem 11.2). This is the main observation of this paper.

(0.6) The first purpose of this paper is to show that properties of X and those of Y are deeply connected, and we can get information of Y from that on the action of G on X, and vice versa.

(0.7) This kind of argument, sometimes called codimension-two argument, is not new here. We give one example which shows that our construction is a generalization of a very important notion in invariant theory.

Let G be a finite group acting on a variety X over a field k. Then for $g \in G$, we define $X_g := \{x \in X \mid gx = x\}$. If $\operatorname{codim}_X X_g = 1$, then we say that g is a pseudoreflection. If $\varphi : X \to Y$ is an algebraic quotient by G, then we have that φ is an almost principal G-bundle if and only if the action of G is small, that is, G does not have a pseudoreflection (Proposition 14.8). We will show that some of the known important results on the invariant subrings under the action of finite groups without pseudoreflections can be generalized to the results on (rational) almost principal bundles.

(0.8) The second purpose of this paper is to show that (rational) almost principal bundles are so ubiquitous in invariant theory, algebraic geometry, and commutative algebra. As an application, we give new and short proofs to various known results. Some of them are generalized using our approach.

(0.9) In what follows, we list the results on the first purpose, that is, the general results on the comparison of properties of X and Y, for a rational almost principal G-bundle (1). These are proved in Chapter 1 (sections 10–13).

For simplicity, in the following list, we assume that X and Y are normal varieties over an algebraically closed base field k.

(0.10) $\mathrm{Ref}(Y)$ and $\mathrm{Ref}(G,X)$ are equivalent, as we mentioned.

(0.11) $\mathrm{Cl}(Y) \cong \mathrm{Cl}(G,X)$ (Theorem 11.2). This is a consequence of the equivalence $\mathrm{Ref}(Y) \cong \mathrm{Ref}(G,X)$. This result has essential overlap with the work of Waterhouse [Wat2, Theorem 4] in the affine case. Our approach also enables us to establish an exact sequence

$$0 \to H^1_{\mathrm{alg}}(G, \mathcal{O}_X^\times) \to \mathrm{Cl}(Y) \to \mathrm{Cl}(X)^G \to H^2_{\mathrm{alg}}(G, \mathcal{O}_X^\times),$$

see Theorem 11.5. The proof depends on the corresponding exact sequence for the Picard groups developed in [Has9].

(0.12) Let ω_Y be the canonical module of Y, and ω_X be the G-canonical module of X. Then $\omega_X \cong i_*\rho^*j^*\omega_X \otimes_k \Theta_G^*$, and $\omega_Y \cong (j_*\rho_*i^*\omega_X)^G \otimes_k \Theta_G$, where $\Theta_G = \Theta_{G,k}$ is a certain one-dimensional representation of G determined only by G (Theorem 11.18). If G is smooth, then $\Theta_G = \bigwedge^{\mathrm{top}} \mathrm{Lie}\, G$. If G is connected reductive or abelian, then Θ_G is trivial. This kind of relationship between ω_X and ω_Y can be found in the work of Knop [Knp] on an action of an algebraic group over an algebraically closed field of characteristic zero. See also [Pes]. As we also work over characteristic $p > 0$ and treat non-reduced group schemes too, the description of ω_X and Θ_G depends on the theory of equivariant twisted inverse developed in [Has5].

Although the situation is different, Θ_G plays a similar role as the differential character (or different character) $\chi_{B,A}^{-1}$ (in the notation of [FlW], where $X = \mathrm{Spec}\, B$ and $Y = \mathrm{Spec}\, A$) played in the study of finite group actions in [Bro] and [FlW]. In the case that the group G is étale, $X = \mathrm{Spec}\, B$ and Y are affine, and $\varphi : X \to Y$ is an almost principal G-bundle with B a UFD, where our settings overlap with theirs, $\Theta_G = \chi_{B,A}^{-1}$ is trivial (this case is treated in [Bra]). In these papers, the assumption that B is a UFD (or a polynomial ring) was important in order to make the different module $\mathcal{D}_{B/A}$ rank-one free. We are free from the assumption that B is a UFD, and we can treat higher dimensional and non-reduced group schemes.

(0.13) Let the characteristic of k be $p > 0$. If $\mathcal{M} \in \mathrm{Ref}(G,X)$, then the Frobenius pushforward $F_*^e({}^e\mathcal{M})$ also lies in $\mathrm{Ref}(G,X)$, see for the notation, section 8. This simple observation suggests that (rational) almost principal bundles are useful in studying Frobenius pushforwards and related properties and invariants of algebraic varieties in characteristic $p > 0$. Let $\mathcal{N} \in \mathrm{Ref}(Y)$ corresponds to $\mathcal{M} \in \mathrm{Ref}(G,X)$ under the equivalence $\mathrm{Ref}(Y) \cong \mathrm{Ref}(G,X)$ above, and $e \geq 1$. If G is k-smooth, then the Frobenius pushforward $F_*^e({}^e\mathcal{N})$

corresponds to the invariance $(F_*^e({}^e\mathcal{M}))^{G_e}$, where G_e is the eth Frobenius kernel of G (Theorem 12.6). If G is a finite group, then G_e is trivial. Applying this result, Nakajima and the author recently gave a description of the generalized F-signatures of maximal Cohen–Macaulay modules over the invariant subrings under the action of finite groups without pseudoreflections [HasN].

(0.14) Using the correspondence in (0.13) of Frobenius pushforwards, we get information on the direct-sum decomposition of the Frobenius pushforwards $F_*^e({}^e\mathcal{O}_Y)$ from the information of the decomposition of $(F_*^e({}^e\mathcal{O}_X))^{G_e}$. The author expects that this observation will be useful in studying the problem of finite F-representation type defined by Smith and van den Bergh [SmVdB]. As an application, we will give a short proof of a generalization of Thomsen's result [Tho] on the decomposition of Frobenius pushforwards of the structure sheaf of toric varieties (Theorem 16.4).

(0.15) Another related result on Frobenius maps is the heredity of global F-regularity. We say that an integral Noetherian $\mathbb{F}_p$-scheme X is globally F-regular if for any invertible sheaf $\mathcal{L}$ on X and any nonzero section $s : \mathcal{O}_X \to \mathcal{L}$ of $\mathcal{L}$, there exists some $e \geq 1$ such that $sF^e : \mathcal{O}_{X^{(e)}} \to \mathcal{L}$ splits as an $\mathcal{O}_{X^{(e)}}$-linear map. This was defined by Smith [Smi] for projective varieties, and this definition is its obvious extension. She applied this notion to prove a vanishing theorem on a GIT quotient of a complex Fano variety with rational Gorenstein singularities. We prove that when G is linearly reductive and both X and Y have ample invertible sheaves, X is globally F-regular if and only if Y is globally F-regular. Note that as we work over characteristic $p > 0$, if G is affine and linearly reductive, then the identity component G° of G is diagonalizable [Swe2], and G/G° is a finite group whose order is not divisible by the characteristic of k.

This is the end of the list of our general results.

(0.16) Some of the results above are proved under more general settings, see the text. We tried to study not only smooth algebraic groups but also non-reduced group schemes, as long as possible. In fact, our base scheme S is basically general, and our group scheme G is basically general, except that it is almost always assumed to be flat. Some additional assumptions are added case-by-case.

When we consider a torus action, our main construction has some applications to algebraic geometry and commutative algebra (sections 15–17). For a finitely generated torsion-free abelian group $\Lambda \cong \mathbb{Z}^s$, a ring B with the action of the torus $G = \operatorname{Spec} \mathbb{Z}\Lambda$ is nothing but a Λ-graded ring. Then the Veronese subring B_Γ with respect to a subgroup $\Gamma \subset \Lambda$ is nothing but B^N, where N is the diagonalizable group scheme $\operatorname{Spec} \mathbb{Z}(\Lambda/\Gamma)$. If the characteristic of the base field k is $p > 0$ and Λ/Γ has a p-torsion, this is a non-reduced group scheme. So the invariant theory of non-reduced group schemes arises naturally in algebraic geometry and commutative algebra in characteristic $p > 0$. As an application of our main construction, we give a characterization of a standard graded k-algebra B of dimension greater than or equal to two whose Veronese subalgebra $B_{d\mathbb{Z}}$ is quasi-Gorenstein (Proposition 17.9). Although their (seeming) statement was a little bit weaker, Goto and K.-i. Watanabe [GW, (3.2.1)] already proved it (exactly the same proof works).

Recently, the author gave a classification of the linearly reductive finite subgroup schemes of SL_2 [Has10]. This enables us to write any complete local F-rational Gorenstein ring of dimension two over an algebraically closed field of positive characteristic as an invariant subring of such a subgroup scheme. This has known to be true for sufficiently large characteristic, but now the bad characteristics have also be covered, using non-reduced group schemes. In this paper, we define the smallness of the action of a group scheme, generalizing the action of finite groups without pseudoreflections, and show that the canonical action of a linearly reductive finite subgroup scheme of SL_n on $k[[x_1, \ldots, x_n]]$ is small (Proposition 17.14). Applying the general results listed above to the action of SL_2 on $k[[x, y]]$, we get some basic and known results on the two-dimensional F-rational Gorenstein singularities (such as finite representation type property), see Theorem 17.17.

The author expects that the generalization from groups to group schemes will give interesting new aspects to invariant theory.

(0.17) The codimension-two argument on reflexive sheaves works comfortably on Noetherian normal schemes. However, as applications to commutative algebra are important motivations, we mainly work on locally Krull schemes when we discuss class groups, as in [Has9] and [Has11]. A generalization to different direction is the coherent sheaves $\mathcal{M}$ which satisfy Serre's (S_2') condition (that is, $\operatorname{depth}_{\mathcal{O}_{Z,z}} \mathcal{M}_z \geq \min(2, \dim \mathcal{O}_{Z,z})$ for $z \in Z$) on a quasi-normal locally Noetherian scheme Z. Quasi-normality is a notion which generalizes a normal scheme (a little bit more generally, schemes which satisfy $(T_1) + (S_2)$ ((T_1) is 'Gorenstein in codimension one,' it is also written as (G_1) by some authors. Our notation is after [GM])) and a locally equidimensional scheme with a dualizing complex simultaneously, see (7.36). In particular, we generalize [Hart4, (1.12)]. A quasi-Gorenstein locally Noetherian scheme is quasi-normal, and the generalization to this direction is used to reprove the theorem of Goto–Watanabe mentioned in (0.16).

(0.18) As in [Has11], instead of working on a single group scheme G, we mostly work on a short exact sequence

$$1 \to N \to G \xrightarrow{f} H \to 1$$

of group schemes. That is, $f : G \to H$ is a qfpqc homomorphism of group schemes, and $N = \operatorname{Ker} f$. For qfpqc morphisms, see [Has11, section 2]. We say that a rational almost principal N-bundle (1) is G-*enriched* if it is also a diagram of G-morphisms.

For example, let $B = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k, and N a finite subgroup of GL_n without pseudoreflection, acting on $X = \operatorname{Spec} B$ in a natural way. Let $H = \mathbb{G}_m$ be the one-dimensional torus, and $G = N \times H$. As the action of N on B preserves grading, G acts on B. As the inclusion $A \hookrightarrow B$ preserves grading, the almost principal N-bundle $\varphi : X = \operatorname{Spec} B \to \operatorname{Spec} B^N = Y$ is G-enriched. By our general result, $\varphi^* : \operatorname{Ref}(H, Y) \to \operatorname{Ref}(G, X)$ is an equivalence whose quasi-inverse is $(?)^N \varphi_*$ (Theorem 11.2). Note that $\operatorname{Ref}(H, Y)$ is the category of graded reflexive $\mathcal{O}_Y$-modules, and $\operatorname{Ref}(G, X)$ is the category of graded reflexive $(N, \mathcal{O}_X)$-modules. So the auxiliary action of H gives us the graded version of the invariant theory.

Another example of an auxiliary action is that of Galois groups. Let k be a field, N_1 an étale k-group scheme, and $\varphi : X \to Y$ an almost principal N_1-bundle. Let k' be a finite Galois extension of k with the Galois group H such that the base change $k' \otimes_k N_1$ is a constant finite group N (such k' always exists). Then H acts on N by group automorphisms, and we can define the semidirect product $G := N \rtimes H$, and the base change $\varphi' : X' \to Y'$, where the base field is still k, not k', is a G-enriched almost principal N-bundle (H-equivariant almost principal N-bundle). Even if the original N_1 is not constant, G and H are constant finite groups, and we can utilize the usual group theory to study φ'.

Yet another example can be found in the study of the Cox rings of toric varieties, see Proposition 16.1. See also Lemma 15.36.

(0.19) In Chapter 2 (sections 14–18), we show various examples of (rational) almost principal bundles and give applications.

(0.20) The first example is the finite group schemes. As we have mentioned, an action of a finite group G on an affine algebraic variety $X = \operatorname{Spec} B$ yields an almost principal G-bundle $\varphi : X = \operatorname{Spec} B \to \operatorname{Spec} B^G = Y$ if and only if the action is small (that is, G does not have a pseudoreflection). For a general finite group scheme action, we defined the smallness of the action via the largeness of the free locus of the action, see (14.1). The author does not know how to redefine the smallness using the non-existence of pseudoreflections for general finite subgroup schemes of GL_n, see Remark 17.15.

We call a group scheme $h : G \to S$ on a scheme S is *locally finite free* (LFF for short) if it is finite and the structure sheaf $h_*\mathcal{O}_G$ is a locally free sheaf on S. We work on LFF group schemes (as a generalization of finite group schemes over a field), and prove that an algebraic quotient $\varphi : X = \operatorname{Spec} B \to Y = \operatorname{Spec} B^G$ is an almost principal G-bundle if and only if the action is small (Proposition 14.8). Thus we know that the action of a finite group G without pseudoreflection on an affine variety yields an almost principal G-bundle, as we have already mentioned. This fact is also useful in finding examples of (rational) almost principal bundles with respect to non-reduced finite group schemes in (0.16).

As an application, assuming that X satisfies the (S_2) condition, we give a characterization of an algebraic quotient $\varphi : X \to Y$ by the action of an LFF group scheme G such that Y is (connected Noetherian with a dualizing complex and) quasi-Gorenstein (Theorem 14.24, **6**). If, moreover, G is étale; abelian group scheme over a field; or a linearly reductive group scheme over a field (more generally, a Reynolds group scheme over S, see below), then we have a very simple relationship: $\omega_Y \cong (\varphi_*\omega_X)^G$. If, moreover, X is normal, then we have $\omega_X \cong (\varphi^*\omega_Y)^{**}$ (Theorem 14.24, **4**). When the group scheme is étale, there are considerable overlaps with the results of [Pes], [Bro], [Bra], and [FlW].

Also, well-known formula for the class group of A is generalized to the action of non-reduced finite group schemes, see Example 14.28.

We point out that if the base scheme S, the group scheme G, and the scheme $X = \operatorname{Spec} B$ are affine, to say that $\varphi : X = \operatorname{Spec} B \to \operatorname{Spec} B^G$ is an almost principal G bundle is the same as to say that $B^G \to B$ is a pseudo-Galois extension in the sense of Waterhouse [Wat] by definition. His study on the class group is applicable to finite group schemes also, and our work has many overlaps with his.

(0.21) Next example is a rational almost principal G-bundle arising from a sequence of divisors $D_1, \ldots, D_s$ over a Noetherian normal variety (more generally, a quasi-compact quasi-separated locally Krull integral scheme) Y, where G is the torus $\mathbb{G}_m^s$, and we assume that $\sum_i \mathbb{Z}D_i$ contains an ample Cartier divisor. Let $j : V = Y_{\mathrm{reg}} \hookrightarrow Y$ be the regular locus of Y (for simplicity, assume that Y is Noetherian), and let

$$\rho : U = \underline{\mathrm{Spec}}_V(\bigoplus_{\lambda \in \mathbb{Z}^s} \mathcal{O}_Y(\sum_i \lambda_i D_i)|_V \cdot t^\lambda) \to V$$

be the canonical map. Also, let

$$X := \operatorname{Spec} \Gamma(U, \mathcal{O}_U) = \operatorname{Spec}(\bigoplus_{\lambda \in \mathbb{Z}^s} \Gamma(Y, \mathcal{O}_Y(\sum_i \lambda_i D_i)) \cdot t^\lambda),$$

and $i : U \to X$ be the canonical map. Then

$$X \xleftarrow{\ i\ } U \xrightarrow{\ \rho\ } V \xhookrightarrow{\ j\ } Y$$

is a rational almost principal G-bundle (Theorem 15.28). No map is defined from X to Y here, and this gives an example of a rational almost principal bundle which is not an almost principal bundle. This construction already essentially appeared in [HasK] without the formal definition of rational almost principal bundles. By our main theorem (Theorem 11.2), we get an equivalence between the categories $\mathrm{Ref}(Y)$ and $\mathrm{Ref}(G, X)$ in an explicit way (Corollary 15.29). Also, the description of the canonical module of a multisection ring (where Y is a projective normal variety over a field) in [HasK] is generalized to a result on Noetherian normal integral schemes (Proposition 15.33). A part of the results on the class group of the multisection ring in [EKW] is also reproved as a theorem on locally Krull schemes (Proposition 15.32).

Let $\Lambda = \mathbb{Z}^s$ so that $G = \operatorname{Spec} \mathbb{Z}\Lambda \times_{\operatorname{Spec} \mathbb{Z}} S$. Let Γ be a subgroup of Λ, and set $H = \operatorname{Spec} \mathbb{Z}\Gamma \times_{\operatorname{Spec} \mathbb{Z}} S$, and $f : G \to H$ the canonical map. Then $N := \operatorname{Ker} f$ is nothing but $\operatorname{Spec} \mathbb{Z}(\Lambda/\Gamma) \times_{\operatorname{Spec} \mathbb{Z}} S$. Let B be the multisection ring $\bigoplus_{\lambda \in \Lambda} \Gamma(Y, \mathcal{O}_Y(\sum_i \lambda_i D_i)) \cdot t^\lambda$. Then B^N is the Veronese subring $B_\Gamma = \bigoplus_{\lambda \in \Gamma} \Gamma(Y, \mathcal{O}_Y(\sum_i \lambda_i D_i)) \cdot t^\lambda$. We show that the canonical algebraic quotient $\theta : X = \operatorname{Spec} B \to \operatorname{Spec} B^N = X'$ is a G-enriched almost principal N-bundle (Lemma 15.36). Consequently, we can prove some results which connect X and X'.

(0.22) When we apply the construction explained in (0.21) to the Cox ring of a toric variety (with a torsion-free class group), then we get some basic information on toric varieties, such as the description of the canonical module (Corollary 16.2), and the global F-regularity (Proposition 16.5). Also, we prove that for a toric variety Y over a perfect field k, there exist finitely many equivariant rank-one reflexive modules $\mathcal{M}_1, \ldots, \mathcal{M}_u$ on Y (equivariant with respect to the torus action) such that any Frobenius pushforward $F_*^e({}^e\mathcal{O}_Y)$ is a finite direct sum of copies of them, as $\mathcal{O}_Y$-modules, generalizing the theorem of Thomsen [Tho] on non-singular toric varieties. This has been known also for affine toric varieties [Bru2].

(0.23) Although we can construct a rational almost principal bundle from a set of divisors on a normal variety, it seems difficult to find a rational almost principal bundle from a given multigraded ring B. But this is relatively easy when B is *surjectively graded*. This notion first appeared in [Has3] for the case that B is a domain. We modify this to a usable definition for the case that B is not a domain, and give an easy way to get many rational almost principal bundles from multigraded rings (Lemma 17.6).

The most typical example is a standard graded algebra $B = \bigoplus_{n \geq 0} B_n$ with $\dim B \geq 2$. Then letting $X = \operatorname{Spec} B$, $U = X \setminus 0$, and $Y = \operatorname{Proj} B$, we get a rational almost principal bundle

$$X \xleftarrow{i} U \xrightarrow{\rho} Y \xrightarrow{1_Y} Y ,$$

where 0 is the origin of the affine cone X. From this construction, we get a very short proof of Grothendieck's theorem which tells that any locally free sheaf on $\mathbb{P}^1$ is uniquely a direct sum of $\mathcal{O}(n)$ (Example 17.10). This is simply because $\operatorname{Ref}(\mathbb{P}^1)$ and $\operatorname{Ref}(\mathbb{G}_m, k[x,y])$ are equivalent by our main theorem.

(0.24) There are also examples where the group scheme G is not a torus or a finite group scheme. We point out that determinantal and Pfaffian varieties yield examples of almost principal G-bundles where G is a connected reductive group which are not finite (section 18).

Given a G-algebra Krull domain B and a candidate Krull domain $A \subset B^G$ of B^G, it is sufficient to prove that $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ is an almost principal G-bundle in order to show $A = B^G$ (Theorem 10.13). Thus, proving that $A = B^G$ is reduced to proving that A is a Krull domain, when we know geometric information that φ is an almost principal bundle. This technique essentially appeared in [Has4], and applied to the same examples.

An example of an action of the additive group $\mathbb{G}_a$ is also given (Example 10.14).

(0.25) Some miscellaneous problems are also discussed in this paper, in order to overcome technical difficulties to discuss main ingredients. Most of them are contained in Chapter 0 (sections 1–9). We will see them below.

(0.26) We overview the contents of this paper section by section.

Chapter 0 (sections 1–9) is preliminaries.

In section 1, we review some basic definitions and facts on quotients. In section 2, we discuss the compatibility of the invariance functor and some other operations on sheaves. In section 3, we discuss the problem of functorial resolutions. In section 4, we discuss the compatibility of the restriction functor and some other operations on sheaves. In section 5, generalizing linearly reductive group schemes over a field, we define Reynolds group schemes and discuss basic properties. This class of group schemes includes linearly reductive group schemes over a field, finite groups with the order invertible in the base ring, and diagonalizable group schemes (including split tori) over an arbitrary base ring (Examples 5.15–5.17). In section 6, we discuss the base-change map of twisted inverse pseudofunctors. In section 7, we discuss the (equivariant) canonical modules. As a generalization of a special case of Knop's work over a field of characteristic zero, we give the correspondence between ω_X and ω_Y for a principal G-bundle $X \to Y$. Also, generalizing the facts on the category of reflexive sheaves on normal varieties, we show that the category of sheaves $\mathcal{M}$ which satisfy the (S_2') condition (that is, $\operatorname{depth} \mathcal{M}_z \geq \min(2, \dim \mathcal{O}_{Z,z})$ for $z \in Z$) behaves very similarly on quasi-normal Noetherian schemes Z, and we show that codimension-two argument works. We generalize [Hart4, (1.12)]. On the way, we generalize the well-knwon result on the equivalence of (S_2'), reflexive, and being a second syzygy due to Evans and Griffith [EvG, Theorem 3.6], using the new notion of 2-canonical modules (Lemma 7.28). Recently, similar results in slightly different contexts are obtained by Dibaei–Sadeghi [DiS] and Araya–Iima [ArI]. In section 8, we define a new category to treat Frobenius twists and Frobenius kernels effectively. This enables us to discuss the Frobenius kernels of a group scheme over an arbitrary $\mathbb{F}_p$-scheme. In section 9, we discuss when B is finite over B^G, and when B^G is F-finite for an affine algebraic group scheme G over a field k of characteristic $p > 0$ and a G-algebra B. The main result is Lemma 9.6. If G is a constant finite group, then the lemma is a special case of [Fog, Theorem].

(0.27) After these preliminaries, in Chapter 1 (sections 10–13), we do the main definitions and discuss general properties of rational almost principal bundles.

In section 10, we give main definitions, and discuss the problem of base change. We also prove Theorem 10.13 mentioned above. In section 11, we discuss the behavior of the (equivariant) class group, using the results obtained in [Has9] and [Has11]. We also discuss the behavior of the canonical modules, using the results in sections 6 and 7. We discuss the behavior of the Frobenius pushforwards with respect to rational almost principal bundles in section 12. The author expects some future applications on the problems in invariant theory related to the characteristic p commutative algebra. The paper [HasN] is a trial toward this direction. Section 13 is on the global F-regularity. As our construction uses open subschemes, we discuss global F-regularity of schemes which may not be projective.

(0.28) After proving general results, we give examples and applications of rational almost principal bundles in Chapter 2 (sections 14–18). In section 14, we discuss finite group schemes (more precisely, LFF group schemes). In particular, we prove a similar result to the results on the canonical modules for the finite group actions due to Broer [Bro] and Fleischmann–Woodcock [FlW]. In section 15, we construct a rational almost principal bundle from a sequence of divisors on a locally Krull scheme, and prove a generalization of the theorem of Kurano and the author which describes the canonical module of the multisection ring. As an application, we prove some known and new results on toric varieties, using the Cox ring in section 16. In section 17, we give a way to construct rational almost principal bundles from a multigraded rings. This enables us to study the Veronese subring using our approach. In section 18, we show that determinantal and Pfaffian varieties treated by De Concini and Procesi [DeCP] give examples of almost principal bundles.

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Chapter 0. Preliminaries

1. Actions and quotients

(1.1) This paper is a continuation of [Has9] and [Has11]. We follow the notation and terminology of these papers. In particular, for the notation and terminology on sheaves over diagrams of schemes and equivariant modules, we follow [Has5], [HasO], and [HasO2], unless otherwise specified. Unexplained notation and terminologies on commutative algebra, algebraic geometry, algebraic groups, representations of algebraic groups, and Hopf algebras that are not in these should be found in [Mat], [Hart3], [Gro], [Bor], [Jan], [Swe], or [Has]. Throughout this paper, S denotes a (base) scheme.

(1.2) Let G be an S-group scheme, and $\varphi : X \to Y$ a G-invariant morphism. The *secondary map* associated with G and φ is the map

$$\Psi = \Psi_{G,\varphi} : G \times X \to X \times_Y X$$

given by $(g, x) \mapsto (gx, x)$. This map is independent of the choice of S in the sense that when we replace S by Y and G by $G_Y = G \times Y$, then we get the same map (over the base scheme Y). If $h : Y' \to Y$ is an S-morphism between S-schemes with trivial G-actions, then $\Psi_{G,\varphi'} : G \times X' \to X' \times_{Y'} X'$ is identified with $1_{Y'} \times \Psi_{G,\varphi} : Y' \times_Y (G \times X') \to Y' \times_Y (X \times_Y X)$.

(1.3) Let G be an S-group scheme. A G-invariant morphism $\varphi : X \to Y$ is said to be a *categorical quotient* if for any G-invariant morphism $\psi : X \to Z$, there exists some unique S-morphism $\theta : Y \to Z$ such that $\psi = \theta\varphi$. The categorical quotient is unique (in the category of G-schemes under X).

(1.4) A G-invariant morphism $\varphi : X \to Y$ is said to be an *algebraic quotient* or *affine quotient* by the action of G if it is an affine morphism, and $\bar{\eta} : \mathcal{O}_Y \to (\varphi_* \mathcal{O}_Y)^G$ is an isomorphism. An algebraic quotient need not be surjective. It need not be a categorical quotient either in general, see Example 10.14 below. However, if $S = \operatorname{Spec} k$ is a field and G is a semireductive k-group scheme (see (9.3) below), it is a categorical quotient (see Lemma 9.5).

(1.5) A morphism of schemes $h : Z \to W$ is said to be submersive if h is surjective, and for any subset U of W, U is open if and only if $h^{-1}(U)$ is open in Z. A G-invariant morphism $\varphi : X \to Y$ is said to be a *geometric quotient* if it is submersive, $\mathcal{O}_Y \to (\varphi_* \mathcal{O}_X)^G$ is an isomorphism, and $\Psi_{G,\varphi}$ is surjective. A geometric quotient is a categorical quotient [MuFK, (0.0.1)]. By definition, an affine geometric quotient is an algebraic quotient. However, a geometric quotient need not be an algebraic quotient in general. For example, let $G = SL_n$ with $n \geq 2$ over an algebraically closed field k, and consider the structure map $\varphi : X = G/B \to \operatorname{Spec} k = Y$, where B is the subgroup of the upper triangular matrices in G. It is a geometric quotient by G, but is not an affine morphism, since G/B is a projective variety of dimension one or more [Bor, (11.1)]. So in particular, we have an example of a categorical quotient which is not an affine morphism.

(1.6) We say that $\varphi : X \to Y$ is a universal (resp. uniform) categorical quotient by G if for any S-morphism (resp. any flat S-morphism) $Y' \to Y$, the base change $\varphi' : X' \to Y'$ is a categorical quotient by G. A similar definition is done for algebraic and geometric quotients. An algebraic quotient is uniform under very mild conditions, see Corollary 2.22 below.

(1.7) Let G be an S-group scheme acting on an S-scheme X. Let us consider $\Psi = \Psi_{G,h_X} : G \times X \to X \times X$, where $h_X : X \to S$ is the structure map. It is easy to see that $\phi : \Psi^{-1}(X) \to X$ induced by Ψ is an X-subgroup scheme of $G \times X$, where X is embedded in $X \times X$ via the diagonal map (if h_X is separated, then it is a closed subgroup). $\mathcal{S}_X := \Psi^{-1}(X)$ is called the *stabilizer* of the action of G on X. If $\mathcal{S}_X$ is trivial as an X-group scheme, then we say that the action of G on X is *free*. We say that the action of G on X is *GIT-free* if Ψ is a closed immersion. Obviously, a GIT-free action is free.

Lemma 1.8. *Let G be an S-group scheme, and $\psi : X' \to X$ be a G-morphism which is a morphism of schemes. Then there is an inclusion $Q : \mathcal{S}_{X'} \to \mathcal{S}_X \times_X X'$ of X'-subgroup schemes of $G \times X'$. In particular, if the action of G on X is free, then so is the action of G on X'. If ψ is a monomorphism (e.g., an immersion), then Q is an isomorphism.*

Proof. Note that both $\mathcal{S}_{X'}$ and $\mathcal{S}_X \times_X X'$ are X'-subgroup schemes of $G \times X'$. For an S-scheme W,

$$\mathcal{S}_{X'}(W) = \{(g, x') \in G(W) \times X'(W) \mid gx' = x'\}$$

and

$$(\mathcal{S}_X \times_X X')(W) = \{(g, x') \in G(W) \times X'(W) \mid \psi(gx') = \psi(x')\}.$$

So $\mathcal{S}_X \times_X X'$ contains $\mathcal{S}_{X'}$.

If the action of G on X is free, then $\mathcal{S}_X$ is trivial. So $\mathcal{S}_{X'}$ is also trivial by the discussion above, and the action of G on X' is free. The argument above also shows that if ψ is a monomorphism, then $\mathcal{S}_{X'} = \mathcal{S}_X \times_X X'$. $\square$

(1.9) Let G be an S-group scheme acting on X, and $\psi : X' \to X$ any monomorphism of S-schemes (X' need not be a G-scheme). Then we define the stabilizer at X' of the action of G on X by the X'-group scheme $\mathcal{S}_{X'} := \mathcal{S}_X \times_X X'$. This definition does not cause a confusion by Lemma 1.8. If x is a point of X, the stabilizer $\mathcal{S}_x$ is a $\kappa(x)$-subgroup scheme of $G \times x$.

(1.10) A G-invariant morphism $\varphi : X \to Y$ is a principal G-bundle if and only if it is qfpqc and $\Psi_{G,\varphi}$ is an isomorphism [Has11, (2.8)]. A principal G-bundle is a universal geometric quotient [Has11, (6.2)]. If, moreover, G is a normal subgroup scheme of an S-group scheme $\tilde{G}$ and φ is also a $\tilde{G}$-morphism, then we say that φ is *$\tilde{G}$-enriched*.

Lemma 1.11. *Let G be an S-group scheme, and $\varphi : X \to Y$ a G-invariant submersive* (*resp. universally submersive*) *morphism such that $\Psi : G \times X \to X \times_Y X$ given by $\Psi(g, x) = (gx, x)$ is surjective. If U is a G-stable open subset of X, then $\varphi(U)$ is an open subset, and $\varphi^{-1}(\varphi(U)) = U$. If, moreover, G is universally open, then φ is open* (*resp. universally open*).

Proof. This is essentially [MuFK, (0.2), Remark (4)]. $\square$

2. Compatibility of G-invariance and direct and inverse images

(2.1) Let G be an S-group scheme and Z an S-scheme on which G acts trivially. Set $(?)^G = (?)_{-1} R_{\Delta_M} : \mathrm{Mod}(G, Z) \to \mathrm{Mod}(Z)$, and $\mathcal{L} = (?)_{\Delta_M} L_{-1} : \mathrm{Mod}(Z) \to \mathrm{Mod}(G, Z)$. Note that $\mathcal{L}$ is left adjoint to $(?)^G$.

Using the description of R_{Δ_M} [Has5, (6.14)], $(?)^G \mathcal{M} = \mathcal{M}^G$ is the kernel of the map

$$\mathcal{M}_0 \xrightarrow{\beta_{\delta_0} - \beta_{\delta_1}} p_* \mathcal{M}_1,$$

where $p : G \times Z \to Z$ is the second projection, which equals the action (because the action is assumed to be trivial), and $\delta_i : [0] = \{0\} \to [1] = \{0, 1\}$

is the map given by $\delta_i(0) = 1 - i$ (for the notation on simplicial objects, see [Has5, Chapter 9]). We call $\mathcal{M}^G \in \mathrm{Mod}(Z)$ the *G-invariance* of $\mathcal{M}$ [Has5, (30.1)], [Has11, (5.30)]. The natural inclusion

$$\mathcal{M}^G \hookrightarrow \mathcal{M}_0$$

is denoted by γ.

Lemma 2.2. *For $\mathcal{M} \in \mathrm{Mod}(G, Z)$, the following are equivalent.*

1 *$\mathcal{M} \cong \mathcal{L}\mathcal{N}$ for some $\mathcal{N} \in \mathrm{Mod}(Z)$.*

2 *$\mathcal{M}$ is equivariant, and $\beta_{\delta_0} = \beta_{\delta_1}$.*

3 *$\mathcal{M}$ is G-trivial, that is, $\mathcal{M}$ is equivariant, and $\gamma : \mathcal{M}^G \to \mathcal{M}_0$ is an isomorphism* (*see* [Has5, (30.4)])*.*

4 *The counit of adjunction $\varepsilon : \mathcal{L}\mathcal{M}^G \to \mathcal{M}$ is an isomorphism.*

Proof. **1⇒2**. Since $[-1]$ is the initial object of Δ_M^+, $\mathcal{M} \cong \mathcal{L}\mathcal{N}$ is equivariant by [Has5, (6.38)]. By definition, $(\mathcal{L}\mathcal{N})_n = (\tilde{B}_G^M(Z))^*_{\varepsilon(n)}\mathcal{N}$, where $\varepsilon(n) : [-1] = \emptyset \to [n]$ is the unique map. The map $\beta_i : (\mathcal{L}\mathcal{N})_0 = \mathcal{N} \to p_*(\mathcal{L}\mathcal{N})_1 = p_*p^*\mathcal{N}$ is the unit map, and is independent of i.

2⇔3 is trivial.

3⇒4. By **1⇒2**, $\mathcal{L}\mathcal{M}^G$ is equivariant, and $\mathcal{M}$ is assumed to be equivariant. Hence it suffices to prove that $\varepsilon_0 : (\mathcal{L}\mathcal{M}^G)_0 \to \mathcal{M}_0$ is an isomorphism, since the restriction $(?)_0 : \mathrm{EM}(G, Z) \to \mathrm{Mod}(Z)$ is faithful. However, this map is identified with $\gamma : \mathcal{M}^G \to \mathcal{M}_0$.

4⇒1. This is trivial. □

(2.3) Let G be an S-group scheme and $h : Z' \to Z$ be a morphism of S-schemes on which G acts trivially. Then the canonical map

$$\epsilon : h^*\mathcal{M}^G \to (h^*\mathcal{M})^G$$

is defined to be the composite

$$\begin{aligned}\epsilon : h^*\mathcal{M}^G = h^*(?)_{-1}R_{\Delta_M}\mathcal{M} &\xrightarrow{\theta} (?)_{-1}\tilde{B}_G^M(h)^*R_{\Delta_M}\mathcal{M} \\ &\xrightarrow{\mu} (?)_{-1}R_{\Delta_M}B_G^M(h)^*\mathcal{M} = (?)^G B_G^M(h)^*\mathcal{M} = (h^*\mathcal{M})^G,\end{aligned}$$

see [HasO, (7.4)]. It is an isomorphism between functors from $\mathrm{Lqc}(G, Z)$ to $\mathrm{Qch}(Z')$ if G is quasi-compact quasi-separated and h is flat [HasO, (7.5)] (the flatness of G is assumed there, but that assumption is unnecessary). Note that as in [HasM, (2.18)], $B_G^M(h)^*\mathcal{M}$ is abbreviated as $h^*\mathcal{M}$, by abuse of notation (although $\mathcal{M}$ may not be quasi-coherent here). Note that ϵ is a natural transformation between the functors from $\mathrm{Mod}(G, Z)$ to $\mathrm{Mod}(Z')$.

Lemma 2.4. *The diagram*

$$\begin{tikzcd} h^*(?)^G \arrow[r, "\gamma"] \arrow[d, "\epsilon"] & h^*(?)_0 \arrow[d, "\theta"] \\ (?)^G h^* \arrow[r, "\gamma"] & (?)_0 h^* \end{tikzcd}$$

is commutative.

Proof. Follows easily from the commutative diagram in [Has5, (6.27)]. □

Corollary 2.5. *Let $h' : Z'' \to Z'$ and $h : Z' \to Z$ be a sequence of G-morphisms. Then the composite*

$$(hh')^*(?)^G \xrightarrow{d^{-1}} (h')^*h^*(?)^G \xrightarrow{\epsilon} (h')^*(?)^G h^* \xrightarrow{\epsilon} (?)^G(h')^*h^* \xrightarrow{d} (?)^G(hh')^G$$

agrees with ϵ.

Proof. This follows easily from Lemma 2.4 and [Has5, (1.23)]. □

(2.6) Let $\varphi : X \to Y$ and $\psi : Y \to Z$ be morphisms of ringed sites. Then the two functors $(\psi\varphi)_*$ and $\psi_*\varphi_*$ are equal, and the standard natural isomorphism $c : (\psi\varphi)_* \to \psi_*\varphi_*$ is nothing but the identity map.

(2.7) Let I be a small category, and X an I^{op}-diagrams of schemes. For $\mathcal{M} \in \mathrm{Mod}(X)$ and $i \in I$, $\mathcal{M}_i \in \mathrm{Mod}(X_i)$. For each $\phi \in I(i, j)$, $\beta_\phi : \mathcal{M}_i \to (X_\phi)_*\mathcal{M}_j$ is induced, and the composite

$$\mathcal{M}_i \xrightarrow{\beta_\phi} (X_\phi)_*\mathcal{M}_j \xrightarrow{\beta_\psi} (X_\phi)_*(X_\psi)_*\mathcal{M}_k \xrightarrow{c} (X_{\psi\phi})_*\mathcal{M}_k \tag{2}$$

agrees with $\beta_{\psi\phi}$ for any sequence of morphisms

$$i \xrightarrow{\phi} j \xrightarrow{\psi} k \tag{3}$$

in I, see [Has5, (4.10)]. We call the collection $((\mathcal{M}_i)_{i\in I}, (\beta_\phi)_{\phi\in\mathrm{Mor}(I)})$ the *structure data* of $\mathcal{M}$. This data exactly determines $\mathcal{M}$ (not up to isomorphisms).

Conversely, if $\mathcal{M}_i \in \mathrm{Mod}(X_i)$ for $i \in I$, $\beta_\phi : \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}_i, (X_\phi)_*\mathcal{M}_j)$ for $\phi : i \to j$, and the composite (2) agrees with $\beta_{\psi\phi}$ for (3), then it is a structure data for a unique $\mathcal{M} \in \mathrm{Mod}(X)$.

(2.8) Let $f : \mathcal{M} \to \mathcal{N}$ be a morphism in $\mathrm{Mod}(X)$. Then $f_i : \mathcal{M}_i \to \mathcal{N}_i$ is a morphism in $\mathrm{Mod}(X_i)$ such that $\beta_\phi \circ f_i = (X_\phi)_* f_j \circ \beta_\phi$ for each $\phi : i \to j$. Conversely, such a collection (f_i) gives a unique morphism $f : \mathcal{M} \to \mathcal{N}$. We call (f_i) the structure data of f.

(2.9) Let $\varphi : X \to Y$ be a morphism of I^{op}-diagrams of schemes, and $\mathcal{M} \in \mathrm{Mod}(X)$. Then the structure data of $\varphi_*\mathcal{M}$ is as follows. $(\varphi_*\mathcal{M})_i = (\varphi_i)_*\mathcal{M}_i$, and $\beta_\phi(\varphi_*\mathcal{M})$ is the composite

$$(\varphi_i)_*\mathcal{M}_i \xrightarrow{\beta_\phi} (\varphi_i)_*(X_\phi)_*\mathcal{M}_j \xrightarrow{c} (Y_\phi)_*(\varphi_j)_*\mathcal{M}_j,$$

as can be seen easily from the direct computation (note that c is the identity map).

(2.10) Let I be a small category, X be an I^{op}-diagram of schemes, J be a subcategory of I, and $\mathcal{N} \in \mathrm{Mod}(X_J)$. Then $R_J\mathcal{N} \in \mathrm{Mod}(X)$ is given by its structure data. $(R_J\mathcal{N})_i = \varprojlim (X_\phi)_*\mathcal{N}_j$, where the limit is taken over the comma category $(i \downarrow J)$, see [Mac, (II.6)] (it is I_i^J in [Has5]). Here, for a morphism $\psi : j \to j'$ in $(i \downarrow J)$ from $\phi : i \to j$ to $\psi\phi$, the map

$$(X_\phi)_*\mathcal{N}_j \xrightarrow{\beta_\psi} (X_\phi)_*(X_\psi)_*\mathcal{N}_{j'} \xrightarrow{c} (X_{\psi\phi})_*\mathcal{N}_{j'}$$

is the structure map.

For a morphism $\phi : i \to i'$ in I, $\beta_\phi : (R_J\mathcal{N})_i \to (X_\phi)_*(R_J\mathcal{N})_{i'}$ is given by

$$\varprojlim_{\psi\in(i\downarrow J)} (X_\psi)_*\mathcal{N}_j \to (X_{\psi'\phi})_*\mathcal{N}_{j'} \xrightarrow{c} (X_\phi)_*(X_{\psi'})_*\mathcal{N}_{j'}$$

for an object $\psi' : i' \to j'$ in $(i' \downarrow J)$.

(2.11) As in [Has5, Chapter 5,6], R_J is right adjoint to the restriction functor $(?)_J$. The unit of adjunction $u : \mathrm{Id} \to R_J(?)_J$ is given by the map $\mathcal{M}_i \to \varprojlim_{\phi\in(i\downarrow J)}(X_\phi)_*\mathcal{M}_j$ induced by $\beta_\phi : \mathcal{M}_i \to (X_\phi)_*\mathcal{M}_j$ for each j. The counit of adjunction $\varepsilon : (?)_J R_J \to \mathrm{Id}$ is the projection $\varprojlim_{(\psi:j\to j')\in(j\downarrow J)}(X_\psi)_*\mathcal{M}_{j'} \to \mathcal{M}_j = (X_{\mathrm{id}_j})_*\mathcal{M}_j$ (thus if J is a full subcategory, then ε is an isomorphism [Has5, (6.15)]).

(2.12) For a morphism $f : X \to Y$ and $J \subset I$, we have that $(?)_J f_*$ and $(f_J)_*(?)_J$ are identical, and $c : (?)_J f_* \to (f_J)_*(?)_J$ is the identity.

(2.13) Combining these, it is easy to describe the canonical isomorphism

$$\xi : f_* R_J \to R_J(f_J)_*$$

(see [Has5, (6.26)]) via the structure data.

$$\xi_i : (f_* R_J \mathcal{N})_i \to (R_J(f_J)_*\mathcal{N})_i$$

is given by

$$(f_i)_* \varprojlim (X_\phi)_*\mathcal{N}_j \cong \varprojlim (f_i)_*(X_\phi)_*\mathcal{N}_j \xrightarrow{c} \varprojlim (Y_\phi)_*(f_j)_*\mathcal{N}_j.$$

(2.14) Let G be an S-group scheme, and $h : Z' \to Z$ an S-morphism between S-schemes on which G acts trivially. We denote the composite isomorphism

$$h_*(?)^G = h_*(?)_{-1}R_{\Delta_M} \xrightarrow{c^{-1}} (?)_{-1}\tilde{B}_G^M(h)_* R_{\Delta_M} \xrightarrow{\xi} (?)_{-1}R_{\Delta_M}B_G^M(h)_* = (?)^G h_*$$

by $e : h_*(?)^G \to (?)^G h_*$ as in [HasO, (7.3)]. (here $B_G^M(h)_*$ is abbreviated to be h_*, by abuse of notation). By (2.13) and the fact that c is the identity, we have that e is nothing but the canonical isomorphism from $h_* \operatorname{Ker}\gamma \to \operatorname{Ker} h_*\gamma$. In particular,

Lemma 2.15. *Let the notation be as in* (2.14)*. Then the diagram*

$$\begin{tikzcd} h_*(?)^G \arrow[r, "\gamma"] \arrow[d, "e"] & h_*(?)_0 \arrow[d, "c^{-1}"] \\ (?)^G h_* \arrow[r, "\gamma"] & (?)_0 h_* \end{tikzcd}$$

is commutative. □

Corollary 2.16. *Let $h' : Z'' \to Z'$ and $h : Z' \to Z$ be a sequence of G-morphisms. Then the composite*

$$(hh')_*(?)^G \xrightarrow{c} h_*h'_*(?)^G \xrightarrow{e} h_*(?)^G h'_* \xrightarrow{e} (?)^G h_* h'_* \xrightarrow{c^{-1}} (?)^G(hh')_*$$

agrees with e. □

Lemma 2.17. *Let I be a small category, J its subcategory, $f : X \to Y$ a morphism of I^{op}-diagrams of schemes. Then the composite*

$$R_J \xrightarrow{u} f_* f^* R_J \xrightarrow{\mu} f_* R_J f_J^* \xrightarrow{\xi} R_J (f_J)_* f_J^*$$

is the unit map u.

Proof. Follows easily from the commutativity of the diagram

$$\begin{tikzcd}
R_J \arrow[r, "u"] \arrow[d, "u"] & R_J(f_J)_* f_J^* \arrow[r, "\xi^{-1}"] \arrow[d, "u"] & f_* R_J f_J^* \arrow[d, "u"] \arrow[dd, bend left, "\mathrm{id}"] \\
f_* f^* R_J \arrow[r, "u"] \arrow[drr, bend right, "\mu"] & f_* f^* R_J (f_J)_* f_J^* \arrow[r, "\xi^{-1}"] & f_* f^* f_* R_J f_J^* \arrow[d, "\varepsilon"] \\
& & f_* R_J f_J^*
\end{tikzcd} .$$

□

Lemma 2.18. *Let G be an S-group scheme, and $h : Z' \to Z$ an S-morphism between S-schemes on which G acts trivially. Then the composite*

$$(?)^G \xrightarrow{u} h_* h^* (?)^G \xrightarrow{\epsilon} h_* (?)^G h^* \xrightarrow{e} (?)^G h_* h^*$$

is the unit map u.

Proof. Follows easily from Lemma 2.17 and [Has5, (1.24)]. □

Lemma 2.19. *Let $\mathcal{S}$ be a category, $(?)_*$ be a covariant symmetric monoidal almost pseudofunctor on $\mathcal{S}$* [Has5, (1.28)], *and $(?)^*$ its left adjoint. Let*

(4)
$$\begin{tikzcd}
X' \arrow[r, "g"] \arrow[d, "\varphi'"] & X \arrow[d, "\varphi"] \\
Y' \arrow[r, "h"] & Y
\end{tikzcd}$$

be a commutative diagram on $\mathcal{S}$. Then the diagram

$$\begin{tikzcd}
h^* \mathcal{O}_Y \arrow[rr, "C"] \arrow[d, "\eta"] & & \mathcal{O}_{Y'} \arrow[d, "\eta"] \\
h^* \varphi_* \mathcal{O}_X \arrow[r, "\theta"] & \varphi'_* g^* \mathcal{O}_X \arrow[r, "C"] & \varphi'_* \mathcal{O}_{X'}
\end{tikzcd}$$

is commutative, where we use the notation in [Has5, Chapter 1].

Proof. Prove the commutativity of the diagram

$$\begin{tikzcd}
h^*\mathcal{O}_Y \arrow[rrrr] \arrow[dr, "\eta"] \arrow[dd, "C"'] & & & & h^*\varphi_*\mathcal{O}_X . \arrow[dl, "\eta"'] \arrow[dd, "\theta"] \\
& h^*h_*\mathcal{O}_Y \arrow[r, "\eta"] \arrow[dl, "\varepsilon"'] \arrow[d, "\eta"] & & h^*\varphi_*g_*\mathcal{O}'_X \arrow[dll, "c"'] \arrow[d, "\theta"] & \\
\mathcal{O}_{Y'} \arrow[drrr, bend right, "\eta"'] & h^*h_*\varphi'_*\mathcal{O}_{X'} \arrow[drr, "\varepsilon"] & & \varphi'_*g^*g_*\mathcal{O}_{X'} \arrow[d, "\varepsilon"] & \varphi'_*g^*\mathcal{O}_X \arrow[l, "\eta"'] \arrow[dl, bend left, "C"] \\
& & & \varphi'\mathcal{O}_{X'} &
\end{tikzcd}$$

The details are left to the reader. ☐

(2.20) Let G be an S-group scheme, and $\varphi : X \to Y$ a G-invariant morphism. Then the canonical map $\bar{\eta} : \mathcal{O}_Y \to (\varphi_*\mathcal{O}_X)^G$ is nothing but the composite

$$\mathcal{O}_Y \xrightarrow{\gamma^{-1}} \mathcal{O}_Y^G \xrightarrow{\eta} (\varphi_*\mathcal{O}_X)^G,$$

where $\gamma : \mathcal{O}_Y^G \to \mathcal{O}_Y$ is an isomorphism as $\mathcal{O}_Y$ is G-trivial, and $\eta : \mathcal{O}_Y \to \varphi_*\mathcal{O}_X$ is the standard map. As γ is a natural map, it is easy to see that the composite

$$\mathcal{O}_Y \xrightarrow{\bar{\eta}} (\varphi_*\mathcal{O}_X)^G \xrightarrow{\gamma} \varphi_*\mathcal{O}_X$$

is η.

Lemma 2.21. *Let G be an S-group scheme, and* (4) *a commutative diagram of G-schemes such that G acts on Y and Y' trivially. Then*

1 *The diagram*

(5)
$$\begin{tikzcd}
h^*\mathcal{O}_Y \arrow[rrr, "C"] \arrow[d, "\bar{\eta}"] & & & \mathcal{O}_{Y'} \arrow[d, "\bar{\eta}"] \\
h^*(\varphi_*\mathcal{O}_X)^G \arrow[r, "\epsilon"] & (h^*\varphi_*\mathcal{O}_X)^G \arrow[r, "\theta"] & (\varphi'_*g^*\mathcal{O}_X)^G \arrow[r, "C"] & (\varphi'_*\mathcal{O}_{X'})^G
\end{tikzcd}$$

is commutative.

2 *Assume that* (4) *is cartesian, φ is quasi-compact quasi-separated, h is flat, and $G \to S$ is quasi-compact quasi-separated. Then $\bar{\eta} : \mathcal{O}_{Y'} \to (\varphi'_*\mathcal{O}_{X'})^G$ is an isomorphism if and only if $h^*\bar{\eta} : h^*\mathcal{O}_Y \to h^*(\varphi_*\mathcal{O}_X)^G$ is an isomorphism.*

3 *In addition to the assumption of* **2**, *assume that $\eta : \mathcal{O}_Y \to h_*\mathcal{O}_{Y'}$ and $\eta : \mathcal{O}_X \to g_*\mathcal{O}_{X'}$ are isomorphisms. Then $\bar{\eta} : \mathcal{O}_Y \to (\varphi_*\mathcal{O}_X)^G$ is an isomorphism if and only if $\bar{\eta} : h_*\mathcal{O}_{Y'} \to h_*(\varphi'_*\mathcal{O}_{X'})^G$ is an isomorphism.*

Proof. **1**. By (2.4), (2.19), and (2.20), the diagram

$$\begin{tikzcd}
 & h^*\mathcal{O}_Y \arrow[rr, "C"] \arrow[dl, "\bar{\eta}"'] \arrow[d, "\eta"] & & \mathcal{O}_{Y'} \arrow[d, "\eta"] \arrow[dd, bend left=60, "\bar{\eta}"] \\
h^*(\varphi_*\mathcal{O}_X)^G \arrow[r, "\gamma"] \arrow[dr, "\epsilon"'] & h^*\varphi_*\mathcal{O}_X \arrow[r, "\theta"] & \varphi'_*g^*\mathcal{O}_X \arrow[r, "C"] & \varphi'_*\mathcal{O}_{X'} \\
 & (h^*\varphi_*\mathcal{O}_X)^G \arrow[r, "\theta"] \arrow[u, "\gamma"'] & (\varphi'_*g^*\mathcal{O}_X)^G \arrow[r, "C"] \arrow[u, "\gamma"'] & (\varphi'_*\mathcal{O}_{X'})^G \arrow[u, "\gamma"']
\end{tikzcd}$$

is commutative. As $\gamma : (\varphi'_*\mathcal{O}_{X'})^G \to \varphi'_*\mathcal{O}_{X'}$ is a monomorphism, the result follows.

2. Note that ϵ in (5) is an isomorphism by [HasO, (7.5)] (note that in [HasO, section 7], G is assumed to be flat, but this assumption is unnecessary in proving [HasO, (7.5)]). θ in (5) is also an isomorphism by [Has5, (7.12)]. As the two C's are isomorphisms, the result follows from **1**.

3 follows easily from **1**, the proof of **2**, Lemma 2.18, and [Has5, (1.24)]. □

Corollary 2.22. *Let G be a quasi-compact quasi-separated S-group scheme, and $\varphi : X \to Y$ an algebraic quotient by G. Then φ is a uniform algebraic quotient.*

Proof. Obvious by Lemma 2.21, **2**. □

Lemma 2.23. *Let G be a quasi-compact quasi-separated S-group scheme, X an S-scheme on which G acts trivially, and $\mathcal{M}$ a locally quasi-coherent $(G, \mathcal{O}_X)$-module. Then $\mathcal{M}^G \in \mathrm{Qch}(X)$. Moreover, $(?)^G : \mathrm{Mod}(G, X) \to \mathrm{Mod}(X)$ preserves direct sums.*

Proof. Let $p : G \times X \to X$ be the second projection. Then p_* preserves quasi-coherence. So if $\mathcal{M}$ is locally quasi-coherent, then the kernel $\mathcal{M}^G$ of $\mathcal{M}_0 \to p_*\mathcal{M}_1$ is quasi-coherent. Moreover, $p_* : \mathrm{Mod}(G \times X) \to \mathrm{Mod}(X)$ preserves the direct sums [Kem, Theorem 8]. As the kernel also preserves the direct sums, $(?)^G$ preserves the direct sums. □

(2.24) Let G be an S-group scheme, and X an S-scheme on which G acts trivially. Then for $\mathcal{M} \in \mathrm{Mod}(G, X)$, we have

$$\Gamma(X, \mathcal{M}^G) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, (?)_{-1} R_{\Delta_M} \mathcal{M}) \cong \\ \mathrm{Hom}_{\mathcal{O}_{B_G^M(X)}}(\mathcal{O}_{B_G^M(X)}, \mathcal{M}) = \Gamma(\mathrm{Zar}(B_G^M(X)), \mathcal{M}).$$

For $\mathcal{M}, \mathcal{N} \in \mathrm{Mod}(G, X)$, we denote $\underline{\mathrm{Hom}}_{\mathcal{O}_{B_G^M(X)}}(\mathcal{M}, \mathcal{N})$ by $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$, and $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})^G$ by $\underline{\mathrm{Hom}}_{G,\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$. In particular,

$$\Gamma(X, \underline{\mathrm{Hom}}_{G,\mathcal{O}_X}(\mathcal{M}, \mathcal{N})) \cong \Gamma(\mathrm{Zar}(B_G^M(X)), \underline{\mathrm{Hom}}_{\mathcal{O}_{B_G^M(X)}}(\mathcal{M}, \mathcal{N})) \\ = \mathrm{Hom}_{\mathcal{O}_{B_G^M(X)}}(\mathcal{M}, \mathcal{N}),$$

which we denote by $\mathrm{Hom}_{G,\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$.

3. Functorial resolutions

Lemma 3.1. *Let $\mathcal{A}$ be an abelian category, and assume that for each complex $\mathbb{F} \in C(\mathcal{A})$, a K-injective resolution $i_{\mathbb{F}} : \mathbb{F} \to \mathbb{I}_{\mathbb{F}}$ is chosen. Then there is a unique functor $\mathbb{I} : K(\mathcal{A}) \to K\text{-inj}(\mathcal{A})$, to the thick subcategory of K-injective objects, such that $\mathbb{I}(\mathbb{F}) = \mathbb{I}_{\mathbb{F}}$ for each $\mathbb{F}$, and that $i : \mathrm{Id} \to j\mathbb{I}$ is a natural transformation, where $j : K\text{-inj}(\mathcal{A}) \hookrightarrow K(\mathcal{A})$ is the inclusion.*

We call such a pair $(\mathbb{I}, i)$ of a functor and a natural map a *functorial K-injective resolution.* Note that the dual assertion of the lemma is the existence of a functorial K-projective resolution.

Proof. Let $h : \mathbb{F} \to \mathbb{G}$ be a morphism in $C(\mathcal{A})$. As $i_{\mathbb{F}}$ is a quasi-isomorphism and $\mathbb{I}_{\mathbb{G}}$ is K-injective,

$$i_{\mathbb{F}}^* : K(\mathbb{I}_{\mathbb{F}}, \mathbb{I}_{\mathbb{G}}) \to K(\mathbb{F}, \mathbb{I}_{\mathbb{G}})$$

is an isomorphism. So it is necessary to define $\mathbb{I}(h)$ to be $((i_{\mathbb{F}})^*)^{-1}(i_{\mathbb{G}} h)$ to make i a natural transformation, and the uniqueness follows.

The fact that $\mathbb{I}$ is a functor and i is a natural transformation with this definition is easy, and is left to the reader. $\square$

(3.2) Let $\mathcal{C}$ be a Grothendieck category. Then for each $\mathbb{F} \in C(\mathcal{C})$, the category of complexes in $\mathcal{C}$, there is an injective strictly injective resolution $i_{\mathbb{F}} : \mathbb{F} \to \mathbb{I}$ [Fra]. That is, $i_{\mathbb{F}}$ is a monomorphism and is a quasi-isomorphism, $\mathbb{I}_{\mathbb{F}}$ is K-injective, and $\mathbb{I}_{\mathbb{F}}^i$ is an injective object for each i. Thus we have

Lemma 3.3. *Let $\mathcal{C}$ be a Grothendieck category. Then there is a functorial K-injective resolution $i_{\mathbb{F}} : \mathbb{F} \to \mathbb{I}_{\mathbb{F}}$ for $\mathbb{F} \in K(\mathcal{C})$ which is a monomorphism for each $\mathbb{F}$.* $\square$

Lemma 3.4. *Let $\mathcal{C}$ be an abelian category, and $(\mathbb{I}, i)$ and $(\mathbb{I}', i')$ be functorial K-injective resolutions of $\mathcal{C}$. Then there is a unique natural isomorphism $\lambda : \mathbb{I} \to \mathbb{I}'$ such that $(j\lambda) \circ i = i'$.*

Proof. For each $\mathbb{F} \in K(\mathcal{C})$,

$$K(\mathcal{C})(\mathbb{I}\mathbb{F}, \mathbb{I}'\mathbb{F}) \xrightarrow{j} K(\mathcal{C})(j\mathbb{I}\mathbb{F}, j\mathbb{I}'\mathbb{F}) \xrightarrow{i^*} K(\mathcal{C})(\mathbb{F}, j\mathbb{I}'\mathbb{F})$$

are isomorphisms. $\square$

(3.5) Let $\mathcal{T}$ be a triangulated category. A triangulated subcategory (see [Nee, Definition 1.5.1] for the definition) $\mathcal{T}'$ is said to be *localizing* if it is closed under small direct sums. For a set of objects (or a full subcategory) $\mathcal{F}$ of $\mathcal{T}$, there is a smallest localizing subcategory $\mathrm{Loc}(\mathcal{F})$ of $\mathcal{T}$ containing $\mathcal{F}$.

(3.6) Let $\mathcal{C}$ be an abelian category which satisfies the (AB3) condition. For $n \in \mathbb{Z}$, let $C(\mathcal{C})_{\leq n}$ be the full subcategory of $C(\mathcal{C})$ of the category of complexes in $\mathcal{C}$ consisting of complexes $\mathbb{F}$ with $\mathbb{F}^i = 0$ for $i > n$.

Lemma 3.7. *Let $\mathcal{F}$ be a full subcategory of $\mathcal{C}$ closed under small direct sums. Assume that there is a pair $(\mathfrak{F}, \mathfrak{f})$ such that $\mathfrak{F} : \mathcal{C} \to \mathcal{F}$ is a functor, and $\mathfrak{f} : j'\mathfrak{F} \to \mathrm{Id}$ is a natural map which is epic objectwise, where $j' : \mathcal{F} \hookrightarrow \mathcal{C}$ is the inclusion. Assume also that $\mathfrak{F}(0) = 0$. Then*

1 *For $\mathbb{F} \in C(\mathcal{C})_{\leq n}$, there is a functorial resolution $(\mathfrak{G}, \mathfrak{g})$ such that $\mathfrak{G}(\mathbb{F})$ is in $C(\mathcal{F}) \cap C(\mathcal{C})_{\leq n}$.*

2 *There is a functorial inductive system $\mathfrak{G}_n(\mathbb{F})$ of complexes, functorial on $\mathbb{F} \in C(\mathcal{C})$, and a quasi-isomorphism $\mathfrak{g}_n : \mathfrak{G}_n(\mathbb{F}) \to \mathbb{F}_{\leq n}$ such that*

  a $\mathfrak{G}_{-1}(\mathbb{F}) = 0$;

b *For $n \in \mathbb{Z}$, $s_n : \mathfrak{G}_n(\mathbb{F}) \to \mathfrak{G}_{n-1}(\mathbb{F})$ is a semisplit epi (that is, $s_n^i : \mathfrak{G}_n(\mathbb{F})^i \to \mathfrak{G}_{n-1}(\mathbb{F})^i$ is a split epimorphism for each $i \in \mathbb{Z}$);*

c *$\mathfrak{H}_n := \operatorname{Ker} s_n$ is in $C(\mathcal{F}) \cap C(\mathcal{C})_{\leq n}$ for each $n \geq 0$.*

3 *If $\mathcal{C}$ satisfies the* (AB4) *condition, then there is a functorial resolution $\mathfrak{g} : \mathfrak{G}(\mathbb{F}) \to \mathbb{F}$, functorial on $\mathbb{F} \in C(\mathcal{C})$ with $\mathfrak{G}(\mathbb{F}) \in \operatorname{Ob}(\operatorname{Loc}(\mathcal{F}'))$, where $\mathcal{F}'$ is the full subcategory of $K(\mathcal{C})$ whose object set is the same as $\mathcal{F}$.*

Proof. For

$$\mathbb{F} : \cdots \to \mathbb{F}^i \xrightarrow{\partial^i_{\mathbb{F}}} \mathbb{F}^{i+1} \to \cdots$$

in $C(\mathcal{C})$, let $\mathfrak{F}(\mathbb{F})$ be

$$\cdots \to \mathfrak{F}(\mathbb{F}^i) \xrightarrow{\mathfrak{F}(\partial^i_{\mathbb{F}})} \mathfrak{F}(\mathbb{F}^{i+1}) \to \cdots .$$

Note that

$$\mathfrak{F}(\partial^{i+1}_{\mathbb{F}})\mathfrak{F}(\partial^i_{\mathbb{F}}) = \mathfrak{F}(\partial^{i+1}_{\mathbb{F}}\partial^i_{\mathbb{F}}) = \mathfrak{F}(0) = 0,$$

since $0 : \mathbb{F}^i \to \mathbb{F}^{i+2}$ factors through the null object 0, and hence $\mathfrak{F}(0) : \mathfrak{F}(\mathbb{F}^i) \to \mathfrak{F}(\mathbb{F}^{i+2})$ also factors through the null object $\mathfrak{F}(0) = 0$, and hence $\mathfrak{F}(0)$ is the zero map.

Let $\mathfrak{f}(\mathbb{F}) : \mathfrak{F}(\mathbb{F}) \to \mathbb{F}$ be the obvious natural map. It is an epic chain map. Let $\mathfrak{K}(\mathbb{F}) := \operatorname{Ker} \mathfrak{f}(\mathbb{F})$. Then defining $\mathfrak{K}(0) := \mathbb{F}$, $\mathfrak{G}(m) := \mathfrak{F}(\mathfrak{K}(m))$, and $\mathfrak{K}(m+1) := \mathfrak{K}(\mathfrak{K}(m))$, we have a resolution

$$\cdots \to \mathfrak{G}(m+1) \to \mathfrak{G}(m) \to \cdots \to \mathfrak{G}(0) \to 0$$

of $\mathbb{F}$.

Assume that $\mathbb{F} \in C(\mathcal{C})_{\leq n}$ for some n, and let $\mathfrak{G}(\mathbb{F})$ be the total complex of this resolution, and $\mathfrak{g} : \mathfrak{G}(\mathbb{F}) \to \mathbb{F}$ be the canonical map. Then $\mathfrak{g}$ is the desired resolution, and we have proved **1**.

2 is proved by the same proof as in [Spa, Lemma 3.3], except that everything here is functorial.

3 This is proved similarly to (the dual assertion of) [BN, Application 2.4], using **2**. □

(3.8) Let $(\mathbb{X}, \mathcal{O}_{\mathbb{X}})$ be a ringed site with a small basis of topology B. For $\mathcal{M} \in \operatorname{Mod}(\mathbb{X})$, $x \in B$ and $c \in \Gamma(x, \mathcal{M})$, there corresponds a map

$$(e(c) : \mathcal{O}_x \to \mathcal{M}) \in \operatorname{Hom}_{\mathcal{O}_{\mathbb{X}}}(\mathcal{O}_x, \mathcal{M}) \cong \operatorname{Hom}_{\mathcal{O}_{\mathbb{X}}|_x}(\mathcal{O}_{\mathbb{X}}|_x, \mathcal{M}|_x) \cong \Gamma(x, \mathcal{M}),$$

corresponding to $c \in \Gamma(x, \mathcal{M})$. So

$$\mathfrak{f}(\mathcal{M}) := \sum e(c) : \mathfrak{F} := \bigoplus_{x \in B} \bigoplus_{0 \neq c \in \Gamma(x,\mathcal{M})} \mathcal{O}_{x,c} \to \mathcal{M}$$

is an epimorphism, where each $\mathcal{O}_{x,c}$ is a copy of $\mathcal{O}_x$, see [Has5, (2.23), (3.19)]. Note that for $y \in \mathbb{X}$, $\Gamma(y, \mathcal{O}_{x,c}) = \bigoplus_{s \in \mathbb{X}(y,x)} \Gamma(y, \mathcal{O}_{\mathbb{X}})_{c,s}$, where $\Gamma(y, \mathcal{O}_{\mathbb{X}})_{c,s}$ is a copy of $\Gamma(y, \mathcal{O}_{\mathbb{X}})$. Then $\mathfrak{f}(\mathcal{M})$ is the unique map such that $1 \in \Gamma(x, \mathcal{O}_{\mathbb{X}})_{c,1_x}$ is mapped to c for each x and $c \neq 0$.

Let $h : \mathcal{M} \to \mathcal{N}$ be a map in $\mathrm{Mod}(\mathbb{X})$. Then mapping $1 \in \Gamma(x, \mathcal{O}_{\mathbb{X}})_{c,1_x}$ to $1 \in \Gamma(x, \mathcal{O}_{\mathbb{X}})_{h(c),1_x}$ if $h(c) \neq 0$, and to 0 if $h(c) = 0$, we get a map $\mathfrak{F}(h) : \mathfrak{F}(\mathcal{M}) \to \mathfrak{F}(\mathcal{N})$ such that $\mathfrak{F} : \mathrm{PM}(\mathbb{X}) \to \mathfrak{W}$ is a functor, where $\mathfrak{W}$ is the full subcategory of $\mathrm{PM}(\mathbb{X})$ consisting of the direct sum of copies of $\mathcal{O}_x$ with $x \in \mathbb{X}$, and that $\mathfrak{f} : j'\mathfrak{F} \to \mathrm{Id}$ is a natural transformation, where $j' : \mathfrak{W} \hookrightarrow \mathrm{PM}(\mathbb{X})$ is the inclusion. Moreover, $\mathfrak{F}(0) = 0$.

Lemma 3.9. *Let $(\mathbb{X}, \mathcal{O}_{\mathbb{X}})$ be a ringed site with a small basis of the topology. Then there is an endofunctor $\mathfrak{F} = \mathfrak{F}_{\mathbb{X}} : C(\mathrm{Mod}(\mathbb{X})) \to \mathcal{L}$ and a functorial $\mathcal{L}$-resolution $\mathfrak{f} = \mathfrak{f}_{\mathbb{X}} : j'\mathfrak{F} \to \mathrm{Id}$, where $\mathcal{L}$ is the full subcategory of $C(\mathrm{Mod}(\mathbb{X}))$ whose object set is the set of strongly K-flat complexes* [Has5, (3.19)].

Proof. As $\mathrm{Mod}(\mathbb{X})$ is a Grothendieck category (so (AB5) is satisfied), Lemma 3.7 and the discussion above are applicable.

By definition, $\mathcal{O}_x$ is strongly K-flat for $x \in \mathbb{X}$. Now $\mathfrak{F}(\mathbb{F})$ is strongly K-flat for $\mathbb{F} \in C(\mathrm{Mod}(\mathbb{X}))$ by construction. □

4. The restriction and other operations on quasi-coherent sheaves

(4.1) Let G' and G be flat S-group schemes, and $h : G' \to G$ a homomorphism of S-group schemes. Let Z be an S-scheme on which G acts. Then $\mathrm{res}^G_{G'} : \mathrm{Mod}(G, Z) \to \mathrm{Mod}(G', Z)$ is defined to be the inverse image functor $B^M_h(Z)^*$, see [Has11, (2.45)].

Lemma 4.2. $\mathrm{res}^G_{G'}$ *is a faithful exact functor from* $\mathrm{Qch}(G, Z)$ *to* $\mathrm{Qch}(G', Z)$.

Proof. By [Has5, (7.22)], $\mathrm{res}^G_{G'}$ is a functor from $\mathrm{Qch}(G, Z)$ to $\mathrm{Qch}(G', Z)$. Then as functors from $\mathrm{Qch}(G, Z)$ to $\mathrm{Qch}(Z)$, we have $(?)_0 \,\mathrm{res}^G_{G'} \cong (?)_0$, where the left $(?)_0$ is from $\mathrm{Qch}(G', Z)$ to $\mathrm{Qch}(Z)$, and the right $(?)_0$ is from $\mathrm{Qch}(G, Z)$ to $\mathrm{Qch}(Z)$. As the both $(?)_0$ are faithful exact, $\mathrm{res}^G_{G'}$ is also faithful exact. □

Lemma 4.3. *If $\mathcal{M} \in \mathrm{Qch}(G, Z)$, then $(L_i \operatorname{res}^G_{G'})\mathcal{M} = 0$ for $i > 0$, where L_i denotes the ith left derived functor $D(G, Z) \to \mathrm{Mod}(G', Z)$.*

Proof. By [Has5, (8.20), (8.21)], we have that $(L_i \operatorname{res}^G_{G'})\mathcal{M}$ is quasi-coherent. As the restriction functor $(?)_0 : \mathrm{Qch}(G, Z) \to \mathrm{Qch}(Z)$ is faithful and exact by [Has5, (12.12)], it suffices to prove that $((L_i \operatorname{res}^G_{G'})\mathcal{M})_0 = 0$ for $i > 0$. But this is $L_i(\mathrm{id}_Z)^*\mathcal{M}_0 = 0$, by [Has5, (8.13)]. □

Lemma 4.4. *Let $\mathbb{G} \in D_{\mathrm{Qch}}(G, Z)$. Then $\mathbb{G}$ is (left) $\operatorname{res}^G_{G'}$-acyclic, in the sense that for any (or equivalently, some) K-flat resolution $\mathbb{P} \to \mathbb{G}$, the map $\operatorname{res}^G_{G'} \mathbb{P} \to \operatorname{res}^G_{G'} \mathbb{G}$ is a quasi-isomorphism. In particular, $L \operatorname{res}^G_{G'} \mathbb{G}$ has quasi-coherent cohomology groups. If Z is locally Noetherian, G and G' are locally of finite type, and $\mathbb{G}$ has coherent cohomology groups, then $\operatorname{res}^G_{G'} \mathbb{G}$ has coherent cohomology groups.*

Proof. If $\mathbb{G} \in D^-_{\mathrm{Qch}}(G, Z)$, then the assertion follows from Lemma 4.3. Consider the general case. From the bounded-above case, it is easy to see that the resolution $\mathfrak{g} : \mathfrak{G}(\mathbb{G}) \to \mathbb{G}$ in Lemma 3.7 is a K-flat resolution such that $\operatorname{res}^G_{G'} \mathfrak{g}$ is a quasi-isomorphism. □

(4.5) Now we can prove that the (derived) restriction is compatible with most of basic operations on sheaves. For the notation, see [Has5].

Lemma 4.6. *Let $f : (\mathbb{X}, \mathcal{O}_{\mathbb{X}}) \to (\mathbb{Y}, \mathcal{O}_{\mathbb{Y}})$ be a morphism of ringed sites. Assume that for each $x \in \mathbb{X}$, the category $(I^f_x)^{\mathrm{op}}$ (see [Has5, (2.6)]) is filtered (that is, connected and pseudofiltered, see [Mil, Appendix A]). Then the canonical map*

$$\Delta : f^*(\mathcal{M} \otimes_{\mathcal{O}_{\mathbb{Y}}} \mathcal{N}) \to f^*\mathcal{M} \otimes_{\mathcal{O}_{\mathbb{X}}} f^*\mathcal{N} \tag{6}$$

is an isomorphism for any $\mathcal{M}, \mathcal{N} \in \mathrm{Mod}(\mathbb{Y})$, and the canonical map

$$\Delta : Lf^*(\mathbb{F} \otimes^L_{\mathcal{O}_{\mathbb{Y}}} \mathbb{G}) \to Lf^*\mathbb{F} \otimes^L_{\mathcal{O}_{\mathbb{X}}} Lf^*\mathbb{G} \tag{7}$$

is also an isomorphism for $\mathbb{F}, \mathbb{G} \in D(\mathbb{Y})$.

Proof. First consider the corresponding map of presheaves

$$\Delta : f^*_{\mathrm{PM}}(\mathcal{M} \otimes^p_{\mathcal{O}_{\mathbb{Y}}} \mathcal{N}) \to f^*_{\mathrm{PM}}\mathcal{M} \otimes^p_{\mathcal{O}_{\mathbb{X}}} f^*_{\mathrm{PM}}\mathcal{N} \tag{8}$$

for $\mathcal{M}, \mathcal{N} \in \mathrm{PM}(\mathbb{Y})$. Then the map between sections at $x \in \mathbb{X}$ is

$$\varinjlim_{x \to fy} \Gamma(x, \mathcal{O}_{\mathbb{X}}) \otimes_{\Gamma(y,\mathcal{O}_{\mathbb{Y}})} (\Gamma(y, \mathcal{M}) \otimes_{\Gamma(y,\mathcal{O}_{\mathbb{Y}})} \Gamma(y, \mathcal{N})) \to$$
$$(\varinjlim_{x \to fy} \Gamma(x, \mathcal{O}_{\mathbb{X}}) \otimes_{\Gamma(y,\mathcal{O}_{\mathbb{Y}})} \Gamma(y, \mathcal{M})) \otimes_{\Gamma(x,\mathcal{O}_{\mathbb{X}})} (\varinjlim_{x \to fy} \Gamma(x, \mathcal{O}_{\mathbb{X}}) \otimes_{\Gamma(y,\mathcal{O}_{\mathbb{Y}})} \Gamma(y, \mathcal{M})),$$

which is an isomorphism by the assumption that $(I_x^f)^{\mathrm{op}}$ is filtered. Thus (8) is an isomorphism.

Now consider $\mathcal{M}, \mathcal{N} \in \mathrm{Mod}(\mathbb{Y})$. It is not so difficult to show that the diagram

$$\begin{tikzcd}
af^*(q\mathcal{M} \otimes^p_{\mathcal{O}_{\mathbb{Y}}} q\mathcal{N}) \arrow[r, "\Delta"] \arrow[d, "u"] & a(f^*q\mathcal{M} \otimes^p_{\mathcal{O}_{\mathbb{X}}} f^*q\mathcal{N}) \arrow[d, "u \otimes u"] \\
af^*qa(q\mathcal{M} \otimes^p_{\mathcal{O}_{\mathbb{Y}}} q\mathcal{N}) \arrow[r, "\Delta"] & a(qaf^*q\mathcal{M} \otimes^p_{\mathcal{O}_{\mathbb{X}}} qaf^*q\mathcal{N})
\end{tikzcd}$$

is commutative. The top horizontal arrow is an isomorphism by the argument for presheaves above, and the vertical arrows are isomorphisms by [Has5, (2.18), (2.34)]. Thus the bottom horizontal arrow, which agrees with the map (6), is an isomorphism.

Now consider $\mathbb{F}, \mathbb{G} \in D(\mathbb{Y})$. Take strongly K-flat resolutions $\mathbb{P} \to \mathbb{F}$ and $\mathbb{Q} \to \mathbb{G}$. Then the map (7) is nothing but the composite

$$Lf^*(\mathbb{F} \otimes^L_{\mathcal{O}_{\mathbb{Y}}} \mathbb{G}) \cong Lf^*(\mathbb{P} \otimes^L_{\mathcal{O}_{\mathbb{Y}}} \mathbb{Q}) \cong f^*(\mathbb{P} \otimes_{\mathcal{O}_{\mathbb{Y}}} \mathbb{Q})$$
$$\xrightarrow{\Delta} f^*\mathbb{P} \otimes_{\mathcal{O}_{\mathbb{X}}} f^*\mathbb{Q} \cong Lf^*\mathbb{F} \otimes^L_{\mathcal{O}_{\mathbb{X}}} Lf^*\mathbb{G}.$$

The second isomorphism comes from the fact that $\mathbb{P} \otimes_{\mathcal{O}_{\mathbb{Y}}} \mathbb{Q}$ is K-flat [Has5, (3.21)]. The last isomorphism comes from the fact that $f^*\mathbb{P}$ and $f^*\mathbb{Q}$ are strongly K-flat [Has5, (3.20)]. Being the composite of isomorphisms, (7) is an isomorphism. □

Lemma 4.7. *Let J be a small category, and $\varphi : X \to Y$ a morphism of J^{op}-diagrams of schemes. Let $f = \varphi^{-1} : \mathrm{Zar}(Y) \to \mathrm{Zar}(X)$ be the corresponding functor between sites* [Has5, (5.3)]. *Then for each $(j, U) \in \mathrm{Zar}(X)$ (where $j \in J$ and $U \in \mathrm{Zar}(X_j)$), the category $(I^f_{(j,U)})^{\mathrm{op}}$ (see* [Has5, (2.6)]*) is filtered.*

Proof. Left to the reader. □

Lemma 4.8. *Let $\mathbb{F}, \mathbb{G} \in D(G, Z)$. Then the canonical map*

$$\Delta : L \operatorname{res}^G_{G'}(\mathbb{F} \otimes^L_{\mathcal{O}_Z} \mathbb{G}) \to (L \operatorname{res}^G_{G'} \mathbb{F}) \otimes^L_{\mathcal{O}_Z} (L \operatorname{res}^G_{G'} \mathbb{G})$$

is an isomorphism (Note that the $\otimes^L_{\mathcal{O}_Z}$ in the left (resp. right) hand side is an abbreviation for $\otimes^L_{\mathcal{O}_{B^M_G(Z)}}$ (resp. $\otimes^L_{\mathcal{O}_{B^M_{G'}(Z)}}$)).

Proof. By Lemma 4.7, Lemma 4.6 is applicable. □

Corollary 4.9. *Let $\mathcal{M}$ and $\mathcal{N}$ be objects in $\operatorname{Qch}(G, Z)$. Then*

$$\operatorname{res}^G_{G'} \underline{\operatorname{Tor}}^{\mathcal{O}_Z}_i(\mathcal{M}, \mathcal{N}) \cong \underline{\operatorname{Tor}}^{\mathcal{O}_Z}_i(\operatorname{res}^G_{G'} \mathcal{M}, \operatorname{res}^G_{G'} \mathcal{N}).$$

□

Lemma 4.10. *Let X be locally Noetherian, and G be locally of finite type. Let $\mathbb{F} \in D^-_{\mathrm{Coh}}(G, Z)$ and $\mathbb{G} \in D^+_{\mathrm{Lqc}}(G, Z)$. Then the canonical map*

$$L \operatorname{res}^G_{G'} R \underline{\operatorname{Hom}}_{\mathcal{O}_Z}(\mathbb{F}, \mathbb{G}) \to R \underline{\operatorname{Hom}}_{\mathcal{O}_Z}(L \operatorname{res}^G_{G'} \mathbb{F}, L \operatorname{res}^G_{G'} \mathbb{G}) \tag{9}$$

is an isomorphism.

Proof. By Lemma 4.4 and [Has5, (13.10)], the two complexes have quasi-coherent cohomology groups. So in order to prove that the map is an isomorphism, we may discuss after applying the functor $(?)_0$. By [Has5, (8.13)], the canonical map

$$\theta : L(\mathrm{id}^*_Z)(?)_0 \to (?)_0 \operatorname{res}^G_{G'}$$

is an isomorphism. So the map (9) applied $(?)_0$ is identified with

$$H_0 : (R \underline{\operatorname{Hom}}_{\mathcal{O}_Z}(\mathbb{F}, \mathbb{G}))_0 \to R \underline{\operatorname{Hom}}_{\mathcal{O}_Z}(\mathbb{F}_0, \mathbb{G}_0).$$

It is an isomorphism by [Has5, (13.9)]. □

Corollary 4.11. *Let $\mathcal{M} \in \operatorname{Coh}(G, Z)$ and $\mathcal{N} \in \operatorname{Qch}(G, Z)$. Then we have*

$$\operatorname{res}^G_{G'} \underline{\operatorname{Ext}}^i_{\mathcal{O}_{B^M_G(Z)}}(\mathcal{M}, \mathcal{N}) \cong \underline{\operatorname{Ext}}^i_{\mathcal{O}_{B^M_{G'}(Z)}}(\operatorname{res}^G_{G'} \mathcal{M}, \operatorname{res}^G_{G'} \mathcal{N}).$$

□

Lemma 4.12. *Let $g : Z' \to Z$ be a concentrated (that is, quasi-compact quasi-separated) G-morphism of G-schemes. Then the canonical map*

$$\theta : L \operatorname{res}^G_{G'} Rf_* \mathbb{F} \to Rf_* L \operatorname{res}^G_{G'} \mathbb{F}$$

is an isomorphism for $\mathbb{F} \in D_{\mathrm{Qch}}(G, Z')$.

Proof. As the complexes have quasi-coherent cohomology groups by [Has5, (8.7)] and Lemma 4.4, we may discuss after applying the functor $(?)_0$, and the rest is easy. □

(4.13) Let $V \subset U \subset Z$ be G-stable open subsets. Assume that the inclusions $f : U \hookrightarrow Z$ and $g : V \hookrightarrow U$ are quasi-compact. For $\mathbb{F} \in D_{\mathrm{Qch}}(G, Z)$, there is a commutative diagram

$$\begin{tikzcd} L\operatorname{res}^G_{G'} R\,\underline{\Gamma}_{U,V}\,\mathbb{F} \arrow[r, "\iota"] & L\operatorname{res}^G_{G'} Rf_*f^*\mathbb{F} \arrow[r, "u"] \arrow[d, "d\theta"] & L\operatorname{res}^G_{G'} Rf_*Rg_*g^*f^*\mathbb{F} \arrow[r] \arrow[d, "dd\theta\theta"] & {} \\ R\,\underline{\Gamma}_{U,V}\,L\operatorname{res}^G_{G'}\mathbb{F} \arrow[r, "\iota"] & Rf_*f^*L\operatorname{res}^G_{G'}\mathbb{F} \arrow[r, "u"] & Rf_*Rg_*g^*f^*L\operatorname{res}^G_{G'}\mathbb{F} \arrow[r] & {} \end{tikzcd} \tag{10}$$

whose rows are distinguished triangles by [HasO, (4.10)]. Then, as the derived category is a triangulated category,

$$\bar{\delta} : L\operatorname{res}^G_{G'} R\,\underline{\Gamma}_{U,V}\,\mathbb{F} \to R\,\underline{\Gamma}_{U,V}\,L\operatorname{res}^G_{G'}\mathbb{F}$$

which completes (10) as a map of triangles. As $d\theta$ and $dd\theta\theta$ are isomorphisms by Lemma 4.12, $\bar{\delta}$ is an isomorphism by [Hart, (I.1.1)].

We can make $\bar{\delta}$ functorial on $\mathbb{F}$. Fix a functorial strictly injective resolution $\mathbb{F} \to \mathbb{I}_{\mathbb{F}}$ (in the category $K_{\mathrm{Qch}}(G, Z)$) as in section 3. Then the composite

$$\begin{aligned} L\operatorname{res}^G_{G'} R\,\underline{\Gamma}_{U,V}\,\mathbb{F} &\cong \operatorname{res}^G_{G'} \underline{\Gamma}_{U,V}\,\mathbb{I}_{\mathbb{F}} \cong \operatorname{res}^G_{G'} \operatorname{Cone}(u : f_*f^*\mathbb{I}_{\mathbb{F}} \to f_*g_*g^*f^*\mathbb{I}_{\mathbb{F}})[-1] \\ &\cong \operatorname{Cone}(u : f_*f^*\operatorname{res}^G_{G'}\mathbb{I}_{\mathbb{F}} \to f_*g_*g^*f^*\operatorname{res}^G_{G'}\mathbb{I}_{\mathbb{F}})[-1] \cong R\,\underline{\Gamma}_{U,V}\operatorname{res}^G_{G'}\mathbb{F} \end{aligned}$$

is the desired functorial $\bar{\delta}$.

In conclusion,

Lemma 4.14. *There is an isomorphism*

$$\bar{\delta} : L\operatorname{res}^G_{G'} R\,\underline{\Gamma}_{U,V}\,\mathbb{F} \to R\,\underline{\Gamma}_{U,V}\,L\operatorname{res}^G_{G'}\mathbb{F}$$

which is functorial on $\mathbb{F} \in D_{\mathrm{Qch}}(G, Z)$, the diagram

$$\begin{tikzcd} \mathcal{L}R\,\underline{\Gamma}_{U,V} \arrow[r, "\iota"] \arrow[d, "\bar{\delta}"] & \mathcal{L}Rf_*f^* \arrow[r, "u"] \arrow[d, "d\theta"] & \mathcal{L}Rf_*Rg_*g^*f^* \arrow[r] \arrow[d, "dd\theta\theta"] & \mathcal{L}R\,\underline{\Gamma}_{U,V}[1] \arrow[r] \arrow[d, "\bar{\delta}[1]"] & {} \\ R\,\underline{\Gamma}_{U,V}\,\mathcal{L} \arrow[r, "\iota"] & Rf_*f^*\mathcal{L} \arrow[r, "u"] & Rf_*Rg_*g^*f^*\mathcal{L} \arrow[r] & R\,\underline{\Gamma}_{U,V}\,\mathcal{L}[1] \arrow[r] & {} \end{tikzcd}$$

is commutative, where $\mathcal{L} = L\operatorname{res}^G_{G'}$, and the diagram

$$\begin{tikzcd} (?)_0\mathcal{L}R\,\underline{\Gamma}_{U,V}\,\mathbb{F} \arrow[r, "\cong"] \arrow[d, "(?)_0\bar{\delta}"] & (?)_0R\,\underline{\Gamma}_{U,V}\,\mathbb{F} \arrow[r, "\cong"] & R\,\underline{\Gamma}_{U,V}\,\mathbb{F}_0 \\ (?)_0R\,\underline{\Gamma}_{U,V}\,\mathcal{L}\mathbb{F} \arrow[r, "\cong"] & R\,\underline{\Gamma}_{U,V}(?)_0\mathcal{L}\mathbb{F} \arrow[ur, "\cong"] & \end{tikzcd}$$

is also commutative.

Proof. Follows easily from the construction above. $\square$

Lemma 4.15. *Let $h : G' \to G$ be a homomorphism of S-group schemes. Let X be an S-scheme with a trivial G-action. Then for $\mathcal{M} \in \mathrm{Mod}(G, X)$, $\mathcal{M}^G \subset \mathcal{M}^{G'} = (\mathrm{res}^G_{G'} \mathcal{M})^{G'}$. If h is faithfully flat, then $\mathcal{M}^G = \mathcal{M}^{G'}$.*

Proof. Let p (resp. p') be the second projection $G \times X \to X$ (resp. $G' \times X \to X$). Then $\mathcal{M}^G$ is the kernel of the map

$$\mathcal{M} \xrightarrow{\beta_{\delta_0} - \beta_{\delta_1}} p_* \mathcal{M}_1.$$

As $p' = p(h \times 1)$, $\mathcal{M}^{G'}$ is the kernel of the map

$$\mathcal{M} \xrightarrow{\beta_{\delta_0} - \beta_{\delta_1}} p_* \mathcal{M}_1 \xrightarrow{p_* u} p_*(h \times 1)_*(h \times 1)^* \mathcal{M}_1.$$

So $\mathcal{M}^G \subset \mathcal{M}^{G'}$. If h is faithfully flat, then $p_* u$ is a monomorphism, and hence $\mathcal{M}^G = \mathcal{M}^{G'}$. $\square$

(4.16) Let $h : G' \to G$ be a homomorphism between flat S-group schemes of finite type, and X a Noetherian G-scheme with a G-dualizing complex $\mathbb{I}_X$.

Lemma 4.17. *$L \mathrm{res}^G_{G'} \mathbb{I}_X$ is a G'-dualizing complex of X.*

Proof. Follows easily from [Has5, (8.20)] and [Has5, (31.17)]. $\square$

(4.18) By abuse of notation, we will sometimes write the G'-dualizing complex $\mathrm{res}^G_{G'} \mathbb{I}_X$ by $\mathbb{I}_X$ or $\mathbb{I}_X(G')$.

5. Groups with the Reynolds operators

(5.1) Let G be a flat quasi-compact quasi-separated S-group scheme, and Y an S-scheme on which G acts trivially. Let $\gamma = \gamma_{G,Y} : (?)^G \to \mathrm{Id}$ be the inclusion between the functors from $\mathrm{Qch}(G, Y)$ to itself (although $(?)^G$ is a functor from $\mathrm{Qch}(G, Y)$ to $\mathrm{Qch}(Y)$, we regard $\mathcal{M}^G$ as a trivial G-module, and then $(?)^G$ can be viewed as a functor from $\mathrm{Qch}(G, Y)$ to itself. So γ in (2.1) is γ_0 here. This abuse of notation does not cause a problem).

We say that G has a functorial Reynolds operator on Y if there is a natural transformation $p = p_{G,Y} : \mathrm{Id} \to (?)^G$ such that $p\gamma = \mathrm{id}$. The natural map p (or sometimes $\mathfrak{R} := \gamma p$) is called the *Reynolds operator* of G on Y. Note that $\mathfrak{R}^2 = \mathfrak{R}$.

(5.2) Let G, Y be as above, and assume that G has a Reynolds operator on Y. For $\mathcal{M} \in \mathrm{Qch}(G,Y)$, we define $U_G(\mathcal{M}) = \operatorname{Ker} p_{G,Y}(\mathcal{M}) = \operatorname{Im}(\mathrm{id} - \mathfrak{R})$, and call it the *anti-invariance* of $\mathcal{M}$. We say that $\mathcal{M}$ is *anti-trivial* if $U_G(\mathcal{M}) = \mathcal{M}$, or equivalently, $\mathcal{M}^G = 0$. If $\mathcal{M}, \mathcal{N} \in \mathrm{Qch}(G,Y)$, and $\mathcal{M}$ is anti-trivial and $\mathcal{N}$ is trivial, then $\operatorname{Hom}_{G,\mathcal{O}_Y}(\mathcal{M},\mathcal{N}) = 0$. Indeed, if $h \in \operatorname{Hom}_{G,\mathcal{O}_Y}(\mathcal{M},\mathcal{N})$, then $h = \mathcal{R}h(\mathrm{id} - \mathcal{R}) = h\mathcal{R}(\mathrm{id} - \mathcal{R}) = 0$. Similarly, we have $\operatorname{Hom}_{G,\mathcal{O}_Y}(\mathcal{N},\mathcal{M}) = 0$.

(5.3) Note that $U_G(\mathcal{M})$ is the sum of all the quasi-coherent submodules $\mathcal{N}$ of $\mathcal{M}$ such that $\mathcal{N}^G = 0$, and is determined only by G, Y, and $\mathcal{M}$, and is independent of the choice of p. As the Reynolds operator p is the projection with respect to the direct sum decomposition $\mathcal{M} = \mathcal{M}^G \oplus U_G(\mathcal{M})$, it is unique, if exists. So $\mathfrak{R}$ is also unique, if exists.

Definition 5.4. We say that an S-group scheme G is *Reynolds*, if G is flat quasi-compact quasi-separated, and for any affine open subscheme U of S, the Reynolds operator of G on U exists.

Lemma 5.5. *Let G be a flat quasi-compact quasi-separated S-group scheme, Y an S-scheme on which G acts trivially, and U its open subset. If G has a Reynolds operator on Y and if the inclusion $j : U \hookrightarrow Y$ is quasi-compact, then G has a Reynolds operator on U.*

Proof. For $\mathcal{M} \in \mathrm{Qch}(G,U)$, define $p_{G,U} : \mathcal{M} \to \mathcal{M}^G$ to be the composite

$$\mathcal{M} \xrightarrow{\varepsilon^{-1}} j^*j_*\mathcal{M} \xrightarrow{p_{G,Y}} j^*(?)^G j_*\mathcal{M} \xrightarrow{e^{-1}} j^*j_*(?)^G\mathcal{M} \xrightarrow{\varepsilon} (?)^G\mathcal{M}.$$

It is easy to see that this is the identity map on $\mathcal{M}^G$. $\square$

Corollary 5.6. *Let G be a flat quasi-compact quasi-separated S-group scheme. Assume that G has a Reynolds operator over S, and S is quasi-separated. Then G is Reynolds.* $\square$

Lemma 5.7. *Let G be a flat quasi-compact quasi-separated S-group scheme, Y an S-scheme on which G-acts trivially, and assume that G has a Reynolds operator on Y. Then*

1 $(?)^G$ *and* U_G *are exact functors on* $\mathrm{Qch}(G,Y)$ *which preserve direct sums. In particular,* $H^i(G,\mathcal{M}) = 0$ *for* $\mathcal{M} \in \mathrm{Qch}(G,Y)$ *and* $i > 0$*, where* $H^i(G,?) = R^i(?)^G$*, the derived functor of the functor* $\mathrm{Qch}(G,Y) \to \mathrm{Qch}(Y)$ *(not a functor from* $\mathrm{Mod}(G,Y)$*).*

2 *The full subcategory of the G-trivial quasi-coherent $(G, \mathcal{O}_Y)$-modules is closed under extensions and subquotients. Similarly for the full subcategory of G-anti-trivial quasi-coherent $(G, \mathcal{O}_Y)$-modules.*

Proof. **1** As we have $\mathrm{Id} = (?)^G \oplus U_G$, we have that both $(?)^G$ and U_G are exact. Let $(\mathcal{M}_i)_{i \in I}$ be a family of objects in $\mathrm{Qch}(G, Y)$. Then we have

$$\bigoplus_i \mathcal{M}_i = (\bigoplus_i \mathcal{M}_i^G) \oplus (\bigoplus_i U_G(\mathcal{M}_i)). \tag{11}$$

As $(?)^G$ is compatible with the direct sum by Lemma 2.23, we have that $(\bigoplus_i \mathcal{M}_i^G)^G = \bigoplus_i ((\mathcal{M}_i)^G)^G = \bigoplus_i \mathcal{M}_i^G$, and hence $\bigoplus \mathcal{M}_i^G$ is G-trivial. As $(\bigoplus_i U_G(\mathcal{M}_i))^G = \bigoplus_i U_G(\mathcal{M}_i)^G = 0$, we have that $\bigoplus U_G(\mathcal{M}_i)$ is anti-trivial. So by the decomposition (11), we have that $\bigoplus_i \mathcal{M}_i^G = (\bigoplus_i \mathcal{M}_i)^G$ and $\bigoplus_i U_G(\mathcal{M}_i) = U_G(\bigoplus_i \mathcal{M}_i)$.

2 follows easily by the five lemma and **1**. □

Lemma 5.8. *Let the notation be as in* Lemma 5.7. *Assume that $Y = \operatorname{Spec} R$ is affine. If V is a G-trivial R-module and M is a G-module, then $(V \otimes_R M)^G = V \otimes_R M^G$ and $U_G(V \otimes_R M) = V \otimes_R U_G(M)$.*

Proof. There is a G-trivial free R-module F and a surjection $F \to V$. Then $F \otimes_R M^G$ is a direct sum of copies of M^G as a (G, R)-module, and hence it is G-trivial. Being a homomorphic image of $F \otimes_R M^G$, $V \otimes_R M^G$ is G-trivial. Similarly, $F \otimes_R U_G(M)$ is G-anti-trivial, and hence so is $V \otimes_R U_G(M)$. As we have the decomposition

$$V \otimes_R M = V \otimes_R M^G \oplus V \otimes_R U_G(M),$$

we must have $(V \otimes_R M)^G = V \otimes_R M^G$ and $U_G(V \otimes_R M) = V \otimes_R U_G(M)$. □

Lemma 5.9. *Let the notation be as in* Lemma 5.8. *Let R' be an R-algebra (on which G acts trivially), and set $h : Y' = \operatorname{Spec} R' \to \operatorname{Spec} R = Y$ be the associated map. Then*

1 *G also has a Reynolds operator on Y'.*

2 *$(?)^G h_*$ and $U_G h_*$ are canonically identified with $h_*(?)^G$ and $h_* U_G$, respectively.*

3 *$h_* p_{G,Y'} = p_{G,Y} h_*$.*

4 *$(?)^G h^*$ and $U_G h^*$ are canonically identified with $h^*(?)^G$ and $h^* U_G$, respectively.*

5 *For a (G,R)-module M, the diagram*

$$\begin{tikzcd} M \arrow[r, "\eta_M"] \arrow[d, "p"] & R' \otimes_R M \arrow[d, "1\otimes p"] \arrow[dr, "p"] & \\ M^G \arrow[r, "\eta_{M^G}"] & R' \otimes_R M^G \arrow[r, "\cong"] & (R' \otimes_R M)^G \end{tikzcd}$$

is commutative.

Proof. Let M be a (G,R')-module. Then we can decompose $M = M^G \oplus U_G(M)$ as (G,R)-modules. On the other hand, As $R' \otimes_R M^G$ is G-trivial, it is mapped to M^G by the product $a_M : R' \otimes_R M \to M$. Similarly, $R' \otimes_R U_G(M)$ is mapped to $U_G(M)$. Hence M^G and $U_G(M)$ are (G,R')-submodules of M, and the decomposition $M = M^G \oplus U_G(M)$ shows **1, 2, 3**.

Next, let M be a (G,R)-module. Then by Lemma 5.8, $R' \otimes_R M^G$ is identified with $(R' \otimes_R M)^G$. As a (G,R)-module, $U_G(R' \otimes_R M)$ is identified with $R' \otimes_R U_G(M)$. However, by **3**, this is an identification also as (G,R')-modules. Now **4** and **5** are clear. □

Lemma 5.10. *Let G be a flat quasi-compact quasi-separated S-group scheme, Y an S-scheme on which G acts trivially. Let $Y = \bigcup_i U_i$ be an affine open covering such that G has a Reynolds operator over U_i for each i. Then for any Y-scheme Y' (with a trivial action), G has a Reynolds operator.*

Proof. First, we prove that G has a Reynolds operator on Y. Let $\mathcal{C} = \mathrm{Zar}(Y)$, and let $\mathcal{D}$ be the full subcategory of $\mathcal{C}$ consisting of affine open subsets W of Y such that $W \subset U_i$ for some i. For $W \in \mathcal{D}$, G has a Reynolds operator on W by Lemma 5.9. So defining $\bar{p} : \mathcal{M}|_{\mathcal{D}} \to \mathcal{M}^G|_{\mathcal{D}}$ by

$$\Gamma(W, \mathcal{M}|_{\mathcal{D}}) = \Gamma(W, \mathcal{M}|_W) \xrightarrow{p_{G,W}} \Gamma(W, (\mathcal{M}|_W)^G) \xrightarrow{\epsilon^{-1}} \Gamma(W, (\mathcal{M}^G)|_W) = \Gamma(W, \mathcal{M}^G|_{\mathcal{D}}),$$

we get a functorial splitting of $\bar{i} : \mathcal{M}^G|_{\mathcal{D}} \to \mathcal{M}|_{\mathcal{D}}$ by Lemma 5.9, **4, 5**. As the restriction from $\mathcal{C}$ to $\mathcal{D}$ gives an equivalence $\mathrm{Sh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{D})$ by [Has11, (4.6)], there is a unique splitting $p : \mathcal{M} \to \mathcal{M}^G$ of $i : \mathcal{M}^G \to \mathcal{M}$ in $\mathrm{Sh}(\mathcal{C})$ whose restriction to $\mathrm{Sh}(\mathcal{D})$ is $\bar{p}$. By the uniqueness, it is easy to see that the restriction of p to each U_i is a Reynolds operator, and hence p is a morphism in $\mathrm{Qch}(G, Y)$, and p is the Reynolds operator of G on Y.

Next, let $h : Y' \to Y$ be a Y-scheme. Then for each affine open subset W of Y' such that $h(W) \subset U_i$ for some i, G has a Reynolds operator over W. Such W covers Y', and hence Y' has a Reynolds operator by the first step. $\square$

Corollary 5.11. *Let G be a flat quasi-compact quasi-separated S-group scheme. If G is Reynolds, then for any S-scheme Y with the trivial G-action, G has a Reynolds operator over Y.*

Lemma 5.12. *Let G be a Reynolds S-group scheme, and $h : Y \to S$ a morphism. Then we have the following.*

0 *The base change $G_Y = Y \times_S G$ is a Reynolds Y-group scheme.*

1 *The canonical map $\epsilon : h^*(?)^G \to (?)^G h^*$ (see (2.3)) is an isomorphism between functors from $\mathrm{Qch}(G, S)$ to $\mathrm{Qch}(G, Y)$. The composite*

$$\mathrm{Id}h^* = h^* = h^*\mathrm{Id} \xrightarrow{p_{G,S}} h^*(?)^G \xrightarrow{\epsilon} (?)^G h^*$$

agrees with $p_{G,Y}$.

2 *Let h be quasi-compact quasi-separated. Then the composite*

$$\mathrm{Id}h_* = h_* = h_*\mathrm{Id} \xrightarrow{p_{G,Y}} h_*(?)^G \xrightarrow{e} (?)^G h_*$$

is $p_{G,S}$.

3 *Let $\mathcal{V} \in \mathrm{Qch}(S)$ be G-trivial, and $\mathcal{M} \in \mathrm{Qch}(G, S)$. Then $1 \otimes \gamma : \mathcal{V} \otimes \mathcal{M}^G \to \mathcal{V} \otimes \mathcal{M}$ is a monomorphism, and it induces an isomorphism γ' onto $(\mathcal{V} \otimes \mathcal{M})^G$. The composite*

$$\mathcal{V} \otimes \mathcal{M} \xrightarrow{1 \otimes p_{G,S}} \mathcal{V} \otimes \mathcal{M}^G \xrightarrow{\gamma'} (\mathcal{V} \otimes \mathcal{M})^G$$

is $p_{G,S}$. Let $\delta : U_G(\mathcal{M}) \to \mathcal{M}$ be the inclusion. Then $1 \otimes \delta : \mathcal{V} \otimes U_G(\mathcal{M}) \to \mathcal{V} \otimes \mathcal{M}$ induces an isomorphism $\delta' : \mathcal{V} \otimes U_G(\mathcal{M}) \to U_G(\mathcal{V} \otimes \mathcal{M})$.

4 *Let $\mathcal{M}, \mathcal{N} \in \mathrm{Qch}(G, S)$, and assume that $\mathcal{M}$ is trivial and $\mathcal{N}$ is anti-trivial. Then $\underline{\mathrm{Hom}}_{G,\mathcal{O}_S}(\mathcal{M}, \mathcal{N}) = 0 = \underline{\mathrm{Hom}}_{G,\mathcal{O}_S}(\mathcal{N}, \mathcal{M})$.*

5 *Let $\mathcal{V} \in \mathrm{Qch}(G, S)$ be G-trivial, and $\mathcal{M} \in \mathrm{Qch}(G, S)$. Then $\underline{\mathrm{Hom}}_{\mathcal{O}_S}(\mathcal{V}, \mathcal{M}^G)$ is G-trivial, and $\underline{\mathrm{Hom}}_{G,\mathcal{O}_S}(\mathcal{V}, U_G(\mathcal{M})) = 0$. In particular,*

$$\underline{\mathrm{Hom}}_{\mathcal{O}_S}(\mathcal{V}, \mathcal{M}^G) \to \underline{\mathrm{Hom}}_{\mathcal{O}_S}(\mathcal{V}, \mathcal{M})$$

is an isomorphism onto $\underline{\mathrm{Hom}}_{\mathcal{O}_S}(\mathcal{V}, \mathcal{M})^G$. If, moreover, $\underline{\mathrm{Hom}}_{\mathcal{O}_S}(\mathcal{V}, \mathcal{M})$ is quasi-coherent, then

$$\underline{\mathrm{Hom}}_{\mathcal{O}_S}(\mathcal{V}, U_G(\mathcal{M})) \to \underline{\mathrm{Hom}}_{\mathcal{O}_S}(\mathcal{V}, \mathcal{M})$$

is an isomorphism onto $U_G(\underline{\mathrm{Hom}}_{\mathcal{O}_S}(\mathcal{V}, \mathcal{M}))$.

6 *Let $\mathcal{V} \in \mathrm{Qch}(G, S)$ be G-trivial, and $\mathcal{M} \in \mathrm{Qch}(G, S)$. Then $\underline{\mathrm{Hom}}_{\mathcal{O}_S}(\mathcal{M}^G, \mathcal{V})$ is G-trivial, and $\underline{\mathrm{Hom}}_{G,\mathcal{O}_S}(U_G(\mathcal{M}), \mathcal{V}) = 0$. In particular, the monomorphism $p^*_{G,S} : \underline{\mathrm{Hom}}_{\mathcal{O}_S}(\mathcal{M}^G, V) \to \underline{\mathrm{Hom}}_{\mathcal{O}_S}(\mathcal{M}, \mathcal{V})$ is an isomorphism onto $\underline{\mathrm{Hom}}_{\mathcal{O}_S}(\mathcal{M}, \mathcal{V})^G$. If, moreover, $\underline{\mathrm{Hom}}_{\mathcal{O}_S}(\mathcal{M}, \mathcal{V})$ is quasi-coherent, then the monomorphism $q^*_{G,S} : \underline{\mathrm{Hom}}_{\mathcal{O}_S}(U_G(\mathcal{M}), \mathcal{V}) \to \underline{\mathrm{Hom}}_{\mathcal{O}_S}(\mathcal{M}, \mathcal{V})$ is an isomorphism onto $U_G(\underline{\mathrm{Hom}}_{\mathcal{O}_S}(\mathcal{M}, \mathcal{V}))$.*

Proof. **0** is trivial by Corollary 5.11.

1 We prove that ϵ is an isomorphism. The case that h is an open immersion (from a sufficiently small affine open subset) follows from Lemma 5.10. The case that both Y and Y' are affine follows from Lemma 5.9. The general case follows from these, using Corollary 2.5.

The latter part is obvious, because the composite map is the identity on $(?)^G h^*$ by Lemma 2.4.

2 is proved similarly to **1**.

3 As $\mathcal{V} \otimes \mathcal{M}^G$ is G-trivial, it suffices to show that $\mathcal{V} \otimes U_G(\mathcal{M})$ is anti-G-trivial. This is checked locally, and we may assume that Y is affine. Then this is Lemma 5.8.

4 It suffices to show that $\mathrm{Hom}_{G,\mathcal{O}_U}(\mathcal{M}|_U, \mathcal{N}|_U) = 0 = \mathrm{Hom}_{G,\mathcal{O}_U}(\mathcal{N}|_U, \mathcal{M}|_U)$ for any affine open subset U. As $\mathcal{M}|_U$ is trivial and $\mathcal{N}|_U$ is anti-trivial, This is checked in (5.2).

5, 6 follow from **4**. $\square$

Lemma 5.13. *Let G be a Reynolds S-group scheme. Let $\mathcal{B}$ be a quasi-coherent $(G, \mathcal{O}_S)$-algebra, and $\mathcal{A} = \mathcal{B}^G$. Then the Reynolds operator $p_{G,S} : \mathcal{B} \to \mathcal{A}$ is $(G, \mathcal{A})$-linear. In particular, $\mathcal{A}$ is a direct summand subalgebra of $\mathcal{B}$.*

Proof. Set $Y := \underline{\mathrm{Spec}}_S \mathcal{A}$. Then $p_{G,S}$ is identified with $p_{G,Y}$, which is $\mathcal{A}$-linear. □

Example 5.14. Let $S = \operatorname{Spec} k$. We say that an affine algebraic k-group scheme G is linearly reductive if any G-module is completely reducible. In this case, for a G-module M, letting $U_G(M)$ to be the sum of all the nontrivial simple G-submodules, we have a decomposition $M = M^G \oplus U_G(M)$ of G-modules. When we set $p : M \to M^G$ to be the projection with respect to this decomposition, we have that p is a Reynolds operator. Conversely, if G is an affine algebraic k-group scheme which is Reynolds, then G is linearly reductive. Assume the contrary. Then there is a non-semisimple G-module V. Let W be its socle. Then we have that $V/W \neq 0$, and there is a simple submodule E of V/W. Then $0 = H^1(G, E^* \otimes W) \cong \operatorname{Ext}^1_G(E, W) \neq 0$, a contradiction.

Example 5.15. Let R be a commutative ring, and H a flat commutative R-Hopf algebra. We say that H is Reynolds if there is a decomposition $H = R \oplus U$ as an R-coalgebra, where R denotes the image $u(R)$ of the unit map $u : R \to H$. Then for any H-comodule M, the decomposition $M = M^H \oplus \operatorname{ind}^U_H M$ is functorial, and so letting $U_G = \operatorname{ind}^U_H$, we have that $G = \operatorname{Spec} H$ has a Reynolds operator over $S = \operatorname{Spec} R$. Hence G_Y is Reynolds for any R-scheme Y. Conversely, if G has a Reynolds operator on S, then the right regular representation H is decomposed as $H = R \oplus U$, where $U = U_G(H)$. As U is a right subcomodule, $\Delta(U) \subset U \otimes H$. Letting G act trivially on U and right regularly on H, $U \otimes H$ is a G-module, and $\Delta : U \to U \otimes H$ is G-linear. So

$$\Delta(U) \subset U_G(U \otimes H) = U \otimes U$$

by Lemma 5.8. Being a direct summand of H, U is a subcoalgebra of H, and $H = R \oplus U$ is a decomposition as subcoalgebras, and hence H is Reynolds.

Example 5.16. Let G be a finite group of order n, and $R = \mathbb{Z}[n^{-1}]$. Then $\rho = n^{-1} \sum_{g \in G} g \in RG$ is a central idempotent of RG such that $g\rho = \rho = \rho g$ for $g \in G$, where RG is the group algebra. Then we have a decomposition of bimodules $RG = \rho(RG) \oplus (1 - \rho)(RG)$. So we have a decomposition of $R[G]$-bicomodules

$$R[G] = (RG)^* = (\rho(RG))^* \oplus ((1 - \rho)(RG))^* = \rho R[G] \oplus (1 - \rho) R[G].$$

As $\varepsilon : \rho RG \to R$ is an isomorphism (since $\varepsilon(\rho) = 1$), $\rho R[G] = R$, and G is Reynolds. The Reynolds operator p is the action of ρ, and the element $\rho \in RG$ is also called the Reynolds operator.

Example 5.17. Let Λ be an additive abelian group, and R a commutative ring. Then the group ring $R\Lambda = \bigoplus_{\lambda\in\Lambda} Rt^\lambda$ is a Hopf algebra, letting each t^λ ($\lambda \in \Lambda$) group-like (that is, $\Delta(t^\lambda) = t^\lambda \otimes t^\lambda$). Then as $R\Lambda = R \oplus (\bigoplus_{\lambda\neq 0} R\cdot t^\lambda)$, $R\Lambda$ is Reynolds. If $\Lambda \cong \mathbb{Z}^s$, then $G = \operatorname{Spec} R\Lambda$ is a split torus (of relative dimension s), and G is Reynolds.

(5.18) Let Y be an S-scheme, and G an S-group scheme. Let κ be an infinite regular cardinal such that S, G, and Y are κ-schemes [Has11, (3.11)]. As in [Has11, (3.14)], we denote the full subcategory of the category of Y-schemes consisting of κ-morphisms by $(\underline{\mathrm{Sch}}/Y)_\kappa$. We call a presheaf on $(\underline{\mathrm{Sch}}/Y)_\kappa$ a Y-prefaisceau. We denote the structure presheaf of $(\underline{\mathrm{Sch}}/Y)_\kappa$ by $\mathcal{O}$. For an $\mathcal{O}_Y$-module $\mathcal{M}$ (in the Zariski topology), we denote the associated Y-prefaisceau by $\mathcal{M}_a$. That is, $\mathcal{M}_a$ is the $\mathcal{O}$-module given by $\Gamma(h : Z \to Y, \mathcal{M}_a) = \Gamma(Z, h^*\mathcal{M})$. For an $\mathcal{O}_Y$-module $\mathcal{M}$, the Y-prefaisceau of groups $Z \mapsto \operatorname{End}_{\mathcal{O}_Z} \Gamma(Z, \mathcal{M}_a)^\times$ is denoted by $GL(\mathcal{M})$, and called the *general linear group* of $\mathcal{M}$.

(5.19) Let $(\mathcal{M}, \phi)$ be a G-linearized $\mathcal{O}_Y$-module. Then for any S-scheme W, $y \in Y(W)$ and $\alpha, \beta \in G(W)$, the composite

$$(\alpha\beta y)^*\mathcal{M} \xrightarrow{(\alpha,\beta y)^*\phi} (\beta y)^*\mathcal{M} \xrightarrow{(\beta,y)^*\phi} y^*\mathcal{M}$$

agrees with $(\alpha\beta, y)^*\phi$, see [MuFK, (1.3)].

(5.20) Assume that the action of G on Y is trivial. Then we denote $(\alpha, y)^*\phi : y^*\mathcal{M} \to y^*\mathcal{M}$ by $h(\alpha)$. Then by the argument above, $h(\alpha)h(\beta) = h(\alpha\beta)$, and we get a homomorphism between Y-prefaisceaux of groups $h : G_Y \to GL(\mathcal{M})$, where G_Y is the restriction of G to $(\underline{\mathrm{Sch}}/Y)_\kappa$. For a given $\mathcal{O}_Y$-module $\mathcal{M}$ on an S-scheme Y with a trivial G-action, giving a G-linearization ϕ and giving a group homomorphism $G_Y \to GL(\mathcal{M})$ are the same thing. $\mathcal{M}$ is quasi-coherent if and only if $B \otimes_A \Gamma(\operatorname{Spec} A, \mathcal{M}_a) \to \Gamma(\operatorname{Spec} B, \mathcal{M}_a)$ is an isomorphism for any morphism of the form $\operatorname{Spec} B \to \operatorname{Spec} A$ in $(\underline{\mathrm{Sch}}/Y)_\kappa$.

(5.21) If G is S-flat, then we modify the construction above, and we consider the full subcategory $\mathcal{E}$ of $(\underline{\mathrm{Sch}}/Y)_\kappa$ consisting of flat Y-schemes. If $\mathcal{M}$

is a G-equivariant module on Y (in the Zariski topology), then we get a homomorphism $h : G_Y \to GL(\mathcal{M})$ of prefaisceaux of groups on $(\underline{\mathrm{Sch}}/Y)_\kappa$. By restriction, we get $h : G|_{\mathcal{E}} \to GL(\mathcal{M})|_{\mathcal{E}}$. As G is flat, it is easy to see that giving such a homomorphism is the same thing as to give a G-linearization on $\mathcal{M}$.

(5.22) Let G be S-flat, $\mathcal{M}$ a G-equivariant module on Y, and $\mathcal{N}$ its $\mathcal{O}_Y$-submodule. Although $\mathcal{N}_a$ may not be a submodule of $\mathcal{M}_a$, we have that $\mathcal{N}_a|_{\mathcal{E}}$ is a submodule of $\mathcal{M}_a|_{\mathcal{E}}$ by flatness.

Lemma 5.23. *Let the notation be as above. Then $\mathcal{N}$ is a $(G, \mathcal{O}_Y)$-submodule of $\mathcal{M}$ if and only if for each object U in $\mathcal{E}$ which is an affine scheme, $\Gamma(U, \mathcal{N}_a)$ is a $G(U)$-submodule of $\Gamma(U, \mathcal{M}_a)$.*

Proof. The 'only if' part is trivial. We prove the 'if' part.

Let U be any affine open subset of $G \times Y$. Then $U \xrightarrow{j} G \times Y \xrightarrow{p_2} Y$ lies in $\mathcal{E}$ and U is affine. where j is the inclusion, and p_2 is the second projection. Let $g \in G|_{\mathcal{E}}(U)$ be the map $p_1 j : U \to G$, where $p_1 : G \times Y \to G$ is the first projection. The action of g on $\Gamma(p_2 j : U \to Y, \mathcal{M}_a)$ is induced by $j^*\phi : j^* p_2^* \mathcal{M} \to j^* p_2^* \mathcal{M}$, where ϕ is the linearization of $\mathcal{M}$.

By assumption, the actions of g and g^{-1} preserve the submodule $\Gamma(p_2 j : U \to Y, \mathcal{N}_a)$. As U is arbitrary, $p_2^* \mathcal{N}$ is preserved by the linearization ϕ and its inverse ϕ^{-1}. Hence $\mathcal{N}$ is a G-equivariant submodule. $\square$

Lemma 5.24. *Let G be a flat quasi-compact quasi-separated S-group scheme, Y an S-scheme on which G acts trivially, and $\mathcal{M}$ a quasi-coherent $(G, \mathcal{O}_Y)$-module. Then $(\mathcal{M}^G)_a|_{\mathcal{E}}$ is an $\mathcal{O}$-submodule of $\mathcal{M}_a|_{\mathcal{E}}$ given by*

$$\begin{aligned}\Gamma(W, (\mathcal{M}^G)_a) = \{m \in \Gamma(W, \mathcal{M}_a) \mid gm = m \ \textit{in}\ \Gamma(W', \mathcal{M}_a) \\ \textit{for any morphism } W' \to W \textit{ in } \mathcal{E}, \textit{ and any } g \in G(W')\}.\end{aligned}$$

Proof. First we prove the assertion for $W = Y$. Then the left-hand side is

$$\{m \in \Gamma(Y, \mathcal{M}) \mid g_0 m = m \text{ in } \Gamma(G \times Y, \mathcal{M})\},$$

where $g_0 \in G(G \times Y)$ is the first projection. So the right-hand side is contained in the left-hand side. On the other hand, if $h : W' \to Y$ is any morphism in $\mathcal{E}$, then for any $g \in G(W')$, we define $\psi_g : W' \to G \times Y$ by (g, h). Then by definition, $g = \psi_g(g_0)$. So if $g_0 m = m$ in $\Gamma(G \times Y, \mathcal{M})$, then $gm = m$ in $\Gamma(W', \mathcal{M})$, and the equality was proved.

Next consider general $W \in \mathcal{E}$. Then replacing Y by W using [HasO, (7.5)], the problem is reduced to the case $Y = W$, and we are done. $\square$

Proposition 5.25. *Let $f : G \to H$ be a quasi-compact quasi-separated flat homomorphism of S-group schemes with $N = \operatorname{Ker} f$. Then for $\mathcal{M} \in \mathrm{Qch}(G, Y)$, $\mathcal{M}^N$ is a quasi-coherent $(G, \mathcal{O}_Y)$-submodule of $\mathcal{M}$.*

If, moreover, N is Reynolds, then $U_N(\mathcal{M})$ is also a quasi-coherent $(G, \mathcal{O}_Y)$-submodule of $\mathcal{M}$. In particular, the Reynolds operator $p_{N,Y} : \mathcal{M} \to \mathcal{M}^N$ is $(G, \mathcal{O}_Y)$-linear.

Proof. Although the first assertion can be proved in the same line of [Has11, (6.18)], we give a new proof.

For the first assertion, in view of Lemma 5.23 and Lemma 5.24, it suffices to show that for each flat morphism $h : W \to Y$ and any morphism $h' : W' \to W$ such that hh' is flat, $m \in \Gamma(W, (\mathcal{M}^G)_a)$, $g \in G(W)$, and $n \in G(W')$, we have that $ngm = gm$ in $\Gamma(W', \mathcal{M}_a)$. As N is normal in G, $ngm = g(g^{-1}ng)m = gm$. As the quasi-coherence is trivial, the first assertion has been proved.

Assume that N is Reynolds. Let $W = \operatorname{Spec} A \to Y$ be an object of $\mathcal{E}$ such that W is affine, and $g \in G(W)$. Set $U = \Gamma(W, U_N(\mathcal{M})_a)$. We claim that gU is an (N, A)-submodule of $M := \Gamma(W, \mathcal{M}_a)$. In order to show this, it suffices to show that for any flat κ-morphism $W' = \operatorname{Spec} A' \to W$ and $n \in N(W')$, we have $n((gU) \otimes_A A') \subset (gU) \otimes_A A'$ in $M \otimes_A A'$. This is clear, since

$$n(gu \otimes a') = g((g^{-1}ng)(u \otimes a')) \in g(U \otimes_A A') = (gU) \otimes_A A'$$

for $u \in U$ and $a' \in A'$.

Next, we prove that $(gU)^N = 0$. Assume the contrary, and take $u \in U \setminus 0$ such that for each flat κ-morphism $W' \to W$ and $n \in N(W')$, $ngu = gu$. Then $nu = g^{-1}(gng^{-1})gu = g^{-1}gu = u$, and hence $u \in U \cap M^N = 0$, and this is a contradiction. Hence $(gU)^N = 0$. That is, $gU \subset U$. By Lemma 5.23, we have that $U_N(\mathcal{M})$ is a G-equivariant submodule. Quasi-coherence is trivial.

The last assertion is clear from the fact that the decomposition $\mathcal{M} = \mathcal{M}^N \oplus U_N(\mathcal{M})$ is that of a $(G, \mathcal{O}_Y)$-module. $\square$

Lemma 5.26. *Let G be a flat quasi-compact quasi-separated S-group scheme, and X a G-scheme. Then $\mathrm{Qch}(G, X)$ is a Grothendieck category.*

Proof. In view of [Has5, (11.5)], it suffices to show that $\mathrm{Qch}(X)$ is Grothendieck. This is Gabber's theorem [Con, (2.1.7)]. $\square$

Lemma 5.27. *Let S be a scheme, and G a Reynolds group over S. Let Y be an S-scheme on which G acts trivially. Let $\mathcal{I}$ be an injective object of* $\mathrm{Qch}(Y)$*. Then $\mathcal{I}$ viewed as an object of* $\mathrm{Qch}(G, Y)$ *(formally* $\mathrm{res}^e_G \mathcal{I}$*) is an injective object.*

Proof. Set $\mathcal{J} := \mathrm{res}^e_G \mathcal{I}$. Let $i : \mathcal{J} \hookrightarrow \mathcal{M}$ be a monomorphism in $\mathrm{Qch}(G, Y)$. By Lemma 5.26, we have that $\mathrm{Qch}(G, Y)$ has enough injectives, and hence it suffices to show that i splits. As we have that $\mathcal{J} = \mathcal{J}^G$, the image of i is contained in $\mathcal{M}^G$. As $\mathcal{M}^G$ is a direct summand of $\mathcal{M}$, it suffices to show that $i : \mathcal{J} \hookrightarrow \mathcal{M}^G$ splits. Note that $(?)_0 : \mathrm{Qch}_G(G, Y) \to \mathrm{Qch}(Y)$ and $\mathrm{res}^e_N :$ $\mathrm{Qch}(Y) \to \mathrm{Qch}_G(G, Y)$ are quasi-inverse each other, where $\mathrm{Qch}_G(G, Y)$ is the category of G-trivial objects in $\mathrm{Qch}(G, Y)$. So it suffices to show that $\mathcal{I} \hookrightarrow \mathcal{M}^G$ splits in $\mathrm{Qch}(Y)$. This is obvious, since $\mathcal{I}$ is injective by assumption. □

Lemma 5.28. *Let $f : G \to H$ be an fpqc homomorphism of flat S-group schemes with $N = \mathrm{Ker}\, f$. Assume that N is Reynolds. Let Y be a locally Noetherian G-scheme on which N acts trivially. Let $\mathbb{F} \in D^-_{\mathrm{Coh}}(G, Y)$ and $\mathbb{G} \in D^+_{\mathrm{Qch}}(G, Y)$. If for each $i \in \mathbb{Z}$, $H^i(\mathbb{F}) = U_N(H^i(\mathbb{F}))$ and $H^i(\mathbb{G}) = H^i(\mathbb{G})^N$, then $\underline{\mathrm{Ext}}^i_{\mathcal{O}_{B^G_M(Y)}}(\mathbb{F}, \mathbb{G})^N = 0$ for $i \in \mathbb{Z}$.*

Proof. Note that the full subcategory $\mathrm{Coh}_N(G, Y)$ (resp. $U_N(G, Y)$) consisting of N-trivial (resp. N-anti-trivial) coherent (resp. quasi-coherent) G-modules forms a plump subcategory (that is, a full subcategory which is closed under extensions, kernels, and cokernels) of $\mathrm{Mod}(G, Y)$. So we may assume that $\mathbb{F} = \mathcal{M}$ and $\mathbb{G} = \mathcal{N}$ are single coherent sheaf and quasi-coherent sheaf, respectively, by the way-out lemma [Hart].

Note that $\underline{\mathrm{Ext}}^i_{\mathcal{O}_{B^G_M(Y)}}(\mathcal{M}, \mathcal{N})^N = 0$ if and only if

$$(\mathrm{res}^G_N(\underline{\mathrm{Ext}}^i_{\mathcal{O}_{B^G_M(Y)}}(\mathcal{M}, \mathcal{N})))^N \cong \underline{\mathrm{Ext}}^i_{\mathcal{O}_{B^M_N(Y)}}(\mathrm{res}^G_N \mathcal{M}, \mathrm{res}^G_N \mathcal{N})^N = 0$$

by Corollary 4.11. Set $\mathcal{M}' := \mathrm{res}^G_N \mathcal{M}$ and $\mathcal{N}' := \mathrm{res}^G_N \mathcal{N}$.

Let $0 \to \mathcal{N}'_0 \to \mathbb{I}$ be an injective resolution in $\mathrm{Qch}(Y)$. Then applying Lemma 4.2, Lemma 5.27, and [Has5, (15.2)], we have that $0 \to \mathrm{res}^{\{e\}}_N \mathcal{N}'_0 \to \mathrm{res}^{\{e\}}_N \mathbb{I}$ is an $\underline{\mathrm{Hom}}_{\mathcal{O}_{B^M_N(Y)}}(\mathcal{M}', ?)$-acyclic resolution. By assumption, $\mathcal{N}'$ is of the form $\mathrm{res}^{\{e\}}_N \mathcal{L}$ for some $\mathcal{L} \in \mathrm{Qch}(Y)$. Then $\mathrm{res}^{\{e\}}_N \mathcal{N}'_0 \cong \mathrm{res}^{\{e\}}_N \mathcal{L} \cong \mathcal{N}'$. So $\underline{\mathrm{Ext}}^i_{\mathcal{O}_{B^M_N(Y)}}(\mathcal{M}', \mathcal{N}')^N$ is a subquotient of $\underline{\mathrm{Hom}}_{\mathcal{O}_{B^M_N(Y)}}(\mathcal{M}', \mathbb{I}^i)^N$. This is zero by Lemma 5.12, **4**. □

Lemma 5.29. *Let G be a Reynolds S-group scheme, and $\varphi : X \to Y$ an algebraic quotient by G. Then φ is a universal algebraic quotient. If, moreover, φ is an affine universally submersive geometric quotient, then φ is a universal geometric quotient.*

Proof. Let $h : Y' \to Y$ be an S-morphism between S-schemes on which G acts trivially, and consider the fiber square (4) in Lemma 2.19. As φ is affine, $\theta : h^*\varphi_*\mathcal{O}_X \to \varphi'_*g^*\mathcal{O}_X$ is an isomorphism. Now the first assertion follows from Lemma 2.21, **1** and Lemma 5.12, **1**. The second assertion follows immediately. $\square$

6. Base change of twisted inverse

Lemma 6.1. *Let*

(12)
$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "\varphi'"] \arrow[dr, phantom, "\sigma"] & X \arrow[d, "\varphi"] \\ Y' \arrow[r, "h"] & Y \end{tikzcd}$$

be a fiber square of schemes. Assume that φ is quasi-compact quasi-separated. Then the following are equivalent.

1 *Lipman's theta $\theta : Lh^*R\varphi_* \to R\varphi'_*Lg^*$ between the functors $D_{\mathrm{Qch}}(X) \to D_{\mathrm{Qch}}(Y')$ (cf. [Lip, (3.9.1), (3.9.2)]) is an isomorphism.*

2 *The square is tor-independent in the sense that for each $x \in X$ and $y' \in Y'$ such that $\varphi(x) = h(y') = y \in Y$, $\mathrm{Tor}_i^{\mathcal{O}_{Y,y}}(\mathcal{O}_{X,x}, \mathcal{O}_{Y',y'}) = 0$ for $i > 0$.*

Proof. As the question is local both on Y and Y', we may assume that both Y and Y' are affine. This case is [Lip, (3.10.3)]. $\square$

(6.2) Let I be a small category, and (12) be a tor-independent fiber square of I^{op}-diagrams of Noetherian schemes such that $\varphi : X \to Y$ is proper. By Lemma 6.1, Lipman's theta $\theta : Lh^*R\varphi_* \to R\varphi'_*Lg^*$ is an isomorphism (between functors from $D_{\mathrm{Lqc}}(X) \to D_{\mathrm{Lqc}}(Y')$. Then we define $\zeta(\sigma) : Lg^*\varphi^\times \to (\varphi')^\times h^*$ as the composite

$$Lg^*\varphi^\times \xrightarrow{u} (\varphi')^\times R\varphi'_*Lg^*\varphi^\times \xrightarrow{\theta^{-1}} (\varphi')^\times Lh^*R\varphi_*\varphi^\times \xrightarrow{\varepsilon} (\varphi')^\times Lh^*$$

(cf. [Has5, (19.1)]).

Lemma 6.3. *Let*

$$(13)\qquad \begin{tikzcd} X'' \arrow[r, "g'"] \arrow[d, "\varphi''"] \arrow[dr, phantom, "\sigma'"] & X' \arrow[r, "g"] \arrow[d, "\varphi'"] \arrow[dr, phantom, "\sigma"] & X \arrow[d, "\varphi"] \\ Y'' \arrow[r, "h'"] & Y' \arrow[r, "h"] & Y \end{tikzcd}$$

be a commutative diagram of schemes. Assume that σ is a tor-independent cartesian square. Then σ' is tor-independent cartesian if and only if the whole rectangle $\sigma' + \sigma$ is a tor-independent cartesian square.

Proof. As σ is cartesian, σ' is cartesian if and only if $\sigma' + \sigma$ is cartesian.

Let $y'' \in Y''$ and $x \in X$ such that $hh'(y'') = \varphi(x)$. Let $A'' = \mathcal{O}_{Y'',y''}$, $A' = \mathcal{O}_{Y',y'}$, $A = \mathcal{O}_{Y,y}$, and $B = \mathcal{O}_{X,x}$, where $y' = h'(y'')$ and $y = h(y') = \varphi(x)$. There is a spectral sequence

$$E^2_{p,q} = \operatorname{Tor}_p^{A'}(A'', \operatorname{Tor}_q^A(A', B)) \Rightarrow \operatorname{Tor}_{p+q}^A(A'', B).$$

By assumption, $E^2_{p,q} = 0$ for $q \neq 0$. So $\operatorname{Tor}_n^{A'}(A'', A' \otimes_A B) \cong \operatorname{Tor}_n^A(A'', B)$, and the equivalence follows. $\square$

Lemma 6.4. *Let I be a small category, and (13) be a diagram of I^{op}-diagrams of Noetherian schemes. Assume that φ is proper, and σ and σ' are tor-independent fiber squares. Then the composite*

$$L(gg')^*\varphi^\times \xrightarrow{d} L(g')^*Lg^*\varphi^\times \xrightarrow{\zeta(\sigma)} L(g')^*(\varphi')^\times Lh^*$$
$$\xrightarrow{\zeta(\sigma')} (\varphi'')^\times L(h')^*Lh^* \xrightarrow{d} (\varphi'')^\times L(hh')^*$$

agrees with $\zeta(\sigma' + \sigma)$.

Proof. Straightforward, and left to the reader (use [Has5, (1.23)]). $\square$

Lemma 6.5. *Let*

$$(14)\qquad \begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "\psi'"] \arrow[dr, phantom, "\sigma'"] & X \arrow[d, "\psi"] \\ Z' \arrow[r, "g"] \arrow[d, "\varphi'"] \arrow[dr, phantom, "\sigma"] & Z \arrow[d, "\varphi"] \\ Y' \arrow[r, "h"] & Y \end{tikzcd}$$

be a diagram of I^{op}-diagrams of Noetherian schemes. Assume that φ and ψ are proper, and σ and σ' are tor-independent fiber squares. Then the composite

$$Lf^*(\varphi\psi)^\times \xrightarrow{d} Lf^*\psi^\times\varphi^\times \xrightarrow{\zeta(\sigma')} (\psi')^\times Lg^*\varphi^\times$$
$$\xrightarrow{\zeta(\sigma)} (\psi')^\times(\varphi')^\times Lh^* \xrightarrow{d} (\varphi'\psi')^\times Lh^*$$

agrees with $\zeta(\sigma'+\sigma)$.

Proof. Left to the reader (use [Has5, (1.22)]). □

(6.6) Let G be an S-group scheme, Y be a G-scheme, and X a G-stable subscheme. That is, X is a subscheme of Y such that $GX \subset X$. Then X is a closed subscheme of an open subscheme U of Y. Assume that G is universally open (e.g., flat locally of finite presentation) over S. Then GU is a G-stable open subscheme of Y, and $X = GU \setminus G(U \setminus X)$ is a G-stable closed subset in GU. Being a G-stable subscheme of Y, X is a G-stable closed subscheme of GU. Thus if G is universally open, a G-stable subscheme is nothing but a G-stable closed subscheme of a G-stable open subscheme.

(6.7) Let G be a flat S-group scheme of finite type, and $\varphi : X \to Y$ an immersion between Noetherian G-schemes. As we have seen in (6.6), We can factorize φ as

$$\varphi : X \xrightarrow{p} U \xrightarrow{i} Y,$$

where U is a G-stable open subscheme of Y, i the inclusion, and p is a G-stable closed immersion.

Let $h : Y' \to Y$ be a G-morphism between Noetherian G-schemes. Assume that φ and h are tor-independent. Then p and (the base change of) h are also tor-independent.

For the fiber square (12), we define $\bar{\zeta}(\sigma)$ to be the composite

$$h^*\varphi^! = h^*p^\times i^* \xrightarrow{\zeta} (p')^\times g^* i^* \xrightarrow{d} (p')^\times (i')^* h^* = (\varphi')^! h^*,$$

where

(15)
$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "p'"] \arrow[dr, phantom, "\sigma_1"] & X \arrow[d, "p"] \\ U' \arrow[r, "g"] \arrow[d, "i'"] \arrow[dr, phantom, "\sigma_2"] & U \arrow[d, "i"] \\ Y' \arrow[r, "h"] & Y \end{tikzcd}$$

is a commutative diagram with σ_1 and σ_2 are cartesian, and $\varphi' = i'p'$. Using Lemma 6.4, it is easy to see that the definition of $\bar{\zeta}(\sigma)$ depends only on σ, and is independent of the choice of factorization σ_1 and σ_2.

7. Serre's conditions and the canonical modules

(7.1) Let G be an S-group scheme. We say that a G-scheme X is *G-connected* if $X = X_1 \coprod X_2$ with X_1 and X_2 are G-stable open subsets, then either X_1 or X_2 is empty. In this paper, a G-connected G-scheme is required to be nonempty. A connected topological space is also required to be nonempty. If the action of G on X is trivial, then G-connected and connected are the same thing.

Lemma 7.2. *Let G be an S-group scheme, and $\psi : X \to W$ and $u : W \to Y$ be S-morphisms. Assume that $\varphi = u\psi : X \to Y$ is G-invariant, and is a categorical quotient by G. If u is a monomorphism (e.g., an immersion), then u is an isomorphism.*

Proof. As u is a monomorphism, it is easy to see that ψ is also G-invariant. By the definition of the categorical quotient, there exists some $v : Y \to W$ such that $\psi = v\varphi$. Then $1_Y\varphi = \varphi = u\psi = uv\varphi$. As φ is a categorical quotient, $1_Y = uv$ by the uniqueness. As $u1_W = u = 1_Y u = uvu$ and u is a monomorphism, $1_W = vu$. Hence u is an isomorphism. □

Lemma 7.3. *Let G be an S-group scheme, and $\varphi : X \to Y$ a G-morphism.*

1 *If φ is dominating and X is G-connected, then Y is G-connected.*

2 *If φ is G-invariant dominating and X is G-connected, then Y is connected.*

3 *Assume that φ is a categorical quotient or an algebraic quotient. Then X is G-connected if and only if Y is connected.*

Proof. **1**. Assume the contrary, and let $Y = Y_1 \coprod Y_2$ with Y_i are nonempty G-invariant open. Then letting $X_i = \varphi^{-1}(Y_i)$, we have that $X = X_1 \coprod X_2$ with X_i nonempty G-invariant open, and this is a contradiction.

2 is obvious from **1**.

3 We prove the 'only if' part. First consider the case that φ is a categorical quotient. Assume that $Y = Y_1 \coprod Y_2$ with each Y_i nonempty open. Then φ cannot factors through Y_1 or Y_2 by Lemma 7.2. Letting $X_i = \varphi^{-1}(Y_i)$ for $i = 1, 2$, we have that each X_i is nonempty G-stable open and $X = X_1 \coprod X_2$. Next, assume that φ is an algebraic quotient. Then it is easy to see that φ is dominating and G-invariant. By **2**, if X is G-connected, then Y is connected.

We prove the 'if' part. Assume that $X = X_1 \coprod X_2$ with X_i are nonempty G-stable open. First consider the case that φ is a categorical quotient. Then the map $h = h_1 \coprod h_2 : X_1 \coprod X_2 \to S \coprod S$, where $h_i : X_i \to S$ is the structure map, factors through Y. This shows that Y cannot be connected. Next, consider the case that φ is an algebraic quotient. Let $V = \operatorname{Spec} A$ be an affine open subset of Y. Then $U = \varphi^{-1}(V) = \operatorname{Spec} B$ is affine. We have $U = U_1 \coprod U_2$ with $U_i = X_i \cap U$. Set $B_i = \Gamma(U_i, \mathcal{O}_{U_i})$, and $A_i = B_i^G$. Then $A = A_1 \times A_2$, and we can write $V = V_1 \coprod V_2$. This construction is compatible with the localization $A \to A[a^{-1}]$ for $a \in A$. So for $y \in Y$ and two affine open neighborhoods V and W, $y \in V_1$ if and only if $y \in W_1$. So letting $Y_i = \bigcup_V V_i$, we have $Y = Y_1 \coprod Y_2$. As $\varphi^{-1}(Y_i) = X_i$, both Y_1 and Y_2 are nonempty, and Y is disconnected. □

(7.4) For a subset Z of X, we say that Z is a G-stable closed subset of X if $X \setminus Z$ is a G-stable open subset. We say that X is G-Noetherian if any descending chain of G-stable closed subsets of X eventually stabilizes.

(7.5) Let X be a G-Noetherian G-scheme. A G-closed subset Z of X is said to be G-irreducible if Z is nonempty and if $Z = Z_1 \cup Z_2$, Z_1 and Z_2 are G-closed subsets of X, then either $Z = Z_1$ or $Z = Z_2$. Any G-closed subset Z of X is of the form $Z = \bigcup_{i=1}^r Z_i$ for some $r \geq 0$ and G-irreducible closed subsets Z_i. Thus we can write $X = \bigcup_{i=1}^r X_i$ with X_i G-irreducible. Let $\equiv$ be the equivalence relation on $\{1, 2, \ldots, r\}$ generated by the relation $X_i \cap X_j \neq \emptyset$. For any equivalence class γ with respect to $\equiv$, $X_\gamma := \bigcup_{i \in \gamma} X_i$ is called a *G-connected component* of X. Obviously, X_γ is G-stable closed open and G-connected, and $X = \coprod_{\gamma \in \{1,\ldots,r\}/\equiv} X_\gamma$. It is easy to see that a G-

connected component of X is nothing but a maximal G-connected G-stable closed subset of X.

(7.6) Let I be a small category, and X an I^{op}-diagram of Noetherian schemes. Then a connected component (see for the definition, [Has5, (28.1)]) U of X is a cartesian closed open subdiagram of schemes. Indeed, for any $i \in I$, U_i is the union of connected components of X_i, and for any $\phi : i \to j$, $X_\phi^{-1}(U_i) = U_j$ by the definition of connected components. In particular, we have

Lemma 7.7. *Let G be an S-group scheme and X a Noetherian G-scheme. Then a connected component of $B_M^G(X)$ is of the form $B_M^G(X_i)$ with X_i a G-connected component of X. Conversely, $B_M^G(X_i)$ with X_i a G-connected component of X is a connected component of $B_M^G(X)$.* $\square$

(7.8) Let G be an S-group scheme flat of finite type. Let X be a Noetherian G-scheme.

Lemma 7.9. *If $\mathbb{I}_X$ is a G-dualizing complex on X, $n \in \mathbb{Z}$, and $\mathcal{L}$ a G-linearized invertible sheaf on X, then $\mathbb{I}_X \otimes_{\mathcal{O}_X} \mathcal{L}[n]$ is a G-dualizing complex on X. If $\mathbb{I}_X$ and $\mathbb{I}'_X$ are two G-dualizing complex on X and X is G-connected, then $R\operatorname{Hom}_{\mathcal{O}_X}(\mathbb{I}_X, \mathbb{I}'_X) \cong \mathcal{L}[n]$ for some n and $\mathcal{L}$, and we have $\mathbb{I}'_X \cong \mathbb{I}_X \otimes_{\mathcal{O}_X} \mathcal{L}[n]$.*

Proof. This is [Has5, (31.12)]. $\square$

(7.10) Let G be an S-group scheme flat of finite type. Let X be a Noetherian G-scheme. Let $\mathbb{I}_X$ be a G-dualizing complex on X. Letting s be the smallest integer such that $H^i(\mathbb{I}_X) \neq 0$, we define ω_X to be $H^s(\mathbb{I}_X)$. We call ω_X the *G-canonical module* corresponding to $\mathbb{I}_X$. If $X = \coprod X_i$ with each X_i G-connected, then we define ω'_X by $\omega'_X|_{X_i} = \omega_{X_i}$. We call ω'_X the *componentwise G-canonical module* (in [Has5, (31.13)], we called ω'_X the G-canonical module, but this is less useful in this paper, and we change the terminology).

A coherent $(G, \mathcal{O}_X)$-module ω (resp. ω') is called a G-canonical module (resp. a componentwise G-canonical module) if there exists some G-dualizing complex $\mathbb{I}$ on X such that ω (resp. ω') is isomorphic to the G-canonical module (resp. a componentwise G-canonical module) corresponding to $\mathbb{I}$. Thus if ω (resp. ω') is a G-canonical module (resp. a componentwise G-canonical module) and $\mathcal{L}$ is a G-linearized invertible sheaf, then $\omega \otimes_{\mathcal{O}_X} \mathcal{L}$ (resp. $\omega' \otimes_{\mathcal{O}_X} \mathcal{L}$) is a G-canonical module (resp. a componentwise G-canonical module). If ω' and ω'' are two componentwise G-canonical modules on X, then there exists some G-linerized invertible sheaf $\mathcal{L}$ on X such that $\omega'' \cong \omega' \otimes_{\mathcal{O}_X} \mathcal{L}$. If G is trivial, then a G-canonical module and a G-componentwise canonical module (with respect to $\mathbb{I}$) are called a canonical module and a componentwise canonical module (with respect to $\mathbb{I}$), respectively, where $\mathbb{I}$ is a dualizing complex of X.

Lemma 7.11. *Let $h : G' \to G$ be a homomorphism between S-group schemes flat of finite type. Let X be a Noetherian G-scheme. Let $\mathbb{I}_X(G)$ be a G-dualizing complex on X, and set $\mathbb{I}_X(G') := L\operatorname{res}^G_{G'} \mathbb{I}_X(G)$ (see* Lemma 4.17*). Let $\omega_X(G)$ be the G-canonical module corresponding to $\mathbb{I}_X(G)$. Then $\omega_X(G') := \operatorname{res}^G_{G'} \omega_X(G)$ is the G'-canonical module corresponding to the G'-dualizing complex $\mathbb{I}_X(G')$.*

Proof. Let $s = \inf\{i \in \mathbb{Z} \mid H^i(\mathbb{I}_X(G)) \neq 0\}$. By Lemma 4.3, $\omega_X(G') = H^s(\mathbb{I}_X(G'))$ and $H^i(\mathbb{I}_X(G')) = 0$ for $i < s$. As $\operatorname{res}^G_{G'} : \operatorname{Qch}(G,X) \to \operatorname{Qch}(G',X)$ is faithful by Lemma 4.2, $\omega_X(G') \neq 0$, and we are done. □

A similar compatibility with the change of groups does not hold for the componentwise canonical module.

(7.12) Let X be a locally Noetherian scheme. A coherent $\mathcal{O}_X$-module ω is said to be *semicanonical* at $x \in X$ if $\omega_{X,x}$ is either zero, or is the canonical module [Aoy, (1.1)] of the local ring $\mathcal{O}_{X,x}$. We say that ω is semicanonical if it is so at each point. Being a semicanonical module is a local condition. That is, if ω is semicanonical and $U \subset X$ is an open subset, then $\omega|_U$ is semicanonical on U. If (U_i) is an open covering of X and each $\omega|_{U_i}$ is semicanonical, then ω is semicanonical.

Lemma 7.13. *Let the notation be as in* (7.10)*. Then the canonical module ω_X and the componentwise canonical module corresponding to $\mathbb{I}_X$ are semicanonical $\mathcal{O}_X$-modules.*

Proof. Let $x \in X$. Then $\mathbb{I}_{X,x}$ is a dualizing complex in the usual sense for the local ring $\mathcal{O}_{X,x}$. So if $\omega_{X,x} \neq 0$, then by the definition of the canonical module for a local ring [Aoy, (1.1)] and the local duality [Hart, (6.3)], $\omega_{X,x}$ is the canonical module of $\mathcal{O}_{X,x}$. Using this result componentwise, we get a similar result for ω'_X. □

(7.14) For a local ring $(A, \mathfrak{m})$ and an A-module M, we define $\operatorname{depth}_A M = \inf\{n \in \mathbb{Z} \mid H^n_{\mathfrak{m}}(M) \neq 0\}$. For a commutative ring A, an ideal I, and an A-module M, we define

$$\operatorname{depth}_A(I, M) = \operatorname{depth}(I, M) := \inf_{P \in V(I)} \operatorname{depth}_{A_P} M_P.$$

If A is Noetherian and M is finitely generated, then we have

$$\operatorname{depth}_A(I, M) = \inf\{n \in \mathbb{Z} \mid \operatorname{Ext}^n_A(A/I, M) \neq 0\},$$

which also equals the length of a maximal M-sequence in I (provided $M \neq IM$), see [Hart2, (3.4), (3.6), (3.10)].

Lemma 7.15. *Let A be a Noetherian ring, M a finite A-module, and I an ideal of A. Then*

$$\operatorname{depth}_A(I, M) = \inf\{n \in \mathbb{Z} \mid H^n_I(M) \neq 0\}.$$

Proof. If $\operatorname{depth}_A(I, M) \geq n$, then $\operatorname{Ext}^i_A(A/I^j, M) = 0$ for $i < n$ and any $j \geq 1$ by [Mat, (16.6)], and hence $H^i_I(M) = \varinjlim \operatorname{Ext}^i_A(A/I^j, M) = 0$ for $i < n$, and the right-hand side is $\geq n$.

We prove the converse. Let $n \geq 0$. We want to prove that for a finite A-module M, $H^i_I(M) = 0$ for $i < n$ implies that $\operatorname{depth}_A(I, M) \geq n$. We prove this by induction on n. If $n = 0$, this is trivial. Assume that $n > 0$. Then as $\operatorname{Hom}_A(A/I, M) \subset H^0_I(M) = 0$, we have that $\operatorname{depth}_A(I, M) > 0$. So we can find a nonzerodivisor $a \in I$ on M. Then we have a long exact sequence

$$\cdots \to H^i_I(M) \xrightarrow{a} H^i_I(M) \to H^i_I(M/aM) \to H^{i+1}_I(M) \to \cdots$$

of the local cohomology. By assumption, we have that $H^i_I(M/aM) = 0$ for $i < n-1$. By induction, $\operatorname{depth}_A(I, M/aM) \geq n-1$. Hence $\operatorname{depth}_A(I, M) \geq n$. $\square$

(7.16) We define $\dim_A M = \dim M$ to be the dimension of the support $\operatorname{supp}_A M$.

(7.17) Let X be a scheme, $n \geq 0$, and $\mathcal{M}$ a quasi-coherent $\mathcal{O}_X$-module. We say that $\mathcal{M}$ satisfies the (S'_n) (resp. (S_n)) condition if for each $x \in X$, we have $\operatorname{depth} \mathcal{M}_x \geq \min(n, \dim \mathcal{O}_{X,x})$ (resp. $\operatorname{depth} \mathcal{M}_x \geq \min(n, \dim \mathcal{M}_x)$

(here we define $\operatorname{depth} 0 = \infty > n$)). Or equivalently, if $\operatorname{depth} \mathcal{M}_x < n$, then $\operatorname{depth} \mathcal{M}_x = \dim \mathcal{O}_{X,x}$ (resp. $\operatorname{depth} \mathcal{M}_x = \dim \mathcal{M}_x$). If $\mathcal{O}_{X,x}$ is Noetherian and $\mathcal{M}_x$ is a finite module, then $\mathcal{M}_x$ is called *maximal Cohen–Macaulay* (resp. *Cohen–Macaulay*) if $\operatorname{depth} \mathcal{M}_x = \dim \mathcal{O}_{X,x}$ (resp. $\operatorname{depth} \mathcal{M}_x = \dim \mathcal{M}_x$). We say that X satisfies the (S_n) condition if $\mathcal{O}_X$ satisfies the (S_n) condition (or equivalently, (S'_n) condition).

Lemma 7.18. *Let B be a Noetherian local ring with the canonical module K. Then the associated sheaf $\omega := \tilde{K}$ is a semicanonical module on $Z = \operatorname{Spec} B$. In other words, if P is a prime ideal of B and $K_P \neq 0$, then K_P is the canonical module of B_P.*

Proof. This is [Aoy, (4.3)]. $\square$

Lemma 7.19. *Let Z be a locally Noetherian scheme with a semicanonical module ω_Z. Then ω_Z satisfies the (S'_2) condition.*

Proof. If $\operatorname{depth} \omega_{Z,z} < 2$, then $\omega_{Z,z} \neq 0$, and hence $\operatorname{depth} \omega_{Z,z} = \dim \omega_{Z,z}$ by [Aoy, (1.10)]. On the other hand, if ζ is a minimal element of $\operatorname{supp} \omega_{Z,z}$, then the dimension of the closure $\bar{\zeta}$ of ζ in $\operatorname{Spec} \mathcal{O}_{Z,z}$ agrees with $\dim \mathcal{O}_{Z,z}$ by [Aoy, (1.7)]. Hence $\dim \omega_{Z,z} = \dim \mathcal{O}_{Z,z}$, and ω_Z satisfies the (S'_2) condition. $\square$

Corollary 7.20. *Let G be a flat S-group scheme of finite type, and X a Noetherian G-scheme with a G-dualizing complex. If X is locally equidimensional (e.g., X is (S_2), see [Ogo]), then for a componentwise G-canonical module ω'_X, we have that* $\operatorname{supp} \omega'_X = X$.

Proof. We may assume that X is G-connected. Let $\mathbb{I}$ be a G-dualizing complex of X, and let $\omega'_X = H^s(\mathbb{I})$ with $H^i(\mathbb{I}) = 0$ for $i < s$. Let $x \in \operatorname{supp} \omega'_X$. As $\omega'_{X,x}$ satisfies the (S'_1)-condition by Lemma 7.19, any generalization of x is in $\operatorname{supp} \omega'_X$. This shows that $\operatorname{supp} \omega'_X$ is G-stable closed open. As X is G-connected and $\omega'_X \neq 0$, $\operatorname{supp} \omega'_X = X$. $\square$

(7.21) Let X be a scheme and $x \in X$. Then the codimension of x is that of $\{x\}$. Namely, $\operatorname{codim}_X x = \operatorname{codim}_X \{x\} = \dim \mathcal{O}_{X,x}$. We denote the set of points of X of codimension n by $X^{\langle n \rangle}$. As can be seen easily using [Stack, (10.24.4)], any irreducible closed subset Z of X has a unique generic point ζ, and obviously we have $\operatorname{codim}_X Z = \operatorname{codim}_X \zeta$. In particular, $X^{\langle 0 \rangle}$ is in one-to-one correspondence with the set of irreducible components of X by the correspondence $\xi \mapsto \bar{\xi}$.

For $n \in \mathbb{Z}$, a subset of U is said to be *n-large* if $\operatorname{codim}_X(X \setminus U) \geq n+1$. This is equivalent to say that $U \supset X^{\langle 0 \rangle} \cup \cdots \cup X^{\langle n \rangle}$. 0-large is also called *strongly dense*, and 1-large is simply called *large*. Note that a strongly dense subset is dense.

If $U \subset V \subset X$ and U is n-large in X, then V is n-large in X. If $U \subset V \subset X$, V is open in X, U is n-large in V, and V is n-large in X, then U is n-large in X.

(7.22) For a scheme X and $n \geq 0$, let $P^n(X)$ be the set of integral closed subsets of codimension n. Note that $X^{\langle n \rangle}$ is in one-to-one correspondence with $P^n(X)$. An element of $P^0(X)$ is nothing but an irreducible component of X. A set Λ of subsets of X is said to be *locally finite* if for any affine open subset U of X, $U \cap F \neq \emptyset$ for only finitely many elements F of Λ. We say that a scheme X is an LFI-scheme if $P^0(X)$ is locally finite. This is equivalent to say that $\{\{\xi\} \mid \xi \in X^{\langle 0 \rangle}\}$ is locally finite.

A locally Noetherian scheme is LFI. An open subset U of an LFI-scheme X is dense in X if and only if it is strongly dense in X.

(7.23) Let Z be a locally Noetherian scheme. A coherent sheaf ω on Z is said to be n-canonical at $z \in Z$ if the $\mathcal{O}_{Z,z}$-module ω_z satisfies the (S'_n)-condition, and for each generalization $z' \in Z$ with $\operatorname{codim} z' < n$, $\omega_{z'}$ is either zero, or is the canonical module of the local ring $\mathcal{O}_{Z,z'}$. We say that ω is n-canonical if it is n-canonical at each point. Being n-canonical is a local condition.

Lemma 7.24. *Let Z be a locally Noetherian scheme with a 1-canonical module ω. For $\mathcal{M} \in \operatorname{Coh}(Z)$, consider the following conditions.*

1 *$\mathcal{M}$ satisfies the (S'_1) condition, and* $\operatorname{supp} \mathcal{M} \subset \operatorname{supp} \omega$.

2 *$\mathcal{M}$ satisfies the (S_1) condition, and* $(\operatorname{supp} \mathcal{M})^{\langle 0 \rangle} \subset (\operatorname{supp} \omega)^{\langle 0 \rangle}$.

3 *The canonical map $\mathcal{M} \to \mathcal{M}^{\vee\vee}$ is monic, where $(?)^\vee = \underline{\operatorname{Hom}}_{\mathcal{O}_Z}(?, \omega)$.*

4 *$\mathcal{M}$ is isomorphic to a submodule of a finite direct sum of copies of ω.*

Then we have **4**$\Rightarrow$**1**$\Leftrightarrow$**2**$\Leftrightarrow$**3**. *A coherent module of the form $\mathcal{M} = \mathcal{N}^\vee$ with $\mathcal{N} \in \operatorname{Coh}(Z)$ satisfies* **3**. *If $Z = \operatorname{Spec} B$ is affine, then* **3**$\Rightarrow$**4** *holds.*

Proof. As the conditions **1, 2, 3** are local, and the conditions $\mathcal{M} = \mathcal{N}^\vee$ and **4** localizes, we may assume that $Z = \operatorname{Spec} B$ is affine, and we are to prove that the four conditions are equivalent, and $\mathcal{N}^\vee$ satisfies **4** for $\mathcal{N} \in \operatorname{Coh}(Z)$.

Set $M = \Gamma(Z, \mathcal{M})$, $K = \Gamma(Z, \omega)$, and $(?)^\vee = \operatorname{Hom}_B(?, K)$.

1$\Rightarrow$**2**. As M satisfies the (S_1')-condition, it satisfies the (S_1)-condition. Let P be a minimal prime of M. As $\dim M_P = 0$, we have that $\operatorname{depth} M_P = 0$, and hence $\dim B_P = 0$ by the (S_1')-property. As $K_P \neq 0$, $\dim K_P = 0$.

2$\Rightarrow$**3**. First, assume that $P \in \operatorname{supp} M$ and $\dim M_P = 0$. Then by assumption, $P \in \operatorname{supp} K$ and $\dim K_P = 0$. As K_P satisfies (S_1') and $\operatorname{depth} K_P = 0$, we have $\dim B_P = 0$.

In particular, as M satisfies (S_1), it also satisfies (S_1').

Let $D : M \to M^{\vee\vee}$ be the canonical map, and set $F := \operatorname{supp} \operatorname{Ker} D$. If $\operatorname{ht} P = 0$, then we have that $K_P = 0$ or K_P is the canonical module of B_P. If $M_P \neq 0$, then as the B_P-module M_P is maximal Cohen–Macaulay and $K_P \neq 0$, $D_P : M_P \to M_P^{\vee\vee}$ is an isomorphism by [Aoy, (4.4)]. Hence $\operatorname{codim}_Z F \geq 1$. If $\operatorname{Ker} D \neq 0$, then taking an associated prime Q of $\operatorname{Ker} D$, $\operatorname{ht} Q \geq 1$ and Q is an associated prime of M. As $\operatorname{depth} M_Q = 0$ and M satisfies the (S_1') condition, $\dim B_Q = 0$. This contradicts $\operatorname{ht} Q \geq 1$. Hence $\operatorname{Ker} D = 0$, as desired.

Now we prove that $\mathcal{N}^\vee$ satisfies **4**. Set $N = \Gamma(Z, \mathcal{N})$, and take a surjection $B^n \to N$. Taking the dual, we have an injection $N^\vee \to K^n$.

We prove **3**$\Rightarrow$**4**. Set $N := M^\vee$. Then there is an injection $M \to M^{\vee\vee} = N^\vee$ and an injection $N^\vee \to K^n$. So there is an injection $M \to K^n$.

4$\Rightarrow$**1**. As $M \subset K^n$, we have that $\operatorname{supp} M \subset \operatorname{supp} K$. If $\operatorname{depth} M_P = 0$, then P is an associated prime of M, and hence P is an associated prime of K, and $\operatorname{depth} K_P = 0$. As K satisfies the (S_1')-condition, $\operatorname{ht} P = 0$, and hence M satisfies the (S_1')-condition. $\square$

Lemma 7.25. *Let B be a Noetherian local ring, N a finite B-module which satisfies the (S_n) condition. If there is a minimal prime P of N such that* $\dim B/P < n$, *then* $\dim B/P = \operatorname{depth} N = \dim N$. *If, moreover, N satisfies (S_n'), then* $\dim B/P = \dim B$.

Proof. As $\operatorname{Hom}_B(B/P, N) \neq 0$, we have $\operatorname{depth} N < n$ by [Mat, (17.1)]. Hence $\operatorname{depth} N = \dim N$ by the (S_n) property. Hence $\dim B/P = \dim N$ by [Mat, (17.3)]. Assume that N satisfies (S_n'). Since $\operatorname{depth} N = \dim B/P < n$, we have $\dim B/P = \operatorname{depth} N = \dim B$. $\square$

Corollary 7.26. *Let B be a Noetherian local ring, and M a finite B-module which satisfies (S_n). Let N be a finite B-module which satisfies (S_n'). If a minimal prime of M is a minimal prime of N, then M satisfies (S_n').*

Proof. Let $P \in \operatorname{Spec} B$ with $\operatorname{depth} M_P < n$. Then by (S_n) property of M, we have $\dim M_P = \operatorname{depth} M_P$. Let Q be a minimal prime of M such that $Q \subset P$ and $\dim B_P/QB_P = \dim M_P$. Then by assumption, Q is a minimal prime of N. As $\dim B_P/QB_P < n$ and QB_P is a minimal prime of N_P, we have that $\operatorname{depth} M_P = \dim B_P/QB_P = \dim B_P$ by Lemma 7.25 and (S'_n) property of N. □

Corollary 7.27. *Let B be a Noetherian local ring which satisfies (S_n), and M a finite B-module which satisfies (S_n) and (S'_1). Then M satisfies (S'_n).*

Proof. Apply Corollary 7.26 to the case that $N = B$. □

Lemma 7.28. *Let Z be a locally Noetherian scheme with a 2-canonical module ω. For $\mathcal{M} \in \operatorname{Coh}(Z)$, consider the following conditions.*

1 *$\mathcal{M}$ satisfies the (S'_2) condition, and $\operatorname{supp} \mathcal{M} \subset \operatorname{supp} \omega$.*

2 *$\mathcal{M}$ satisfies the (S_2) condition, and $(\operatorname{supp} \mathcal{M})^{\langle 0 \rangle} \subset (\operatorname{supp} \omega)^{\langle 0 \rangle}$.*

3 *The canonical map $\mathcal{M} \to \mathcal{M}^{\vee\vee}$ is isomorphic, where $(?)^\vee = \underline{\operatorname{Hom}}_{\mathcal{O}_Z}(?, \omega)$.*

4 *There is an exact sequence of the form*

$$0 \to \mathcal{M} \to \mathcal{K}^0 \to \mathcal{K}^1$$

such that $\mathcal{K}^i$ is a finite direct sum of copies of ω for $i = 0, 1$.

Then we have **4⇒1⇔2⇔3**. *A coherent module of the form $\mathcal{M} = \mathcal{N}^\vee$ with $\mathcal{N} \in \operatorname{Coh}(Z)$ satisfies* **3**. *If $Z = \operatorname{Spec} B$ is affine, then* **3⇒4** *holds.*

Proof. We may assume that $Z = \operatorname{Spec} B$ is affine, and we need to prove that the four conditions are equivalent, and $\mathcal{N}^\vee$ satisfies **4**. Let M, K, and $(?)^\vee$ be as in the proof of Lemma 7.24.

1⇒2. As M satisfies (S'_2), it satisfies (S_2). $(\operatorname{supp} \mathcal{M})^{\langle 0 \rangle} \subset (\operatorname{supp} \omega)^{\langle 0 \rangle}$ is by Lemma 7.24, **1⇒2**.

2⇒3. By Corollary 7.26, we have that M satisfies (S'_2). Note that $\operatorname{supp} M \subset \operatorname{supp} K$ by Lemma 7.24, **2⇒1**.

Also by Lemma 7.24, **2⇒3**, we have that $D : M \to M^{\vee\vee}$ is monic. Let $C := \operatorname{Coker} D$. Let $P \in \operatorname{Spec} B$ and $\operatorname{depth} M_P < 2$. Then M_P is maximal Cohen–Macaulay by the (S'_2) property of M. We have $K_P \neq 0$ by $\operatorname{supp} M \subset \operatorname{supp} K$. So K_P is the canonical module of B_P, since K is 2-canonical. Hence $C_P = 0$ by [Aoy, (4.4)].

Now assume that $C \neq 0$, and let Q be a minimal prime of C. Then $C_Q \neq 0$, and hence depth $M_Q \geq 2$ by the argument above. Hence $\dim B_Q \geq 2$. On the other hand,

$$0 \to M_Q \to M_Q^{\vee\vee} \to C_Q \to 0$$

is exact. As $M_Q^{\vee\vee}$ satisfies (S'_1) by Lemma 7.24, $\operatorname{depth} M_Q^{\vee\vee} \geq 1$. By the choice of Q, $\operatorname{depth} C_Q = 0$. By the depth lemma, $\operatorname{depth} M_Q = 1$, and this contradicts $\operatorname{depth} M_Q \geq 2$. Hence $C = 0$, as desired.

Now we prove that $\mathcal{N}^\vee$ satisfies **4**. Set $N = \Gamma(Z, \mathcal{N})$, and take a presentation

$$F_1 \to F_0 \to N \to 0$$

with F_0 and F_1 finite free. Dualizing, we get a desired exact sequence.

Now we prove **3**$\Rightarrow$**4**. Set $N := M^\vee$. Then $M \cong M^{\vee\vee} \cong N^\vee$. As $N^\vee$ satisfies **4**, M satisfies **4**, too.

4$\Rightarrow$**1**. Let $P \in \operatorname{Spec} B$ and assume that $\operatorname{depth} M_P < 2$. By the exact sequence, we must have that $\operatorname{depth} K_P = \operatorname{depth} M_P$. As K satisfies the (S'_2)-condition, $\dim B_P = \operatorname{depth} K_P = \operatorname{depth} M_P$, and hence M satisfies the (S'_2)-condition. $\operatorname{supp} \mathcal{M} \subset \operatorname{supp} \omega$ is trivial. $\square$

(7.29) Let Z be a locally Noetherian scheme. We denote the full subcategory of $\operatorname{Coh}(Z)$ consisting of coherent sheaves satisfying the (S'_n) condition by $(S'_n)(Z)$. It is an additive subcategory of $\operatorname{Coh}(Z)$ closed under direct summands, extensions, and epikernels.

Lemma 7.30. *For $\mathcal{M} \in \operatorname{Coh}(Z)$, the following are equivalent.*

1 $\mathcal{M} \in (S'_1)(Z)$.

2 *For any dense open subset $i : U \to Z$ of Z, the canonical map $u : \mathcal{M} \to i_* i^* \mathcal{M}$ is a monomorphism.*

Proof. As the question is local, we may assume that $Z = \operatorname{Spec} B$ is affine. Set $M = \Gamma(Z, \mathcal{M})$.

1$\Rightarrow$**2**. Let I be an ideal of B such that $U = Z \setminus V(I)$. Then we have $\operatorname{ht} I \geq 1$, and hence we have that $\operatorname{depth} \mathcal{M}_z \geq 1$ for each $z \in V(I)$ by the (S'_1) property. Thus $\operatorname{depth}_B(I, M) \geq 1$, and hence $H^0_I(M) = 0$ by Lemma 7.15. Hence $M \to \Gamma(U, \mathcal{M})$ is injective.

2$\Rightarrow$**1**. Assume that P is an associated prime of M with $\operatorname{ht} P \geq 1$. Then letting $U = D(P) = Z \setminus V(P)$, U is dense. However, as P is an associated prime of M, the local cohomology $H^0_P(M)$, which is the kernel of $M \to \Gamma(Z, i_* i^* \mathcal{M})$, is nonzero, and this is a contradiction. Hence if $P \in \operatorname{Ass} M$, then $\operatorname{ht} P = 0$. Namely, $\mathcal{M}$ satisfies the (S_1') condition. $\square$

Lemma 7.31. *Let Z be a locally Noetherian scheme. For $\mathcal{M} \in \operatorname{Coh}(Z)$, the following are equivalent.*

1 $\mathcal{M} \in (S_2')(Z)$.

2 $\mathcal{M} \in (S_1')(Z)$, *and for any large open subset $i : U \to Z$ of Z, the canonical map $u : \mathcal{M} \to i_* i^* \mathcal{M}$ is an isomorphism.*

Proof. As in the proof of Lemma 7.30, we may assume that $Z = \operatorname{Spec} B$ is affine. Let $M = \Gamma(Z, \mathcal{O}_Z)$.

1⇒2. Clearly, $\mathcal{M}$ satisfies the (S_1') condition.

Let I be an ideal of B such that $U = Z \setminus V(I)$. As we have $\operatorname{ht} I \geq 2$, $\operatorname{depth}(I, M) \geq 2$. So the local cohomology $H^i_I(M)$ vanishes for $i = 0, 1$. By the exact sequence

$$H^0_I(M) \to M \to \Gamma(Z, i_* i^* \mathcal{M}) \to H^1_I(M),$$

we are done.

2⇒1. Let $P \in \operatorname{Spec} B$ satisfy $\operatorname{depth} M_P < 2$. If $\operatorname{depth} M_P = 0$, then $\dim B_P = 0$ by the (S_1') assumption. If not, then $\operatorname{depth} M_P = 1$. We want to prove that $\dim B_P = 1$. Assume the contrary. Then $\dim A_P \geq 2$. Then letting $U = Z \setminus V(P)$, we have that U is a large open subset of Z. As we have an exact sequence

$$0 \to H^0_P(M) \to M \xrightarrow{\cong} \Gamma(U, i^* \mathcal{M}) \to H^1_P(M) \to 0$$

by [Hart2, (1.9)], we have that $H^0_P(M) = H^1_P(M) = 0$ by assumption. Hence $H^0_{PB_P}(M_P) = H^1_{PB_P}(M_P) = 0$. Hence $\operatorname{depth} M_P \geq 2$, and this is a contradiction.

Hence $\operatorname{depth} M_P < 2$ implies that $\operatorname{depth} M_P = \dim A_P$. That is, M satisfies the (S_2') condition. $\square$

(7.32) Let Z be a scheme with a quasi-coherent sheaf $\mathcal{M}$. We say that $\mathcal{M}$ is *full* if $\operatorname{supp} \mathcal{M} = Z$.

Lemma 7.33. *If $\varphi : X \to Y$ is a flat morphism of schemes and $\mathcal{M}$ is a quasi-coherent sheaf on Y. If $\mathcal{M}$ is full, then $\varphi^* \mathcal{M}$ is also full. If $\varphi^* \mathcal{M}$ is a full and φ is faithfully flat, then $\mathcal{M}$ is full.*

Proof. Easy. □

Lemma 7.34. *Let Z be a locally Noetherian scheme with a full 2-canonical module ω, and U its large open subset. Let $i : U \hookrightarrow Z$ be the inclusion. If $\mathcal{M} \in (S_2')(U)$, then $i_*\mathcal{M} \in (S_2')(Z)$. Moreover, $i_* : (S_2')(U) \to (S_2')(Z)$ and $i^* : (S_2')(Z) \to (S_2')(U)$ are quasi-inverse each other.*

Proof. If $i_*\mathcal{M} \in (S_2')(Z)$ for $\mathcal{M} \in (S_2')(U)$, then $i_* : (S_2')(U) \to (S_2')(Z)$ and $i^* : (S_2')(Z) \to (S_2')(U)$ are well-defined functors. The counit map $i^*i_* \to \mathrm{Id}$ is obviously an isomorphism. On the other hand, $\mathrm{Id} \to i_*i^*$ is an isomorphism by Lemma 7.31. So the last assertion follows.

So it suffices to show, If $i_*\mathcal{M} \in (S_2')(Z)$ for $\mathcal{M} \in (S_2')(U)$. We can take a coherent subsheaf $\mathcal{Q}$ of $i_*\mathcal{M}$ such that $i^*\mathcal{Q} = i^*i_*\mathcal{M} = \mathcal{M}$ by [Hart3, Exercise II.5.15]. Set $\mathcal{N} := \mathcal{Q}^{\vee\vee}$. Then $\mathcal{N} \in (S_2')(Z)$ by Lemma 7.28. So by Lemma 7.31, $\mathcal{N} \to i_*i^*\mathcal{N} \cong i_*(i^*\mathcal{Q})^{\vee\vee} \cong i_*\mathcal{M}^{\vee\vee} \cong i_*\mathcal{M}$ is an isomorphism, and hence $i_*\mathcal{M}$ is coherent and is in $(S_2')(Z)$. □

Example 7.35. Let Z be a locally Noetherian scheme. In the following cases, Z has a full 2-canonical module ω.

1 Z is normal. Any rank-one reflexive module (e.g., $\mathcal{O}_Z$) can be used as ω.

2 Z is locally equidimensional Noetherian with a dualizing complex. The componentwise canonical module is the desired one.

3 $Z = \operatorname{Spec} B$ with B an equidimensional Noetherian local ring with a canonical module in the local sense. The canonical module is the desired one.

(7.36) A locally Noetherian scheme Z is said to be *quasi-normal* by ω if ω is a full 2-canonical module of Z. It is simply called quasi-normal, if it is quasi-normal by some ω. We say that Z satisfies (T_n) (resp. (R_n)) if $\mathcal{O}_{Z,z}$ is Gorenstein (resp. regular) for $z \in Z$ with $\operatorname{codim} z \leq n$, or equivalently, the Gorenstein (resp. regular) locus of Z is n-large.

Lemma 7.37. *For a locally Noetherian scheme Z, the following are equivalent.*

1 *Z satisfies $(T_1) + (S_2)$.*

2 *Z is quasi-normal by $\mathcal{O}_Z$.*

In particular, if Z is normal (or equivalently, satisfies $(R_1) + (S_2)$), then it is quasi-normal by $\mathcal{O}_Z$.

Proof. Easy. $\square$

Lemma 7.38. *Let Z be a locally equidimensional connected Noetherian scheme with a nonzero semicanonical module ω. Then $\operatorname{supp}\omega = Z$, and we have that Z is universally catenary and quasi-normal by ω.*

Proof. Let $X_1, \ldots, X_r$ be the irreducible components of Z that are contained in $\operatorname{supp}\omega$. Assume that $\operatorname{supp}\omega \neq Z$. Let $Y_1, \ldots, Y_s$ be the irreducible components of Z that are not contained in $\operatorname{supp}\omega$. By assumption, $r \geq 1$ and $s \geq 1$. As Z is connected, there exist some i and j such that $X_i \cap Y_j \neq \emptyset$. Take $z \in (X_i \cap Y_j)^{\langle 0 \rangle}$, and consider the local ring $B = \mathcal{O}_{Z,z}$. As $z \in \operatorname{supp}\omega$, B has a canonical module $K = \omega_z$. By assumption, B is equidimensional. By [Aoy, (1.7)], $\operatorname{supp} K = \operatorname{Spec} B$. Hence $Y_j \subset \operatorname{supp}\omega$, and this is a contradiction. Hence $\operatorname{supp}\omega = Z$, as desired.

Let $z \in Z$ be any point, and set $B = \mathcal{O}_{Z,z}$. As $\operatorname{supp}\omega_z = \operatorname{Spec} B$, we have that $\operatorname{supp}\hat{\omega}_z = \operatorname{Spec}\hat{B}$, where $\hat{B}$ is the completion of B. As $\hat{\omega}_z$ is a canonical module of $\hat{B}$, we have that $\hat{B}$ is equidimensional by [Aoy, (1.7)]. Hence B is universally catenary by [Mat, (31.6)]. Hence Z is universally catenary. $\square$

Corollary 7.39. *Let Z be a Noetherian scheme with a dualizing complex. If Z is locally equidimensional, then the componentwise canonical module ω' is a full semicanonical module, and Z is quasi-normal by ω'.*

Proof. The componentwise canonical module ω' is full by Lemma 7.38. $\square$

Lemma 7.40. *Let $\varphi : X \to Y$ be a flat morphism between locally Noetherian schemes. Let $\mathcal{M}$ be a coherent sheaf on Y, and $n \geq 0$. Then*

1 *If φ is faithfully flat, and $\varphi^*\mathcal{M}$ satisfies the (S'_n) condition (resp. the (S_n) condition), then so does $\mathcal{M}$.*

2 *If the fibers of φ satisfy (S_n) and $\mathcal{M}$ satisfies (S'_n) (resp. (S_n)), then so does $\varphi^*\mathcal{M}$.*

Proof. For the property (S_n), see [Gro2, (6.4.1)]. The assertions for (S'_n) is also proved similarly. $\square$

Lemma 7.41. *Let $\varphi : X \to Y$ be a flat morphism between locally Noetherian schemes. Let ω be a coherent sheaf on Y.*

1 *If φ is faithfully flat and $\varphi^*\omega$ is semicanonical (resp. n-canonical), then so is ω.*

2 *If X is quasi-normal by $\varphi^*\omega$, then Y is quasi-normal by ω.*

3 *If ω is semicanonical and each fiber of φ is Gorenstein, then $\varphi^*\omega$ is semicanonical.*

4 *If ω is n-canonical and each fiber of φ satisfies $(T_{n-1})+(S_n)$, then $\varphi^*\omega$ is n-canonical.*

5 *If Y is quasi-normal by ω and each fiber of φ satisfies $(T_1)+(S_2)$, then X is quasi-normal by $\varphi^*\omega$.*

Proof. **1**. The assertion for the semicanonical property follows from [Aoy, (4.2)]. As the (S_n) property also descends by Lemma 7.40, the assertion for the n-canonical property follows easily.

2 follows immediately by **1**.

3. Let $A \to B$ a flat Gorenstein local homomorphism between Noetherian local rings, and K the canonical module of A. It suffices to prove $B \otimes_A K$ is the canonical module. Let $Q = \mathfrak{m}_B(\hat{A} \otimes_A B)$, where $\mathfrak{m}_B$ is the maximal ideal of B. Note that Q is a maximal ideal of $\hat{A} \otimes_A B$. Set $C := (\hat{A} \otimes_A B)_Q$. Since $\hat{K} = \hat{A} \otimes_A K$ is the lowest nonvanishing cohomology group of the dualizing complex $\mathbb{I}$ of $\hat{A}$, we have that $C \otimes_A K$ is the lowest nonvanishing cohomology group of $C \otimes_{\hat{A}} \mathbb{I}$. As $\hat{A} \to C$ is a flat Gorenstein local homomorphism, $C \otimes_{\hat{A}} \mathbb{I}$ is a dualizing complex by [AvF, (5.1)]. Hence $C \otimes_A K$ is the canonical module of C. By [Aoy, (4.2)], $B \otimes_A K$ is the canonical module of B.

4 and **5** are immediate consequences of **3**. □

(7.42) Let $f : G \to H$ be a quasi-compact flat homomorphism between flat S-group schemes of finite type with $N = \operatorname{Ker} f$. Note that N is also flat of finite type.

Lemma 7.43. *Let $g : Z' \to Z$ be a G-morphism separated of finite type. Assume that Z is Noetherian. Then the flat base change map*

$$\bar{\zeta} : \operatorname{res}^H_G g^! \to g^! \operatorname{res}^H_G$$

(see [Has5, Chapter 21]*) is an isomorphism between the functors $D^+_{\mathrm{Lqc}}(H, Z) \to D^+_{\mathrm{Lqc}}(G, Z')$ (it would be better to write $L\operatorname{res}^H_G$ instead of res^H_G, but as in* [Has5]*, for a left or right derived functor of an exact functor, we omit L or R).*

Proof. This is [Has5, (21.8)]. □

(7.44) Let $f : G \to H$, and N be as in (7.42). Let Y_0 be a fixed Noetherian H-scheme with a fixed H-dualizing complex $\mathbb{I}_{Y_0} = \mathbb{I}_{Y_0}(H)$. The restriction $\operatorname{res}^H_G \mathbb{I}_{Y_0}(H)$ is a G-dualizing complex by [Has5, (31.17)]. We denote it by $\mathbb{I}_{Y_0}$ or $\mathbb{I}_{Y_0}(G)$.

Let $\mathcal{F}(G, Y_0)$ be the category of (G, Y_0)-schemes separated of finite type over Y_0. For $(h_Z : Z \to Y_0) \in \mathcal{F}(G, Y_0)$, *the* G-dualizing complex of Z (or better, of h_Z) is $h_Z^! \mathbb{I}_{Y_0}(G)$ by definition, and we denote it by $\mathbb{I}_Z = \mathbb{I}_Z(G)$.

Lemma 7.45. *Let $h_Z : Z \to Y_0$ be an object of $\mathcal{F}(G, Y_0)$. Assume that the action of N on Z is trivial. Then $Z \in \mathcal{F}(H, Y_0)$. When we set $\mathbb{I}_Z(H) := h_Z^! \mathbb{I}_{Y_0}(H)$, then we have $\mathbb{I}_Z(G) = \operatorname{res}^H_G \mathbb{I}_Z(H)$. In particular, each cohomology group of $\mathbb{I}_Z(G)$ belongs to $\operatorname{Coh}_N(G, Z)$, the full subcategory of $\operatorname{Qch}(G, Z)$ consisting of N-trivial coherent $(G, \mathcal{O}_Z)$-modules.*

Proof. The first assertion is by [Has11, (6.5)]. We have

$$\mathbb{I}_Z(G) = h_Z^! \mathbb{I}_{Y_0}(G) = h_Z^! \operatorname{res}^H_G \mathbb{I}_{Y_0}(H) \cong \operatorname{res}^H_G h_Z^! \mathbb{I}_{Y_0}(H) \cong \operatorname{res}^H_G \mathbb{I}_Z(H)$$

by Lemma 7.43, and the second assertion holds. In particular, being restricted from H, each cohomology group of $\mathbb{I}_Z(G)$ is N-trivial. □

Example 7.46. Let S be Noetherian with a fixed dualizing complex $\mathbb{I}_S$. Then $\mathbb{I}_S(H) = (L_{-1}\mathbb{I}_S)_{\Delta_M}$ is an H-dualizing complex of the H-scheme S by [Has5, (31.17)], where $L_{-1} : \operatorname{Mod}(S) \to \operatorname{Mod}(\tilde{B}^M_H(S))$ is the left induction, and $(?)_{\Delta_M} : \operatorname{Mod}(\tilde{B}^M_H(S)) \to \operatorname{Mod}(H, S) = \operatorname{Mod}(B^M_H(S))$ is the restriction. See for the notation, [Has5]. Letting $Y_0 = S$, we are in the situation of (7.44).

(7.47) Let S, $f : G \to H$ and N be as in (7.42). Y_0, $\mathbb{I}_{Y_0}$, and $\mathcal{F}(G, Y_0)$ be as in (7.44).

Let $Z \in \mathcal{F}(G, Y_0)$. *The* G-canonical module of Z, denoted by ω_Z, is defined to be the G-canonical module corresponding to the G-dualizing complex $\mathbb{I}_Z$. Note that the definition in [Has5, (31.13)] is slightly different, and used the componentwise G-canonical module, see (7.10).

Lemma 7.48. *Let $f : G \to H$, N, Y_0, $\mathbb{I}_{Y_0}$, and $Z \in \mathcal{F}(G, Y_0)$ be as in (7.47). Let U be a G-stable open subset of Z. If $\omega_Z|_U \neq 0$ (e.g., U is dense), then $\omega_Z|_U \cong \omega_U$ as $(G, \mathcal{O}_U)$-modules. If, moreover, U is large in Z, then $\omega_Z \cong i_*\omega_U$, where $i : U \hookrightarrow Z$ is the inclusion.*

Proof. Assume that $\omega_Z = H^s(\mathbb{I}_Z)$. If $\omega_Z|_U = H^s(\mathbb{I}_U) \neq 0$, then as $H^i(\mathbb{I}_U) = 0$ for $i < s$, we have that $\omega_Z|_U \cong \omega_U$ as $(G, \mathcal{O}_U)$-modules.

Assume that U is large in Z. Then since ω_Z satisfies (S_2'), we have that $\omega_Z \to i_* i^* \omega_Z \cong i_* \omega_U$ is an isomorphism by Lemma 7.31. $\square$

(7.49) Let $f : G \to H$ be as in (7.42). Assume that N is smooth over S. Let $\mathcal{I}$ be the defining ideal of the unit element e in N. Then $\mathcal{I}/\mathcal{I}^2$ is a locally free sheaf over S on which G acts via the conjugation. We set $\operatorname{Lie} N := (\mathcal{I}/\mathcal{I}^2)^*$, and $\Theta_N := \bigwedge^{\mathrm{top}} \operatorname{Lie} N$. The following is essentially due to Knop [Knp, Lemma 5].

Proposition 7.50. *Assume that N is smooth over S. If $\varphi : X \to Y$ is a G-enriched principal N-bundle with Y locally Noetherian, then $\Omega_{X/Y}$ is isomorphic to $h_X^*((\operatorname{Lie} N)^*)$, where $h_X : X \to S$ is the structure map.*

Proof. Consider the commutative diagram

$$
\begin{tikzcd}
X \arrow[r, "e\times 1_X"] \arrow[d, "h_X"] & N \times X \arrow[r, "\Psi"] \arrow[d, "p_1"] \arrow[dr, "p_2"] & X \times_Y X \arrow[r, "p_1"] \arrow[d, "p_2"] \arrow[dr, phantom, "\text{(b)}"] & X \arrow[d, "\varphi"] \\
S \arrow[r, "e"] & N \arrow[r, phantom, "\text{(a)}"] \arrow[dr] & X \arrow[r, "\varphi"] \arrow[d, "h_X"] & Y \\
 & & S &
\end{tikzcd}
\tag{16}
$$

of G-schemes, where G acts on N by conjugation action, and $\Psi(n, x) = (nx, x)$. Then as Ψ is an isomorphism and (a) and (b) are fiber squares,

$$
\begin{aligned}
\Omega_{X/Y} &\cong (e \times 1_X)^* \Psi^* p_1^* \Omega_{X/Y} \cong (e \times 1_X)^* \Psi^* \Omega_{X \times_Y X/X} \cong \\
&\quad (e \times 1_X)^* \Omega_{N \times X/X} \cong (e \times 1_X)^* p_1^* \Omega_{N/S} \cong h_X^* e^* \Omega_{N/X} \cong h_X^*((\operatorname{Lie} N)^*),
\end{aligned}
$$

as desired. $\square$

So $\omega_{X/Y} := \bigwedge^{\mathrm{top}} \Omega_{X/Y} = h_X^*(\Theta_N^*) = \Theta_{N,X}^*$, where $\Theta_{N,X} := h_X^* \Theta_N$. We prove a version of this fact which can be used also for the case where N may not be smooth.

(7.51) We say that a morphism of schemes $\varphi : X \to Y$ is of relative dimension d if $\dim_x \varphi = d$ for each $x \in X$, see [Gro3, (17.10.1)] for the notation.

(7.52) Let $f : A \to B$ be a local homomorphism between Noetherian local rings. Let $\hat{f} : \hat{A} \to \hat{B}$ be its completion, and

$$\hat{A} \xrightarrow{g} C \xrightarrow{h} \hat{B} \tag{17}$$

a Cohen factorization [AvFH] of it. That is, g is flat with $C/\mathfrak{m}_A C$ regular, and h is surjective. If so, we define the Avramov–Foxby–Herzog dimension (AFH dimension for short) of f by

$$\mathrm{AFHdim}\, f = \mathrm{AFHdim}_A B = \dim C - \dim A - \mathrm{ht}\, \mathrm{Ker}\, h$$

and the depth of f by $\mathrm{depth}\, f = \mathrm{depth}\, B - \mathrm{depth}\, A$, see [AvFH] and [AvF2]. If f is flat, then $\mathrm{AFHdim}\, f$ is nothing but the dimension of the closed fiber. We define $\mathrm{cmd}\, f = \mathrm{AFHdim}\, f - \mathrm{depth}\, f$, and call it the Cohen–Macaulay defect of f. We say that f is Cohen–Macaulay at $\mathfrak{m}_B$ if $\mathrm{flat.dim}\, f < \infty$ and $\mathrm{cmd}\, f = 0$. This is equivalent to say that $\mathrm{Ker}\, h$ is a perfect ideal. If $\mathrm{Ker}\, h$ is a Gorenstein ideal (that is, $\mathrm{Ker}\, h$ is perfect and $\mathrm{Ext}^c_C(B, C) \cong B$, where $c = \mathrm{ht}\, \mathrm{Ker}\, h$, we say that f is Gorenstein at $\mathfrak{m}_B$. These definitions are independent of the choice of Cohen factorization (17) of $\hat{f}$.

(7.53) A morphism $\varphi : X \to Y$ between locally Noetherian schemes is said to be Cohen–Macaulay (resp. Gorenstein) if $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is Cohen–Macaulay (resp. Gorenstein) at $\mathfrak{m}_x$ for every $x \in X$. $\mathrm{AFHdim}_x\, \varphi$ is $\mathrm{AFHdim}_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}$. If $\mathrm{AFHdim}_x\, \varphi = d$ is independent of $x \in X$, then we say that φ has AFH dimension d.

Lemma 7.54. *Let $\varphi : X \to Y$ be a Cohen–Macaulay separated morphism of finite type between Noetherian schemes. Then φ has a well-defined AFH dimension on each connected component of X.*

If φ is of AFH dimension d, then $H^i(\varphi^!(\mathcal{O}_Y)) = 0$ for $i \neq -d$. If, moreover, φ is Gorenstein, then $\omega_{X/Y} := H^{-d}(\varphi^!(\mathcal{O}_Y))$ is an invertible sheaf. If, moreover, G is a flat S-group scheme of finite type and φ is a G-morphism, then $\omega_{X/Y}$ is a G-linearized invertible sheaf. If, moreover, φ is smooth, then $\omega_{X/Y} = \bigwedge^d \Omega_{X/Y}$.

Proof. By the flat base change [Lip, (4.4.3)], the question is local both on X and Y, and we may assume that $Y = \mathrm{Spec}\, A$ and $X = \mathrm{Spec}\, B$ are both affine. As B is finitely generated, we may write $B = C/I$, where $C = A[x_1, \ldots, x_n]$ is a polynomial ring, and I an ideal of C. By assumption, I is a perfect ideal of codimension $h := n - d$. We have

$$j_*\varphi^! = R\, \underline{\mathrm{Hom}}_{\mathcal{O}_Z}(j_*\mathcal{O}_X, \psi^*(?))[n],$$

where $\psi : Z = \operatorname{Spec} C \to Y$ is the canonical map, and $j : X \to Z$ is the inclusion. As we have $\operatorname{Ext}^i_C(B, C) = 0$ for $i \neq h$, the first assertion follows. If, moreover, φ is Gorenstein, $\operatorname{Ext}^h_C(B, C)$ is rank-one projective as a B-module, and the second assertion follows. The third assertion is trivial. The last assertion follows from [Has5, (28.11)]. $\square$

Definition 7.55. Let G be a flat S-group scheme of finite type, and $\varphi : X \to Y$ be a G-morphism separated of finite type between Noetherian G-schemes. We denote the lowest non-vanishing cohomology group $H^s(\varphi^!\mathcal{O}_Y) \neq 0$ ($H^i(\varphi^!\mathcal{O}_Y) = 0$ for $i < s$) of $\varphi^!\mathcal{O}_Y$ by $\omega_{X/Y}$ or ω_φ if $X \neq \emptyset$ (if $X = \emptyset$, we define $\omega_{X/Y} = 0$), and call $\omega_{X/Y}$ the relative canonical sheaf of φ (or of X/Y).

Lemma 7.56. *Let G and $\varphi : X \to Y$ be as in Definition 7.55. Assume that φ is flat Gorenstein of relative dimension d. Then for any morphism $h : Y' \to Y$ with Y' Noetherian, we have that $\omega_{X'/Y'} \cong h_X^*\omega_{X/Y}$, where $X' = Y' \times_Y X$ and $h_X : X' \to X$ is the second projection.*

Proof. Consider the diagram

$$\begin{tikzcd}
X \arrow[r, "\Delta"] \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, "1_X"'] & X \times_Y X \arrow[r, "p_1"] \arrow[d, "p_2"] & X \arrow[d, "\varphi"] \\
& X \arrow[r, "\varphi"] & Y
\end{tikzcd} . \tag{18}$$

Then by the flat base change,

$$\mathcal{O}_X \cong \Delta^! p_2^! \varphi^* \mathcal{O}_Y \cong \Delta^! p_1^* \varphi^! \mathcal{O}_Y \cong \Delta^! p_1^* \omega_{X/Y}[d].$$

As Δ is a closed immersion, the description of $\Delta^!$ in [Has5, Chapter 27] yields that there is an isomorphism

$$\mathcal{O}_X \cong \Delta^! p_1^* \omega_{X/Y}[d] \cong p_1^* \omega_{X/Y} \otimes \Delta^! \mathcal{O}_{X \times_Y X}[d].$$

Thus $\Delta^! \mathcal{O}_{X \times_Y X} \cong \omega_{X/Y}^{-1}[-d]$. Note that all the maps in (18) are tor-independent to h and its base change.

The argument above applied to $\varphi' : X' \to Y'$ yields $(\Delta')^! \mathcal{O}_{X' \times_{Y'} X'} \cong \omega_{X'/Y'}^{-1}[-d]$. So it suffices to show that the canonical map $\bar{\zeta} : Lh_X^* \Delta^! \mathcal{O}_{X \times_Y X} \to (\Delta')^! Lh_{X \times_Y X}^*$ is an isomorphism. To verify this, we may forget the G-action, and we may assume that G is trivial. Then, as the question is local on X, Y, and Y', we may assume that $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$, $Y' = \operatorname{Spec} A'$ are all affine.

Then by definition, $\bar{\zeta} = \zeta$ is identified with the map

$$H : R\operatorname{Hom}_C(B, C) \otimes^L_A A' \to R\operatorname{Hom}_{C'}(B', C'),$$

where $C = B \otimes_A B$, and B is viewed as a C-algebra via the product map, and $B' = A' \otimes_A B$ and $C' = A' \otimes_A C$.

To compute the map H, let $\mathbb{F}$ be a C-free resolution of B whose terms are finite free. Note that

$$H^i(\mathbb{F}^* \otimes_A A') \cong H^i(\operatorname{Hom}_{C'}(A' \otimes_A \mathbb{F}, C')) \cong \begin{cases} \omega_{X'/Y'} & (i = d) \\ 0 & (\textit{otherwise}) \end{cases}.$$

Letting

$$\mathbb{F}^* : 0 \to G^0 \to \cdots \to G^i \xrightarrow{\partial^i} G^{i+1} \to \cdots$$

and $B^i(\mathbb{F}^*) = \operatorname{Im} \partial^{i-1}$, we have that $H^i(\mathbb{F}^* \otimes A/J) = 0$ for any ideal J of A and any $i > d$. This shows that $\operatorname{Tor}_1^A(B^{d+3}, A/J) = 0$ for any J, and hence B^{d+3} is A-flat, and $B^{d+3} \otimes_A A' \to B^{d+3}(\mathbb{F}^* \otimes_A A')$ is an isomorphism. So it is easy to see that $Z^d = \operatorname{Ker} \partial^d$ is also A-flat and $Z^d \otimes_A A' \to Z^d(\mathbb{F}^* \otimes_A A')$ is an isomorphism. So considering the exact flat complex bounded above

$$0 \to G^0 \to \cdots \to G^{d-1} \to Z^d \to Z^d/B^d \to 0$$

is compatible with the base change. We have that the induced map $\omega_{X/Y} \otimes_A A' \to \omega_{X'/Y'}$ is an isomorphism, as desired. □

Lemma 7.57. *Let G be a flat S-group scheme of finite type, and $\varphi : X \to Y$ and $\psi : Y \to Z$ be flat Gorenstein G-morphisms separated of finite type between Noetherian G-schemes with well-defined relative dimensions. Then $\omega_{X/Z} \cong \varphi^*\omega_{Y/Z} \otimes_{\mathcal{O}_X} \omega_{X/Y}$.*

Proof. Let d and d' be the relative dimensions of φ and ψ, respectively. We have

$$\begin{aligned} \omega_{X/Z} = H^{-d-d'}((\psi\varphi)^!(\mathcal{O}_Z)) &\cong H^{-d-d'}(\varphi^!(\omega_{Y/Z}[d'])) \\ &\cong H^{-d}(\varphi^*\omega_{Y/Z} \otimes^L_{\mathcal{O}_X} \varphi^!(\mathcal{O}_Y)) \cong \varphi^*\omega_{Y/Z} \otimes_{\mathcal{O}_X} \omega_{X/Y}. \end{aligned}$$

□

(7.58) Let $f : G \to H$, N, Y_0, $\mathbb{I}_{Y_0}$, and $Z \in \mathcal{F}(G, Y_0)$ be as in (7.47). Assume that N is separated and has a fixed relative dimension. We define $\Theta = \Theta_{N,Z} := e_{N_Z}^* \omega_{N_Z/Z}^*$, where $N_Z = N \times_S Z$, and $e_{N_Z} : Z \to N_Z$ is the unit element. Letting $h_Z : Z \to Y_0$ is the structure map, $h_Z^* \Theta_{N,Y_0} \cong \Theta_{N,Z}$ by Lemma 7.56. If S is Noetherian, then letting $\Theta_{N,S} = e_N^* \omega_{N/S}^*$, we have that $\Theta_{N,Z} = \bar{h}_Z^* \Theta_{N,S}$, where $\bar{h}_Z : Z \to S$ is the structure map.

Note that Θ is a G-linearized invertible sheaf on Z. If N is smooth of relative dimension d, then $\Theta_{N,Z} \cong \bar{h}_Z^*(\bigwedge^d \operatorname{Lie} N)$, where $\operatorname{Lie} N = \Omega_{N/S}^*$ and $\bar{h}_Z : Z \to S$ is the structure map.

Proposition 7.59. *Let $\varphi : X \to Y$ be a morphism in $\mathcal{F}(G, Y_0)$. Assume that N is separated and has a relative dimension d. If φ is a G-enriched principal N-bundle, then $\omega_{X/Y} \cong \Theta_{N,X}^*$.*

Proof. Let us consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "e\times 1_X"] & N \times X \arrow[r, "\Psi"] \arrow[dr, "p_2"'] & X \times_Y X \arrow[r, "p_1"] \arrow[d, "p_2"] \arrow[dr, phantom, "\text{(a)}"] & X \arrow[d, "\varphi"] \\ & & X \arrow[r, "\varphi"] & Y \end{tikzcd}$$

in $\mathcal{F}(G, Y_0)$.

Note that (a) is cartesian, and φ is flat Gorenstein of relative dimension d. Now

$$\begin{aligned} \omega_{X/Y}[d] &\cong \varphi^! \mathcal{O}_Y \cong L(e \times 1_X)^* \Psi^* p_1^* \varphi^! \mathcal{O}_Y \cong L(e \times 1_X)^* \Psi^* p_2^! \varphi^* \mathcal{O}_Y \\ &\cong L(e \times 1_X)^* p_2^! \mathcal{O}_X \cong L(e \times 1_X)^* \omega_{(N \times X)/X}[d] \cong \Theta_{N,X}^*[d], \end{aligned}$$

and the result follows. □

The following is due to Knop [Knp] when $S = \operatorname{Spec} k$ with k an algebraically closed field of characteristic zero.

Corollary 7.60. *Let $f : G \to H$, N, Y_0, and $\mathbb{I}_{Y_0}$ be as in (7.47). Let $\varphi : X \to Y$ be a G-enriched principal N-bundle which is a morphism in $\mathcal{F}(G, Y_0)$. If N is separated and has a fixed relative dimension, then $\omega_X \cong \varphi^* \omega_Y \otimes_{\mathcal{O}_X} \Theta_{N,X}^*$ as $(G, \mathcal{O}_X)$-modules, and $\omega_Y \cong (\varphi_* \omega_X \otimes_{\mathcal{O}_Y} \Theta_{N,Y})^N$ as $(H, \mathcal{O}_Y)$-modules.*

Proof. The first assertion follows immediately from [Has5, (28.11)] and Proposition 7.50. The second assertion follows from the first one and [Has11, (6.21)], using the equivariant projection formula [Has5, (26.4)]. □

Lemma 7.61. *Let $f : G \to H$, N, and Y_0 be as in* (7.47)*. If N is finite and Reynolds, then $\Theta_{N,Y_0} \cong \mathcal{O}_{Y_0}$. In particular, for any object Y of $\mathcal{F}(G, Y_0)$, $\Theta_{N,Y} \cong \mathcal{O}_Y$.*

Proof. Note that G acts on N by conjugation, and hence we can define the semidirect product $\tilde{G} := G \ltimes N$. Letting G act on Y_0 by the original action and N act on Y_0 as a subgroup of G, $\tilde{G}$ acts on Y_0. The group $\tilde{G}$ acts on N by $(g, n)n' = gnn'g^{-1}$. The first projection $p_1 : N_{Y_0} = N \times Y_0 \to Y_0$ is a $\tilde{G}$-enriched principal N-bundle. As $(G, \mathcal{O}_{Y_0})$-modules,

$$\begin{aligned}\Theta^*_{N,Y_0} = e^*_{N_{Y_0}}\omega_{N_{Y_0}/Y_0} &\cong e^*_{N_{Y_0}} p_1^*(?)^N \omega_{N_{Y_0}/Y_0} \cong \omega^N_{N_{Y_0}/Y_0} \\ &\cong R(?)^N R\,\underline{\mathrm{Hom}}_{\mathcal{O}_{Y_0}}(\mathcal{O}_{N_{Y_0}}, \mathcal{O}_{Y_0}) \cong \underline{\mathrm{Hom}}_{\mathcal{O}_{Y_0}}(\mathcal{O}_{N_{Y_0}}, \mathcal{O}_{Y_0})^N \\ &\cong \underline{\mathrm{Hom}}_{\mathcal{O}_{Y_0}}(\mathcal{O}^N_{N_{Y_0}}, \mathcal{O}_{Y_0}) \cong \underline{\mathrm{Hom}}_{\mathcal{O}_{Y_0}}(\mathcal{O}_{Y_0}, \mathcal{O}_{Y_0}) \cong \mathcal{O}_{Y_0}.\end{aligned}$$

Hence $\Theta_{N,Y_0} \cong \mathcal{O}_{Y_0}$. The last assertion is trivial. □

8. Frobenius twists and Frobenius kernels

(8.1) In this section, S is an $\mathbb{F}_p$-scheme, where p is a prime number, and $\mathbb{F}_p$ is the prime field of characteristic p, unless otherwise specified.

(8.2) Let us consider the ordered set $\mathbb{Z}$ as a category. For $m, n \in \mathbb{Z}$, there is a unique morphism from m to n when $m \leq n$. Otherwise, $\mathbb{Z}(m, n)$ is empty.

Let $F\text{-Sch}_S$ be the category defined as follows. An object of $F\text{-Sch}_S$ is a pair $(h_X : X \to S, n)$ with $h_X : X \to S$ an S-scheme, and $n \in \mathbb{Z}$. The hom-set $F\text{-Sch}_S((X, n), (Y, m))$ is empty if $n < m$. If $n \geq m$, then $F\text{-Sch}_S((X, n), (Y, m))$ is the set of morphisms $\varphi : X \to Y$ (not necessarily S-morphisms) such that $h_Y \varphi = F_S^{n-m} h_X$, where $F_S^{n-m} : S \to S$ is the $(n-m)$th iteration of the (absolute) Frobenius morphism. Note that $\nu : F\text{-Sch}_S \to \mathbb{Z}^{\mathrm{op}}$ given by $\nu(X, n) = n$ is a functor which makes $F\text{-Sch}_S$ a fibered category over $\mathbb{Z}^{\mathrm{op}}$. An object $F\text{-Sch}_S$ is called an (F, S)-scheme, and a morphism of $F\text{-Sch}_S$ is called an (F, S)-morphism.

(8.3) For (F, S)-morphisms $\varphi : (X, n) \to (Y, r)$ and $h : (Y', m) \to (Y, r)$, the fiber product $(X, n) \times_{(Y,r)} (Y', m)$ in $F\text{-Sch}_S$ does not exist in general. However, if $S = \operatorname{Spec} k$ with k a perfect field (of characteristic p), then it does exist. If S is general and $m = r$, then it exists. It is $(X \times_Y Y', n)$ with the structure map

$$X \times_Y Y' \xrightarrow{p_1} X \to S.$$

Similarly, if $n = r$, then the fiber product exists.

(8.4) If $u : S \to S'$ is a morphism of $\mathbb{F}_p$-schemes, then

$$(h_X : X \to S, n) \mapsto (uh_X : X \to S', n)$$

is a functor from F-Sch_S to F-$\mathrm{Sch}_{S'}$. If $v : S' \to S$ is a morphism of $\mathbb{F}_p$-schemes, then $(X, n) \mapsto (S' \times_S X, n)$ is a functor from F-Sch_S to F-$\mathrm{Sch}_{S'}$.

(8.5) A forgetful functor F-$\mathrm{Sch}_S \to \underline{\mathrm{Sch}}/\mathbb{F}_p$ is given by $(X, n) \mapsto X$. X is called the underlying scheme of (X, n). Sheaves and modules over (X, n) are those for its underlying scheme X.

(8.6) For $n \in \mathbb{Z}$, We denote the fiber $\nu^{-1}(n)$ by F-$\mathrm{Sch}_{S,n}$. Note that $i : \underline{\mathrm{Sch}}/S \to F$-$\mathrm{Sch}_{S,0}$ given by $X \mapsto (X, 0)$ is an equivalence. We identify X with $i(X) = (X, 0)$, and $\underline{\mathrm{Sch}}/S$ with F-$\mathrm{Sch}_{S,0}$ via i, and consider that $\underline{\mathrm{Sch}}/S$ is a full subcategory of F-Sch_S.

(8.7) For $r \in \mathbb{Z}$, ${}^r(?) : F$-$\mathrm{Sch}_S \to F$-Sch_S given by ${}^r(X, n) = (X, n + r)$ and ${}^r\varphi = \varphi$ is an autoequivalence of F-Sch_S. ${}^r(?)$ is also denoted by $(?)^{(-r)}$. Thus we will write (X, r) by ${}^rX$.

In what follows, when we consider Frobenius maps, we work over F-Sch_S. The advantage of doing so is, we may consider $X^{(r)}$ for artitrary $r \in \mathbb{Z}$ not only for $S = \operatorname{Spec} k$ with k a perfect field (as in [Jan, (9.2)]), but also for an arbitrary $\mathbb{F}_p$-scheme S.

(8.8) A homomorphism $h : A \to B$ of $\mathbb{F}_p$-algebras is said to be *purely inseparable* if for each $b \in B$, there exists some $e \geq 0$ such that $b^{p^e} \in h(A)$. A morphism of $\mathbb{F}_p$-schemes $\varphi : X \to Y$ is purely inseparable if it is affine, and for each affine open subset U of Y, $\Gamma(U, Y) \to \Gamma(\varphi^{-1}(U), X)$ is purely inseparable. A purely inseparable morphism is a radical morphism, and hence is an integral morphism.

(8.9) Let Z be an S-scheme, $r \in \mathbb{Z}$ and $e \geq 0$. Note that the absolute Frobenius map $F_Z^e : {}^{e+r}Z \to {}^rZ$ is an (F, S)-morphism.

An $\mathcal{O}_Z$-module $\mathcal{M}$, viewed as an $\mathcal{O}_{{}^rZ}$-module (note that ${}^rZ$ is Z, when it is viewed as a scheme), is denoted by ${}^r\mathcal{M}$. The structure sheaf ${}^r\mathcal{O}_X$ is also denoted by $\mathcal{O}_{{}^rX}$.

Let $\psi : Z' \to Z$ be an S-morphism. The map ${}^{e+r}Z' \to {}^{e+r}Z \times_{{}^rZ} {}^rZ'$ given by $z' \mapsto (\psi(z'), F^e(z'))$ is denoted by $\Phi_e(Z, Z')$ or $\Phi_e(\psi)$ for $e \geq 0$. Note that

$\Phi_e(Z, Z')$ is purely inseparable. By abuse of notation, we sometimes denote the map

$$\eta_{\Phi_e(Z,Z')} : \mathcal{O}_{e+r Z \times_{r Z} {}^r Z'} \to \Phi_e(Z, Z')_* \mathcal{O}_{e+r Z'}$$

by $\Phi_e(Z, Z')$ or Φ_e. $\Phi_e(S, Z)$ is denoted by $\Phi_e(Z)$, and is called the *eth relative Frobenius map* or the *S-Frobenius map* of Z.

Φ_e is a natural transformation between the functors from the category of morphisms in $\underline{\mathrm{Sch}}/S$ to the category F-$\mathrm{Sch}_{S,r+e}$, as can be seen easily.

Lemma 8.10. *If G is an S-group scheme, then $\Phi_e(G) : {}^eG \to {}^eS \times_S G$ is a homomorphism of ${}^eS$-group schemes. Similarly, $\Phi_e(G) : G \to S \times_{S^{(e)}} G^{(e)}$ is a homomorphism of S-group schemes.*

Proof. The first assertion is the consequence of the commutativity of the diagram

$$\begin{tikzcd}
{}^eG \times_{{}^eS} {}^eG \arrow[r, "(\Phi_e,\Phi_e)"] \arrow[d, "\cong"] \arrow[dd, bend right=60, "\mu_{{}^eG}"'] & ({}^eS \times_S G) \times_{{}^eS} ({}^eS \times_S G) \arrow[d, "\cong"] \arrow[dd, bend left=60, "\mu_{{}^eS \times_S G}"] \\
{}^e(G \times_S G) \arrow[r, "\Phi_e"] \arrow[d, "{}^e\mu_G"] & {}^eS \times_S (G \times_S G) \arrow[d, "1\times\mu_G"] \\
{}^eG \arrow[r, "\Phi_e"] & {}^eS \times_S G
\end{tikzcd}.$$

The second assertion is also proved similarly. $\square$

(8.11) For a G-scheme Z, ${}^eZ$ is an ${}^eG$-scheme. Also, ${}^eS \times_S Z$ is an ${}^eS \times_S G$-scheme, and hence it is also an ${}^eG$-scheme through $\Phi_e(G)$. It is easy to see that $\Phi_e(Z) : {}^eZ \to {}^eS \times_S Z$ is an ${}^eG$-morphism.

The kernel of $\Phi_e(G) : G \to S \times_{S^{(e)}} G^{(e)}$ is denoted by G_e, and is called the *eth Frobenius kernel* of G, see [Jan, (I.9.4)] (for the case that $S = \operatorname{Spec} k$ with k a perfect field). It is an S-subgroup scheme of G. The kernel of $\Phi_e(G) : {}^eG \to {}^eS \times_S G$ is ${}^eG_e$. We may also call ${}^eG_e$ the Frobenius kernel, by abuse of terminologies.

Lemma 8.12. *Let V and W be locally Noetherian $\mathbb{F}_p$-schemes, and $\psi : V \to W$ a smooth morphism with relative dimension d. Then $\Phi_e(W, V)_*(\mathcal{O}_{{}^eV})$ is a locally free sheaf of ${}^eW \times_W V$ of rank p^{de}.*

Proof. The question is local both on V and W, and we may assume that $V = \operatorname{Spec} B$ and $W = \operatorname{Spec} A$ are both affine, and that there is a factorization

$A \to C = A[x_1, \ldots, x_d] \to B$ such that C is a polynomial ring on d variables over A, and B is étale over C (see [Mil, (I.3.24)]). It suffices to show that ${}^eB$ is a projective ${}^eA \otimes_A B$-module of rank p^{de}. By [Has5, (33.5)], ${}^eB \cong {}^eC \otimes_C B$, and hence we may assume that $B = C$. In this case, ${}^eA \otimes_A B = {}^eA[x_1, \ldots, x_d]$, and ${}^eB = {}^eA[{}^ex_1, \ldots, {}^ex_d]$. So ${}^eB$ is a free ${}^eA \otimes_A B$-module with the basis $\{{}^ex_1^{i_1} \cdots {}^ex_d^{i_d} \mid 0 \leq i_1, \ldots, i_d < p^e\}$. $\square$

Lemma 8.13. *Let G be an S-group scheme and N its normal subgroup scheme. Let $\psi : V \to W$ be a G-enriched principal N-bundle, and assume that S and N are locally Noetherian, and N is regular over S (that is, flat with geometrically regular fibers). Then $\Phi_e(W, V) : {}^eV \to {}^eW \times_W V$ is an ${}^eG$-enriched principal ${}^eN_e$-bundle.*

Proof. It is obvious that $\Phi_e(W, V)$ is an ${}^eG$-morphism. So it suffices to prove that $\Phi_e(W, V)$ is a principal ${}^eN_e$-bundle, assuming that $G = N$. Since N is flat over S, ψ is fpqc.

Let W' be the S-scheme V with the trivial N-action, and $h : W' \to W$ be ψ. Then since ψ is a principal N-bundle, the base change $\psi' : V' \to W'$ of ψ by h is a trivial N-bundle. As the base change

$$\Phi_e(W, V)' : W' \times_W {}^eV \xrightarrow{1_{W'} \times \Phi_e(W,V)} W' \times_W ({}^eW \times_W V)$$

is identified with $\Phi_e(W', V') = \Phi_e(\psi')$ (see [Has2, Lemma 4.1, **4**]), it suffices to prove that $\Phi_e(W', V')$ is a principal ${}^eN_e$-bundle by [Has11, (2.11)], since h is fpqc.

As ψ' is a trivial bundle, $\Phi_e(\psi')$ is identified with $1_{W'} \times \Phi_e(N)$. By [Has11, (2.11)] again, it suffices to prove that $\Phi_e(N) : {}^eN \to {}^eS \times_S N$ is a principal ${}^eN_e$-bundle. By the theorem of Radu and André [Rad], [And], [Dum], $\Phi_e(W, V) = \Phi_e(N)$ is flat. Being a homeomorphism, it is fpqc. Note that $\operatorname{Ker} \Phi_e(N) = {}^eN_e$. Being an fpqc homomorphism, Φ_e is a principal ${}^eN_e$-bundle (as in [Has11, (6.4)]), as required. $\square$

9. Semireductive group schemes

Lemma 9.1. *Let $A \subset B$ be a finite extension of commutative rings. A is Noetherian if and only if B is Noetherian. A is Noetherian F-finite if and only if B is Noetherian F-finite.*

Proof. If A is Noetherian, then B is Noetherian by Hilbert's basis theorem. The converse is known as Eakin–Nagata theorem [Mat, Thereom 3.7].

We prove the second assertion. If A is F-finite, then B is F-finite, since B is F-finite over A [Has8, Lemma 2, Example 3]. We prove the converse. We have $A^p \subset B^p \subset B$, and B^p is A^p-finite and B is B^p-finite, where $A^p = \{a^p \mid a \in A\} = F_A(A) \subset A$ (F_A is the Frobenius map). So B is A^p-finite. As A^p is Noetherian and A is a submodule of the finite A^p-module B, A is a finite A^p-module, and A is F-finite. $\square$

Lemma 9.2. *Let k be a field of characteristic $p > 0$, and G a finite group. Then there exists some $e \geq 0$ such that for any finite dimensional G-module V and any $v \in V^G \setminus \{0\}$, there exists some $h \in \mathrm{Sym}_{p^e} V^*$ such that $h(v) = 1$.*

Proof. Let P be a Sylow p-subgroup of G of order p^e. Let $\sigma_1, \ldots, \sigma_n$ be a complete set of representatives of G/P. Note that n is invertible in k. Let $\psi \in V^*$ be any element such that $\psi(v) = 1$. Then $h = n^{-1} \sum_{i=1}^{r} \sigma_i \prod_{g \in P} g\psi$ is the desired element. $\square$

(9.3) Let k be a field of arbitrary characteristic, and G be an affine algebraic k-group scheme. We say that G is *semireductive* if $\bar{G}^\circ_{\mathrm{red}}$ is (connected) reductive, where $\bar{k}$ is the algebraic closure of k, and $\bar{G} = \bar{k} \otimes_k G$. That is, the radical of $\bar{G}_{\mathrm{red}}$ is a torus. We say that G is a *semitorus* if $\bar{G}^\circ_{\mathrm{red}}$ is a torus. Note that a semireductive and linearly reductive are equivalent in characteristic zero. A linearly reductive affine algebraic k-group scheme in characteristic $p > 0$ is a semitorus, as can be seen easily from Nagata's theorem [Nag, Theorem 1]. See also [Swe2].

Lemma 9.4. *Let k be a field of arbitrary characteristic, and G a semireductive affine algebraic k-group scheme. Let B be a G-algebra, and I a G-ideal. Then for each $b \in (B/I)^G$, there exists some r such that $b^r \in B^G/I^G$. More precisely,*

1 *If the characteristic of k is zero, r can be taken to be* 1.

2 *If the characteristic of k is $p > 0$, then r can be taken to be a power of p.*

3 *In* **2***, if G is a semitorus, then there exists some e_0 which depends only on G and independent of B or b, such that r can be taken to be p^{e_0}.*

In case **2**, $(B/I)^G$ *is purely inseparable over* B^G/I^G. *In any case, the canonical map* $\operatorname{Spec}(B/I)^G \to \operatorname{Spec}(B^G/I^G)$ *is a universal homeomorphism.*

Proof. We may assume that k is algebraically closed. If the characteristic of k is zero, then G is linearly reductive, and $B^G \to (B/I)^G$ is surjective, and the assertion **1** is obvious. So we may assume that the characteristic is $p > 0$.

Take $c \in B$ such that c modulo I equals b. Let $e \geq 0$ be the number such that G/G_e is reduced (hence is smooth and is isomorphic to $G_{\mathrm{red}}^{(e)}$). Note that the connected reductive group G°_{red} over the algebraically closed field k is defined over $\mathbb{F}_p$ (see [Jan, (II.1)] and references therein), and hence $(G_{\mathrm{red}}^{(e)})^\circ = (G^\circ_{\mathrm{red}})^{(e)} \cong G^\circ_{\mathrm{red}}$ is reductive. It is easy to see that $c^{p^e} \in B^{G_e}$. Note that we can take this e depending only on G. Replacing b by b^{p^e}, B by B^{G_e}, I by I^{G_e}, and G by $G/G_e \cong G_{\mathrm{red}}^{(e)}$, we may assume that G is smooth.

Then by Haboush's theorem (the Mumford conjecture) [Jan, (II.10.7)] and [MuFK, (A.1.2)], we have that there exists some e such that b^{p^e} is in $(B^{G^\circ}/I^{G^\circ})^{G/G^\circ}$. The choice of e may depend on b this time, but if **3** is assumed, then G° is a torus, which is linearly reductive, and we can take $e = 0$, which depends only on G.

Then replacing G by G/G°, we may assume that G is a finite group. This case is proved by the same proof as in [MuFK, (A.1.2)], using Lemma 9.2. $\square$

Lemma 9.5. *Let k be a field of arbitrary characteristic and G a semireductive k-group scheme, and $\varphi : X \to Y$ be an algebraic quotient by G. Then φ is a universally submersive categorical quotient. If, moreover, φ is a geometric quotient, then it is universally open.*

Proof. We may assume that k is of characteristic $p > 0$ by [MuFK, Theorem 1.1]. In the proof of [MuFK, Theorem A.1.1] in Appendix to Chapter 1, C., it is proved that φ is a submersive categorical quotient. We prove the first assertion. We only need to prove that φ is universally submersive. So we may assume that $Y = \operatorname{Spec} A$ is affine. Then $B = \operatorname{Spec} B$ is also affine and $A = B^G$. It suffices to show that for any A-algebra A', the base change $X' = \operatorname{Spec} B' \to \operatorname{Spec} A' = Y'$ is submersive.

There is a sequence of maps

$$A \xrightarrow{\alpha} A'' \xrightarrow{\beta} A'$$

such that α is flat and β is surjective. Indeed, for each $a \in A'$, consider a variable x_a, and set $A'' = A[x_a \mid a \in A']$. Then $(B'')^G = A''$, where

$B'' = A'' \otimes_A B$. So we know that $\varphi'' = X'' = \operatorname{Spec} B'' \to \operatorname{Spec} A'' = Y''$ is submersive. So replacing A by A'', we may assume that $A \to A' = A/I$ is surjective.

We want to prove that the canonical map $\gamma : Z' = \operatorname{Spec}(B')^G \to \operatorname{Spec} A' = Y'$ is a homeomorphism. As we know that φ is surjective, $\varphi' : X' = \operatorname{Spec} B' \to \operatorname{Spec} A' = Y'$ is also surjective. As φ' factors through γ, we have that γ is surjective. On the other hand, by Lemma 9.4, $A' \to (B')^G$ is purely inseparable. Thus γ is injective and closed, and hence is a homeomorphism. As $\delta : X' = \operatorname{Spec} B' \to \operatorname{Spec}(B')^G = Z'$ is known to be submersive, $\varphi' = \gamma\delta$ is also submersive.

Now the last assertion follows from Lemma 1.11. □

Lemma 9.6. *Let k be a field, and G an affine algebraic k-group scheme. Let B be a Noetherian G-algebra. Set $A := B^G$. Assume either*

- **a** *k is of characteristic zero and G is semireductive;*
- **b** *k is a field of characteristic $p > 0$, G is a semitorus, and B is F-finite; or*
- **c** *k is a field of characteristic $p > 0$, G is semireductive, and there is a Noetherian k-algebra R and a k-algebra map $R \to A$ such that B is of finite type over R.*

Then

- **1** *For any B-finite (G, B)-module M, M^G is a finite A-module.*
- **2** *A is Noetherian.*
- **3** *If G is finite, then B is finite over A.*
- **4** *In the case of* **b**, *A is F-finite.*
- **5** *In the case of* **c**, *A is of finite type over R.*

Proof. We prove the lemma by the Noetherian induction. The cases are divided, and when we consider the case **b** (resp. **c**), the inductive hypothesis **5** (resp. **4**) will never be used.

We may assume that for any nonzero G-ideal I of B and any $(G, B/I)$-module M, M^G is $(B/I)^G$-finite, $(B/I)^G$ is Noetherian, and if **b** is assumed,

$(B/I)^G$ is F-finite. If **c** is assumed, then we may assume that $(B/I)^G$ is of finite type over R.

Consider the case that **b** is assumed. Note that there exists some e_0 such that $((B/I)^G)^{p^{e_0}} \subset B^G/I^G \subset (B/I)^G$ by Lemma 9.4, **3**. As $(B/I)^G$ is F-finite, $(B/I)^G$ is a finite B^G/I^G-module for any nonzero G-ideal I of B. If **c** is assumed, then we have that $(B/I)^G$ is integral over B^G/I^G by Lemma 9.4, **2**. As we assume that $(B/I)^G$ is of finite type over R, $(B/I)^G$ is finite over B^G/I^G. If **a** is assumed, then $B^G/I^G = (B/I)^G$ by the linear reductivity, and obviously $(B/I)^G$ is B^G/I^G-finite. Thus $(B/I)^G$ is finite over B^G/I^G in either case.

In either case, we have that B^G/I^G is a Noetherian ring by Lemma 9.1. Also by the induction hypothesis, M^G is a Noetherian A-module if M is a B-finite (G, B)-module with $\operatorname{ann} M \neq 0$.

We prove that for any B-finite (G, B)-module M, M^G is a Noetherian A-module. This proves **1** and **2**.

If B is not a G-domain (see [HasM]) and $IJ = 0$ for some nonzero G-ideals I and J, then

$$0 \to (IM)^G \to M^G \to (M/IM)^G$$

is exact and $(IM)^G$ and $(M/IM)^G$ are Noetherian A-modules (since J and I respectively annihilate IM and M/IM), M^G is a Noetherian A-module.

So we may assume that B is a G-domain. We prove that M^G is a Noetherian A-module by the induction on the length of M_P, where P is any fixed minimal prime ideal of B.

By a G-torsion submodule of M, we mean a (G, B)-submodule of M whose annihilator is nonzero. We define the G-torsion part M_{tor} of M to be the sum of all the G-torsion submodules of M. Note that M_{tor} is the largest G-torsion submodule of M. As $(M_{\mathrm{tor}})^G$ is a Noetherian A-module and

$$0 \to (M_{\mathrm{tor}})^G \to M^G \to (M/M_{\mathrm{tor}})^G$$

is exact, replacing M by M/M_{tor}, we may assume that M is G-torsion-free, that is, M does not have a nonzero G-torsion submodule.

If $M^G = 0$, then M^G is a Noetherian module. If $M^G \neq 0$, then there is an injective (G, B)-linear map $B \to M$. As $(M/B)^G$ is Noetherian by induction, it suffices to prove that $A = B^G$ is a Noetherian A-module, that is, A is a Noetherian ring.

Let J be an ideal of A. We want to show that J is finitely generated. If $J = 0$, then J is finitely generated. So we may assume that $J \neq 0$. Let

$a \in J \setminus \{0\}$. Since aB is a nonzero G-ideal of B and B is a G-domain, we have that $0 :_B aB = 0$. That is, a is a nonzero divisor in B. So $B \subset B[a^{-1}]$, and hence $A = B^G = B \cap B[a^{-1}]^G$. So if $c = ab \in aB \cap A$, then $b = ca^{-1} \in B \cap B[a^{-1}]^G = A$. So $c \in aA$, and we have that $(aB)^G = aB \cap A = aA$. Hence A/aA is a Noetherian ring, and J/aA is finitely generated. Hence J is finitely generated, as desired.

Next, we prove **3**. Let B_0 be the k-algebra B with the trivial G-action. Then the coaction $\omega_B : B \to B_0 \otimes k[G]$ is a G-algebra map. Thus $B_0 \otimes k[G]$ is a B-finite (G, B)-module. Clearly, $(B_0 \otimes k[G])^G = B_0 \otimes k = B_0$ as A-algebras, and B_0 is isomorphic to B as A-algebras, as can be seen easily. On the other hand, $B_0 = (B_0 \otimes k[G])^G$ is A-finite by **1**. Thus B is A-finite, and **3** has been proved.

So the case **a** has been completed, because **4** and **5** are trivial in this case.

If **c** is assumed, then $\operatorname{Spec} B \to \operatorname{Spec} A$ is universally submersive by Lemma 9.5. Now the assertion **5** follows from **2**, which has already been proved, by [Alp, (6.2.1)]. So the proof of **c**$\Rightarrow$**1,2,3,5** has been completed (as we have emphasized, we have not used the inductive hypothesis **4** for the case **c** at all).

If G is finite, then **4** follows from **3** and Lemma 9.1.

Thus the lemma has also been completely proved for the case that G is finite.

We prove the lemma for the general case. It suffices to prove **1, 2, 3, 4** assuming **b**.

Replacing k by its finite purely inseparable extension, we may assume that G_{red} is k-smooth. Then for some $e \geq 0$, $G/G_e \cong k \otimes_{k^{p^e}} G_{\mathrm{red}}^{(e)}$ is k-smooth, see for the notation on the Frobenius twist, see section 12. Note that $\bar{k} \otimes_k (G/G_e) \cong \bar{k} \otimes_{k^{p^e}} G_{\mathrm{red}}^{(e)} \cong \bar{G}_{\mathrm{red}}^{(e)}$. As the torus $\bar{G}_{\mathrm{red}}^{\circ}$ is defined over $\mathbb{F}_p$, we have that G/G_e is a semitorus. As the lemma is already proved for the finite group scheme G_e, replacing G by G/G_e, we may assume that G is smooth.

Next, replacing k by some finite Galois extension, we may assume that G° is a split torus. If the lemma is proved for the split torus G°, then as the lemma is already proved for the finite group scheme G/G°, the proof of the lemma completes. Thus we may assume that G is a split torus. Then replacing k by $\mathbb{F}_p$, we may assume that $k = \mathbb{F}_p$, which is perfect. Then the absolute Frobenius map $F : G \to G^{(1)}$ is a faithfully flat homomorphism of k-group schemes by the theorem of Kunz [Kun2, Theorem 2.1], G acts on

$B^{(1)}$, and $(B^{(1)})^G = (B^{(1)})^{G^{(1)}} = A^{(1)}$ by Lemma 4.15.

Now we repeat the inductive argument above. Then as above, we reach the situation that **1, 2, 3** are already proved, and we prove **4**. Then **1, 2, 3** are also true for the action of $G^{(1)}$ on $B^{(1)}$. Hence **1, 2, 3** are also true for the action of G on $B^{(1)}$ through the Frobenius map $G \to G^{(1)}$.

As B is a $B^{(1)}$-finite $(G, B^{(1)})$-module by F-finiteness, $A = B^G$ is a finite $A^{(1)}$-module by **1**, which has already been proved. Now by induction, we have proved **1, 2, 3, 4** for the case **b**. This finishes the proof of the lemma. □

Chapter 1. Main Results

10. Almost principal fiber bundles

(10.1) Let $f : G \to H$ be a qfpqc homomorphism of S-group schemes with $N = \operatorname{Ker} f$.

Definition 10.2. A diagram of G-schemes

$$X \xleftarrow{i} U \xrightarrow{\rho} V \xrightarrow{j} Y$$

is said to be a *G-enriched rational n-almost principal N-bundle* if the following six conditions hold.

1 N acts on Y trivially.

2 i is an open immersion.

3 j is an open immersion.

4 $i(U)$ is n-large in X.

5 $j(V)$ is n-large in Y.

6 ρ is a G-enriched principal N-bundle.

A G-enriched rational n-almost principal G-bundle is simply called a rational n-almost principal G-bundle. In these definitions, we may simply say 'almost' instead of saying '1-almost.'

Definition 10.3. A G-morphism $\varphi : X \to Y$ is said to be a *G-enriched n-almost principal N-bundle* with respect to U and V if U is a G-stable open subset of X and V is an H-stable open subset of Y, $\varphi(U) \subset V$, and

$$X \xleftarrow{i} U \xrightarrow{\rho} V \xrightarrow{j} Y$$

is a G-enriched rational n-almost principal N-bundle, where $\rho : U \to V$ is the restriction of φ, and i and j are inclusions. We simply say that φ is a G-enriched n-almost principal N-bundle, if it is so with respect to some U and V. We may omit the epithet 'G-enriched' if $G = N$. We may use 'almost' as a synonym of '1-almost.'

Lemma 10.4. *Let S be a scheme, and $h : M \to N$ be a morphism of S-schemes. Then the following conditions are equivalent.*

1 *h is qfpqc.*

2 *For any S-scheme W and an S-morphism $\alpha : W \to N$, there exists some qfpqc morphism $\beta : W' \to W$ such that $\alpha\beta \in N(W')$ is in the image of $h(W') : M(W') \to N(W')$.*

Proof. **1⇒2**. Replacing M if necessary, we may assume that h is fpqc. Let $W' = M \times_N W$, and let $\beta : W' \to W$ be the second projection, and let $\gamma \in M(W')$ be the first projection. Then $h\gamma = \alpha\beta$ is in the image of $h(W')$.

2⇒1. Let α be the identity morphism of N. Then there exists some qfpqc morphism $\gamma : W' \to M$ such that $\beta = h\gamma : W' \to N$ is qfpqc. Thus h is also qfpqc, as desired. □

Lemma 10.5. *Let $h : M \to N$ be a qfpqc monomorphism of S-schemes. Then h is an isomorphism.*

Proof. Letting $W = N$ in Lemma 10.4, **2**, there is a qfpqc morphism $\beta : W' \to N$ which is in the image of $h(W') : M(W') \to N(W')$. Replacing β if necessary, we may assume that β is fpqc. There is $\gamma \in M(W')$ such that $h\gamma = \beta$. Let $p_i : W' \times_N W' \to W'$ be the ith projection. Then $h\gamma p_1 = \beta p_1 = \beta p_2 = h\gamma p_2$. As h is a monomorphism, $\gamma p_1 = \gamma p_2$. By [Vis, (2.55)], there is $g : N \to M$ such that $g\beta = \gamma$. Then $hg\beta = h\gamma = \beta$. By [Has11, (2.9)] (or by [Vis, (2.55)] again), $hg = 1_N$. So $hgh = 1_N h = h1_M$. As h is a monomorphism, $gh = 1_M$. So $g = h^{-1}$, and h is an isomorphism, as desired. □

(10.6) Let E be an S-group scheme, and G its subgroup scheme. Let $f : G \to H$ be a qfpqc homomorphism between S-group schemes, and $N := \operatorname{Ker} f$. Assume that G and N are normal in E. As G is normal in E, E acts on G by the conjugation. That is, the action is given by $(a, g) \mapsto aga^{-1}$ for $a \in E$ and $g \in G$.

Lemma 10.7. *Let the notation be as in* (10.6). *There exists a unique action of E on H such that f is an E-morphism. The action is by group automorphisms.*

Proof. Consider the following diagram.

$$\begin{tikzcd} E \times G \times_H G \arrow[r, shift left, "1_E \times p_1"] \arrow[r, shift right, "1_E \times p_2"'] & E \times G \arrow[r, "1_E \times f"] \arrow[d, "a_G"] & E \times H \arrow[d, dotted, "a_H"] \\ & G \arrow[r, "f"] & H \end{tikzcd}$$

We want to find a unique arrow a_H such that the diagram is commutative. As the representable functor $H = \operatorname{Hom}_{\underline{\mathrm{Sch}}/S}(?, H)$ is a sheaf with respect to the fpqc topology [Vis, (2.55)], $H(E \times H)$ is the difference kernel of

$$H(E \times G) \underset{(1_E \times p_2)^*}{\overset{(1_E \times p_1)^*}{\rightrightarrows}} H(E \times G \times_H G) .$$

So a_H which makes the diagram commutative is unique.

To show the existence, it suffices to show that $fa_G(1_E \times p_1) = fa_G(1_E \times p_2)$. Let $b \in E$, and $(g_1, g_2) \in G \times_H G$. Then

$$\begin{aligned}(fa_G(1_E \times p_1)(b, g_1, g_2))(fa_G(1_E \times p_2)(b, g_1, g_2))^{-1} = \\ (fa_G(b, g_1))(fa_G(b, g_2))^{-1} = f(bg_1b^{-1})f(bg_2b^{-1})^{-1} = f(bg_1g_2^{-1}b^{-1}).\end{aligned}$$

As $g_1g_2^{-1} \in N$ and N is normal in E, $f(bg_1g_2^{-1}b^{-1})$ is trivial, and we have $fa_G(1_E \times p_1) = fa_G(1_E \times p_2)$.

We show that the action a_H is by group automorphisms. We want to show that the two maps $\alpha_1, \alpha_2 : E \times H \times H \to H$ given by $\alpha_1(b, h_1, h_2) = b \cdot (h_1h_2)$ and $\alpha_2(b, h_1, h_2) = (b \cdot h_1)(b \cdot h_2)$ agree. As the qfpqc morphism $1_E \times f \times f : E \times G \times G \to E \times H \times H$ is an epimorphism, it suffices to prove that $\alpha_1(1_E \times f \times f) = \alpha_2(1_E \times f \times f)$. This is left to the reader. $\square$

Lemma 10.8. *Let E, G, $f : G \to H$, and N be as in* (10.6)*. Let X, Y, and Z be E-schemes. Let $\varphi : X \to Y$ be a G-morphism which is N-invariant. Let $\psi : Y \to Z$ be an H-invariant morphism. Consider the following conditions.*

a *φ is an E-enriched principal N-bundle.*

b *ψ is an E-enriched principal H-bundle.*

c *$\psi\varphi$ is an E-enriched principal G-bundle.*

Then we have

1 **a** *and* **b** *together imply* **c**.

2 **a** *and* **c** *together imply* **b**.

3 *If φ is an E-morphism, then* **b** *and* **c** *together imply* **a**.

Proof. **1**. First, we prove that $\Psi_G : G \times X \to X \times_Z X$ given by $\Psi_G(g, x) = (gx, x)$ is a monomorphism. That is, $\Psi_G(W) : G(W) \times X(W) \to X(W) \times_{Z(W)} X(W)$ is injective for any S-scheme W. Let $(g, x), (g_1, x_1) \in G(W) \times X(W)$ such that $\Psi_G(W)(g, x) = \Psi_G(W)(g_1, x_1)$. Then $x = x_1$, and $gx = g_1x$. Letting $g_1^{-1}g = u$, $ux = x$. As $f(u)\varphi(x) = \varphi(x)$ and hence $\Psi_H(e, \varphi(x)) = \Psi_H(f(u), \varphi(x))$, we have that $f(u) = e$ by **b**. That is, $u \in N$.

As $\Psi_N(e, x) = \Psi_N(u, x)$, we have $u = e$ by **a**, and hence Ψ_G is injective.

Next, we prove that for any $(x', x) \in X(W) \times_{Z(W)} X(W)$, there exists some qfpqc morphism $\alpha : W' \to W$ such that $(x'\alpha, x\alpha) \in X(W') \times_{Z(W')} X(W')$ is in the image of $\Psi_G(W')$.

Note that $(\varphi x', \varphi x) \in Y(W) \times_{Z(W)} Y(W)$ is in the image of $\Psi_H(W)$. That is, there exists some $h \in H(W)$ such that $h\varphi x = \varphi x'$. As $f : G \to H$ is qfpqc, there exists some qfpqc morphism $\beta : W' \to W$ and $g \in G(W')$ such that $f(g) = h\beta$. Then we have $\varphi(g(x\beta)) = \varphi(x'\beta)$. As $(g(x\beta), x'\beta) \in X(W') \times_{Y(W')} X(W')$ is in the image of $\Psi_N(W')$, there exists some $n \in N(W')$ such that $ng(x\beta) = x'\beta$. This shows that $\Psi_G(ng, x\beta) = (x'\beta, x\beta)$, and hence the image of (x', x) in $(X \times_Z X)(W')$ is in the image of $\Psi_G(W')$. Hence Ψ_G is qfpqc by Lemma 10.4. Being a qfpqc monomorphism, Ψ_G is an isomorphism by Lemma 10.5. As φ and ψ are qfpqc E-morphisms, $\psi\varphi$ is a qfpqc E-morphism by [Has11, (2.3)]. By [Vis, (4.43)], $\psi\varphi$ is an E-enriched principal G-bundle.

2. Let W be any S-scheme, $y \in Y(W)$, $h \in H(W)$ such that $hy = y$. Then there is a qfpqc morphism $\beta : W' \to W$, $x \in X(W')$, $g \in G(W')$ such that $\varphi(x) = y\beta$ and $f(g) = h\beta$. Thus $f(g)(\varphi(x)) = \varphi(x)$. This implies $(gx, x) \in X(W') \times_{Y(W')} X(W')$. So there exists some $n \in N(W')$ such that $(nx, x) = (gx, x)$. As Ψ_G is a monomorphism, $g = n \in N(W')$. Hence $h\beta = f(g) = e = e\beta$. As β is an epimorphism, $h = e$, and hence Ψ_H is a monomorphism.

Next, let $(y', y) \in Y(W) \times_{Z(W)} Y(W)$. Take an fpqc morphism $\beta : W' \to W$, $x' \in X(W')$, $x \in X(W')$ such that $\varphi(x') = y'\beta$ and $\varphi(x) = y\beta$. As $\Psi_G(W')$ is bijective, there exists some $g \in G(W')$ such that $gx = x'$. Then $\Psi_H(fg, y\beta) = (y'\beta, y\beta)$. By Lemma 10.5, Ψ_H is an isomorphism. As $\psi\varphi$ is qfpqc, ψ is qfpqc. As $\psi\varphi$ is an E-morphism and φ is a qfpqc E-morphism, it is easy to see that ψ is an E-morphism. Thus ψ is an E-enriched principal H-bundle.

3. Consider the diagram

$$\begin{tikzcd}
G\times X \arrow[rr, "a_{G,X}"] \arrow[d, "f\times 1_X"] \arrow[drr, phantom, "\text{(a)}"] & & X \arrow[d, "\varphi"] \\
H\times X \arrow[r, "1_H\times\varphi"] \arrow[d, "p_2^X"] \arrow[dr, phantom, "\text{(b)}"] & H\times Y \arrow[r, "a_{H,Y}"] \arrow[d, "p_2^Y"] \arrow[dr, phantom, "\text{(c)}"] & Y \arrow[d, "\psi"] \\
X \arrow[r, "\varphi"] & Y \arrow[r, "\psi"] & Z
\end{tikzcd},$$

where p_2^X and p_2^Y are the second projections. It is easy to check that the diagram is commutative. As ψ is a principal H-bundle, the square (c) is cartesian. Similarly, as $\psi\varphi$ is a principal G-bundle, the whole square ((a)+(b)+(c)) is also cartesian. (b) is also cartesian, and hence it is easy to see that (a) is cartesian. On the other hand, letting N act on $H \times X$ trivially and on $G \times X$ by $n(g, x) = (ng, x)$, (a) is a commutative diagram of N-schemes.

As $f \times 1_X$ is a principal N-bundle and it is a base change of φ by $\psi\varphi$, which is qfpqc, we have that φ is also a principal N-bundle by [Has11, (2.11)]. As we assume that φ is an E-morphism, we have that φ is an E-enriched principal N-bundle. □

(10.9) In Lemma 10.8, **3**, the assumption that φ is an E-morphism is indispensable. Let $S = \operatorname{Spec} k = Z$ with k a field, $Y = H = N = E_0 = \mathbb{Z}/2\mathbb{Z}$ (the constant group), $X = G = H \times N$, $\varphi = f : G \to H$ the first projection, and $E = E_0 \times G$. Let E act on X by $(e_0, h, n)(h', n') = (e_0hh', nn')$ for $e_0 \in E_0$ and $(h, n), (h', n') \in G$. Let E act on Y by $(e_0, h, n)y = hy$, and on Z trivially. Then **b** and **c** are satisfied, but φ is not an E-morhphism.

Lemma 10.10. *Let $\psi : Z' \to Z$ be a flat morphism of schemes, and $W \subset Z$ be an open subset. Set $W' := \psi^{-1}(W)$.*

1 *If W is n-large in Z, then W' is n-large in Z'.*

2 *Assume that ψ is qfpqc, and that Z' is locally Krull. If W' is large in Z', then W is large in Z.*

3 *Assume that ψ is qfpqc, and that Z' is locally Noetherian. If W' is n-large in Z', then W is n-large in Z.*

Proof. **1**. Let $z' \in Z' \backslash W'$. Then $z := \psi(z') \in Z \backslash W$, and hence $\dim \mathcal{O}_{Z,z} \geq n$. As the going-down theorem holds between $\mathcal{O}_{Z,z}$ and $\mathcal{O}_{Z',z'}$ by flatness, we have $\dim \mathcal{O}_{Z',z'} \geq \dim \mathcal{O}_{Z,z} \geq n$, and hence W' is n-large in Z'.

2, 3. The question is local on Z, and we may assume that $Z = \operatorname{Spec} A$ is affine. Replacing Z', we may assume that $Z' = \operatorname{Spec} A'$ is also affine. Note that A' is faithfully flat over A.

We prove **2**. A' is locally Krull. As A' is a finite direct product of Krull domains, so is A by [Has9, (5.8)]. So we may further assume A is a domain. Then the result follows from [Has9, (5.13)].

We prove **3**. A' is Noetherian. Then A is also Noetherian. Let $P \in \operatorname{Spec} A$ with $\operatorname{ht} P < n$. If P' is a minimal prime of the ideal PB, then $\operatorname{ht} P' < n$ by [Mat, Theorem 15.1]. So the assertion follows. $\square$

Remark 10.11. Let A be the DVR $k[y]_{(y)}$, B the DVR $k(y)[x]_{(x)}$, and $A' := A + xB$. Note that A' is the composite of B and A [Mat, section 10]. Set $Z' := \operatorname{Spec} A'$, and $Z := \operatorname{Spec} A$. Let $W = Z \setminus \{(y)\} = D(y) \subset Z$. Let $\psi : Z' \to Z$ be the morphism associated with the inclusion $A \hookrightarrow A'$. Then although A is a DVR and A' is a valuation ring faithfully flat over A, the conclusion of **2** or **3** in (10.10) does not hold. So the Noetherian or Krull hypothesis on A' is indispensable.

Lemma 10.12. *Let $f : G \to H$ be a qfpqc homomorphism between S-group schemes with $N = \operatorname{Ker} f$, and $\varphi : X \to Y$ a G-morphism which is N-invariant. Let $U \subset X$ and $V \subset Y$ be open subsets. Let $h : Y' \to Y$ be a flat G-morphism such that Y' is N-trivial. Let $\varphi' : X' := Y' \times_Y X \to Y'$ be the base change, and set $U' = Y' \times_Y U$, and $V' = Y' \times_Y V$. Then*

a *If φ is a G-enriched n-almost principal N-bundle with respect to U and V, then φ' is a G-enriched n-almost principal N-bundle with respect to U' and V'.*

b *Assume that h is fpqc, and that both X' and Y' are locally Krull* (*resp. locally Noetherian*). *If φ' is a G-enriched almost* (*resp. n-almost*) *principal N-bundle with respect to U' and V', then φ is a G-enriched almost* (*resp. n-almost*) *principal N-bundle with respect to U and V.*

Proof. Clearly, φ' is also a G-morphism which is N-invariant.

a Let $\rho : U \to V$ be the restriction of φ, and $\rho' : U' \to V'$ be its base change. Each of the six conditions in Definition 10.2 for X', $\rho' : U' \to V'$ and U' is proved using the corresponding condition for X, $\rho : U \to V$ and U. This is trivial for the conditions **1**, **2**, and **3**. The conditions **4** and **5** follow from Lemma 10.10, **1**. The conditions **6** follows from [Has11, (2.7)].

b The image of the composite

$$G \times U' \hookrightarrow G \times X' \xrightarrow{a} X' \to X$$

is contained in U by assumption. This map agrees with

$$G \times U' \xrightarrow{1 \times h|_{U'}} G \times U \hookrightarrow G \times X \xrightarrow{a} X.$$

As $1 \times h|_{U'}$ is faithfully flat and hence is surjective, U is G-stable. Similarly, V is H-stable.

Now we check the six conditions in Definition 10.2 for X, $\rho : U \to V$ and U. The conditions **1, 2, 3** are assumed.

4 and **5** are the consequences of Lemma 10.10, **2** (resp. **3**). **6** follows from [Has11, (2.11)]. □

Theorem 10.13. *Let G be a quasi-compact quasi-separated flat S-group scheme, and $\varphi : X \to Y$ a quasi-compact quasi-separated almost principal G-bundle. Assume that X is locally Krull. Then the following are equivalent.*

1 *The canonical map $\bar{\eta} : \mathcal{O}_Y \to (\varphi_* \mathcal{O}_X)^G$ is an isomorphism.*

2 *Y is a locally Krull scheme.*

Proof. By Lemma 2.21, **2**, the question is local on Y, and hence we may assume that Y is affine. So X is quasi-compact quasi-separated, and **1**⇒**2** follows from [Has9, (6.3)].

We prove **2**⇒**1**. Let φ be an almost principal G-bundle with respect to $i : U \hookrightarrow X$ and $j : V \hookrightarrow Y$. Let $\rho : U \to V$ be the restriction of φ, which is a principal G-bundle. Then applying Lemma 2.21, **3** to the cartesian square

$$\begin{tikzcd} \varphi^{-1}(V) \arrow[r, "i'"] \arrow[d, "\varphi'"] & X \arrow[d, "\varphi"] \\ V \arrow[r, "j"] & Y \end{tikzcd} ,$$

the result follows if $\bar{\eta} : \mathcal{O}_V \to (\varphi'_*\mathcal{O}_{\varphi^{-1}(V)})^G$ is an isomorphism, where $\varphi' : \varphi^{-1}(V) \to V$ is the restriction of φ, and $i' : \varphi^{-1}(V) \to X$ is the inclusion. So we may assume that $V = Y$ (and hence $\varphi^{-1}(V) = X$). As $\eta : \mathcal{O}_X \to i_*\mathcal{O}_U$ is an isomorphism by [Has9, (5.28)], it suffices to show that the composite

$$\mathcal{O}_Y \xrightarrow{\bar{\eta}} (\varphi_*\mathcal{O}_X)^G \xrightarrow{\eta} (\varphi_* i_*\mathcal{O}_U)^G = (\rho_*\mathcal{O}_U)^G,$$

which equals $\bar{\eta}$ for ρ, is an isomorphism. As ρ is a principal G-bundle, this is [Has11, (5.31)]. $\square$

Example 10.14. Theorem 10.13 can be used to check that a candidate of the invariant subring is certainly the one.

Let $S = \operatorname{Spec}\mathbb{C}$, $X = \mathbb{A}^4_{\mathbb{C}}$, $Y = \mathbb{A}^3_{\mathbb{C}}$, and $G = \mathbb{G}_a = \operatorname{Spec}\mathbb{C}[\tau]$. Let G act on X by $t(x_1, x_2, x_3, x_4) = (x_1 + tx_2, x_2, x_3 + tx_4, x_4)$. Let $\varphi : X \to Y$ be the map given by $\varphi(x_1, x_2, x_3, x_4) = (x_2, x_4, x_1x_4 - x_2x_3)$. Let $B = \Gamma(X, \mathcal{O}_X) = \mathbb{C}[\xi_1, \xi_2, \xi_3, \xi_4]$, and $A = \Gamma(Y, \mathcal{O}_Y) = \mathbb{C}[\xi_2, \xi_4, w]$, where $\xi_i(x_1, x_2, x_3, x_4) = x_i$, and $w = \xi_1\xi_4 - \xi_2\xi_3$. Then it is easy to verify that φ is G-invariant. Let $V = D(\xi_2, \xi_4) = Y \setminus V(\xi_2, \xi_4)$, and $U = \varphi^{-1}(V) = X \setminus V(\xi_2, \xi_4)$. Obviously, V is a large open subset of Y, and U is a large G-stable open subset of X. Let $\rho : U \to V$ be the restriction of φ. Since $B[\xi_2^{-1}] = A[\xi_2^{-1}][-\xi_2^{-1}\xi_1]$ and $t(-\xi_2^{-1}\xi_1) = -\xi_2^{-1}\xi_1 + t$, $\operatorname{Spec} B[\xi_2^{-1}] \to \operatorname{Spec} A[\xi_2^{-1}]$ is a trivial G-bundle. Similarly, $\operatorname{Spec} B[\xi_4^{-1}] \to \operatorname{Spec} A[\xi_4^{-1}]$ is also a trivial G-bundle, and hence $\rho : U \to V$ is a principal G-bundle.

Hence $\varphi : X \to Y$ is an almost principal G-bundle with respect to U and V. By Theorem 10.13, we have that $A = B^G$. So φ is an algebraic quotient.

Note that φ is not surjective. Indeed, $(0, 0, 1)$ is not in the image of φ. This also shows that φ is not a categorical quotient. Indeed, if φ is a categorical quotient, then letting $W = Y \setminus \{(0, 0, 1)\}$, $\psi : X \to W$ the same as φ, and $u : W \hookrightarrow Y$ the inclusion, we have that $\varphi = u\varphi'$. By Lemma 7.2, u must be an isomorphism, and this is absurd.

11. The behavior of the class groups and the canonical modules with respect to rational almost principal bundles

(11.1) Let $f : G \to H$ be an fpqc homomorphism between flat S-group schemes with $N = \operatorname{Ker} f$.

Theorem 11.2. *Let*

$$X \xleftarrow{i} U \xrightarrow{\rho} V \xhookrightarrow{j} Y$$

be a G-enriched rational almost principal N-bundle. Assume that both X and Y are locally Krull. Then $i_*\rho^*j^* : \operatorname{Ref}(H,Y) \to \operatorname{Ref}(G,X)$ is an equivalence, and $(j_*\rho_*i^*?)^N$ is its quasi-inverse. This equivalence induces an equivalence $\operatorname{Ref}_n(H,Y) \cong \operatorname{Ref}_n(G,X)$ for each $n \geq 0$, where Ref_n denotes the category of reflexive modules of rank n. It also induces an isomorphism $\operatorname{Cl}(H,Y) \cong \operatorname{Cl}(G,X)$.

Proof. This follows immediately from [Has11, (7.4)] and [Has9, (5.31)]. $\square$

Lemma 11.3. *Let $\varphi : X \to Y$ be a G-enriched almost principal N-bundle with respect to the open subsets U and V. Let $i : U \to X$ and $j : V \to Y$ be the inclusion, and $\rho : U \to V$ the restriction of φ. Assume that X and Y are locally Krull. Then the equivalence $i_*\rho^*j^*$ agrees with $(?)^{**}\varphi^*$ as functors from $\operatorname{Ref}(H,Y)$ to $\operatorname{Ref}(G,X)$, and is independent of the choice of U or V. Its quasi-inverse $(j_*\rho_*i^*?)^N$ agrees with $(\varphi_*?)^N$ as functors from $\operatorname{Ref}(G,X)$ to $\operatorname{Ref}(H,Y)$, and is also independent of U or V.*

Proof. As functors from $\operatorname{Ref}(H,Y)$ to $\operatorname{Ref}(G,X)$,

$$(?)^{**}\varphi^* \cong i_*i^*(?)^{**}\varphi^* \cong i_*(?)^{**}i^*\varphi^* \cong i_*(?)^{**}\rho^*j^* \cong i_*\rho^*j^*$$

by [Has9, (5.28), (5.20), (5.9)]. As functors from $\operatorname{Ref}(G,X)$ to $\operatorname{Ref}(H,Y)$,

$$(?)^N\varphi_* \cong (?)^N\varphi_*i_*i^* \cong (?)^N j_*\rho_*i^*.$$

$\square$

Corollary 11.4. *If $\varphi : X \to Y$ is a G-enriched almost principal N-bundle, and X and Y are locally Krull, then $(?)^N \circ \varphi_* : \operatorname{Ref}(G,X) \to \operatorname{Ref}(H,Y)$ is an equivalence, and $(?)^{**} \circ \varphi^* : \operatorname{Ref}(H,Y) \to \operatorname{Ref}(G,X)$ is its quasi-inverse. In particular, $(\varphi_*\mathcal{O}_X)^G \cong \mathcal{O}_Y$ in $\operatorname{Ref}(H,Y)$. This equivalence also induces an isomorphism $\operatorname{Cl}(H,Y) \cong \operatorname{Cl}(G,X)$.* $\square$

In the next theorem, consider that $G = N$ and $f : G \to H = e = S$ is the trivial homomorphism.

Theorem 11.5. *Let G be a flat S-group scheme, and let*

$$X \xleftarrow{i} U \xrightarrow{\rho} V \xhookrightarrow{j} Y$$

be a rational almost principal G-bundle. Assume that both X and Y are locally Krull, X is quasi-compact quasi-separated, and Y is quasi-compact. Assume that ρ is quasi-compact (e.g., $G \to S$ is quasi-compact) and universally open (e.g., $G \to S$ or ρ is locally of finite presentation). Then there is an exact sequence

$$0 \to H^1_{\mathrm{alg}}(G, \mathcal{O}_X^\times) \to \mathrm{Cl}(Y) \to \mathrm{Cl}(X)^G \to H^2_{\mathrm{alg}}(G, \mathcal{O}_X^\times),$$

where $\mathrm{Cl}(X)^G$ is the subgroup of $\mathrm{Cl}(X)$ consisting of $[\mathcal{M}]$ with $\mathcal{M} \in \mathrm{Ref}_1(X)^G$, where $\mathrm{Ref}_1(X)^G$ is the full subcategory of $\mathrm{Qch}(X)$ consisting of rank-one reflexive sheaves $\mathcal{M}$ such that $a^*\mathcal{M} \cong p_2^*\mathcal{M}$ in $\mathrm{Ref}_1(G \times X)$, where $a : G \times X \to X$ is the action, and $p_2 : G \times X \to X$ is the second projection.

Proof. Let $\mathcal{C}$ be the set of quasi-compact large open subsets of V. Let $\mathcal{D}_1$ be the set of quasi-compact large open subsets of U. Let $\mathcal{D}$ be the set of G-stable open subsets Z of U such that $Z \in \mathcal{D}_1$.

First, for $Z_1 \in \mathcal{D}_1$, we have that $\rho(Z_1) \in \mathcal{C}$. Indeed, as ρ is universally open, $\rho(Z_1)$ is an open subset of V. As Z_1 is quasi-compact, so is $\rho(Z_1)$. As Z_1 is large in U and $\rho^{-1}(\rho(Z_1)) \supset Z_1$, we have that $\rho^{-1}(\rho(Z_1))$ is also large in U. As U is locally Krull and ρ is fpqc, we have that $\rho(Z_1)$ is large in V by Lemma 10.10, **2**. Thus $\rho(Z_1) \in \mathcal{C}$.

Next, for $W \in \mathcal{C}$, we have that $\rho^{-1}(W) \in \mathcal{D}$. As ρ is a G-invariant morphism, $\rho^{-1}(W)$ is a G-stable open subset of U. As ρ is quasi-compact, $\rho^{-1}(W)$ is quasi-compact. By Lemma 10.10, **1**, we have that $\rho^{-1}(W)$ is large in U, and hence $\rho^{-1}(W) \in \mathcal{D}$.

As ρ is a principal G-bundle, $\Psi : G \times U \to U \times_V U$ is a U-isomorphism, where $U \times_V U$ is a U-scheme via the second projection. As ρ is quasi-compact, $U \times_V U$ is quasi-compact over U, and hence so is $G \times U$. Thus for each $Z_1 \in \mathcal{D}_1$, $G \times Z_1$ is quasi-compact.

For $Z_1 \in \mathcal{D}_1$, $Z := \rho^{-1}(\rho(Z_1))$ lies in $\mathcal{D}$ by the argument above. Let $a_1 : G \times Z_1 \to Z$ be the action. As $\Psi : G \times U \to U \times_V U$ is surjective (since ρ is a principal G-bundle), a_1 is surjective. As a_1 is flat surjective, $G \times Z_1$ is quasi-compact, and Z is quasi-separated, we have that a_1 is fpqc.

Note that $Z \mapsto \rho(Z)$ and $W \mapsto \rho^{-1}(W)$ gives an order-preserving bijection between $\mathcal{D}$ and $\mathcal{C}$. Indeed, as ρ is surjective, $\rho(\rho^{-1}(W)) = W$. On the other hand, for $Z \in \mathcal{D}$, as $G \times Z \to \rho^{-1}(\rho(Z))$ is surjective and Z is G-stable, we have that $\rho^{-1}(\rho(Z)) = Z$.

Note that

$$\varinjlim_{Z \in \mathcal{D}} \operatorname{Pic}(\rho(Z)) \cong \varinjlim_{W \in \mathcal{C}} \operatorname{Pic}(W) \cong \operatorname{Cl}(Y) \tag{19}$$

by [Has9, (5.33)].

Let $Z \in \mathcal{D}$. Then $i^* : \operatorname{Cl}(X) \to \operatorname{Cl}(Z)$ induces an isomorphism between $\operatorname{Cl}(X)^G$ and $\operatorname{Cl}(Z)^G$, where $i : Z \hookrightarrow X$ is the inclusion. Indeed, if $\mathcal{M} \in \operatorname{Ref}_1(X)^G$, then

$$a_Z^* i^* \mathcal{M} \cong (1_G \times i)^* a^* \mathcal{M} \cong (1_G \times i)^* p_2^* \mathcal{M} \cong (p_2^Z)^* i^* \mathcal{M},$$

and i^* maps $\operatorname{Cl}(X)^G$ to $\operatorname{Cl}(Z)^G$. Let $\mathcal{N} \in \operatorname{Ref}_1(Z)^G$. As i is quasi-compact quasi-separated and a and p_2 are flat,

$$a^* i_* \mathcal{N} \cong (1 \times i)_* a_Z^* \mathcal{N} \cong (1 \times i)_* (p_2^Z)^* \mathcal{N} \cong p_2^* i_* \mathcal{N},$$

and $i_* \mathcal{N} \in \operatorname{Ref}_1(X)^G$. So $(i^*)^{-1} = i_*$ maps $\operatorname{Cl}(Z)^G$ to $\operatorname{Cl}(X)^G$.

On the other hand, for any $\mathcal{M} \in \operatorname{Ref}_1(X)^G$, there exists some $Z \in \mathcal{D}$ such that $\mathcal{M}|_Z$ is an invertible sheaf. Indeed, first take a large open subset Z_1 of U such that $\mathcal{M}|_{Z_1}$ is an invertible sheaf. This is possible as in the proof of [Has9, (5.33)]. By [Has9, (5.29)], replacing Z_1 if necessary, we may assume that Z_1 is quasi-compact, and $Z_1 \in \mathcal{D}_1$. Let $Z = \rho^{-1}(\rho(Z_1)) \in \mathcal{D}$. As $\mathcal{M}|_{Z_1}$ is an invertible sheaf,

$$p_2^*(\mathcal{M}|_{Z_1}) \cong (p_2^* \mathcal{M})|_{G \times Z_1} \cong (a^* \mathcal{M})|_{G \times Z_1} \cong a_1^*(\mathcal{M}|_Z)$$

is also an invertible sheaf. So $\mathcal{M}|_Z$ is also an invertible sheaf, since a_1 is fpqc as we have seen.

Combining these, we have that

$$\varinjlim_{Z \in \mathcal{D}} \operatorname{Pic}(Z)^G \cong \operatorname{Cl}(X)^G. \tag{20}$$

Next, we have that $H^i_{\mathrm{alg}}(G, \mathcal{O}_X^\times) \to H^i_{\mathrm{alg}}(G, \mathcal{O}_Z^\times)$ is an isomorphism for $Z \in \mathcal{D}$. In order to prove this, it suffices to prove the canonical chain map

$$\begin{array}{ccccccc}
0 \longrightarrow & \Gamma(X, \mathcal{O}^\times) & \xrightarrow{d_0 - d_1} & \Gamma(G \times X, \mathcal{O}^\times) & \xrightarrow{d_0 - d_1 + d_2} & \Gamma(G \times G \times X, \mathcal{O}^\times) & \longrightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \Gamma(Z, \mathcal{O}^\times) & \xrightarrow{d_0 - d_1} & \Gamma(G \times Z, \mathcal{O}^\times) & \xrightarrow{d_0 - d_1 + d_2} & \Gamma(G \times G \times Z, \mathcal{O}^\times) & \longrightarrow \cdots
\end{array}$$

is a chain isomorphism. To verify this, it suffices to prove that the canonical restriction $\Gamma(G^i \times X, \mathcal{O}_X) \to \Gamma(G^i \times Z, \mathcal{O}_Z)$ is an isomorphism. Let $i : Z \hookrightarrow X$ be the inclusion. Then as Z is large in X, $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is an isomorphism. As G^i is flat over S and i is quasi-compact quasi-separated,

$$\mathcal{O}_{G^i \times X} \cong p_2^*\mathcal{O}_X \cong p_2^* i_* \mathcal{O}_Z \cong (1 \times i)_* p_2^* \mathcal{O}_Z \cong (1 \times i)_* \mathcal{O}_{G^i \times Z}.$$

Taking the global section, we get the desired isomorphism.

Thus we have proved that the canonical map $H^i_{\mathrm{alg}}(G, \mathcal{O}_X^\times) \to H^i_{\mathrm{alg}}(G, \mathcal{O}_Z^\times)$ is an isomorphism. In particular, we have that

$$H^i_{\mathrm{alg}}(G, \mathcal{O}_X^\times) \cong \varinjlim_Z H^i_{\mathrm{alg}}(G, \mathcal{O}_Z^\times). \tag{21}$$

By [Has9, (3.14)], there is an exact sequence

$$0 \to H^1_{\mathrm{alg}}(G, \mathcal{O}_Z^\times) \to \mathrm{Pic}(G, Z) \to \mathrm{Pic}(Z)^G \to H^2_{\mathrm{alg}}(G, \mathcal{O}_Z^\times).$$

The proof of [Has9, (3.14)] shows that the sequence is functorial on Z. On the other hand, there is a natural isomorphism $\mathrm{Pic}(G, Z) \cong \mathrm{Pic}(\rho(Z))$. This is obvious by [Has7, (3.13)]. Taking the inductive limit $\varinjlim_{Z \in \mathcal{D}}$, and using the isomorphisms (19), (20), and (21),

$$0 \to H^1_{\mathrm{alg}}(G, \mathcal{O}_X^\times) \to \mathrm{Cl}(Y) \to \mathrm{Cl}(X)^G \to H^2_{\mathrm{alg}}(G, \mathcal{O}_X^\times)$$

is exact, as desired. $\square$

(11.6) Let $f : G \to H$ be an fpqc homomorphism between flat S-group schemes with $N = \mathrm{Ker}\, f$.

Let X be a locally Noetherian G-scheme. We denote the full subcategory of $\mathrm{Coh}(G, X)$ consisting of $\mathcal{M} \in \mathrm{Coh}(G, X)$ which satisfy the (S'_n) condition as an $\mathcal{O}_X$-modules by $(S'_n)(G, X)$.

Lemma 11.7. *Let X be as above, and U a large open subset of X. If X has a full 2-canonical module, then $i_* : (S'_2)(G, U) \to (S'_2)(G, X)$ is an equivalence whose quasi-inverse is $i^* : (S'_2)(G, X) \to (S'_2)(G, U)$.*

Proof. Follows easily from Lemma 7.34. $\square$

Proposition 11.8. *Let $f : G \to H$ be an fpqc homomorphism between flat S-group schemes, and*

$$X \xleftarrow{i} U \xrightarrow{\rho} V \xhookrightarrow{j} Y$$

be a G-enriched rational almost principal N-bundle. Assume that X and Y are locally Noetherian, and have full 2-canonical modules (e.g., they are normal; or Noetherian locally equidimensional and have dualizing complexes). If ρ has (S_2) fibers (e.g., N is of finite type or X is (S_2)), then $i_*\rho^*j^* : (S_2')(H,Y) \to (S_2')(G,X)$ is an equivalence, and $(j_*\rho_*i^*?)^N$ is its quasi-inverse.

Proof. By Lemma 11.7, $i_* : (S_2')(G,U) \to (S_2')(G,X)$ is an equivalence with the quasi-inverse i^*. Similarly, $j_* : (S_2')(H,V) \to (S_2')(H,Y)$ is an equivalence with the quasi-inverse j^*. In view of [Has11, (6.21)], it suffices to show that for $\mathcal{M} \in \mathrm{Coh}(V)$, $\rho^*\mathcal{M}$ satisfies (S_2') if and only if $\mathcal{M}$ does. This is proved easily using (7.40). $\square$

Lemma 11.9. *Let $h : A \to B$ be a ring homomorphism, and assume that B is rank-one free as an A-module. Then h is an isomorphism.*

Proof. Note that $\operatorname{Ker} h = \operatorname{ann} B = \operatorname{ann} A = 0$, and h is injective. So it suffices to show that h is surjective. So we may assume that $(A, \mathfrak{m})$ is local. By Nakayama's lemma, we may assume that A is a field. Then $A = B$, since $\dim_A A = \dim_A B = 1$. $\square$

Lemma 11.10. *Let $\varphi : X \to Y$ be a G-enriched almost principal N-bundle with respect to the open subsets U and V. Let $i : U \to X$ and $j : V \to Y$ be the inclusion, and $\rho : U \to V$ the restriction of φ. Assume that X and Y are locally Noetherian.*

1 *The functor $(j_*\rho_*i^*?)^N : (S_2')(G,X) \to \mathrm{Qch}(H,Y)$ agrees with $(\varphi_*?)^N$.*

2 *If $\mathcal{M}_X$ is a coherent $(G, \mathcal{O}_X)$-module which is a full 2-canonical module as an $\mathcal{O}_X$-module, then the functor $i_*\rho^*j^* : (S_2')(H,Y) \to (S_2')(G,X)$ agrees with $(?)^{\vee\vee}\varphi^*$, where $(?)^\vee = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(?, \mathcal{M}_X)$.*

3 *Assume that both X and Y are quasi-normal. If N is of finite type or X is (S_2), then $(\varphi_*?)^N : (S_2')(G,X) \to (S_2')(H,Y)$ is an equivalence whose quasi-inverse is $(?)^{\vee\vee}\varphi^*$.*

4 *In **3**, if X and Y satisfy (S_2), then $\bar{\eta} : \mathcal{O}_Y \to (\varphi_*\mathcal{O}_X)^N$ is an isomorphism.*

5 *If either X and Y satisfy $(T_1) + (S_2)$; or N is of finite type and X and Y are Noetherian (S_2) with dualizing complexes, then $\bar{\eta}$ in **4** is an isomorphism.*

Proof. **1** and **2** are proved similarly to Lemma 11.3, and is left to the reader. **3** is immediate by **1** and Proposition 11.8.

4. As X satisfies (S_2), we have that $\mathcal{O}_X \cong i_*i^*\mathcal{O}_X \cong i_*\mathcal{O}_U \cong i_*\rho^*j^*\mathcal{O}_Y$. As $\mathcal{O}_Y \in (S_2')(H,Y)$, we have that $(\varphi_*\mathcal{O}_X)^N \cong \mathcal{O}_Y$ as $(H,\mathcal{O}_Y)$-modules by **3**. By Lemma 11.9, $\bar{\eta}$ is an isomorphism.

5 is immediate by **3** and **4**, in view of Corollary 7.39. □

(11.11) An S-group scheme G is said to be *locally finite free* (*LFF* for short), if the structure map $h_G : G \to S$ is finite and $(h_G)_*\mathcal{O}_G$ is locally free. Using the results of [Stack, (10.129)], it is not so difficult to show that G is LFF if and only if it is flat finite of finite presentation (hence is finite syntomic by [Has5, (31.14)]).

Lemma 11.12. *Let G be an LFF S-group scheme, and $\psi : X \to Y$ be an algebraic quotient by the action of G. Then ψ is a surjective integral universally open morphism which is a universally submersive geometric quotient. If the action of G on X is free, it is a principal G-bundle.*

Proof. Replacing Y by its affine open subset, we may assume that Y is affine. Then X is affine, since ψ is assumed to be affine. Now ψ is integral and the map $\Psi : G \times X \to X \times_Y X$ is surjective by [DemG, (III.§2,n°4)]. By the same theorem, ψ is a principal G-bundle if the action is free.

Being an algebraic quotient, it is dominating. Being integral, it is universally closed. Being dominating and closed, it is surjective. Being surjective and universally closed, it is universally submersive. By Lemma 1.11, it is universally open. Now it is clear that φ is a geometric quotient. □

(11.13) The following generalizes [Has5, (32.4), **3**].

Proposition 11.14. *Let $f : G \to H$ be as in* (7.42)*. Assume that N is finite and Reynolds. Let Y_0, $\mathbb{I}_{Y_0}$, and $\mathcal{F}(G, Y_0)$ be as in* (7.44)*. Let $\varphi : X \to Y$ be a morphism in $\mathcal{F}(G, Y_0)$. Assume that it is also an algebraic quotient by the action of N. Then φ is finite, and $(\varphi_*\omega_X)^N \cong \omega_Y$ as $(G, \mathcal{O}_Y)$-modules.*

Proof. As N is finite flat and Y_0 is Noetherian, the Y_0-group scheme $N \times_S Y_0$ is LFF. So φ is integral by Lemma 11.12. Being a morphism in $\mathcal{F}(G, Y_0)$, φ is of finite type. So φ is finite.

Let $s := \inf\{i \mid H^i(\mathbb{I}_Y) \neq 0\}$. We may assume that $\mathbb{I}_Y = \mathbb{I}_Y(G)$ consists of injective $\mathcal{O}_{B_G^M(Y)}$-modules, and $\mathbb{I}_Y^i = 0$ for $i < s$.

Then by definition, $\omega_Y = H^s(\mathbb{I}_Y)$. Let $\mathbb{I}_X = \rho^!(\mathbb{I}_Y)$ be the G-equivariant dualizing complex of X. We may assume that $\mathbb{I}_X$ is bounded below and consists of injective $\mathcal{O}_{B_G^M(X)}$-modules. Then using G-Grothendieck's duality [Has5, (29.5)],

$$\begin{aligned}(\varphi_*\mathbb{I}_X)^N &= (R\varphi_* R\,\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X, \varphi^!\mathbb{I}_Y))^N \cong (R\,\underline{\mathrm{Hom}}_{\mathcal{O}_Y}(R\varphi_*\mathcal{O}_X, \mathbb{I}_Y))^N \\ &\cong \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\varphi_*\mathcal{O}_X, \mathbb{I}_Y)^N \cong \underline{\mathrm{Hom}}_{\mathcal{O}_Y}((\varphi_*\mathcal{O}_X)^N \oplus U_N(\varphi_*\mathcal{O}_X), \mathbb{I}_Y)^N.\end{aligned}$$

By Lemma 5.12, **4** and Corollary 11.4, this is

$$\underline{\mathrm{Hom}}_{\mathcal{O}_Y}((\varphi_*\mathcal{O}_X)^N, \mathbb{I}_Y) \cong \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathbb{I}_Y) \cong \mathbb{I}_Y.$$

If $i < s$, then

$$\varphi_* H^i(\mathbb{I}_X) \cong \underline{\mathrm{Ext}}^i_{\mathcal{O}_Y}(\varphi_*\mathcal{O}_X, \mathbb{I}_Y) = 0.$$

As $\varphi_* : \mathrm{Qch}(X) \to \mathrm{Qch}(Y)$ is faithful and $\varphi_*\omega_X$ is nonzero, we must have $\omega_X = H^s(\mathbb{I}_X)$, and $(\varphi_*\omega_X)^N = \omega_Y$. □

(11.15) A Noetherian local ring A is said to be *quasi-Gorenstein* if A is the canonical module of A. A locally Noetherian scheme is said to be quasi-Gorenstein if all of its local rings are quasi-Gorenstein.

Lemma 11.16. *Let Z be a locally Noetherian scheme. Then the following are equivalent.*

1 *Z is quasi-Gorenstein.*

2 *$\mathcal{O}_Z$ is a semicanonical module of Z.*

3 *There exists some invertible sheaf on Z which is also a semicanonical module of Z.*

4 *A coherent sheaf on Z is an invertible sheaf if and only if it is a full semicanonical module.*

Proof. Trivial. □

Lemma 11.17. *A quasi-Gorenstein locally Noetherian scheme is quasi-normal by $\mathcal{O}_Z$.*

Proof. Follows immediately by Lemma 11.16 and Lemma 7.19. □

Theorem 11.18. *Let $f : G \to H$ and N be as in* (7.42)*. Y_0, $\mathbb{I}_{Y_0}$, and $\mathcal{F}(G, Y_0)$ be as in* (7.44)*. Let*

$$X \xleftarrow{i} U \xrightarrow{\rho} V \xhookrightarrow{j} Y$$

be a G-enriched rational almost principal N-bundle which is also a diagram in $\mathcal{F}(G, Y_0)$. Assume that N is separated and has a fixed relative dimension. Then there exist isomorphisms of $(G, \mathcal{O}_X)$-modules

$$\omega_X \cong i_*\rho^*j^*\omega_Y \otimes_{\mathcal{O}_X} \Theta^*_{N,X} \cong i_*\rho^*j^*(\omega_Y \otimes_{\mathcal{O}_Y} \Theta^*_{N,Y}) \tag{22}$$

and isomorphisms of $(H, \mathcal{O}_Y)$-modules

$$\omega_Y \cong (j_*\rho_*i^*(\omega_X \otimes_{\mathcal{O}_X} \Theta_{N,X}))^N \cong ((j_*\rho_*i^*\omega_X) \otimes_{\mathcal{O}_Y} \Theta_{N,Y})^N. \tag{23}$$

Proof. We prove the first isomorphism of (22). By Lemma 7.48, $j^*\omega_Y \cong \omega_V$, and hence we may assume that $V = Y$. As $i_*\rho^*\omega_Y \otimes_{\mathcal{O}_X} \Theta^*_{N,X} \cong i_*(\rho^*\omega_Y \otimes_{\mathcal{O}_U} \Theta^*_{N,U})$ by the equivariant projection formula [Has5, (26.4)] and $i_*\omega_U \cong \omega_X$ by Lemma 7.48, we may assume that $U = X$. Now the assertion follows from Corollary 7.60. The second isomorphism follows easily using the equivariant projection formula. We prove the first isomorphism of (23). As $(?)^N \circ j_* \cong j_* \circ (?)^N$ by [HasO, (7.3)] and $j_*\omega_V \cong \omega_Y$, we may assume that $Y = V$. As $i^*(\omega_X \otimes_{\mathcal{O}_X} \Theta_{N,X}) \cong \omega_U \otimes_{\mathcal{O}_U} \Theta_{N,U}$, we may assume that $X = U$. Now the assertion follows from Corollary 7.60. The second isomorphism follows easily from the equivariant projection formula. □

Corollary 11.19 (Watanabe type theorem)**.** *Let the assumptions be as in* Theorem 11.18. *Then for an H-linerized invertible sheaf $\mathcal{L}$ on Y, the following conditions are equivalent.*

- **a** $\omega_X \cong i_*\rho^*j^*\mathcal{L} \otimes_{\mathcal{O}_X} \Theta^*_{N,X} \cong i_*\rho^*j^*(\mathcal{L} \otimes_{\mathcal{O}_Y} \Theta^*_{N,Y})$ *(resp. $\omega_X \cong i_*\rho^*j^*\mathcal{L}$) in* $\mathrm{Qch}(G, X)$*, and Y satisfies the (S_2) condition.*
- **b** $\omega_Y \cong \mathcal{L}$ *in* $\mathrm{Qch}(H, Y)$*.*

If, moreover, $\Theta_{N,X}$ is trivial, then the following are equivalent.

- **c** $\omega_X \cong \mathcal{O}_X$ *and Y is (S_2)*
- **d** $\omega_Y \cong \mathcal{O}_Y$ *and X is (S_2).*

If these conditions are satisfied, then both X and Y are quasi-Gorenstein.

Proof. **a**$\Rightarrow$**b**. We easily have that $\omega_Y \cong j_* j^* \mathcal{L}$ by Theorem 11.18 and the assumption. As Y is (S_2), $\mathcal{L} \cong j_* j^* \mathcal{L}$, and $\omega_Y \cong \mathcal{L}$.

b$\Rightarrow$**a**. As the semicanonical module ω_Y is an invertible sheaf, Y is quasi-Gorenstein. In particular, Y is (S_2). The isomorphisms follow from Theorem 11.18 and the assumption.

Now assume that $\Theta_{N,X}$ is trivial.

c$\Rightarrow$**d**. As the semicanonical module ω_X is invertible, we have that X is quasi-Gorenstein and (S_2). So $\mathcal{O}_X \cong i_* i^* \mathcal{O}_X \cong i_* \mathcal{O}_U \cong i_* \rho^* j^* \mathcal{O}_Y$. By **a**$\Rightarrow$**b** above, we have that $\omega_Y \cong \mathcal{O}_Y$.

d$\Rightarrow$**c**. Then Y is quasi-Gorenstein, and so Y is (S_2). By **b**$\Rightarrow$**a** above, we have that $\omega_X \cong i_* \rho^* j^* \mathcal{O}_Y \cong i_* i^* \mathcal{O}_X \cong \mathcal{O}_X$. $\square$

Corollary 11.20. *Let the assumptions be as in* Theorem 11.18*. Then the following are equivalent.*

- **a** $\omega_X \cong i_* \rho^* j^* \mathcal{L}$ *for some H-linearized invertible sheaf $\mathcal{L}$ on Y, and Y satisfies the (S_2) condition.*
- **b** ω_Y *is an invertible sheaf.*

These conditions imply

- **c** Y *is quasi-Gorenstein.*

If, moreover, Y is connected, then **a, b, c** *are equivalent.*

Proof. **a**$\Rightarrow$**b**. We have

$$\omega_X \cong i_* \rho^* j^* \mathcal{L} \cong i_* \rho^* j^* (\mathcal{L} \otimes_{\mathcal{O}_Y} \Theta_{N,Y}) \otimes_{\mathcal{O}_X} \Theta^*_{N,X},$$

and hence $\omega_Y \cong \mathcal{L} \otimes_{\mathcal{O}_Y} \Theta_{N,Y}$ is an invertible sheaf by Corollary 11.19.

b$\Rightarrow$**a,c**. Letting $\mathcal{L} = \omega_Y \otimes_{\mathcal{O}_Y} \Theta^*_{N,Y}$, we have that $\mathcal{L}$ is an H-linearized invertible sheaf on Y. We have $\omega_X \cong i_* \rho^* j^* \mathcal{L}$ by Corollary 11.19. As ω_Y is an invertible sheaf, Y is quasi-Gorenstein, and hence is (S_2).

Now assume that Y is connected and **c** is satisfied. Then Y is (S_2) and has a dualizing complex. Hence it is locally equidimensional by Ogoma's theorem [Ogo]. By Lemma 7.38, the semicanonical module ω_Y is full. As Y is quasi-Gorenstein, ω_Y is an invertible sheaf and so **c**$\Rightarrow$**b** holds. $\square$

Remark 11.21. Let the assumptions be as in Theorem 11.18. In view of Corollary 11.19, it is important to know when $\Theta_{N,Y_0} \cong \mathcal{O}_{Y_0}$ holds.

1 If N is étale, then $\Theta_{N,Y_0} \cong \mathcal{O}_{Y_0}$, since $\Theta_{N,Y_0} \cong q_{Y_0}^*(\bigwedge^0 \Omega^*_{N/S})$, where $q_{Y_0} : Y_0 \to S$ is the structure map.

2 If N is finite and Reynolds, then $\Theta_{N,Y_0} \cong \mathcal{O}_{Y_0}$, see Lemma 7.61.

3 If $G = N$ and N is split reductive, then $\Theta_{N,Y_0} \cong \mathcal{O}_{Y_0}$. To verify this, as G is defined over $\mathbb{Z}$, we may assume that $S = Y_0 = \mathbb{Z}$. As the positive and the negative roots cancel out in $\bigwedge^{\mathrm{top}} \operatorname{Lie} G$, Θ is a rank-one free representation whose weight is zero. Similarly, if $S = \operatorname{Spec} k$ with k a field, $G = N$, and N is reductive, then $\Theta_{N,S} = \mathcal{O}_S$.

3 If $S = \operatorname{Spec} k$ with k a field and N is contained in the center of G, then the action of G on N is trivial, and hence $\omega_{N/S}$ is G-trivial. Hence Θ is a G-trivial one-dimensional representation, that is, $\Theta \cong k$.

4 Even if $S = Y_0 = \operatorname{Spec} k$, $G = N$, and the identity component N° of N is reductive, Θ may not be trivial. For example, if the characteristic of k is not two and $N = O(2)$, the orthogornal group, then Θ is not trivial, see [Knp, Bemerkung 4 after Korollar 2].

Corollary 11.22. *Let $f : G \to H$ and N be as in* (7.42). *Y_0, $\mathbb{I}_{Y_0}$, and $\mathcal{F}(G, Y_0)$ be as in* (7.44). *Let $\varphi : X \to Y$ be a G-enriched almost principal N-bundle which is also a morphism in $\mathcal{F}(G, Y_0)$. Assume that N is separated with a fixed relative dimension. Then we have the following.*

1 *There exist isomorphisms of $(H, \mathcal{O}_Y)$-modules*

$$\omega_Y \cong (\varphi_*(\omega_X \otimes_{\mathcal{O}_X} \Theta_{N,X}))^N \cong (\varphi_*\omega_X \otimes_{\mathcal{O}_Y} \Theta_{N,Y})^N. \tag{24}$$

If, moreover, X has a coherent $(G, \mathcal{O}_X)$-module $\mathcal{M}_X$ which is a full 2-canonical module, then there exist isomorphisms of $(G, \mathcal{O}_X)$-modules

$$\omega_X \cong (\varphi^*\omega_Y)^{\vee\vee} \otimes_{\mathcal{O}_X} \Theta^*_{N,X} \cong (\varphi^*(\omega_Y \otimes_{\mathcal{O}_Y} \Theta^*_{N,Y}))^{\vee\vee}, \tag{25}$$

where $(?)^\vee = \underline{\operatorname{Hom}}_{\mathcal{O}_X}(?, \mathcal{M}_X)$.

2 (Watanabe type theorem) *Let $\mathcal{L}$ be an H-linearized invertible sheaf on Y. Then the following are equivalent.*

a $\omega_X \cong \varphi^*\mathcal{L} \otimes_{\mathcal{O}_X} \Theta^*_{N,X} \cong \varphi^*(\mathcal{L} \otimes_{\mathcal{O}_Y} \Theta^*_{N,Y})$ *(resp.* $\omega_X \cong \varphi^*\mathcal{L}$*), and* Y *satisfies* (S_2)*.*

b $\omega_Y \cong \mathcal{L}$*, and* X *satisfies* (S_2)*.*

3 *The following are equivalent.*

a $\omega_X \cong \varphi^*\mathcal{L}$ *for some* H*-linearized invertible sheaf* $\mathcal{L}$ *on* Y*, and* Y *satisfies the* (S_2) *condition.*

b ω_Y *is an invertible sheaf on* Y*, and* X *satisfies the* (S_2) *condition.*

These conditions imply that both X *and* Y *are quasi-Gorenstein, and hence we have*

c Y *is quasi-Gorenstein and* X *satisfies the* (S_2) *condition.*

If, moreover, Y *is connected, then* **a, b, c** *are equivalent.*

Proof. Let $\varphi : X \to Y$ be a G-enriched almost principal bundle with respect to U and V. Let $i : U \to X$ and $j : V \to V$ be the inclusion, and $\rho : U \to V$ be the restriction of φ.

1. As $\omega_X \otimes_{\mathcal{O}_X} \Theta_{N,X}$ satisfies the (S_2')-condition by Lemma 7.19, the first isomorphism of (24) is immediate from (23) in Theorem 11.18, **1** and Lemma 11.10, **1**. The second isomorphism is by the equivariant projection formula [Has5, (26.4)].

The first isomorphism of (25) follows from (22) in Theorem 11.18, **1** and Lemma 11.10, **2**. The second isomorphism follows easily by the equivariant projection formula.

2. If **a** is assumed, then the semicanonical module ω_X is an invertible sheaf, and hence X is quasi-Gorenstein. In particular, X satisfies (S_2). If **b** is assumed, X is (S_2) by assumption. So in either case, we have

$$\varphi^*\mathcal{L} \cong i_*i^*\varphi^*\mathcal{L} \cong i_*\rho^*j^*\mathcal{L}.$$

By Corollary 11.19, the assertion follows.

3 follows easily from **2**. □

12. Frobenius pushforwards

Lemma 12.1. *Let $A \to B$ be a homomorphism between Noetherian rings whose fibers are zero-dimensional (e.g., an integral homomorphism), and M a (possibly infinite) B-module. Then for a prime ideal $\mathfrak{p}$ of A,*

$$\operatorname{depth}_{A_\mathfrak{p}} M_\mathfrak{p} = \inf_{P \cap A = \mathfrak{p}} \operatorname{depth}_{B_P} M_P.$$

If, moreover, the going-down theorem holds between A and B, and M satisfies the (S'_n) condition as a B-module, then M satisfies the (S'_n) condition as an A-module.

Proof. We have

$$H^i_{\mathfrak{p}A_\mathfrak{p}} M_\mathfrak{p} = H^i_{\mathfrak{p}B_\mathfrak{p}} M_\mathfrak{p} = \bigoplus_{P \cap A = \mathfrak{p}} H^i_{PB_P} M_P,$$

and the first assertion follows.

We prove the second assertion. Let $\mathfrak{p}$ be a prime ideal of A such that $\operatorname{depth}_{A_\mathfrak{p}} M_\mathfrak{p} < n$. Then there exists some $P \in \operatorname{Spec} B$ such that $P \cap A = \mathfrak{p}$ and $\operatorname{depth}_{B_P} M_P = \operatorname{depth}_{A_\mathfrak{p}} M_\mathfrak{p} < n$. So $\operatorname{depth}_{B_P} M_P = \dim B_P$ by the (S'_n) property as a B-module. As the fibers are zero dimensional, $\dim A_\mathfrak{p} \geq \dim B_P$. By the going-down, $\dim A_\mathfrak{p} \leq \dim B_P$. Hence $\dim A_\mathfrak{p} = \dim B_P = \operatorname{depth}_{B_P} M_P = \operatorname{depth}_{A_\mathfrak{p}} M_\mathfrak{p}$, and M satisfies the (S'_n) condition as an A-module. □

Lemma 12.2. . *Let G be a flat S-group scheme which is quasi-compact over S, and X be a locally Noetherian S-scheme on which G acts trivially. Let $\mathcal{M}$ be a quasi-coherent $(G, \mathcal{O}_X)$-module which satisfies the (S'_2) condition. Then $\mathcal{M}$ satisfies the (S'_2) condition.*

Proof. This is proved in the same line as [Has9, (5.34)]. □

(12.3) Until the end of this section, S is an $\mathbb{F}_p$-scheme, where p is a prime number, and $\mathbb{F}_p$ is the prime field of characteristic p, unless otherwise specified.

Lemma 12.4. *Let*

$$X \xleftarrow{i} U \xrightarrow{\rho} V \xhookrightarrow{j} Y$$

be a diagram of S-schemes. Assume that i and j are open immersions whose images are large in X and Y, respectively. Assume that Y is locally Noetherian and (S_2), and ${}^eS \times_S Y$ is locally Noetherian with a full 2-canonical module for some $e \geq 1$. Let X be locally Noetherian, and assume that ρ is faithfully flat and reduced (that is, flat with geometrically reduced fibers). Assume that ${}^{e'}S \times_S X$ is locally Noetherian for some $e' \geq 1$. If X is F-finite over S (that is, $\Phi_1(X)$ is a finite morphism, see [Has8]*), then Y is F-finite over S.*

Proof. As the open immersion $i : U \to X$ is F-finite, U is also F-finite over S. As $\rho : U \to V$ is faithfully flat and reduced, it is easy to see that V is also F-finite over S by [Has8, Theorem 21]. So for each $e \geq 0$, $\Phi_e(V)_*(\mathcal{O}_{{}^eV})$ is coherent. As Y satisfies the (S_2) condition, ${}^eV$ satisfies the (S_2) condition. Hence $\Phi_e(V)_*(\mathcal{O}_{{}^eV})$ satisfies the (S_2') condition by Lemma 12.1.

Now take $e \geq 1$ so that ${}^eS \times_S Y$ is locally Noetherian with a full 2-canonical module. As V is large in Y and ${}^eS \times_S V$ is large in ${}^eS \times_S Y$,

$$\Phi_e(Y)_*(\mathcal{O}_{{}^eY}) \cong \Phi_e(Y)_*{}^ej_*(\mathcal{O}_{{}^eV}) \cong (1_{{}^eS} \times j)_*\Phi_e(V)_*(\mathcal{O}_{{}^eV})$$

is coherent. As $\Phi_e(Y)$ is affine, it is finite. By [Has8, Lemma 2], Y is F-finite over S. $\square$

Lemma 12.5. *Let $A \to B$ be an F-finite reduced homomorphism between Noetherian rings of characteristic p. Then ${}^eA \otimes_A B$ is Noetherian for any $e \geq 1$.*

Proof. By Dumitrescu's theorem [Dum2], the relative Frobenius map $\Phi_e(A, B) : {}^eA \otimes_A B \to {}^eB$ is ${}^eA$-pure. In particular, it is injective. It is also finite by assumption. By Eakin–Nagata theorem [Mat, Theorem 3.7], the assertion follows. $\square$

Theorem 12.6. *Let $f : G \to H$ be a qfpqc homomorphism between S-group schemes with $N = \operatorname{Ker} f$. Let S be an $\mathbb{F}_p$-scheme, and assume that S is locally Noetherian and quasi-normal by a full 2-canonical module $\mathcal{M}_S$. Let*

$$X \xleftarrow{i} U \xrightarrow{\rho} V \xrightarrow{j} Y$$

be a G-enriched rational almost principal N-bundle. Assume that X and Y are locally Noetherian and flat with $(R_0) + (T_1) + (S_2)$-fibers over S, and that X is F-finite over S. Then

1 *${}^eS \times_S X$ is locally Noetherian and quasi-normal by $p_X^* {}^e\mathcal{M}_S$ for each $e \geq 0$, where $p_X : {}^eS \times_S X \to {}^eS$ is the first projection.*

2 *If N is reduced over S, then Y is F-finite over S.*

3 *If Y is F-finite over S, then ${}^eS \times_S Y$ is locally Noetherian and quasi-normal by $p_Y^* {}^e\mathcal{M}_S$ for $e \geq 0$, where $p_Y : {}^eS \times_S Y \to {}^eS$ is the first projection.*

4 *For each $e \geq 0$,*

$$ {}^eS \times_S X \xleftarrow{1\times i} {}^eS \times_S U \xrightarrow{1\times\rho} {}^eS \times_S V \xrightarrow{1\times j} {}^eS \times_S Y $$

is an ${}^eH \times_H G$-enriched rational almost principal ${}^eS \times_S N$-bundle (where the base scheme is ${}^eS$, and not S).

5 *Assume further that G is flat over S, and f is regular (that is, flat with geometrically regular fibers). If, moreover, either N is of finite type; or ${}^eS \times_S X$ and ${}^eS \times_S Y$ satisfy $(T_1) + (S_2)$, then $(S_2')({}^eH \times_H G, {}^eS \times_S X)$ and $(S_2')({}^eH, {}^eS \times_S Y)$ are equivalent under the equivalence in* Proposition 11.8.

6 *Let the assumptions be as in* **5**. *Let $\mathcal{M} \in (S_2')(G, X)$, and let $\mathcal{N}$ be the corresponding sheaf $(j_*\rho_*i^*\mathcal{M})^N \in (S_2')(H, Y)$ (by the correspondence in* Proposition 11.8). *Then for each $e \geq 0$, ${}^eN_e$ is ${}^eS$-flat and the sheaf*

$$ (1_{{}^eS} \times i)_*(1_{{}^eS} \times \rho)^*(1_{{}^eS} \times j)^*\Phi_e(Y)_*({}^e\mathcal{N}) \in (S_2')({}^eH \times_H G, {}^eS \times_S X), $$

which corresponds to the Frobenius pushforward $\Phi_e(Y)_*({}^e\mathcal{N})$ by the equivalence in Proposition 11.8, *is isomorphic to $(\Phi_e(X)_*({}^e\mathcal{M}))^{{}^eN_e}$.*

Proof. **1** Local Noetherian property follows from Lemma 12.5. Quasi-Normality follows from Lemma 7.41, **5**, applied to the map ${}^eS \times_S X \to {}^eS$.

2 follows from **1** and Lemma 12.4.

3 is proved similarly to **1**.

4 and **5** are trivial.

6 As N is flat, f is fpqc. Since G is S-flat, H is also S-flat. As f is regular, N is regular over S. Note that

$$ 1 \to {}^eN_e \to {}^eN \xrightarrow{\Phi_e} {}^eS \times_S N \to 1 $$

is exact (that is, Φ_e is qfpqc and ${}^eN_e = \operatorname{Ker}\Phi_e$) by Lemma 8.13. By the theorem of Radu and André [Rad], [And], [Dum], Φ_e is flat. Hence ${}^eN_e$ is flat. Note that

$$1 \to {}^eN_e \to {}^eG \xrightarrow{\Phi_e(H,G)} {}^eH \times_H G \to 1$$

is exact (that is, $\Phi_e(H,G)$ is qfpqc and ${}^eN_e = \operatorname{Ker}\Phi_e(H,G)$) by Lemma 8.13. As ${}^eN_e$ is flat, $\Phi_e(H,G)$ is flat. Being flat and qfpqc, it is fpqc. In particular, ${}^eH \times_H G$ is flat over ${}^eS$.

Note that $\Phi_e(X)_*({}^e\mathcal{M})$ satisfies the (S_2') condition. So

$$\begin{aligned}\Phi_e(X)_*({}^e\mathcal{M})^{{}^eN_e} &\cong ((1\times i)_*(1\times i)^*\Phi_e(X)_*({}^e\mathcal{M}))^{{}^eN_e}\\ &\cong (1\times i)_*(\Phi_e(U)_*{}^ei^{*e}\mathcal{M})^{{}^eN_e} \cong (1\times i)_*(\Phi_e(U)_*({}^e(i^*\mathcal{M})))^{{}^eN_e}.\end{aligned}$$

So we may assume that $X = U$.

On the other hand,

$$(1\times j)^*\Phi_e(Y)_*({}^e\mathcal{N}) \cong \Phi_e(V)_*({}^ej^{*e}\mathcal{N}) \cong \Phi_e(V)_*({}^e(j^*\mathcal{N})),$$

as can be seen easily. So we may assume that $Y = V$, and hence $\varphi : X \to Y$ is a G-enriched principal N-bundle.

As ${}^e\mathcal{M} \in (S_2')({}^eG, {}^eX)$ and Φ_e is a finite homeomorphism, we have that $\Phi_e(X)_*({}^e\mathcal{M}) \in (S_2')({}^eG, {}^eS \times_S X)$ by Lemma 12.1. So $(\Phi_e(X)_*({}^e\mathcal{M}))^{{}^eN_e}$ belongs to $(S_2')({}^eH \times_H G, {}^eS \times_S X)$ by Lemma 12.2. So in view of Theorem 11.2, it remains to prove that

$$((1\times\varphi)_*(\Phi_e(X)_*({}^e\mathcal{M}))^{{}^eN_e})^{{}^eS\times_S N} \cong \Phi_e(Y)_*({}^e\mathcal{N}).$$

This is clear, since

$$\begin{aligned}((1\times\varphi)_*(\Phi_e(X)_*({}^e\mathcal{M}))^{{}^eN_e})^{{}^eS\times_S N} &\cong (((1\times\varphi)_*\Phi_e(X)_*({}^e\mathcal{M}))^{{}^eN_e})^{{}^eS\times_S N}\\ &\cong (\Phi_e(Y)_*{}^e\varphi_*({}^e\mathcal{M}))^{{}^eN} \cong \Phi_e(Y)_*({}^e(\varphi_*\mathcal{M}))^{{}^eN} \cong \Phi_e(Y)_*({}^e\mathcal{N}).\end{aligned}$$

$\square$

(12.7) Let $S = \operatorname{Spec} k$ with k a perfect field, H and N be S-group schemes. Assume that H and N are locally Noetherian and regular. Let H act on N by the group automorphisms, and let G be the semidirect product $H \ltimes N$. Let X be a locally Noetherian F-finite G-scheme. We say that X is of *finite*

(H,N)-F-*representation type* by $\mathcal{M}_1,\dots,\mathcal{M}_r \in \mathrm{Coh}({}^{e_0}H \ltimes N, X)$ if for any $e \geq 1$, we can write $(F_*^e\mathcal{O}_{{}^eX})^{{}^eN_e} \cong \mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_u$ by some $\mathcal{N}_1,\dots,\mathcal{N}_u \in \mathrm{Coh}({}^eH \ltimes N, X)$ such that for each $j = 1,\dots,u$, there exists some $l(j)$ such that $\mathcal{N}_j \cong \mathcal{M}_{l(j)}$ as $(N,\mathcal{O}_X)$-modules (not as $({}^eH \ltimes N, \mathcal{O}_X)$-modules). If X is finite (H,e)-F-representation type, then we say that X is finite H-F-*representation type*, where $e = \operatorname{Spec} k$ denotes the trivial group. If, moreover, H is also trivial, then we say that X is finite F-representation type. If $H = \mathbb{G}_m^s$, the split s-torus, and $G = H \times N$, the direct product, then finite (H,N)-F-representation type is called graded finite F-representation type modulo N. If, moreover, N is trivial, we say that X is graded finite F-representation type.

From Theorem 12.6, we immediately have the following.

Corollary 12.8. *Let the assumptions be as in* Theorem 12.6, **5, 6**. *Assume further that $S = \operatorname{Spec} k$ with k a perfect field, and $G = H \ltimes N$ is a semidirect product. Then Y is of finite H-F-representation type by $\mathcal{N}_1,\dots,\mathcal{N}_u$ if and only if X is of finite (H,N)-F-representation type by $\mathcal{M}_1,\dots,\mathcal{M}_u$, where $\mathcal{M}_l = i_*\rho^*j^*\mathcal{N}_l$ for $l = 1,\dots,u$.* □

13. Global F-regularity

(13.1) Let X be a scheme and $h : \mathcal{M} \to \mathcal{N}$ an $\mathcal{O}_X$-linear map between $\mathcal{O}_X$-modules. We say that h is generically monic if $h_\xi : \mathcal{M}_\xi \to \mathcal{N}_\xi$ is injective for each $\xi \in X^{\langle 0 \rangle}$.

(13.2) Let S be an $\mathbb{F}_p$-scheme, G an S-group scheme, and $\varphi : X \to Y$ a G-morphism. For $e \geq 0$, the scheme $Y \times_{Y^{(e)}} X^{(e)}$ is simply denoted by $X_Y^{(e)}$. As $X_Y^{(e)} = Y \times_{Y_S^{(e)}} X_S^{(e)}$, it is a G-scheme in a natural way, and the relative Frobenius map $\Phi_e(Y,X) : X \to X_Y^{(e)}$ is a G-morphism.

Definition 13.3. Let S be an $\mathbb{F}_p$-scheme, G an S-group scheme, and X a G-scheme. Assume that $S = \operatorname{Spec} k$ with k a perfect field.

1 We say that X is G-*globally F-regular* if for any G-linearized invertible sheaf $\mathcal{L}$ on X and any G-invariant generically monic section $s : \mathcal{O}_X \to \mathcal{L}$, there exists some $e \geq 1$ such that the composite

$$sF^e : \mathcal{O}_{X^{(e)}} \xrightarrow{F^e} F_*^e\mathcal{O}_X \xrightarrow{s} F_*^e\mathcal{L} \tag{26}$$

splits as a $(G, \mathcal{O}_{X^{(e)}})$-linear map.

2 We say that X is *G-F-split* if for any (or equivalently, some) $e \geq 1$,

$$F^e : \mathcal{O}_{X^{(e)}} \to F^e_* \mathcal{O}_X$$

splits as a $(G, \mathcal{O}_{X^{(e)}})$-linear map.

If G is trivial, then we simply say that X is globally F-regular or F-split.

(13.4) A G-globally F-regular scheme is G-F-split.

(13.5) Let $\mathcal{L}$ be an ample invertible sheaf on a Noetherian $\mathbb{F}_p$-scheme X. Assume that for any $r \geq 0$ and a monic section $s : \mathcal{O}_X \to \mathcal{L}^{\otimes r}$, there exists some $e \geq 1$ such that $sF^e : \mathcal{O}_{X^{(e)}} \to F^e_* \mathcal{L}^{\otimes r}$ splits as an $\mathcal{O}_{X^{(e)}}$-linear map. Then X is globally F-regular. This is proved similarly to [Has3, Theorem 2.6].

Lemma 13.6. *Let S be an $\mathbb{F}_p$-scheme, and Z a smooth S-scheme. Then the relative Frobenius map $\Phi_e : Z \to Z^{(e)}_S$ is affine and $(\Phi_e)_*(\mathcal{O}_{Z^{(e)}_S})$ is locally free. In particular, Φ_e is faithfully flat. If, moreover, Z is étale over S, then Φ_e is an isomorphism.*

Proof. We prove that $(\Phi_e)_*(\mathcal{O}_{Z^{(e)}_S})$ is locally free. As the question is local both on S and Z, we may assume that $S = \operatorname{Spec} R$ and $Z = \operatorname{Spec} A$ are affine. Then by [Stack, (10.131.14)], there exists some finitely generated $\mathbb{F}_p$-subalgebra R_0 of R and a smooth R_0 algebra A_0 such that $A \cong R \otimes_{R_0} A_0$. As $R_0 \to A_0$ is a regular homomorphism between Noetherian $\mathbb{F}_p$-algebras, we have that A_0 is $(A_0)^{(e)}_{R_0} := A_0^{(e)} \otimes_{R_0^{(e)}} R_0$-flat by Radu–André theorem [Rad], [And], [Dum]. As A_0 is F-finite, A_0 is a finite projective $(A_0)^{(e)}_{R_0}$-module (see Lemma 8.12). Taking the base change $? \otimes_{R_0} R$, we get the desired result.

If, moreover, Z is étale, then we can take A_0 to be étale over R_0. Then by [Has5, (33.5)], $(A_0)^{(e)}_{R_0} \to A_0$ is an isomorphism. By the base change, we have that Φ_e is an isomorphism. □

Lemma 13.7. *Let $S = \operatorname{Spec} k$ with k a field of characteristic $p > 0$, $f : G \to H$ be an fpqc homomorphism between S-group schemes, and $N = \operatorname{Ker} f$. Assume that N is smooth over S. Let $\varphi : X \to Y$ be a G-morphism which is N-invariant. Assume that $\varphi(X^{\langle 0 \rangle}) \subset Y^{\langle 0 \rangle}$. Assume that $\bar{\eta} : \mathcal{O}_Y \to (\varphi_* \mathcal{O}_X)^N$ is an isomorphism. If X is G-globally F-regular* (*resp. G-F-split*)*, then Y is H-globally F-regular* (*resp. H-F-split*).

Proof. We prove the assertion for the global F-regularity. The case of F-splitting is similar.

Let $\mathcal{L}$ be an H-linearized invertible sheaf on Y, and $s : \mathcal{O}_Y \to \mathcal{L}$ an H-invariant generically monic section. As $\varphi(X^{\langle 0\rangle}) \subset Y^{\langle 0\rangle}$, it is easy to see that $s : \mathcal{O}_X \to \varphi^*\mathcal{L}$ is a G-invariant generically monic section. So there exists some $e \geq 1$ and a G-invariant splitting $\pi : F^e_*(\varphi^*\mathcal{L}) \to \mathcal{O}_{X^{(e)}}$ of $sF^e : \mathcal{O}_{X^{(e)}} \to F^e_*(\varphi^*\mathcal{L})$.

Obviously,

$$\bar{\eta} : \mathcal{O}_{Y^{(e)}} \to (\varphi^{(e)}_* \mathcal{O}_{X^{(e)}})^{N^{(e)}}$$

is an isomorphism. On the other hand, as N is S-smooth, the Frobenius map $F^e : N \to N^{(e)}$ is faithfully flat, and hence the restriction $(?)^{N^{(e)}}$ agrees with $(?)^N$ by Lemma 4.15.

So

$$\bar{\eta} : \mathcal{O}_{Y^{(e)}} \to (\varphi^{(e)}_* \mathcal{O}_{X^{(e)}})^{N}$$

is an isomorphism.

On the other hand,

$$\varphi^{(e)}_*(F^e_X)_*(\varphi^*\mathcal{L}))^N \cong (F^e_Y)_*(\varphi_*(\varphi^*\mathcal{L}))^N \cong (F^e_Y)_*\mathcal{L}.$$

So applying $(\varphi^{(e)}_*(?))^N$ to the sequence

$$\mathcal{O}_{X^{(e)}} \xrightarrow{sF^e} F^e_*(\varphi^*\mathcal{L}) \xrightarrow{\pi} \mathcal{O}_{X^{(e)}},$$

we get

$$\mathcal{O}_{Y^{(e)}} \xrightarrow{sF^e} F^e_*\mathcal{L} \xrightarrow{\pi} \mathcal{O}_{Y^{(e)}}$$

whose composite is the identity. Hence Y is H-globally F-regular. □

Corollary 13.8 (cf. [HaraWY, Proposition 1.2, (2)]). *Let $\varphi : X \to Y$ be a morphism between integral $\mathbb{F}_p$-schemes such that $\eta : \mathcal{O}_Y \to \varphi_*\mathcal{O}_X$ is an isomorphism. If X is globally F-regular (resp. F-split), then so is Y.*

Proof. Consider $S = \operatorname{Spec} \mathbb{F}_p = Z$ and the trivial G, H, and N. Then apply Lemma 13.7. As η is an isomorphism, φ is dominating and $\varphi(X^{\langle 0\rangle}) \subset Y^{\langle 0\rangle}$. The results follow from Lemma 13.7 easily. □

Proposition 13.9. *Let $S = \operatorname{Spec} k$ with k a perfect field of characteristic $p > 0$. Let H and N be S-group schemes. Let H act on N by group automorphisms, and $G := N \rtimes H$. Assume that N is a linearly reductive affine* *algebraic k-group scheme. Let X be an F-finite Noetherian H-globally F-regular* (*resp. H-F-split*) *G-scheme. Then X is G-globally F-regular* (*resp. G-F-split*).

Proof. We prove the assertion for the global F-regularity. The assertion for the F-splitting is similar.

Let $\mathcal{L}$ be a G-linearized invertible sheaf on X, and $s : \mathcal{O}_X \to \mathcal{L}$ a generically monic section. Then there exists some $e \geq 1$ and an $(H, \mathcal{O}_{X^{(e)}})$-linear map $\pi : F^e_* \mathcal{L} \to \mathcal{O}_{X^{(e)}}$ such that $\pi s F^e = \mathrm{id}$.

As $F^e : X \to X^{(e)}$ is finite, $\mathcal{H}_0 = \underline{\mathrm{Hom}}_{\mathcal{O}_{X^{(e)}}}(\mathcal{O}_{X^{(e)}}, F^e_* \mathcal{L})$ and $\mathcal{H}_1 = \underline{\mathrm{Hom}}_{\mathcal{O}_{X^{(e)}}}(F^e_* \mathcal{L}, \mathcal{O}_{X^{(e)}})$ are coherent $(G, \mathcal{O}_{X^{(e)}})$-modules.

Let $h : X^{(e)} \to S = \operatorname{Spec} k$ be the structure map, which is quasi-compact quasi-separated by assumption (if $q : X \to S$ is the structure map, then h is the composite

$$X^{(e)} \xrightarrow{q^{(e)}} S^{(e)} \xrightarrow{F^{-e}} S).$$

So we have a direct sum decomposition of quasi-coherent $(G, \mathcal{O}_S)$-modules $h_* \mathcal{H}_1 = (h_* \mathcal{H}_1)^N \oplus U_N(h_* \mathcal{H}_1)$ by Proposition 5.25. Applying $\Gamma(S, ?) \circ (?)^H$, we get the direct sum decomposition of abelian groups

$$\begin{aligned} (27) \quad & \mathrm{Hom}_{H, \mathcal{O}_{X^{(e)}}}(F^e_* \mathcal{L}, \mathcal{O}_{X^{(e)}}) \\ & = \mathrm{Hom}_{G, \mathcal{O}_{X^{(e)}}}(F^e_* \mathcal{L}, \mathcal{O}_{X^{(e)}}) \oplus \Gamma(S, U_N(h_* \underline{\mathrm{Hom}}_{\mathcal{O}_{X^{(e)}}}(F^e_* \mathcal{L}, \mathcal{O}_{X^{(e)}}))^H). \end{aligned}$$

By the product

$$h_* \mathcal{H}_1 \otimes_{\mathcal{O}_S} h_* \mathcal{H}_0 \to h_*(\mathcal{H}_1 \otimes_{\mathcal{O}_{X^{(e)}}} \mathcal{H}_0) \to h_*(\underline{\mathrm{Hom}}_{\mathcal{O}_{X^{(e)}}}(\mathcal{O}_{X^{(e)}}, \mathcal{O}_{X^{(e)}})),$$

$U_N(h_* \mathcal{H}_1) \otimes_{\mathcal{O}_S} (h_* \mathcal{H}_0)^N$ is mapped to $U_N(h_*(\underline{\mathrm{Hom}}_{\mathcal{O}_{X^{(e)}}}(\mathcal{O}_{X^{(e)}}, \mathcal{O}_{X^{(e)}})))$ by Lemma 5.12, **3**. Hence when we decompose $\pi = \pi_0 + \pi_1$ according to the decomposition (27),

$$\pi_0(sF^e) \in \mathrm{Hom}_{G, \mathcal{O}_{X^{(e)}}}(\mathcal{O}_{X^{(e)}}, \mathcal{O}_{X^{(e)}})$$

and

$$\pi_1(sF^e) \in \Gamma(S, U_N(h_*(\underline{\mathrm{Hom}}_{\mathcal{O}_{X^{(e)}}}(\mathcal{O}_{X^{(e)}}, \mathcal{O}_{X^{(e)}})))^H).$$

As we can decompose the identity of $\mathcal{O}_{X^{(e)}}$ in two ways as

$$\mathrm{id} = \pi_0(sF^e) + \pi_1(sF^e) = \mathrm{id} + 0,$$

we must have $\pi_0(sF^e) = \mathrm{id}$ and $\pi_1(sF^e) = 0$ by the uniqueness of the decomposition. Hence $\pi_0 : F^e_*\mathcal{L} \to \mathcal{O}_{X^{(e)}}$ is the desired $(G, \mathcal{O}_{X^{(e)}})$-linear splitting of sF^e, and the proof of the proposition has been completed. $\square$

Corollary 13.10. *Let $S = \operatorname{Spec} k$, H, N, and G be as in* Proposition 13.9. *Assume that N is smooth. Let $\varphi : X \to Y$ be a G-morphism which is N-invariant. Assume that $\bar{\eta} : \mathcal{O}_Y \to (\varphi_*\mathcal{O}_X)^N$ is an isomorphism. Assume that X is Noetherian normal and F-finite. If X is H-globally F-regular* (*resp. H-F-split*), *then Y is also H-globally F-regular* (*resp. H-F-split*).

Proof. By [Has9, (6.3)], $\varphi(X^{\langle 0\rangle}) \subset Y^{\langle 0\rangle}$. The assertion follows easily from Proposition 13.9 and Lemma 13.7. $\square$

Lemma 13.11. *Let $\varphi : X \to Y$ be a globally F-regular F-finite Noetherian $\mathbb{F}_p$-scheme with an ample invertible sheaf $\mathcal{A}$. Then any open subscheme U is also globally F-regular. In particular, X is F-regular in the sense that each local ring of X is strongly F-regular. In particular, X is Cohen–Macaulay normal.*

Proof. Let $r \geq 0$ and $s \in \Gamma(U, \mathcal{A}^{\otimes r})$ a generically monic section. Take $r' \geq 0$ and a section $u \in \Gamma(X, \mathcal{A}^{\otimes r'})$ which is generically monic such that $X_u \subset U$. Then by [Stack, (27.24.6)], there exists some $n \geq 0$ such that $u^n s \in \Gamma(X, \mathcal{A}^{\otimes(r+nr')})$. Then there exists some $e \geq 1$ such that $u^n sF^e : \mathcal{O}_{X^{(e)}} \to F^e_* A^{\otimes(r+nr')}$ has a splitting. Restricting to U, $u^n sF^e$ also has a splitting over U. Hence sF^e also has a splitting over U. Thus U is globally F-regular.

In particular, any affine open $U = \operatorname{Spec} A$ is globally F-regular, and the F-finite Noetherian ring A is strongly F-regular [HocH]. So any local ring of X is also strongly F-regular. As an F-finite Noetherian ring is excellent [Kun2] and a strongly F-regular ring is weakly F-regular [HocH, (3.1)], X is Cohen–Macaulay normal by [Hun, (4.2)]. $\square$

Lemma 13.12. *Let S be an $\mathbb{F}_p$-scheme, G an S-group scheme, and X be an G-F-split scheme. If $h : U \to X$ is an étale G-morphism, then U is a G-F-split G-scheme.*

Proof. There exists some $e \geq 1$ and a $(G, \mathcal{O}_{X^{(e)}})$-linear splitting $\pi : F^e_*\mathcal{O}_X \to \mathcal{O}_{X^{(e)}}$ of F^e. Applying $(h^{(e)})^*$, we have that

$$\eta_{p_2} = (h^{(e)})^* F^e_X : \mathcal{O}_{U^{(e)}} \to (h^{(e)})^* F^e_* \mathcal{O}_X \cong (p_2)_* p_1^* \mathcal{O}_X \cong (p_2)_* \mathcal{O}_{U^{(e)}_X}$$

has a $(G, \mathcal{O}_{U^{(e)}})$-linear splitting, where $p_1 : U_X^{(e)} \to X$ is the first projection, and $p_2 : U_X^{(e)} \to U^{(e)}$ is the second projection. As h is étale, $\Phi_e(X, U)$ is an isomorphism by Lemma 13.6. As the composite

$$U \xrightarrow{\Phi_e(X,U)} U_X^{(e)} \xrightarrow{p_2} U^{(e)}$$

is F_U^e, $F_U^e : \mathcal{O}_{U^{(e)}} \to F_*^e(\mathcal{O}_U)$ has a $(G, \mathcal{O}_{U^{(e)}})$-linear splitting, as desired. □

Lemma 13.13. *Let X be a Noetherian quasi-normal $\mathbb{F}_p$-scheme, and U its large open subset. Then X is F-finite if and only if U is F-finite.*

Proof. Assume that X is F-finite. Then $F_X : X \to X^{(1)}$ is finite. Taking the base change by $U \to X$, $F_U : U \cong X \times_{X^{(1)}} U^{(1)} \to U^{(1)}$ is finite.

Assume that U is F-finite. Then $F_*\mathcal{O}_U$ is a coherent (S_2') $\mathcal{O}_U$-module. So letting $i : U \hookrightarrow X$ be the inclusion, $(i^{(1)})_* F_* \mathcal{O}_U \cong F_* i_* \mathcal{O}_U \cong F_* \mathcal{O}_X$ is a coherent sheaf. □

Theorem 13.14. *Let $S = \operatorname{Spec} k$ with k a perfect field of characteristic $p > 0$. Let G be a smooth linearly reductive affine algebraic k-group scheme. Let the diagram*

$$X \xleftarrow{\ i\ } U \xrightarrow{\ \rho\ } V \xhookrightarrow{\ j\ } Y$$

be a rational almost principal G-bundle. Assume that X and Y are Noetherian normal schemes. Then we have the following.

1 *X is F-finite if and only if Y is F-finite.*

2 *Assume that X and Y have ample invertible sheaves and are F-finite. Then X is globally F-regular (resp. F-split) if and only if Y is globally F-regular (resp. F-split).*

Proof. **1** In view of Lemma 13.13, we may assume that $X = U$ and $Y = V$. If Y is F-finite, then X is F-finite, since $\rho : X \to Y$ is of finite type. If X is F-finite, then Y is F-finite, since ρ is an algebraic quotient by N and N is linearly reductive, see Lemma 9.6.

2 We only prove the assertion for the global F-regularity. Let $\mathcal{L}$ be an invertible sheaf on Y, and $s \in \Gamma(Y, \mathcal{L})$ a generically monic section. Then assuming that V is globally F-regular, there exists some $e \geq 1$ such that there is a splitting π of sF^e on V. Note that $\pi \in \Gamma(V, \underline{\mathrm{Hom}}_{\mathcal{O}_Y^{(e)}}(F_*^e \mathcal{O}_Y, \mathcal{O}_{Y^{(e)}}))$. As Y is F-finite Noetherian normal, $\underline{\mathrm{Hom}}_{\mathcal{O}_Y^{(e)}}(F_*^e \mathcal{O}_Y, \mathcal{O}_{Y^{(e)}})$ is a reflexive sheaf, and hence π is defined over Y. This shows that Y is globally F-regular. On the other hand, if Y is globally F-regular, then by Lemma 13.11, V is globlly F-regular. Similarly, X is globally F-regular if and only if U is so. Hence we may assume that $X = U$ and $Y = V$.

If X is globally F-regular, then by Corollary 13.10, Y is globally F-regular.

Let Y be globally F-regular. Let $\mathcal{L}$ be an ample invertible sheaf on Y. We can take some $r \geq 1$ and a generically monic section $a \in \Gamma(Y, \mathcal{L}^{\otimes r})$ such that Y_a is regular and affine. Replacing $\mathcal{L}$ by $\mathcal{L}^{\otimes r}$ if necessary, we may assume that $r = 1$. As Y is globally F-regular, there exists some $e_0 \geq 1$ and $\pi_0 : F_*^{e_0}\mathcal{L} \to \mathcal{O}_{Y^{(e_0)}}$ such that $\pi_0 a F^{e_0} = \mathrm{id}$.

As N is smooth, ρ is smooth, and hence $X_a = \rho^{-1}(Y_a)$ is regular. By Lemma 10.10, **1**, X_a is dense in X. That is, $a \in \Gamma(X, \rho^*\mathcal{L})$ is generically monic. By [Stack, (28.38.7)], $\rho^*\mathcal{L}$ is an ample invertible sheaf on X.

Let $n > 0$ and $s \in \Gamma(X, \rho^*\mathcal{L}^{\otimes n})$ be a generically monic section. As X_a is affine regular F-finite, it is globally F-regular, and hence there exists some $e_1 \geq 1$ and a splitting $\pi_1 : F_*^{e_1}\rho^*\mathcal{L}^{\otimes n}|_{X_a} \to \mathcal{O}_{X_a^{(e_1)}}$ of $sF^{e_1} : \mathcal{O}_{X_a^{(e_1)}} \to F_*^{e_1}\rho^*\mathcal{L}^{\otimes n}|_{X_a}$. As $\underline{\mathrm{Hom}}_{\mathcal{O}_{X^{(e_1)}}}(F_*^{e_1}\rho^*\mathcal{L}^{\otimes n}, \mathcal{O}_{X^{(e_1)}})$ is coherent, there exists some $e_2 \geq 0$ such that $\pi_2 = (a^{p^{e_2}})^{(e_1)}\pi_1$ lies in $\mathrm{Hom}_{\mathcal{O}_X^{(e_1)}}(F_*^{e_1}\rho^*\mathcal{L}^{\otimes n}, (\mathcal{L}^{\otimes p^{e_2}})^{(e_1)})$. Then $\pi_2 s F^{e_1} = (a^{p^{e_2}})^{(e_1)}$.

As Y is F-split, there exists some $\pi_3 : F_*^{e_2}\mathcal{O}_Y \to \mathcal{O}_{Y^{(e_2)}}$ such that $\pi_3 F^{e_2} = \mathrm{id}$. Then

$$
\begin{aligned}
(28)\quad \pi_0^{(e_1+e_2)}\pi_3^{(e_1)}\pi_2 s F^{e_0+e_1+e_2} &= \pi_0^{(e_1+e_2)}\pi_3^{(e_1)}(a^{p^{e_2}})^{(e_1)}F^{e_0+e_2} \\
&= \pi_0^{(e_1+e_2)}\pi_3^{(e_1)}F^{e_2}a^{(e_1+e_2)}F^{e_0} = \pi_0^{(e_1+e_2)}a^{(e_1+e_2)}F^{e_0} = \mathrm{id},
\end{aligned}
$$

and $sF^{e_0+e_1+e_2}$ has a splitting. This shows that X is globally F-regular. □

Chapter 2. Examples and Applications

14. Finite group schemes

(14.1) Let G be an S-group scheme acting on X. If there is a G-stable open subset U of X such that the action of G on U is free and U is n-large in X then we say that the action of G on X is *n-small.* 0-small is also called *generically free.* 1-small is also called *small.*

Lemma 14.2. *Let G be an S-group scheme, and $\psi : Z' \to Z$ a flat G-morphism. If the action of G on Z is n-small, then the action of G on Z' is also n-small.*

Proof. Obvious from Lemma 1.8 and Lemma 10.10. □

(14.3) Letting G act on $G \times X$ by $g(g_1, x) = (gg_1g^{-1}, gx)$ and on $X \times X$ diagonally, $\Psi : G \times X \to X \times X$ and the diagonal map $\Delta : X \to X \times X$ are G-morphisms. So the structure map $\phi : \mathcal{S}_X \to X$ is also a G-morphism, where $\mathcal{S}_X$ is the stabilizer of the action of G on X.

If there is a separated G-invariant morphism $\varphi : X \to Y$ (e.g., X is S-separated or φ is affine), then $\mathcal{S}_X \to G \times X$ is a closed immersion. If, moreover, G is finite, then ϕ is finite.

(14.4) Assume that ϕ is finite. Then the cokernel of the split monomorphism $\eta : \mathcal{O}_X \to \phi_* \mathcal{O}_{\mathcal{S}_X}$ is a quasi-coherent $(G, \mathcal{O}_X)$-module which is finite-type as an $\mathcal{O}_X$-module. The complement $\mathcal{U}_X$ of the support of $\operatorname{Coker} \eta$ is the largest G-stable open subset of X on which G acts freely. We call $\mathcal{U}_X$ the free locus of the action.

Example 14.5. Let G be a finite (constant) group, and X be Noetherian and irreducible. Assume that there is a separated G-invariant morphism $\varphi : X \to Y$. Then ϕ is finite, and the free locus exists. More precisely, for $g \in G$, set $X_g := \{x \in X \mid gx = x\}$. Note that X_g is a closed subscheme of X. It is easy to see that $\mathcal{U}_X = X \setminus \bigcup_{g \neq e} X_g$. So the action is generically free if and only if the action is faithful (that is, the action of g on X is not the identity if $g \neq e$). If $\operatorname{codim}_X X_g = 1$, then we say that g is a *pseudoreflection*. The action is small if and only if there is no pseudoreflection. However, see Remark 17.15 below.

(14.6) Let G be a finite group scheme over a field k of characteristic $p > 0$. Then the smallest nonnegative integer $e \geq 0$ such that $G_e = G^\circ$ is called the *exponent* of G. If I is the nilradical of $k[G]$, then the exponent e of G is the smallest nonnegative integer such that $I^{[p^e]} = 0$, where $I^{[p^e]} = I^{(e)} k[G]$. The exponent is not changed by the extension of the base field.

Proposition 14.7. *Let k be a field, G a finite k-group scheme acting on a reduced artinian k-algebra L, and set $K = L^G$. Let $\varphi : X = \operatorname{Spec} L \to \operatorname{Spec} L^G = Y$ be the canonical map. Assume that X is G-connected. Then we have that K is a field, and $\dim_K L \leq \dim_k k[G]$ in general. In particular, $\dim_K L$ is finite. Moreover, the following are equivalent.*

1 *φ is a principal G-bundle.*

2 *The action of G on X is free.*

3 *The action of G on X is generically free.*

4 $\dim_K L = \dim_k k[G]$.

Proof. **1⇒2** As Ψ is an isomorphism, its base change $\phi : \mathcal{S}_X \to X$ is an isomorphism.

2⇒1 is Lemma 11.12.

2⇔3 is obvious, since X is an artinian scheme.

1⇒4. Compare the L-dimension of the two isomorphic spaces $L \otimes_K L$ and $k[G] \otimes_k L$.

By [Has5, (32.6)], K is a finite direct product of normal domains. As X is G-connected, Y is connected by Lemma 7.3. Hence K is a domain. As L is an integral extension of K by Lemma 11.12 and L is of Krull dimension zero, K is also zero dimensional. Being a zero dimensional domain, K is a field.

It remains to prove the assertions (i) $\dim_K L \leq \dim_k k[G]$; and (ii) **4⇒(1, 2, or 3)**.

Replacing k by K, G by $K \otimes_k G$, and not changing L, we may assume that $k = K$.

We claim that if N is a closed normal subgroup scheme and the proposition is true for N and G/N, then (i) and (ii), hence the proposition is also true for G. Indeed, $M = L^N = M_1 \times \cdots \times M_r$ is a finite direct product of fields. Applying the proposition for the action of N on $L_i = M_i \otimes_M L$, $\dim_{M_i} L_i \leq \dim_k k[N]$. On the other hand, $\dim_K M \leq \dim_k k[G/N]$. So

$$\begin{aligned}(29)\quad \dim_K L = \sum_i \dim_K L_i &= \sum_i \dim_K M_i \dim_{M_i} L_i \\ &\leq \dim_k k[N] \sum_i \dim_K M_i = \dim_k k[N] \dim_K M \\ &\leq \dim_k k[N] \dim_k k[G/N] = \dim_k k[G].\end{aligned}$$

The equality $\dim_K L = \dim_k k[G]$ holds if and only if the equality holds everywhere in (29). If so, $\dim_K M = \dim_k k[G/N]$ and $\dim_{M_i} L_i = \dim_k k[N]$ for each i. As the proposition is assumed to be true for N and G/N, we have that $\operatorname{Spec} M \to \operatorname{Spec} K = Y$ is a principal G/N-bundle, and $\operatorname{Spec} L_i \to$

$\operatorname{Spec} M_i$ is a principal N-bundle for each i. In particular, $X = \operatorname{Spec} L \to \operatorname{Spec} M$ is a G-enriched principal N-bundle. By Lemma 10.8, $\varphi : X \to Y$ is a principal G-bundle, and the claim has been proved.

Assume that G is infinitesimal of exponent one. That is, G equals its first Frobenius kernel G_1. As G is geometrically connected, X is also connected, and hence L is a field in this case.

Let $\mathfrak{g} := \operatorname{Lie} G$ be the Lie algebra of G. It is a restricted Lie algebra over k. There is a canonical map

$$\theta : L \otimes \mathfrak{g} \to \operatorname{End}_k L$$

given by $(\theta(\alpha \otimes D))(\beta) = \alpha D(\beta)$ for $D \in \mathfrak{g}$ and $\alpha, \beta \in L$. Obviously, the image $\mathcal{D} := \operatorname{Im} \theta$ is contained in $\operatorname{Der}_k(L, L)$, the space of k-derivations. Moreover, we have

$$\begin{aligned}&[\theta(\alpha \otimes D), \theta(\beta \otimes D_1)] \\ &\qquad = \theta(\alpha\beta \otimes [D, D_1] + \alpha(D(\beta)) \otimes D_1 - \beta(D_1(\alpha)) \otimes D) \in \mathcal{D}\end{aligned}$$

and

$$(\theta(\alpha \otimes D))^p = \theta(\alpha^p \otimes D^p + ((\alpha D)^{p-1}(\alpha)) \otimes D) \in \mathcal{D}$$

by [Mat, Exercise 25.1] and [Mat, (25.5)]. This shows that $\mathcal{D}$ is a restricted L-Lie subalgebra of $\operatorname{Der}_k(L, L)$ in the sense of Jacobson [Jac, (IV.8)].

Note that $\mathfrak{g}$ generates $k[G]^*$ as a k-algebra. To verify this, we may assume that k is algebraically closed, and this case is shown in [Jan, (I.9.6)]. Let $\mathcal{A}$ be the k-subalgebra of $\operatorname{End}_k L$ generated by $\mathcal{D}$. By [Jac, (IV.8), Theorem 19], $[L : k]$ is finite, and $\dim_L \operatorname{End}_k L = [L : k] = \dim_L \mathcal{A}$. After all, $\mathcal{A} = \operatorname{End}_k L$. It is easy to see that $\tilde{\theta} : L \otimes k[G]^* \to \mathcal{A}$ induced by θ is surjective, and hence we have $\dim_k L \leq \dim_k k[G]$. Moreover, if the equality holds (i.e., **4** holds), then $\tilde{\theta} : L \otimes k[G]^* \to \operatorname{End}_k L \cong \operatorname{Hom}_L(L \otimes_k L, L)$ is an isomorphism. Then its L-dual $L \otimes L \to L \otimes k[G]$ given by $\alpha \otimes \beta \mapsto \sum_{(\beta)} \alpha\beta_{(0)} \otimes \beta_{(1)}$ is also an isomorphism, where we are using the Sweedler's notation [Swe, (1.2)]. This is equivalent to say that $\Psi : G \times X \to X \times_Y X$ is an isomorphism, and hence **4**$\Rightarrow$**1** has been proved. So the proposition has been proved for the case that G is infinitesimal of exponent one.

Next, consider the case that G is infinitesimal. We prove the proposition for this case by the induction on the exponent e of G. The case that $e \leq 1$ is already done by above. Let $e \geq 2$. Then there is an exact sequence

$$1 \to G_1 \to G \xrightarrow{\pi} G/G_1 \to 1.$$

Note that $\pi^{-1}(G/G_1)_i = \Phi_i^{-1}(k \otimes_{k^{(e)}} G_1^{(e)})$, where $\Phi_i : G \to k \otimes_{k^{(e)}} G^{(e)}$ is the realtive Frobenius map. The right-hand side is the whole G for $i = e-1$, and hence the exponent of G/G_1 is at mose $e-1$. By induction, the proposition is true for G_1 and G/G_1, and hence the proposition is true for G.

Now we consider the general case. By the exact sequence

$$1 \to G^\circ \to G \to G/G^\circ \to 1,$$

replacing G by G/G°, we may assume that G is étale, since the proposition for G° is already proved by the infinitesimal case. Replacing $K = k$ by its suitable finite Galois extension k' and L by $L' = k' \otimes_k L$, we may assume that G is a constant finite group.

Let $e_1, \ldots, e_r$ be the set of primitive idempotents of L. As X is G-connected, G acts transitively on this set. Let H be the stabilizer of e_1. Then $[G : H] = r$. Let $\sigma_1, \ldots, \sigma_r$ be the complete set of representatives of G/H in G, where we choose the index so that $\sigma_i(e_1) = e_i$. Set $L_i = Le_i = \sigma_i(L_1)$. The image of $K \to L \to L_1$ ($\alpha \mapsto e_1\alpha$) is contained in L_1^H. On the other hand, $\sum_i \sigma_i$ maps L_1^H to K, and $e_1 : K \to L_1^H$ has an inverse $\sum_i \sigma_i$. By the Galois theory, $[L_1 : K] \leq \#H$ (note that H need not act on L_1 effectively, so the equality need not hold). So $[L : K] = r[L_1 : K] \leq [G : H] \cdot \#H = \#G = \dim_k k[G]$, and in particular, L is K-finite.

It remains to prove that if $\dim_K L = \#G$, then the action of G on X is free. In order to check this, taking the base change by the separable closure k_{sep} of k, we may assume that k is separably closed. Let $e_1, \ldots, e_r$, H, L_i be as above. As L_1 is a purely inseparable extension of k, we have that $L_1 = L_1^H$ this time. So $\dim_k L_i = 1$ for each i, and hence $r = \#G$ by assumption. As $[G : H] = r = \#G$, we have that H is trivial. Then G acts on $e_1, \ldots, e_r$ freely, and hence G acts on X freely, and **4**$\Rightarrow$**2** has been proved. $\square$

Proposition 14.8. *Let G be an LFF S-group scheme, and $\varphi : X \to Y$ an algebraic quotient. Let U be the free locus, and $V := \varphi(U)$. Then*

1 *$\varphi(X^{\langle n \rangle}) = Y^{\langle n \rangle}$ for $n \geq 0$.*

2 *$\rho : U \to V$ is a principal G-bundle, where ρ is the restriction of φ.*

3 *The following are equivalent.*

  a *The action of G on X is n-small.*

  a' *U is n-large.*

b *V is n-large.*

c *φ is an n-almost principal G-bundle with respect to U and V.*

d *φ is an n-almost principal G-bundle.*

Proof. **1** As φ is surjective, it suffices to show that $\varphi(X^{\langle n\rangle}) \subset Y^{\langle n\rangle}$ for $n \geq 0$. To verify this, we may assume that both $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ are affine. Let $x \in X$ and $y = \varphi(x)$. Then as φ is open, the going-down theorem holds for the map $A \to B$ [Stack, (10.38.2)], and hence $\operatorname{codim} x \geq \operatorname{codim} y$. As φ is integral, $\operatorname{codim} x \leq \operatorname{codim} y$. So the assertion follows.

2 By Lemma 1.11, V is an open subset of Y, and $\rho : U = \varphi^{-1}(V) \to V$ is an algebraic quotient. As the action of G on U is free, ρ is a principal G-bundle by Lemma 11.12.

3 Follows easily from **1** and **2**. □

Proposition 14.9. *Let G be an LFF S-group scheme with the well defined rank r, and $\varphi : X \to Y$ an algebraic quotient by the action of G. Assume that X is reduced and LFI. Then for each $\eta \in Y^{\langle 0\rangle}$,*

$$\dim_{\mathcal{O}_{Y,\eta}} (\varphi_* \mathcal{O}_X)_\eta \leq r. \tag{30}$$

Moreover, the action of G on X is generically free if and only if the equality holds in (30) *for each point $\eta \in Y^{\langle 0\rangle}$.*

Proof. We may assume that $Y = \operatorname{Spec} A$ is affine. Then $X = \operatorname{Spec} B$ is affine and $A = B^G$. Then for each minimal prime P of A, B_P is reduced and zero-dimensional (since $\kappa(P) = A_P \to B_P$ is an integral extension). As B has finitely many minimal primes, B_P has finitely many minimal primes, and $\operatorname{Spec} B_P$ is finite. Then it is easy to see that B_P is a finite direct product of fields. To prove that (30) holds, replacing φ by $\varphi_\eta : X_\eta \to \eta$, S by η and G by G_η, we may assume that $S = Y = \operatorname{Spec} k$ is the spectrum of a field (note that φ_η is an algebraic quotient, since A is reduced and hence $\kappa(\eta)$ is merely a localization of A). By Proposition 14.7, the inequality follows.

By Proposition 14.7, $\eta \in Y^{\langle 0\rangle}$ lies in V if and only if the equality in (30) holds. The assertion follows immediately by Proposition 14.8 for the case that $n = 1$. □

Example 14.10. Let V be a finite dimensional k-vector space, and let G be an étale finite subgroup scheme of $GL(V)$. Then the action of G on V is generically free. In order to check this, we may assume that k is algebraically closed, and hence G is a constant subgroup. As G is a subgroup of $GL(V)$, we have that g is a non-identity for $g \neq e$, and hence the action is generically free (see Example 14.5).

If G is not étale, this is not true any more. Let k be an algebraically closed field of characteristic $p > 0$. Let $V = k^2$, and consider $G = GL(V)_1$, the first Frobenius kernel of $GL(V)$. Let $B = k[V] = \operatorname{Sym} V^* = k[x, y]$, $A = B^G$, $K = Q(A)$, and $L = Q(B)$. Then $A = k[x^p, y^p]$. So $\dim_K L = 2 < \dim_k k[G] = 4$. Hence the action is not generically free by Proposition 14.9.

Lemma 14.11. *Let $S = \operatorname{Spec} R$ be affine. Let $G = \operatorname{Spec} \Gamma$ be an LFF S-group scheme. Then the coordinate ring Γ of G is a projective object as a G-module.*

Proof. It is easy to see that there exists some finitely generated $\mathbb{Z}$-subalgebra R_0 of R and an LFF R_0-group scheme G_0 such that $R \otimes_{R_0} G_0 \cong G$. Hence we may assume that R is Noetherian. By [Has, (III.4.1.3)], we may assume that R is a field.

Note that a G-module is nothing but a (right) Γ-comodule, which is the same as a (left) Γ^*-module. By [SkY, (VI.3.6)], $\Gamma \cong \Gamma^*$ as a Γ^*-module, and we are done. $\square$

Lemma 14.12. *Let G be an LFF S-group scheme acting on an S-scheme $X = \operatorname{Spec} B$ which is an affine scheme. Let $A = B^G$. If $\varphi : X = \operatorname{Spec} B \to \operatorname{Spec} A = Y$ is a principal G-bundle, then B is A-finite and (G, A)-projective. If A is a Noetherian Henselian local ring, then $B \cong A[G_A]$ as (G, A)-modules.*

Proof. We may assume that $S = Y = \operatorname{Spec} A$. As G is flat, φ is fpqc. Let $\Gamma = A[G_A] = A[G]$ be the coordinate ring of G. Let B' be the A-algebra B with a trivial G-action. Then $B \otimes_A B' \cong \Gamma \otimes_A B'$. By the descent argument, B is a finite projective A-module. We prove that B is a projective G-module, or a Γ^*-module. There is a surjective Γ^*-linear map $\alpha : (\Gamma^*)^n \to B$, as B is A-finite. We want to prove that this map splits. This is equivalent to the surjectivity of

$$\alpha_* : \operatorname{Hom}_{\Gamma^*}(B, (\Gamma^*)^n) \to \operatorname{Hom}_{\Gamma^*}(B, B). \tag{31}$$

This is checked after tensoring B' over A. But

$$\begin{aligned}\operatorname{Hom}_{\Gamma^*}(B, ?) \otimes_A B' &= \operatorname{Hom}_A(B, ?)^G \otimes_A B' = (\operatorname{Hom}_A(B, ?) \otimes_A B')^G \\ &= \operatorname{Hom}_{B'}(B \otimes_A B', ? \otimes_A B')^G = \operatorname{Hom}_{(G,B')}(\Gamma \otimes_A B', ? \otimes_A B').\end{aligned}$$

This is an exact functor by Lemma 14.11. So (31) is surjective.

Now assume that A is Noetherian Henselian local. Since B' is finite projective as an A-module and A is a local ring, $B' \cong A^n$ for some n. Hence $B^n \cong B \otimes_A B' \cong \Gamma \otimes_A B' \cong \Gamma^n$ as Γ^*-modules. Since A is Henselian, any finite Γ^*-module has a semiperfect endomorphism ring, and the Krull–Schmidt theorem holds in the category of finite Γ^*-modules (the fact that a mofule-finite algebra over a Noetherian Henselian local ring is semiperfect follows easily from [Mil, (I.4.2)]). So $B \cong \Gamma$ as G-modules, as desired. □

Example 14.13. Let k be a field, and G a finite k-group scheme acting on a k-algebra B. Even if B is a DVR (discrete valuation ring) and the action of G on $X = \operatorname{Spec} B$ is generically free, the action may not be free (so it is not a small action either). We give an example of a finite group in characteristic zero and an infinitesimal group scheme in characteristic p.

1 If $k = \mathbb{C}$, $G = \mathbb{Z}_2 = \langle \sigma \rangle$ (the cyclic group of order two with the generator σ), $B = k[[x]]$ with $\sigma(x) = -x$, then the stabilizer at the vertex $\operatorname{Spec} k = \operatorname{Spec} B/(x)$ is G, and is nontrivial. So the action is not free. The action is generically free by Proposition 14.9, since $[Q(B) : Q(B^G)] = [k((x)) : k((x^2))] = 2 = \#G$.

2 Let k be a field of characteristic p, and $B = k[x]_{(x)}$, the localization of the polynomial ring $k[x]$ at the prime ideal (x). Let D be the k-derivation $x^p \dfrac{d}{dx}$ of B. Note that $D^p = 0$. So $G = \alpha_p := (\mathbb{G}_a)_1$, the first Frobenius kernel of the additive group $\mathbb{G}_a$, acts on $X = \operatorname{Spec} B$. The algebra map $B \to k[G] \otimes B$ associated with the action $G \times X \to X$ is the map $k[x]_{(x)} \to k[t]/(t^p) \otimes k[x]_{(x)} = k[t, x]_{(x)}/(t^p)$ given by

$$f \mapsto \exp(D(-t))(f) = \sum_{i=0}^{p-1} D^i(f)(-t)^i/i!.$$

So $B^G = B^D = \{f \in B \mid Df = 0\} = k[x^p]_{(x^p)}$. As $[Q(B) : Q(B^G)] = p = k[G]$, the action is generically free. It is easy to see that the stabilizer at $\operatorname{Spec} k = \operatorname{Spec} B/(x)$ is G, and the action is not free.

(14.14) Let $f : G \to H$ be an fppf finite homomorphism between flat S-group schemes, and $N = \operatorname{Ker} f$. Note that N is fppf finite over S, that is, LFF.

Lemma 14.15. *Let the notation be as above. Let $\varphi : X \to Y$ be a G-morphism which is an algebraic quotient by the action of N. Then the free locus U of the action of N on X is G-stable in X. In particular, $\rho : U \to V = \varphi(U)$ is a G-enriched principal N-bundle. If, moreover, the action of N on X is n-small, then φ is a G-enriched n-almost principal N-bundle.*

Proof. In view of Proposition 14.8, it suffices to show that U is G-stable.

Let G act on $X \times_Y X$ diagonally and on $N \times X$ by $g(n, x) = (gng^{-1}, gx)$. Then $\Psi : G \times X \to X \times_Y X$ defined by $\Psi(g, x) = (gx, x)$ and the diagonal map $\Delta_X : X \to X \times_Y X$ are G-morhpisms. So $\phi : \mathcal{S}_X \to X$ is also a G-morphism, and hence U is G-stable. □

(14.16) Let G be a flat S-group scheme. For a G-scheme X, we define the *G-radical* of X by

$$\operatorname{rad}_G(X) := (\bigcap_{\mathfrak{M} \in \operatorname{Max}(G,X)} \mathfrak{M})^*,$$

the sum of all the quasi-coherent G-ideals of $\mathcal{O}_X$ contained in $\bigcap_{\mathfrak{M} \in \operatorname{Max}(G,X)} \mathfrak{M}$, where $\operatorname{Max}(G, X)$ is the set of G-maximal G-ideals of $\mathcal{O}_X$. We define the *G-nilradical* of X to be $\sqrt[G]{0}$, the G-radical of the zero ideal, see [HasM, (4.25)]. Note that $\sqrt[G]{0} \subset \sqrt{0}$ [HasM, (4.30)]. If X is quasi-compact, then by [HasM, (4.27)], we have that $\operatorname{rad}_G(X) \supset \sqrt[G]{0}$. Note that even if $S = \operatorname{Spec} k$ and both G and X are k-varieties, $\operatorname{rad}(X)$ may not contain $\operatorname{rad}_G(X)$. For example, when we consider the action of $G = \mathbb{G}_m$ on $B = k[x]$ with the grading $\deg x = 1$, then the ideal (x) is the unique G-maximal ideal, and so $\operatorname{rad}_G(B) = (x) \not\subset \operatorname{rad}(B) = (0)$.

Lemma 14.17 (G-Nakayama's lemma)**.** *Let G be a flat S-group scheme and X a quasi-compact G-scheme. Let $\mathcal{M}$ be a quasi-coherent $(G, \mathcal{O}_X)$-module of finite type. If $\operatorname{rad}_G(X)\mathcal{M} = \mathcal{M}$, then $\mathcal{M} = 0$.*

Proof. Assume the contrary. Let $\mathcal{I} = 0 :_{\mathcal{O}_X} \mathcal{M}$ be the annihilator of $\mathcal{M}$. Note that $\mathcal{I}$ is a quasi-coherent G-ideal (the proof is the same as that of [HasM, (4.2)]). As $\mathcal{I} \neq \mathcal{O}_X$ by assumption, there exists some $\mathfrak{M} \in \operatorname{Max}(G, X)$ containing $\mathcal{I}$ by [HasM, (4.28)]. Let $\mathfrak{m}$ be a maximal quasi-coherent ideal of $\mathcal{O}_X$ containing $\mathfrak{M}$, and x the closed point of X corresponding to $\mathfrak{m}$. Since $\mathfrak{m}$ contains $\mathcal{I}$ and $\mathcal{M}$ is of finite type, $\mathcal{M}_x \neq 0$. By Nakayama's lemma, $\mathcal{M}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x) \neq 0$. Similarly, $(\mathcal{O}_X / \operatorname{rad}_G(X))_x \otimes_{\mathcal{O}_{X,x}} \kappa(x) \neq 0$, since $\mathfrak{m} \supset \operatorname{rad}_G(X)$. Taking the tensor product, $(\mathcal{M} / \operatorname{rad}_G(X)\mathcal{M})_x \otimes_{\mathcal{O}_{X,x}} \kappa(x) \neq 0$. Hence $\mathcal{M} \neq \operatorname{rad}_G(X)\mathcal{M}$. This is a contradiction. □

Lemma 14.18. *Let $f : G \to H$ be an fppf finite homomorphism between flat S-group schemes, and $N = \operatorname{Ker} f$. Let $\varphi : X \to Y$ be a G-morphism which is an algebraic quotient by the action of N. Let $\mathcal{I}$ be a quasi-coherent G-ideal of $\mathcal{O}_X$ contained in $\operatorname{rad}_G(X)$, and $Z = V(\mathcal{I})$ the corresponding closed G-subscheme of X. If the action of N on Z is free, then the action of N on X is also free, and hence φ is a principal G-bundle.*

Proof. Let $\mathcal{C}_X$ be the cokernel of $\mathcal{O}_X \to \phi_*\mathcal{O}_{\mathcal{S}_X}$. This is a finite-type quasi-coherent $(G, \mathcal{O}_X)$-module. It suffices to prove that $\mathcal{C}_X = 0$. By Lemma 14.17, it suffices to show that $j^*\mathcal{C}_X = 0$, where $j : Z \hookrightarrow X$ is the inclusion. As N acts on Z freely, $\mathcal{C}_Z = 0$, and hence it suffices to show that $\mathcal{S}_Z = \mathcal{S}_X \times_X Z$ in a natural way. This follows from Lemma 1.8, as $j : Z \to X$ is a monomorphism. □

Lemma 14.19. *Let G be a smooth S-group scheme.*

1 *If X is a G-scheme, then the reduction X_{red} has a unique G-scheme structure such that the inclusion* red $: X_{\mathrm{red}} \hookrightarrow X$ *is a G-morphism. Hence $\sqrt{0} = \sqrt[G]{0}$. If, moreover, X is an LFI-scheme, then the normalization X^ν* [Stack, (28.48.12)] *has a unique G-scheme structure such that the morphism $\nu : X^\nu \to X$ is a G-morphism.*

Let $\varphi : X \to Y$ be a G-morphism.

2 *If φ is a G-morphism (resp. a principal G-bundle), then $\varphi_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$ is a G-morphism (resp. a principal G-bundle). If, moreover, φ is a morphism between LFI-schemes such that $\varphi(X^{\langle 0 \rangle}) \subset Y^{\langle 0 \rangle}$, then $\varphi^\nu : X^\nu \to Y^\nu$ is a G-morphism (resp. a principal G-bundle).*

3 *If $\varphi : X \to Y$ is an algebraic quotient and $\varphi_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$ is a principal G-bundle, then φ is a principal G-bundle.*

Proof. **1** As $G \times X_{\mathrm{red}}$ is reduced by [Stack, (10.149.6)], the composite

$$G \times X_{\mathrm{red}} \xrightarrow{1_G \times \mathrm{red}} G \times X \xrightarrow{a} X$$

uniquely factors through red $: X_{\mathrm{red}} \to X$. As X_{red} is G-stable in X, $\sqrt{0} = \sqrt{0}^* = \sqrt[G]{0}$ by [HasM, (4.30)]. The latter part is proved similarly, using [Stack, (10.149.7)] and [Stack, (28.48.15)].

2 Consider the diagram

$$\begin{tikzcd} G \times X_{\mathrm{red}} \arrow[r, "1_G \times \varphi_{\mathrm{red}}"] \arrow[d, "a"] \arrow[dr, phantom, "\text{(a)}"] & G \times Y_{\mathrm{red}} \arrow[r, "1_G \times \mathrm{red}_Y"] \arrow[d, "a"] \arrow[dr, phantom, "\text{(b)}"] & G \times Y \arrow[d, "a"] \\ X_{\mathrm{red}} \arrow[r, "\varphi_{\mathrm{red}}"] & Y_{\mathrm{red}} \arrow[r, "\mathrm{red}_Y"] & Y \end{tikzcd}$$

As the square (b) and the whole rectangle (a)+(b) commutes and red_Y is a monomorphism, (a) commutes and φ_{red} is a G-morphism. If φ is a principal G-bundle, then φ is smooth, and hence $X \times_Y Y_{\mathrm{red}}$ is reduced. As $1_X \times \mathrm{red}_Y : X \times_Y Y_{\mathrm{red}} \to X \times_Y Y = X$ is a surjective closed immersion, $X \times_Y Y_{\mathrm{red}} = X_{\mathrm{red}}$, and $\varphi_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$ is a base change of φ. Hence it is a principal G-bundle.

The case of normalization is similar and left to the reader. Note that if $\varphi : X \to Y$ is a principal G-bundle (which is a morphism between LFI-schemes such that $\varphi(X^{\langle 0 \rangle}) \subset Y^{\langle 0 \rangle}$), then $X \times_Y Y^\nu$ is normal, and $(1_X \times \nu_Y)_{\mathrm{red}} : X \times_Y Y^\nu \to X_{\mathrm{red}}$ is integral and birational (for the definition of birational morphisms, see [Stack, (28.9.1)]).

3. We may assume that Y is affine. Then X is also affine, and hence $\sqrt[G]{0} = \sqrt{0}$ is contained in the G-radical $\mathrm{rad}_G(X)$ of X by **1** and [HasM, (4.27)]. By assumption, the action of G on X_{red} is free, and hence the action of G on X is also free by Lemma 14.18. So φ is a principal G-bundle by Lemma 11.12. $\square$

Lemma 14.20. *Let G be an LFF S-group scheme, and $\varphi : X \to Y$ be an algebraic quotient. Let U be the free locus, and $V = \varphi(U)$. For $y \in Y$, the following are equivalent.*

1 *$y \in V$.*

2 *$X_y := \varphi^{-1}(y) \to y$ is a principal G-bundle.*

3 *The action of G on X_y is free.*

Proof. **1⇒2**. We have that $\rho : U \to V$ is a principal G-bundle, and its base change $X_y \to y$ is also a principal G-bundle.

2⇒3. This is trivial.

3⇒1. We may assume that $Y = \operatorname{Spec} A$ is affine, and then $X = \operatorname{Spec} B$ is affine and $A = B^G$. Let M be the coordinate ring of $\mathcal{S}_X$, and let C be the cokernel of $B \to M$. Let P be the prime ideal corresponding to y. Since $A_P \to B_P$ is integral, PB_P is contained in the radical of B_P. As $X_y \to X$ is a monomorphism, we have that $C \otimes_B (B_P/PB_P) = 0$ by the assumption that the action of G on X_y is free and Lemma 1.8. As C_P is a finite B_P-module, $PB_P \subset \operatorname{rad}(B_P)$, and $PC_P = C_P$, we have that $C_P = 0$ by Nakayama's lemma. So no point of X_y supports C. That is, $X_y \subset U$. Hence $y \in V$. $\square$

Proposition 14.21. *Let G be an étale finite S-group scheme (in particular, LFF). Let $\varphi : X \to Y$ be an algebraic quotient by G. Assume that X and Y are locally Noetherian and φ is finite (for example, let $S = \operatorname{Spec} k$ with k a field, X be locally Noetherian, and if the characteristic of k is positive, assume further that X is F-finite, see* Lemma 9.6*). Let U be the free locus of the action of G, and assume that the action is generically free. Then U agrees with the étale locus of φ.*

Proof. We may assume that $Y = \operatorname{Spec} A$ is affine and connected. So $X = \operatorname{Spec} B$ is affine G-connected and $A = B^G$.

Set $V = \varphi(U)$. Let $\rho : U \to V$ be the restriction of φ. Then $\rho : U \to V$ is a principal G-bundle. As G is étale, ρ is étale, and hence U is contained in the étale locus.

We prove the opposite incidence. Let U' be the étale locus of φ. Then it is a G-stable open subset. We have shown that $U \subset U'$. So to prove that $U' = U$, replacing X by U', we may assume that φ is étale, and we need to prove that $U = X$. Again, we may assume that $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ are affine with $A = B^G$, and Y is connected. We may assume that $S = Y$. Note that B is a finite projective A-module. As Y is connected, it has a well-defined rank, say r. On the other hand, the coordinate ring Γ of A has a finite projective module. Let r' be its rank. As the action of G on X is generically free, we have $r = r'$ by Proposition 14.9. Let $y \in Y$. Consider the base change of the map

$$\Psi : G \times X \to X \times X \qquad ((g, x) \mapsto (gx, x))$$

by $y \to Y = S$. It is

$$\Psi_y : G_y \times_y X_y \to X_y \times_y X_y.$$

This map is surjective, since Ψ is. The map Ψ_y is a map between affine schemes corresponding to the $\kappa(y)$-algebra map between étale $\kappa(y)$-algebras

$$I : B(y) \otimes_{\kappa(y)} B(y) \to \Gamma(y) \otimes_{\kappa(y)} B(y),$$

where $?(y)$ means $? \otimes_A \kappa(y)$. This map I is injective, since the algebras are reduced and the corresponding map Ψ_y is surjective. On the other hand, the source and the target of I are both of dimension r^2 over $\kappa(y)$. So I must be an isomorphism. This shows that $X_y \to y$ is a principal G-bundle. By Lemma 14.20, $y \in V$. As y is an arbitrary point of Y, we have that $V = Y$, and hence $U = X$. $\square$

Corollary 14.22. *Under the assuptions of* Proposition 14.21*, assume further that X is regular and the action of G is small. Then the singular locus of Y is $Y \setminus \varphi(U)$.*

Proof. Set $V = \varphi(U)$. Let V' be the regular locus of Y. We want to prove that $V = V'$.

As U is regular and $U \to V$ is faithfully flat, V is regular. So $V \subset V'$.

Let $y \in V'$, $A = \mathcal{O}_{Y,y}$, and $B = (\varphi_* \mathcal{O}_X)_y$. Then by the smallness assumption, the branch locus of B over A has codimension at least two. By the purity of branch locus [Gro4, (X.3.1)], B is étale over A. That is, $y \in V$, and $V \supset V'$. $\square$

Lemma 14.23. *Let G be an LFF S-group scheme, and $\varphi : X \to Y$ an algebraic quotient by G. Assume that X and Y are locally Noetherian.*

1 *If $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$-module which satisfies (S'_n), then $\varphi_* \mathcal{M}$ satisfies (S'_n).*

2 *If a quasi-coherent $(G, \mathcal{O}_X)$-module $\mathcal{M}$ on X satisfies the (S'_2) condition, then $(\varphi_* \mathcal{M})^G$ satisfies the (S'_2) condition.*

Proof. **1** follows from Lemma 12.1 and Lemma 11.12. **2** follows from **1** and Lemma 12.2. $\square$

From the results we obtained so far, we can state the following.

Theorem 14.24. *Let S be a scheme, and $f : G \to H$ an fppf finite homomorphism between flat S-group schemes, and $N = \operatorname{Ker} f$. Let $\varphi : X \to Y$ be a G-morphism which is an algebraic quotient by the action of N. Assume that the action of N on X is small. Let U be the free locus of the action of N on X, and $V := \varphi(U)$. Then we have the following.*

0 *φ is a G-enriched almost principal N-bundle with respect to U and V.*

1 *(cf. [HasN, Theorem 2.4]) Assume that X is locally Krull. Then Y is also locally Krull, and $(\varphi^*?)^{**} : \mathrm{Ref}(H,Y) \to \mathrm{Ref}(G,X)$ and $(\varphi_*?)^N : \mathrm{Ref}(G,X) \to \mathrm{Ref}(H,Y)$ are quasi-inverse each other. In particular, for $\mathcal{L},\mathcal{M} \in \mathrm{Ref}(G,X)$, $\mathcal{L} \cong \mathcal{M}$ if and only if $(\varphi_*\mathcal{L})^N \cong (\varphi_*\mathcal{M})^N$ in $\mathrm{Ref}(H,Y)$. $\mathcal{M}$ is indecomposable in $\mathrm{Ref}(G,X)$ if and only if $(\varphi_*\mathcal{M})^N$ is so in $\mathrm{Ref}(H,Y)$. This equivalence induces an isomorphism between $\mathrm{Cl}(H,Y)$ and $\mathrm{Cl}(G,X)$.*

2 *Assume that X is quasi-compact quasi-separated and locally Krull. Then there is an exact sequence*

$$0 \to H^1(N,\mathcal{O}_X^\times) \to \mathrm{Cl}(Y) \to \mathrm{Cl}(X)^N \to H^2(N,\mathcal{O}_X^\times).$$

3 *Assume that G is of finite type. Let Y_0 be a fixed Noetherian H-scheme with a fixed H-dualizing complex $\mathbb{I}_{Y_0}$, and assume that φ is a morphism in $\mathcal{F}(G,Y_0)$. Then φ is finite, and we have*

$$\omega_Y \cong (\varphi_*(\omega_X \otimes_{\mathcal{O}_X} \Theta_{N,X}))^N \cong (\varphi_*\omega_X \otimes_{\mathcal{O}_Y} \Theta_{N,Y})^N$$

as $(H,\mathcal{O}_Y)$-modules. If, moreover, X has a coherent $(G,\mathcal{O}_X)$-module $\mathcal{M}_X$ which is a full 2-canonical module, then we have

$$\omega_X \cong (\varphi^*\omega_Y)^{\vee\vee} \otimes_{\mathcal{O}_X} \Theta^*_{N,X} \cong (\varphi^*(\omega_Y \otimes_{\mathcal{O}_Y} \Theta^*_{N,Y}))^{\vee\vee}$$

as $(G,\mathcal{O}_X)$-modules.

4 *In* **3***, If $\Theta_{N,Y_0} \cong \mathcal{O}_{Y_0}$ (e.g., N is étale, N is Reynolds, or $S = \mathrm{Spec}\, k$ with k a field and G centralizes N) , then $\omega_Y \cong (\varphi_*\omega_X)^N$ as $(H,\mathcal{O}_Y)$-modules. If, moreover, X has a coherent $(G,\mathcal{O}_X)$-module $\mathcal{M}_X$ which is a full 2-canonical module, then we have $\omega_X \cong (\varphi^*\omega_Y)^{\vee\vee}$ as $(H,\mathcal{O}_X)$-modules.*

5 *In* **3***, assume that $\Theta_{N,Y_0} \cong \mathcal{O}_{Y_0}$. Let $\mathcal{L}$ be an H-linearized invertible sheaf on Y. Then $\omega_X \cong \varphi^*\mathcal{L}$ as $(G,\mathcal{O}_X)$-modules if and only if $\omega_Y \cong \mathcal{L}$ as $(H,\mathcal{O}_Y)$-modules and X satisfies the (S_2) condition. If so, both X and Y are quasi-Gorenstein.*

6 *Let the assumptions be as in* **3***. Then $\omega_X \cong \varphi^*\mathcal{L}$ for some H-linearized invertible sheaf on Y if and only if ω_Y is an invertible sheaf and X satisfies the (S_2) condition. If so, then both X and Y are quasi-Gorenstein. If, moreover, Y is connected, then these conditions are equivalent to say that Y is quasi-Gorenstein and X satisfies (S_2).*

Proof. **0** See Lemma 14.15.

1 Note that Y is also locally Krull by [Has9, (6.3)]. Now the result follows from Corollary 11.4.

2 follows from Theorem 11.5.

Considering the fact that X is (S_2) implies Y is (S_2) (by Lemma 14.23), **3, 4, 5, 6** are immediate from Corollary 11.22. For the conditions for Θ_{N,Y_0} to be trivial, see Remark 11.21. $\square$

(14.25) Let k be a field, and $B = \bigoplus_{n\in\mathbb{Z}} B_n$ be a finitely generated positively graded k-algebra (that is, $B_n = 0$ for $n < 0$ and $B_0 = k$). Let ω_B denote the canonical module of B. The base scheme S is $\operatorname{Spec} k$, the group G is the torus $\mathbb{G}_m$, and the G-dualizing complex of S is fixed to k (concentrated in degree zero). Then ω_B is a finitely generated $\mathbb{Z}$-graded nonzero B-module. So

$$a = a(B) = -\min\{n \in \mathbb{Z} \mid \omega_{B,n} \neq 0\}$$

is well-defined. The integer a is called the *a-invariant* of Goto–Watanabe [GW, (3.1.4)].

(14.26) If B is a quasi-Gorenstein Noetherian $\mathbb{Z}^n$-graded k-algebra such that there exists some homomorphism $h : \mathbb{Z}^n \to \mathbb{Z}$ such that B is positively graded with respect to the induced $\mathbb{Z}$-grading. Then there exists some unique $a \in \mathbb{Z}^n$ such that $\omega_B \cong B(a)$ as $\mathbb{Z}^n$-graded B-modules. We also call a the a-invariant of B. This definition is consistent with the one in (14.25) when $n = 1$ and B is positively graded.

Example 14.27. Let k be a field, and N a finite k-group scheme. Let $G = N \times \mathbb{G}_m$ and $H = \mathbb{G}_m$.

Let B be a G-algebra, and $\varphi : X = \operatorname{Spec} B \to \operatorname{Spec} A = Y$ be the algebraic quotient, where $A = B^N$. Assume that the action of N on X is small. Note that B is a $\mathbb{Z}$-graded N-algebra. Assume that B is positively graded (that is, $B = \bigoplus_{n\geq 0} B_n$ with $B_0 = k$). Let $k(a)$ be the one-dimensional G-module which is the one-dimensional $\mathbb{G}_m$-module concentrated in degree $-a$ and is trivial as an N-module, and set $B(a) = B \otimes k(a)$.

If N is either étale; linearly reductive; or abelian, then $\omega_B^N = \omega_A$. By Theorem 14.24, **5**, $\omega_B \cong B(a)$ as (G, B)-modules if and only if B is (S_2) and $\omega_A \cong A(a)$ as graded A-modules, that is, B is (S_2) and A is quasi-Gorenstein with the a-invariant a.

Even if G is a general finite k-group scheme, $\omega_B \cong B$ as (N, B)-modules if and only if A is quasi-Gorenstein and B is (S_2), by Theorem 14.24, **6**. The author does not know if the a-invariants of A and B agree in general.

Example 14.28. Let B be the polynomial ring $k[x_1, \ldots, x_d]$ with $\deg x_i = 1$ in Example 14.27 above. As above, assume that G acts on B (that is, N acts on B linearly). As in Example 14.27, assume that the action of N on X is small. Set $A = B^N$. Then by Theorem 14.24,

1 $\mathrm{Cl}(A) \cong H^1(N, B^\times)$, since $\mathrm{Cl}(B) = 0$. If, moreover, N is étale, then $\mathrm{Cl}(A) \cong \mathcal{X}(N)$ by [Has9, (4.13)].

2 $\omega_A \cong (B \otimes \bigwedge^d B_1 \otimes \Theta_{N,k})^N$ as graded A-modules, and

$$(B \otimes_A \omega_A)^{**} \cong B \otimes_k (\bigwedge^d B_1 \otimes_k \Theta_{N,k})$$

as graded (N, B)-modules.

3 The following are equivalent.

- **a** $\bigwedge^d B_1 \cong \Theta^*_{N,k}$ as N-modules.
- **b** $\bigwedge^d B_1 \cong \Theta^*_{N,k} \otimes k(-d)$ as G-modules.
- **c** A is quasi-Gorenstein.
- **d** A is quasi-Gorenstein of the a-invariant $-d$.

4 (cf. [Bro], [Bra], [FlW]) Assume that N is either étale; linearly reductive; or abelian. Then $\Theta_{N,k}$ is trivial, and the following are equivalent.

- **a** $N \subset SL(B_1)$.
- **b** A is quasi-Gorenstein.
- **c** A is quasi-Gorenstein with the a-invariant $-d$.

Proof. We only prove **3**. **a**$\Rightarrow$**b**$\Rightarrow$**d**$\Rightarrow$**c** is easy. If A is quasi-Gorentein, then $\omega_A \cong A(a)$ for some $a \in \mathbb{Z}$ as (H, A)-modules. So $B(a) \cong B \otimes_k (\bigwedge^d B_1 \otimes_k \Theta_{N,k})$. Tensoring B/B_+, where B_+ is the irrelevant ideal of B, we get $k \cong \bigwedge^d B_1 \otimes_k \Theta_{N,k}$ as N-modules, and **c**$\Rightarrow$**a** follows. $\square$

15. Multisection rings

(15.1) Let X be a locally Krull scheme. We define a divisor on X and the $\mathcal{O}_X$-module $\mathcal{O}_X(D)$ for a divisor on X.

Recall that $P^1(X)$ denotes the set of integral closed subschemes of codimension one (7.22). Set $\mathcal{F} = \prod_{W\in P^1(X)} \mathbb{Z}\cdot W$. An element of $\mathcal{F}$ is called a *formal divisor*. $W \in P^1(X)$ as an element of $\mathcal{F}$ is called a prime divisor. For $D = (a_{D,W}W)_{W\in X} \in \mathcal{F}$, the support $\operatorname{supp} D$ of D is $\{W \in P^1(X) \mid a_{D,W} \neq 0\}$. For $D = (a_{D,W}W)$ and $D' = (a'_{D,W}W)$ in $\mathcal{F}$, we say that $D \geq D'$ if $a_{D,W} \geq a'_{D,W}$ for any W. We say that D is *effective* if $D \geq 0$. If $\operatorname{supp} D$ is locally finite (see (7.22)) in X, we say that D is a *divisor* on X. The set of divisors $\operatorname{Div}(X)$ on X forms a subgroup of $\mathcal{F}$. If X is quasi-compact, $\operatorname{Div}(X) = \bigoplus_{W\in P^1(X)} \mathbb{Z}\cdot W$. For $D \in \operatorname{Div}(X)$ and an open subset U of X, we define the restriction $D|_U$ of D to to be $(a_{D,\bar{W}}W)_{W\in P^1(U)}$, where $\bar{W}$ is the closure of W in X.

(15.2) Let X be integral. Then for $f \in K^\times$ and $W \in P^1(X)$, we define $a_{f,W}$ to be the order of f in the DVR $\mathcal{O}_{X,W} \subset K = \kappa(\xi)$, where ξ is the generic point of X, and K is the function field. We define $\operatorname{div} f := (a_{f,W}W) \in \mathcal{F}$. It is easy to see that $\operatorname{div} f$ is a divisor. Note that $\operatorname{div} : K^\times \to \operatorname{Div}(X)$ is a homomorphism. Its image $\operatorname{div}(K^\times)$ is denoted by $\operatorname{Prin}(X)$. An element of $\operatorname{Prin}(X)$ is called a *principal divisor*.

(15.3) Let X be integral. Let ξ be its generic point, and $j : \xi \to X$ the inclusion. The quasi-coherent sheaf $j_*j^*\mathcal{O}_X$ is denoted by $\mathcal{K}$. It is the constant sheaf of $K = \kappa(\xi)$. Let $D = (a_{D,W}W) \in \operatorname{Div}(X)$.

We define an $\mathcal{O}_X$-submodule $\mathcal{O}_X(D)$ of $\mathcal{K}$ by

$$\Gamma(U, \mathcal{O}_X(D)) = \{0\} \cup \{f \in K^\times \mid (\operatorname{div} f + D)|_U \geq 0\}.$$

Note that $\mathcal{O}_X(D)$ is a rank-one reflexive quasi-coherent sheaf.

(15.4) In general, a locally Krull scheme $X = \coprod_i X_i$ is the disjoint union of its irreducible components. We define

$$\operatorname{Prin}(X) := \prod_i \operatorname{Prin}(X_i) \subset \prod_i \operatorname{Div}(X_i) = \operatorname{Div}(X).$$

We define the (geometric) class group $\operatorname{Cl}'(X)$ to be $\operatorname{Div}(X)/\operatorname{Prin}(X)$. Thus $\operatorname{Cl}'(X) = \prod_i \operatorname{Cl}'(X_i)$.

For $D \in \mathrm{Div}(X)$, $\mathcal{O}_X(D)$ is also defined componentwise. It is easy to see that $\mathcal{O}_X : D \mapsto \mathcal{O}_X(D)$ gives a homomorphism from $\mathrm{Div}(X)$ to $\mathrm{Cl}(X)$.

Lemma 15.5. *$\mathcal{O}_X$ induces an isomorphism $\mathcal{O}_X : \mathrm{Cl}'(X) \to \mathrm{Cl}(X)$.*

Proof. We may assume that X is integral. First, we prove that $\mathcal{O}_X$ is surjective. Let $\mathcal{M}$ be a rank-one reflexive sheaf on X. Let ξ be the generic point of ξ, and $j : \xi \to X$ the inclusion. Note that $\mathcal{K} = j_* j^* \mathcal{O}_X$. As $\mathcal{M}$ is rank-one reflexive, there is a monomorphism

$$\mathcal{M} \xrightarrow{u} j_* j^* \mathcal{M} \cong j_* j^* \mathcal{O}_X \cong \mathcal{K},$$

and we can identify $\mathcal{M}$ with a subsheaf of $\mathcal{K}$. Then it is easy to see that there exists some $D \in \mathrm{Div}(X)$ such that $\mathcal{M} \cong \mathcal{O}_X(D)$. That is, $\mathcal{O}_X : \mathrm{Div}(X) \to \mathrm{Cl}(X)$ is surjective.

Assume that $f : \mathcal{O}_X \cong \mathcal{O}_X(D)$. Then $f \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X(D)) = \Gamma(X, \mathcal{O}_X(D)) \subset K$. As f is an isomorphism, $f \in K^\times$ and $\mathrm{div}\, f + D = 0$, and hence D is principal. It is obvious that $\mathcal{O}_X(\mathrm{div}\, f) = f^{-1}\mathcal{O}_X \cong \mathcal{O}_X$. So $D \in \mathrm{Ker}(\mathcal{O}_X : \mathrm{Div}(X) \to \mathrm{Cl}(X))$ if and only if $D \in \mathrm{Prin}(X)$.

Thus the isomorphism $\mathcal{O}_X : \mathrm{Cl}'(X) \to \mathrm{Cl}(X)$ is induced. □

With this isomorphism, we identify $\mathrm{Cl}'(X)$ and $\mathrm{Cl}(X)$.

(15.6) Let S be a scheme, Λ a finitely generated $\mathbb{Z}$-module, and $G = \mathrm{Spec}\, \mathbb{Z}\Lambda \times_{\mathrm{Spec}\, \mathbb{Z}} S$, where $\mathbb{Z}\Lambda$ is the group algebra $\bigoplus_{\lambda \in \Lambda} \mathbb{Z} t^\lambda$ with each t^λ group-like. Let $\varphi : X \to Y$ be an affine G-invariant morphism. So $X = \underline{\mathrm{Spec}}_Y\, \mathcal{A}$ with $\mathcal{A} = \bigoplus_{\lambda \in \Lambda} \mathcal{A}_\lambda$ a graded quasi-coherent $\mathcal{O}_Y$-algebra.

Lemma 15.7. *The following are equivalent.*

1 *φ is a principal G-bundle.*

2 *$\mathcal{O}_Y \to \mathcal{A}_0$ is an isomorphism, each $\mathcal{A}_\lambda$ is an invertible sheaf, and the product $\mathcal{A}_\lambda \otimes_{\mathcal{O}_Y} \mathcal{A}_\mu \to \mathcal{A}_{\lambda+\mu}$ is an isomorphism.*

2' *$\mathcal{O}_Y \to \mathcal{A}_0$ is an isomorphism, and the product $\mathcal{A}_\lambda \otimes_{\mathcal{O}_Y} \mathcal{A}_\mu \to \mathcal{A}_{\lambda+\mu}$ is surjective for any $\lambda, \mu \in \Lambda$.*

3 *For each $y \in Y$, there exists some affine open neighborhood $y \in U = \mathrm{Spec}\, R$ and a faithfully flat R-algebra R' such that $A' = \bigoplus_\lambda A'_\lambda$ with $A'_\lambda = R' \otimes_R \Gamma(U, \mathcal{A}_\lambda)$ is isomorphic to the group algebra $R'\Lambda = \bigoplus_\lambda R' t^\lambda$.*

If, moreover, Λ is torsion-free, then φ is a principal G-bundle in the Zariski topology. That is, we can take $R' = R$ in **3**.

Proof. **1⇒2** is obvious from the descent argument.

2⇒2' is trivial. **2'⇒3**. Take any affine open neighborhood $U = \operatorname{Spec} R$ of y. Then $A = \Gamma(\varphi^{-1}(U), \mathcal{O}_X)$ is a graded algebra with $A_0 = R$. We may assume that

$$\Lambda = \mathbb{Z}/(m_1) \oplus \mathbb{Z}/(m_2) \oplus \cdots \oplus \mathbb{Z}/(m_s)$$

with $m_i \geq 0$, $m_i \neq 1$. Let λ_i be a generator of $\mathbb{Z}/(m_i)$. So $\lambda_1, \ldots, \lambda_s$ together generate Λ.

For each i, there exists some expression $1 = \sum_{l=1}^{m_i} u_{i,l} v_{i,l}$ with $u_{i,l} \in A_{\lambda_i}$ and $v_{i,l} \in A_{-\lambda_i}$. Then contracting U if necessary, we may assume that for each i, there exists some l_i such that $u_{i,l_i} v_{i,l_i}$ is invertible in A_0. So each $t_i := u_{i,l_i}$ are units of A. If Λ is torsion-free, then $m_i = 0$ for each i, and $A = R[t_1^{\pm 1}, \ldots, t_s^{\pm s}]$. So the last assertion has been proved.

We consider the case that Λ may have a torsion. We may assume that $m_1, \ldots, m_r \geq 2$ and $m_i = 0$ for $i > r$. Set

$$R' = R[T_1, \ldots, T_r]/(T_1^{m_1} - t_1^{m_1}, \ldots, T_r^{m_r} - t_r^{m_r}),$$

where $T_1, \ldots, T_r$ are new variables of degree zero. As R' is a nonzero free R-module, R' is faithfully flat over R. Letting $t_i' := t_i \bar{T}_i^{-1}$, we have that

$$A' := R' \otimes_R A = R'[t_1', \ldots, t_r', t_{r+1}^{\pm 1}, \ldots, t_s^{\pm 1}]/((t_1')^{m_1} - 1, \ldots, (t_r')^{m_r} - 1) \cong R'\Lambda,$$

as desired.

3⇒1 is trivial. $\square$

(15.8) Let $\varphi : X \to Y$ be a morphism between locally Krull schemes such that $\varphi(X^{\langle 0 \rangle}) \subset Y^{\langle 0 \rangle}$ and $\varphi(X^{\langle 1 \rangle}) \subset Y^{\langle 0 \rangle} \cup Y^{\langle 1 \rangle}$. We define the pull-back $\varphi^* : \operatorname{Div}(Y) \to \operatorname{Div}(X)$ by φ by $\varphi^*(a_V V)_{V \in P^1(Y)} = (b_W W)_{W \in P^1(X)}$, where $b_W = a_{\overline{\varphi(W)}} \operatorname{length}_{\mathcal{O}_{X,w}}(\mathcal{O}_{X,w}/\mathfrak{m}_{\varphi(w)}\mathcal{O}_{X,w})$ for the generic point w of $W \in P^1(X)$ if $w \in Y^{\langle 1 \rangle}$ ($\mathfrak{m}_{\varphi(w)}$ is the maximal ideal of the DVR $\mathcal{O}_{Y,\varphi(w)}$), and $b_W = 0$ if $w \in Y^{\langle 0 \rangle}$. It is easy to see that φ^* is a homomorphism, and $\varphi^*(\operatorname{div}(f)) = \operatorname{div}(\varphi^* f)$ if both X and Y are integral and $f \in K^\times$, where K is the function field of Y. So $\varphi^* : \operatorname{Cl}'(Y) \to \operatorname{Cl}'(X)$ is induced.

Lemma 15.9. *Let φ be as in* (15.8)*. Then $[\mathcal{M}] \mapsto [(\varphi^*\mathcal{M})^{**}]$ gives a homomorphism $\varphi^* : \mathrm{Cl}(Y) \to \mathrm{Cl}(X)$. Moreover, the diagram*

$$\begin{tikzcd} \mathrm{Cl}'(Y) \arrow[r, "\mathcal{O}_Y"] \arrow[d, "\varphi^*"] & \mathrm{Cl}(Y) \arrow[d, "\varphi^*"] \\ \mathrm{Cl}'(X) \arrow[r, "\mathcal{O}_X"] & \mathrm{Cl}(X) \end{tikzcd}$$

is commutative.

Proof. To prove the commutativity of the diagram, we may assume that both X and Y are integral. It suffices to prove that $(\varphi^*\mathcal{O}_Y(D))^{**} \subset \varphi^*\mathcal{K} = \mathcal{L}$ agrees with $\mathcal{O}_X(\varphi^*D)$, where $\mathcal{K}$ and $\mathcal{L}$ are the constant sheaves of the rational function fields of Y and X, respectively. As $(\varphi^*\mathcal{O}_Y(D))^{**}$ is reflexive, it suffices to prove that $(\varphi^*\mathcal{O}_Y(D))_x^{**} = (\varphi^*\mathcal{O}_Y(D))_x \subset L$ agrees with $\mathcal{O}_X(\varphi^*D)_x$ for each $x \in X^{\langle 1 \rangle}$, where L is the function field of X. Let $y = \varphi(x)$. First consider the case that $\operatorname{codim} y = 1$. If the coefficient of $\bar{y}$ in D is a and the ramification index $\mathrm{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x})$ is b, then the coefficient of $\bar{x}$ in φ^*D is the product ab by definition. So $\mathcal{O}_X(\varphi^*D)_x = \mathfrak{m}_x^{-ab}$. On the other hand,

$$(\varphi^*\mathcal{O}_Y(D))_x = \mathcal{O}_Y(D)_y\mathcal{O}_{X,x} = \mathfrak{m}_y^{-a}\mathcal{O}_{X,x} = \mathfrak{m}_x^{-ab} = \mathcal{O}_X(\varphi^*D)_x.$$

Next, consider the case that $\operatorname{codim} y = 0$. Then the coefficient of $\bar{x}$ in φ^*D is zero by definition. So $\mathcal{O}_X(\varphi^*D)_x = \mathcal{O}_{X,x}$. On the other hand,

$$(\varphi^*O_Y(D))_x = \mathcal{O}_Y(D)_y\mathcal{O}_{X,x} = K\mathcal{O}_{X,x} = \mathcal{O}_{X,x} = \mathcal{O}_X(\varphi^*D)_x,$$

where K is the rational function field of Y. Hence $(\varphi^*\mathcal{O}_Y(D))^{**} = \mathcal{O}_X(\varphi^*D)$ as subsheaves of $\mathcal{L}$. In particular, $\varphi^*[\mathcal{O}_Y(D)] = [\mathcal{O}_X(\varphi^*D)]$ in $\mathrm{Cl}(X)$, and the diagram in problem is commutative.

In the diagram, $\mathcal{O}_Y$ and $\mathcal{O}_X$ are group isomorphisms, and the left φ^* is a homomorphism. So the right φ^* is also a homomorphism. $\square$

(15.10) Let X be a locally Krull scheme, and Σ a subset of $P^1(X)$. For $D = (a_W W), D' = (a'_W W) \in \mathrm{Div}(X)$, we say that $D \geq_\Sigma D'$ if $a_W \geq a'_W$ for $W \in P^1(X) \setminus \Sigma$. We define $\mathcal{O}_{X,\Sigma}(D)$ by

$$\Gamma(U, \mathcal{O}_{X,\Sigma}(D)) = \{0\} \cup \{f \in K^\times \mid (\operatorname{div} f + D) \geq_{\Sigma \cup \{W \in P^1(X) \mid W \cap U = \emptyset\}} 0\}.$$

Note that $\mathcal{O}_{X,\Sigma} = \mathcal{O}_{X,\Sigma}(0)$ is a quasi-coherent $\mathcal{O}_X$-algebra. We define $X_\Sigma = \underline{\operatorname{Spec}}\,\mathcal{O}_{X,\Sigma}$. As a subintersection of a Krull domain is again a Krull domain [Fos, (1.5)], it is easy to see that X_Σ is locally Krull.

As in (15.8), the canonical map $j_\Sigma : X_\Sigma \to X$ induces a surjective map $j_\Sigma^* : \operatorname{Div}(X) \to \operatorname{Div}(X_\Sigma)$. As j_Σ is birational, j_Σ^* maps $\operatorname{Prin}(X)$ surjectively onto $\operatorname{Prin}(X_\Sigma)$. By the snake lemma, we get the following easily.

Lemma 15.11 (Nagata's theorem [Fos, (7.1)])**.** *Let X be a locally Noetherian scheme, and Σ a subset of $P^1(X)$. Then $j_\Sigma^* : \operatorname{Cl}'(X) \to \operatorname{Cl}'(X_\Sigma)$ is a surjective map whose kernel is generated by the divisors supported in Σ.* □

Lemma 15.12. *Let $G = \mathbb{G}_m^s$ be a split s-torus. Let $\varphi : X \to Y$ be a principal G-bundle. If X is locally Krull, then the flat pullback $\varphi^* : \operatorname{Cl}'(Y) \to \operatorname{Cl}'(X)$ (see (15.8)) is surjective.*

Proof. We may assume that Y is integral. Let $W \in P^1(Y)$. As G is geometrically integral, we have that $\varphi^{-1}(W)$ is integral. Applying [Has9, (5.13)] to the map of local rings $\mathcal{O}_{Y,W} \to \mathcal{O}_{X,\varphi^{-1}(W)}$, we have that $\varphi^{-1}(W) \in P^1(X)$. By Lemma 15.11, $\operatorname{Cl}'(X) \to \operatorname{Coker}\varphi^*$ factors through the surjection $\operatorname{Cl}'(X) \to \operatorname{Cl}'(X_\Sigma)$, where Σ is the set of prime divisors of the form $\varphi^{-1}(W)$ with $W \in P^1(Y)$. It is easy to see that $X_\Sigma = \varphi^{-1}(\eta) = \operatorname{Spec}\kappa(\eta)[t_1^{\pm 1}, \ldots, t_s^{\pm 1}]$ by Lemma 15.7, where η is the generic point of Y. As $\operatorname{Cl}'(X_\Sigma) = 0$, we are done. □

(15.13) Let $s \geq 0$. A $\mathbb{Z}^s$-graded ring $R = \bigoplus_{\lambda \in \mathbb{Z}^s} R_\lambda$ is called a homogeneous DVR if R is a Krull domain with a unique graded maximal ideal (that is, a maximal element in the set of graded ideals which are not equal to R) P such that $\operatorname{ht} P = 1$. We say that (R, P) is a homogeneous DVR. When we set G to be the split torus $\mathbb{G}_m^s = \operatorname{Spec}\mathbb{Z}[t_1^{\pm 1}, \ldots, t_s^{\pm 1}]$, then (R, P) is G-local.

Lemma 15.14. *Let (R, P) be a $\mathbb{Z}^s$-graded homogeneous DVR. Then P is a principal ideal.*

Proof. As P is generated by homogeneous elements, we can take a homogeneous element $\alpha \in P \setminus P^{(2)}$, where $P^{(2)}$ is the second symbolic power of P. Then a minimal prime of α must be height-one homogeneous, and hence α generates P. □

(15.15) Let (R, P) be a $\mathbb{Z}^s$-graded homogeneous DVR. Set Q to be the localization of R by all the nonzero homogeneous elements. It is the G-total ring of quotients of R [Has7, (3.1)], as can be seen easily.

Lemma 15.16. *$Q = R[\alpha^{-1}]$ and $R = Q \cap R_P$.*

Proof. As $R[\alpha^{-1}]$ does not have a nonzero homogeneous prime ideal, any homogeneous element of $R[\alpha^{-1}]$ is a unit, and we cannot localize homogeneously any more, and so $Q = R[\alpha^{-1}]$ holds.

Obviously, $R \subset Q \cap R_P$. Let $\beta \in Q \cap R_P$. We can write $\beta = b/\alpha^n$ for some $n \geq 0$ and $b \in R$. Take n as small as possible, and assume that $n > 0$. Then $b = \beta\alpha^n \in R \cap PR_P = P$, and b is divisible by α. This is absurd. $\square$

(15.17) Let B be a $\mathbb{Z}^s$-graded Krull domain with the field of fractions K. We say that Λ is a defining family of homogeneous DVR's of B if each element $R \in \Lambda$ is a homogeneous DVR which is a graded subring of $Q_G(B)$, and $B = \bigcap_{R\in\Lambda} R$, where $Q_G(B)$ is the localization of B by all the nonzero homogeneous elements of B.

Lemma 15.18. *Let B be a $\mathbb{Z}^s$-graded Krull domain with the field of fractions K. Set $\Lambda_1 = \{B_{(P)} \mid P$ is homogeneous and of height one$\}$, where $B_{(P)}$ denotes the localization of B by the set of homogeneous elemets of $B \setminus P$. Then Λ_1 is a defining family of homogeneous DVR's of B. If Λ is a defining family of homogeneous DVR's of B, then $\Lambda \supset \Lambda_1$, and thus Λ_1 is the smallest defining family of homogeneous DVR's of B.*

Proof. It is easy to see that $(B_{(P)}, PB_{(P)})$ is a homogeneous DVR and is a graded subring of $Q_G(B) = B_{(0)}$ for a homogeneous height one prime P of B. We prove that $B = \bigcap_{R\in\Lambda_1} R = \bigcap_P B_{(P)}$. $B \subset \bigcap_P B_{(P)}$ is trivial. Let $a/b \in \bigcap_P B_{(P)}$, where $a \in B$ and b is a nonzero homogeneous element of B. As a minimal prime of Bb is homogeneous, $a/b \in B_Q$ for any height one inhomogeneous prime ideal Q. On the other hand, obviously, $a/b \in B_P$ for any P. Thus $a/b \in (\bigcap_P B_P) \cap (\bigcap_Q B_Q) = B$, since B is a Krull domain.

Next, let Λ be a defining family of homogeneous DVR's of B. For $R \in \Lambda$, let $\mathfrak{m}_R$ be the graded maximal ideal of R. Then

$$B = \bigcap_{R\in\Lambda} R = Q_G(B) \cap \bigcap_R R_{\mathfrak{m}_R}$$

by Lemma 15.16.

Let P be a homogeneous height-one prime ideal of B, and assume that $B_P \supset Q_G(B)$. Set $\mathfrak{P} = PB_P \cap Q_G(B)$. Then $\mathfrak{P} \cap B = PB_P \cap B = P$, and hence $\mathfrak{P} = PQ_G(B)$ contains 1, and this is a contradiction. So B_P does not contain $Q_G(B)$. When we express $Q_G(B) = \bigcap_{R'\in\Lambda'} R'$, where Λ' is a set of

DVR's whose field of quotients are K, then $B_P \in \Lambda' \cup \{R_{\mathfrak{m}_R} \mid R \in \Lambda\}$ by [Mat, (12.3)]. As we know that $B_P \notin \Lambda'$, $B_P = R_{\mathfrak{m}_R}$ for some $R \in \Lambda$, and hence $B_{(P)} = Q_G(B) \cap B_P = Q_G(B) \cap R_{\mathfrak{m}_R} = R \in \Lambda$. Hence $\Lambda_1 \subset \Lambda$. $\square$

(15.19) Let Y be a locally Krull integral S-scheme which is quasi-compact and separated over S. Assume that Y has an h_Y-ample Cartier divisor D (that is, D is a Weil divisor on Y such that $\mathcal{O}_Y(D)$ is an h_Y-ample invertible sheaf), where $h_Y : Y \to S$ is the structure map. Set $\mathcal{A} = \mathcal{R}(Y; D) := (h_Y)_*(\bigoplus_{n\geq 0} \mathcal{O}_Y(nD)T^n)$, where T is a variable. By assumption, the canonical morphism $u : Y \to \bar{Y} := \underline{\mathrm{Proj}}\, \mathcal{A}$ is an open immersion [Stack, (27.24.14)]. We identify Y with the image $u(Y)$ of u, and regard Y as an open subset of $\bar{Y}$.

Lemma 15.20. *Y is large in $\bar{Y}$.*

Proof. The question is local on S, and hence we may assume that S is affine. The question is also local on $\bar{Y}$, so it suffices to show that for any $n > 0$ and $0 \neq s \in \mathcal{A}_n = \Gamma(Y, \mathcal{O}(nD))$, Y_s is large in $D_+(sT^n)$, where $Y_s = Y \setminus \mathrm{Supp}(\mathrm{Coker}(s : \mathcal{O} \to \mathcal{O}(nD)))$. Note that $D_+(sT^n)$ is affine with the coordinate ring

$$\{0\} \cup \{f \in K^\times \mid \mathrm{div}\, f + r(\mathrm{div}\, s + nD) \geq 0 \text{ for some } r\} = \Gamma(Y_s, \mathcal{O}_Y),$$

and the inclusion $Y_s \to D_+(sT^n)$ is the obvious map. Set $R = \Gamma(Y_s, \mathcal{O}_Y)$. As $R = \Gamma(Y_s, \mathcal{O}_Y)$ and Y_s is locally Krull integral, we have that $R = \bigcap_{W \in P^1(Y_s)} \mathcal{O}_{\bar{Y},W}$ and R is a Krull domain, where $P^1(?)$ dentoes the set of prime divisors. Then as R is a Krull domain and is the coordinate ring of $D_+(sT^n)$, $R = \bigcap_{W \in P^1(D_+(sT^n))} \mathcal{O}_{\bar{Y},W}$. By [Mat, (12.3)], each W in $P^1(D_+(st^n))$ must intersect Y_s. Namely, Y_s is large in $D_+(sT^n)$. $\square$

(15.21) Let Y and D be as above. Let $s \geq 1$. Set G to be the split torus of relative dimension s. That is, $G = \mathrm{Spec}\, \mathbb{Z}[t_1^{\pm 1}, \ldots, t_s^{\pm s}] \times_{\mathrm{Spec}\, \mathbb{Z}} S$. Let $D_1, \cdots, D_s$ be Weil divisors on Y, and assume that we can write $D = \sum_{i=1}^s \mu_i D_i$ for some $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{Z}^s$.

Set $\mathcal{D} = \bigoplus_{\lambda \in \mathbb{Z}^s} \mathcal{O}(D_\lambda)t^\lambda$, where for $\lambda = (\lambda_1, \ldots, \lambda_s) \in \mathbb{Z}^s$, $D_\lambda := \sum_{i=1}^s \lambda_i D_i$, and $t^\lambda = t_1^{\lambda_1} \cdots t_s^{\lambda_s}$. Note that $\mathcal{D}$ is a quasi-coherent subalgebra of the constant sheaf of algebra $K[t_1^{\pm 1}, \ldots, t_s^{\pm s}]$ over Y, where K is the rational function field of Y.

The (relative) multisection ring of $D_1, \cdots, D_s$ is defined to be

$$\mathcal{B} = \mathcal{R}(Y; D_1, \ldots, D_s) := (h_Y)_*(\mathcal{D}).$$

(15.22) We set $X = \underline{\operatorname{Spec}}_S \mathcal{B}$, and $Z = \underline{\operatorname{Spec}}_Y \mathcal{D}$. Note that $\mathcal{D}$ is $\mathbb{Z}^s$-graded, and hence the canonical map $\pi : Z \to Y$ is a G-invariant morphism. There is a canonical map $v : Z \to X$, since $\mathcal{B} = (h_Y)_*(\mathcal{D})$. It is a G-morphism.

Lemma 15.23. *There is a large open subset V of Y such that $D_i|_V$ is Cartier (that is, $\mathcal{O}_V(D_i|_V)$ is invertible) for $i = 1, \dots, s$.*

Proof. Let $Y = \bigcup_{j \in J} Y_j$ be an affine open covering with Y_j connected. Then by the proof of [Has9, (5.33)], for each j, we can take a large open subset $V_j \subset Y_j$ such that $D_i|_{V_j}$ is Cartier for $i = 1, \dots, s$. Now define $V := \bigcup_{j \in J} V_J$. $\square$

We fix such a V.

Proposition 15.24. *$v : Z \to X$ above is an open immersion. We identify Z by $v(Z)$ and regard Z as an open subscheme of X. Then $U := \pi^{-1}(V) \subset Z$ is large in X.*

Proof. The question is local on S, and we may assume that $S = \operatorname{Spec} R$ is affine and hence $B = \mathcal{B} = \bigoplus_\lambda B_\lambda$ is a graded R-algebra. Let $n > 0$, and $s \in \Gamma(Y, \mathcal{O}_Y(nD))$. Then the degree λ component $B[(st^{n\mu})^{-1}]_\lambda$ of the localization $B[(st^{n\mu})^{-1}]$ is

$$\bigcup_{r \geq 0} (st^{n\mu})^{-r} B_{\lambda + rn\mu} = (\bigcup_r \Gamma(Y, \mathcal{O}_Y(D_\lambda + r(\operatorname{div} s + nD))))t^\lambda = \Gamma(Y_s, \mathcal{O}_Y(D_\lambda)).$$

So if, moreover, Y_s is also affine, then v maps $\pi^{-1}(Y_s)$ isomorphically onto the open subset $X_{st^{n\mu}} = \operatorname{Spec} B[(st^{n\mu})^{-1}]$.

We can take a sufficiently divisible n and $s_1, \dots, s_m \in \Gamma(Y, \mathcal{O}_Y(nD))$ such that each Y_{s_i} is affine and $\bigcup_i Y_{s_i} = Y$. Hence v is an open immersion whose image is $X \setminus V(J)$, where

$$J = (s_1 t^{n\mu}, \dots, s_m t^{n\mu}) \subset B. \tag{32}$$

To prove that U is large, since V is large in Y, replacing Y by V (this does not changes X), we may assume that $V = Y$ (and $U = Z$). It suffices to prove that J has height at least two.

Before we finish the proof, we need some constructions.

(15.25) Let the notation be as above (S is affine). For each $W \in P^1(Y)$, define $P_W = \bigoplus_{\lambda\in\mathbb{Z}^s} \Gamma(Y, \mathcal{O}_Y(D_\lambda - W))t^\lambda$. This is a graded prime ideal of B. For $n > 0$ and $s \in \Gamma(Y, \mathcal{O}_Y(nD))$, the localization $B[(st^{n\mu})^{-1}] \otimes_B P_W$ is $\bigoplus_\lambda \Gamma(Y_s, \mathcal{O}_Y(D_\lambda - W))t^\lambda$. As affine Y_s with $Y_s \cap W \neq \emptyset$ forms a fundamental set of open neighborhoods of the generic point w of W, we have

$$(33)\quad B_{\mathcal{S}} = \bigoplus_\lambda (\{0\} \cup \{f \in K^\times \mid \operatorname{div}(f) + D_\lambda \geq_{P^1(Y)\setminus\{W\}} 0\}) \cdot t^\lambda$$
$$= \mathcal{O}_{Y,W}[(\alpha^{-c_1}t_1)^{\pm 1}, \dots, (\alpha^{-c_s}t_s)^{\pm 1}]$$

and

$$(34)\quad (P_W)_{\mathcal{S}} = \bigoplus_\lambda (\{0\} \cup \{f \in K^\times \mid \operatorname{div}(f) + D_\lambda >_{P^1(Y)\setminus\{W\}} 0\}) \cdot t^\lambda = \alpha B_{\mathcal{S}},$$

where $\mathcal{S}$ is the homogeneous multiplicatively closed subset of B given by

$$\mathcal{S} = \{st^{n\mu} \mid n > 0,\ s \in \Gamma(Y, \mathcal{O}_X(nD)),\ Y_s \text{ is affine},\ Y_s \cap W \neq \emptyset\} \cup \{1\},$$

α is the generator of the maximal ideal of $\mathcal{O}_{Y,W}$, and c_i is the coefficient of W in the divisor D_i. So $B_{\mathcal{S}}$ is a homogeneous DVR with the graded maximal ideal $(P_W)_{\mathcal{S}}$. In particular, P_W is a homogeneous height one prime of B, and $B_{\mathcal{S}}$ is the homogeneous localization $B_{(P_W)}$.

Lemma 15.26. *The map $W \mapsto P_W$ gives a bijection between $P^1(Y)$ and the set $HP^1(B)$ of homogeneous height one prime ideals of B. We have $B = \bigcap_{W\in P^1(Y)} B_{(P_W)}$. B is a Krull domain.*

Proof. By the first equality of (33), it is easy to see that $B = \bigcap_{W\in P^1(Y)} B_{(P_W)}$. As each $B_{(P_W)}$ is a homogeneous DVR, B is (graded) Krull. We know that $P_W \in HP^1(B)$ for $P^1(Y)$. As Y is a separated scheme, the description of (34) shows that $W \mapsto P_W$ is injective. By Lemma 15.18 and the fact $B = \bigcap_{W\in P^1(Y)} B_{(P_W)}$ (with $Q(B) = Q(B_{(P_W)}) = K(t_1^{\pm 1}, \dots, t_s^{\pm 1})$), $W \mapsto P_W$ is surjective. $\square$

The following lemma finishes the proof of Proposition 15.24.

Lemma 15.27. *Let J be the ideal of B defined in* (32)*. Then the height of J is at least two.*

Proof. Assume the contrary. As J is graded, J is contained in some P_W. Since $Y = \bigcup_i Y_{s_i}$, there is some i such that Y_{s_i} intersects W. Then $s_i t^{n\mu} \in J$ cannot be an element of P_W by the definition of P_W. A contradiction. $\square$

Now Proposition 15.24 has been proved. $\square$

We have a diagram of S-schemes

$$X \xleftarrow{i} U \xrightarrow{\rho} V \xhookrightarrow{j} Y \ , \tag{35}$$

where $X = \underline{\mathrm{Spec}}_S\,\mathcal{B}$, V is a large open subset of Y such that $D_l|_V$ is Cartier for $l = 1, \ldots, s$, $\pi : Z \to Y$ and $v : Z \to X$ are the canonical maps, $U = \pi^{-1}(V)$, $\rho : U = \pi^{-1}(V) \to V$ the restriction of $\pi : Z \to Y$, $i : U \to X$ the composite $U \hookrightarrow Z \xrightarrow{v} X$, and $j : V \to Y$ the inclusion.

Theorem 15.28. *Let S be a scheme, $h_Y : Y \to S$ an integral locally Krull S-scheme with an ample Cartier divisor D. Let G be the s-torus*

$$\mathrm{Spec}\,\mathbb{Z}[t_1^{\pm 1}, \ldots, t_s^{\pm 1}] \times_{\mathrm{Spec}\,\mathbb{Z}} S$$

over S. Let $s \geq 1$, and $D_1, \ldots, D_s$ divisors on Y such that $D \in \sum_i \mathbb{Z} D_i$. Let the diagram (35) *be constructed as above. Then it is a rational almost principal G-bundle. X is a locally Krull scheme.*

Proof. By construction, $\pi : Z \to Y$ is G-invariant, and $v : Z \to X$ is a G-morphism. V is large in Y by construction. As $U \hookrightarrow Z$ is the base change of $V \hookrightarrow Y$, U is a G-stable open subset of Z. So the diagram is a diagram of G-schemes, and G acts on Y and V trivially.

i is an open immersion and $i(U)$ in X is large by Proposition 15.24.

The fact that ρ is a principal G-bundle follows from Lemma 15.7 easily.

To prove the last assertion, we may assume that S is affine, and this case is done in Lemma 15.26. $\square$

Corollary 15.29. *Let the notation be as above. Then $\gamma = i_* \rho^* j^* : \mathrm{Ref}(Y) \to \mathrm{Ref}(G, X)$ is an equivalence. The quasi-inverse is given by $\delta = (?)^G j_* \rho_* i^* : \mathrm{Ref}(G, X) \to \mathrm{Ref}(Y)$. For a divisor E on Y, $\mathcal{O}_Y(E)$ corresponds to*

$$(h_Y)_*\Big(\bigoplus_{\lambda \in \mathbb{Z}^s} \mathcal{O}_Y(D_\lambda + E) t^\lambda\Big).$$

In particular, $\mathcal{O}_Y(D_\nu)$ corresponds to $\mathcal{O}_X(\nu)$, where $?(\nu)$ denotes the shift of degree. γ is equivalent to $v_*(?)^{**}\pi^*$, and δ is equivalent to $(?)^G \pi_* v^*$, where $(?)^{**}$ denotes the double dual.

Proof. Follows easily from Theorem 15.28 and Theorem 11.2. $\square$

Lemma 15.30. *Let the notation be as above, and let $\mathcal{N} \in \mathrm{Ref}(Y)$ corresponds to $\mathcal{M} \in \mathrm{Ref}(G, X)$. Namely, set*

$$\mathcal{M} = \bigoplus_{\lambda \in \mathbb{Z}^s} (h_Y)_*(\mathcal{N}(D_\lambda)))t^\lambda.$$

Then the G-local cohomology $\underline{H}^i_{X \setminus Z}(\mathcal{M})$ is zero for $i = 0, 1$, and

$$\underline{H}^i_{X \setminus Z}(\mathcal{M}) \cong \bigoplus_{\lambda \in \mathbb{Z}^s} (R^{i-1} h_Y)_*(\mathcal{N}(D_\lambda))t^\lambda$$

for $i \geq 2$, where $\mathcal{N}(D_\lambda)$ denotes the reflexive sheaf $(\mathcal{N} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(D_\lambda))^{**}$.

Proof. From [HasO, (4.10)], the sequence

$$0 \to \underline{H}^0_{X \setminus Z}(\mathcal{M}) \to \mathcal{M} \xrightarrow{u} v_* v^* \mathcal{M} \to \underline{H}^1_{X \setminus Z}(\mathcal{M}) \to 0$$

is exact, and $R^{i-1} v_*(v^* \mathcal{M}) \cong \underline{H}^i_{X \setminus Z}(\mathcal{M})$ for $i \geq 2$. As $\mathcal{M}$ is reflexive and Z is large in X, $u : \mathcal{M} \to v_* v^* \mathcal{M}$ is an isomorphism. The result follows. $\square$

(15.31) Let R be a commutative ring, M a finitely generated abelian group, and $G := \operatorname{Spec} RM$, where $RM = \bigoplus_{m \in M} Rt^m$ is a group algebra of M over R. Letting each t^m group-like, G is an R-group scheme. A G-module is nothing but an RM-comodule. If V is a G-module, then $V = \bigoplus_{m \in M} V_m$ as a G-module, where

$$V_m = \{v \in V \mid \omega_V(v) = v \otimes t^m\}. \tag{36}$$

Conversely, if $V = \bigoplus_{m \in M} V_m$ as a graded R-module, then letting (36) the definition, V is a G-module, and a G-module and an RM-comodule and an M-graded R-module are the same thing.

So a G-algebra is an M-graded R-algebra $B = \bigoplus_m B_m$, where $\omega_B(b) = b \otimes t^m$ for $b \in B_m$. We follow the convention that if G acts on an affine R-scheme $X = \operatorname{Spec} B$, then B is a G-module by $(gb)(x) = b(g^{-1}x)$. That is, $\alpha(b) = t^{-m} \otimes b$ for $b \in B_m$ (since the antipode of RM sends t^m to t^{-m}), where $\alpha : B \to RM \otimes_R B$ is the map corresponding to the action $G \times X \to X$.

For a (G, B)-module N, the G-linearization

$$\phi : (RM \otimes_R B)_\alpha \otimes_B N \to (RM \otimes_R B)_\beta \otimes_B N$$

maps $(1 \otimes 1) \otimes n$ to $(t^{-m} \otimes 1) \otimes n$ for $n \in N_m$, where β is given by $\beta(b) = 1 \otimes b$, and corresponds to the second projection $p_2 : G \times X \to X$.

Proposition 15.32 (cf. [EKW, (1.1), (3)]). *In* Theorem 15.28*, assume that S is quasi-compact quasi-separated. Then we have an exact sequence*

$$0 \to \mathcal{X}(G,X) \xrightarrow{\alpha} \mathcal{X}(G) \xrightarrow{\beta} \mathrm{Cl}(Y) \xrightarrow{\gamma} \mathrm{Cl}(X) \to 0,$$

where $\mathcal{X}(G) = \mathbb{Z}^s$, $\mathcal{X}(G,X) = \{\lambda \in \mathcal{X}(G) \mid B^\times \cap B_\lambda \neq \emptyset\}$ (where $B = \Gamma(X, \mathcal{O}_X)$), α is the inclusion, $\beta(\varepsilon_i) = \mathcal{O}_Y(D_i)$ ($\varepsilon_i = (0,\ldots,0,1,0,\ldots,0)$ with 1 at the ith place), and $\gamma = (i^*)^{-1}\rho^* j^*$.

Proof. The map $j^* : \mathrm{Cl}(Y) \to \mathrm{Cl}(V)$ is an isomorphism. The map $\rho^* : \mathrm{Cl}(V) \to \mathrm{Cl}(U)$ is surjective by Lemma 15.12. As the map $i^* : \mathrm{Cl}(X) \to \mathrm{Cl}(U)$ is an isomorphism, we have that $\gamma = (i^*)^{-1}\rho^* j^* : \mathrm{Cl}(Y) \to \mathrm{Cl}(X)$ is surjective.

Note that $\gamma(\mathcal{O}_Y(D_\lambda)) = \mathcal{O}_X(\lambda)$ in $\mathrm{Cl}(G,X)$ by Corollary 15.29. In particular, it is zero in $\mathrm{Cl}(X)$.

Note that the map $\gamma : \mathrm{Cl}(Y) \to \mathrm{Cl}(G,X)$ is an isomorphism by Theorem 11.2, and $\gamma(\mathcal{O}_Y(D_i)) = t_i^{-1}\mathcal{O}_X = \mathcal{O}_X(\varepsilon_i)$ in $\mathrm{Cl}(G,X)$.

The kernel of the forgetful map $r : \mathrm{Cl}(G,X) \to \mathrm{Cl}(X)$ is the algebraic first G-cohomology group $H^1_{\mathrm{alg}}(G, \mathcal{O}_X^\times)$, see [Dol, Theorem 7.1] and Theorem 11.5. It is explained as follows. An element of $\operatorname{Ker} r$ is the isomorphism class of a rank-one free module $\mathcal{O}_X$ equipped with a G-linearization $\Phi : a^*\mathcal{O}_X \to p_2^*\mathcal{O}_X$. However, both $a^*\mathcal{O}_X$ and $p_2^*\mathcal{O}_X$ are identified with $\mathcal{O}_{G\times X}$, and Φ is nothing but a unit element of the ring $C = \Gamma(G \times X, \mathcal{O}_{G\times X}) = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_s^{\pm 1}] \otimes_{\mathbb{Z}} B$ (by the projection formula [Lip, (3.9.4)]). As B is a domain, we can write $\Phi = t^\lambda \otimes b$ with $b \in B^\times$ and $\lambda \in \mathbb{Z}^s$. From the cocycle condition on Φ, we have that $\Phi = t^\lambda \otimes 1$. Conversely, $t^\lambda \otimes 1$ satisfies the cocycle condition, and the group of 1-cocycles $Z^1_{\mathrm{alg}}(G, \mathcal{O}_X^\times)$ is the character group $\mathcal{X}(G)$. By definition,

$$B^1_{\mathrm{alg}}(G, \mathcal{O}_X^\times) = \{\phi(gx)/\phi(x) \mid \phi \in B^\times\} \subset Z^1_{\mathrm{alg}}(G, \mathcal{O}_X^\times).$$

As ϕ is a homogeneous element, it has a degree, say λ. Then $\phi(gx)/\phi(x) = t^\lambda \otimes 1$, and thus $B^1_{\mathrm{alg}}(G, \mathcal{O}_X^\times) = \mathcal{X}(G,X)$.

Then as in (15.31), the linearization of $t_i^{-1}B$ corresponds to $t_i \otimes 1 \in Z^1_{\mathrm{alg}}(G, \mathcal{O}_X^\times)$, and the exact sequence has been proved. $\square$

Proposition 15.33 (cf. [HasK, (1.2), (1.3)]). *Let the notation be as in* Theorem 15.28*. Let S be Noetherian with a fixed dualizing complex $\mathbb{I}_S$, and assume that Y and X are of finite type over S. Then*

$$\omega_X \cong \bigoplus_{\lambda \in \mathbb{Z}^s} (h_Y)_* \omega_Y(D_\lambda) t^\lambda.$$

$\omega_Y \cong \mathcal{O}_Y(D_\lambda)$ *as* $\mathcal{O}_Y$*-modules if and only if* $\omega_X \cong \mathcal{O}_X(\lambda)$ *as* $(G, \mathcal{O}_X)$*-modules.*

Proof. As $G_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[t_1^{\pm 1}, \ldots, t_s^{\pm 1}]$ is a $\mathbb{Z}$-smooth abelian group, $\operatorname{Lie}(G_{\mathbb{Z}})$ is trivial, and hence $\Theta_S = h_S^*(\bigwedge^s \operatorname{Lie}(G_{\mathbb{Z}}))$ is trivial, where $h_S : S \to \operatorname{Spec} \mathbb{Z}$ is the structure map. Now the result follows from Theorem 11.18 immediately. $\square$

(15.34) Let Λ be the abelian group $\mathbb{Z}^s$, and Γ its subgroup. Let H be the torus $\operatorname{Spec} \mathbb{Z}\Gamma \times_{\operatorname{Spec} \mathbb{Z}} S$. The inclusion of group rings $\mathbb{Z}\Gamma \hookrightarrow \mathbb{Z}\Lambda$ induces an fppf homomorphism of S-group schemes $f : G \to H$. We set $N = \operatorname{Ker} f = \operatorname{Spec} \mathbb{Z}(\Lambda/\Gamma) \times_{\operatorname{Spec} \mathbb{Z}} S$.

Let Y be an S-scheme on which G acts trivially, and $\mathcal{M} = \bigoplus_{\lambda \in \Lambda} \mathcal{M}_\lambda$ a $(G, \mathcal{O}_Y)$-module. Then $\mathcal{M}^N = \bigoplus_{\lambda \in \Gamma} \mathcal{M}_\lambda$ is nothing but the Veronese submodule of $\mathcal{M}$.

(15.35) Let the assumptions be as in Theorem 15.28. Let M, Γ, $f : G \to H$, and N be as above. Set

$$X' = \underline{\operatorname{Spec}}_S \mathcal{B}^N = \underline{\operatorname{Spec}}_S (h_Y)_* (\bigoplus_{\lambda \in \Gamma} \mathcal{O}(D_\lambda) t^\lambda).$$

If $\lambda_1, \ldots, \lambda_{s'}$ is a $\mathbb{Z}$-basis of Γ and when we set $D'_l := D_{\lambda_l}$, then we have

$$\mathcal{B}^N = \mathcal{R}(Y; D'_1, \ldots, D'_{s'}) = (h_Y)_* (\bigoplus_{\alpha \in \mathbb{Z}^{s'}} \mathcal{O}(\alpha_1 D'_1 + \cdots + \alpha_{s'} D'_{s'}) t^{\sum_i \alpha_i \lambda_i}).$$

The schemes and morphisms constructed from the divisors $D'_1, \ldots, D'_{s'}$ instead of $D_1, \ldots, D_s$ are denoted by $\pi' : Z' \to Y$ and $\rho' : U' = (\pi')^{-1}(V) \to V$. Thus $Z' = \underline{\operatorname{Spec}}_Y \mathcal{D}^N$. Let $\tau : Z \to Z'$ be the map induced by the map of $\mathcal{O}_Y$-algebras $\mathcal{D}^N \hookrightarrow \mathcal{D}$. It is an algebraic quotient by N. Note that $\pi'\tau = \pi$ and $\tau^{-1}(U') = U$. Let $\upsilon : U \to U'$ be the restriction of τ. Let $\theta : X \to X'$ be the map corresponding to the map of $\mathcal{O}_S$-algebras $\mathcal{B}^N \hookrightarrow \mathcal{B}$. Thus we get the commutative diagram

$$\begin{tikzcd}
X \arrow[d, "\theta"] & U \arrow[l, hook', "i"'] \arrow[r, "\rho"] \arrow[d, "\upsilon"] & V \arrow[r, hook, "j"] \arrow[d, "\mathrm{id}_V"] & Y \arrow[d, "\mathrm{id}_Y"] \\
X' & U' \arrow[l, hook', "i'"'] \arrow[r, "\rho'"] & V \arrow[r, hook, "j"] & Y
\end{tikzcd}$$

whose first and second rows are rational almost principal G- and H-bundles, respectively.

Lemma 15.36. *Let the notation be as above. Then $\theta : X \to X'$ is a G-enriched almost principal N-bundle with respect to U and U'.*

Proof. It suffices to show that $\upsilon : U \to U'$ is a principal N-bundle. Let $p : \Lambda \to \Lambda/\Gamma$ be the canonical projection. Let us write $\rho_* \mathcal{O}_U = \bigoplus_{\lambda \in \Lambda} \mathcal{A}_\lambda$. Then the N-action on $\rho_* \mathcal{O}_U$ is given by the grading $\rho_* \mathcal{O}_U = \bigoplus_{\bar{\lambda} \in \Lambda/\Gamma} \mathcal{A}_{\bar{\lambda}}$, where $\mathcal{A}_{\bar{\lambda}} = \bigoplus_{\lambda \in p^{-1}(\bar{\lambda})} \mathcal{A}_\lambda$. Then $\rho'_* \mathcal{O}_{U'} = (\rho_* \mathcal{O}_U)^N \to \mathcal{A}_{\bar{0}}$ is an isomorphism, and $\mathcal{A}_{\bar{\lambda}} \otimes_{\mathcal{A}_{\bar{0}}} \mathcal{A}_{\bar{\mu}} \to \mathcal{A}_{\bar{\lambda}+\bar{\mu}}$ is surjective for $\bar{\lambda}, \bar{\mu} \in \Lambda/\Gamma$. By Lemma 15.7, we have that υ is a principal N-bundle. □

Lemma 15.37. *Let the notation be as above. Then there is an exact sequence*

$$0 \to \mathcal{X}(N, X) \xrightarrow{\bar{\alpha}} \mathcal{X}(N) \xrightarrow{\bar{\beta}} \mathrm{Cl}(X') \xrightarrow{\bar{\gamma}} \mathrm{Cl}(X) \to 0,$$

where $\mathcal{X}(N) = \Lambda/\Gamma$, $\mathcal{X}(N, X) = \{\bar{\lambda} \in \mathcal{X}(N) \mid B^\times \cap B_{\bar{\lambda}} \neq \emptyset\}$, $\bar{\alpha}$ the inclusion, $\bar{\beta}(\bar{\lambda}) = \gamma'(\mathcal{O}_Y(D_\lambda))$ (where $p(\lambda) = \bar{\lambda}$, and this definition is independent of the choice of $\lambda \in p^{-1}(\bar{\lambda})$), and $\bar{\gamma}(\mathcal{M}) = (\theta^* \mathcal{M})^{**}$, where $B = \Gamma(X, \mathcal{O}_X)$.

Proof. Using Lemma 15.36, we may repeat the proof of Proposition 15.32. Here we give a proof which use the result of Proposition 15.32. As B is a domain, a unit of B is homogeneous. It is easy to see that $\mathcal{X}(H, X') = \mathcal{X}(H, X)$. As

$$0 \to \mathcal{X}(H) \to \mathcal{X}(G) \to \mathcal{X}(N) \to 0$$

and

$$0 \to \mathcal{X}(H, X') \to \mathcal{X}(G, X) \to \mathcal{X}(N, X) \to 0$$

are exact, we have that the sequence

$$0 \to \mathcal{X}(H)/\mathcal{X}(H, X') \to \mathcal{X}(G)/\mathcal{X}(G, X) \to \mathcal{X}(N)/\mathcal{X}(N, X) \to 0$$

is exact. Now the result follows from the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{X}(G)/\mathcal{X}(G, X) \arrow[r, "\alpha"] & \mathrm{Cl}(Y) \arrow[r, "\gamma"] & \mathrm{Cl}(X) \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{X}(H)/\mathcal{X}(H, X) \arrow[r, "\alpha'"] \arrow[u] & \mathrm{Cl}(Y) \arrow[r, "\gamma'"] \arrow[u, "\mathrm{id}"] & \mathrm{Cl}(X') \arrow[r] \arrow[u, "\bar{\gamma}"] & 0
\end{tikzcd}$$

and the snake lemma. □

Lemma 15.38. *Let the notation be as above. Let S be Noetherian with a fixed dualizing complex $\mathbb{I}_S$, and assume that Y and X are of finite type. Then we have*

$$\omega_{X'} \cong (\theta_* \omega_X)^N$$

as $(H, \mathcal{O}_{X'})$-modules.

Proof. Let $\Gamma' \subset \Lambda$ be the subgroup such that $\Gamma \subset \Gamma'$, Γ'/Γ is a torsion module, and Λ/Γ' is torsion-free. Then comparing X and $X'' = \operatorname{Spec} B_{\Gamma'}$ and then X'' and X', we may assume either that N is a torus or finite. If N is a torus, we may use Corollary 11.22, **1** (note that $\Theta_{N,S}$ is trivial in both cases). □

16. The Cox rings of toric varieties

We give an example of toric varieties.

Let $M = \mathbb{Z}^n$ be the free $\mathbb{Z}$-module of rank n. Let k be a field, and Y a toric variety determined by a fan Δ in $M^* = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ [Ful]. Let H be the torus $\operatorname{Spec} kM$, where kM is the group algebra of M (with each element of M group-like). Let $\Delta(1)$ be the set of one-dimensional faces of Δ. Note that $\Delta(1)$ is in one-to-one correspondence with the set of H-stable prime divisors. For each $\sigma \in \Delta(1)$, m_σ^* denotes the generator of $\sigma \cap M^*$. Let $W = \bigoplus_{\sigma \in \Delta(1)} \mathbb{Z} D_\sigma$ be the $\mathbb{Z}$-free module with the basis $\Delta(1)$ so that W is the group of H-stable divisors, where D_σ is the G-stable prime divisor corresponding to σ. An element of M is a rational function on Y, and $\operatorname{div} m = \sum_\sigma \langle m, m_\sigma^* \rangle D_\sigma \in W$. We assume that the map $\operatorname{div} : M \to W$ is injective, and $\operatorname{Cl}(Y) = W/M$ is torsion-free. This is equivalent to say that $\{m_\sigma^* \mid \sigma \in \Delta(1)\}$ generates M^*. We set $G = \operatorname{Spec} kW$. The inclusion $\operatorname{div} : M \hookrightarrow W$ induces a surjective map $f : G \to H$ with $N := \operatorname{Ker} f = \operatorname{Spec} k \operatorname{Cl}(Y)$. Let $B = k[x_\sigma \mid \sigma \in \Delta(1)]$ be the Cox ring of Y [Cox], where x_σ are variables, and B is a polynomial ring. Letting each x_σ of degree σ, B is W-graded, and hence is a G-algebra. We set $X = \operatorname{Spec} B$. We choose $\sigma_1, \ldots, \sigma_s \in \Delta(1)$ so that $[D_1], \ldots, [D_s]$ forms a $\mathbb{Z}$-basis of $\operatorname{Cl}(Y)$, where $D_i = D_{\sigma_i}$. This gives a splitting $\operatorname{Cl}(Y) \to W$ (given by $[D_i] \mapsto D_i$) and the direct docompositions $W = M \oplus \operatorname{Cl}(Y)$ and $G = N \times H$. Then by [Cox, (1.1)], B is identified with

$$R(Y; D_1, \ldots, D_s) = \bigoplus_{\lambda \in \mathbb{Z}^s} \Gamma(Y, \mathcal{O}_Y(D_\lambda)) t^\lambda.$$

Note that $mt^\lambda \in R(Y; D_1, \dots, D_s)_\lambda$ corresponds to $x^{\operatorname{div} m + D_\lambda}$ for $m \in M$ and $\lambda \in \mathbb{Z}^s$, and thus this identification $B = R(Y; D_1, \dots, D_s)$ respects the W-grading.

We set $V = Y_{\mathrm{reg}}$. This particular choice is consistent with our main discussion in section 15. That is, V is a large open subset of Y such that $D_i|_V$ is Cartier for each i. Not only that, V is an H-open subset. Obviously, $\pi : Z \to V$ is a G-morphism which is N-invariant. So we have

Proposition 16.1. *Let the notation be as above. Assume that Y is quasi-projective. Then* (35) *is a G-enriched rational almost principal N-bundle.*

Proof. We have already seen that the diagram is that of G-schemes. As Y is quasi-projective, it has an ample Cartier divisor D. D lies in $\sum_i \mathbb{Z} D_i$, because this group is the whole $\mathrm{Cl}(Y)$. By Theorem 15.28, the assertion follows. $\square$

Corollary 16.2 (cf. [Ful, (4.3), Proposition], [Stan, (I.13.1)])**.** *Let $M = \mathbb{Z}^n$, and $H = \operatorname{Spec} kM$. Let Y be a toric variety over a field k defined by a fan Δ in M^*. Then the H-canonical module ω_Y of Y is isomorphic to $\mathcal{O}_Y(-\sum_{\sigma \in \Delta(1)} D_\sigma)$.*

Proof. Assume that Δ is not complete. Then extending Δ outside, Y is an H-open subscheme of a complete toric variety $\bar{Y}$ determined by $\bar{\Delta}$. If $\omega_{\bar{Y}} \cong \mathcal{O}_{\bar{Y}}(\sum_{\sigma \in \bar{\Delta}(1)} D_\sigma)$, then restricting to Y, we get

$$\omega_Y \cong \mathcal{O}_{\bar{Y}}(-\sum_{\sigma \in \bar{\Delta}(1)} \bar{D}_\sigma)|_Y = \mathcal{O}_Y((-\sum_{\sigma \in \bar{\Delta}(1)} \bar{D}_\sigma)|_Y) = \mathcal{O}_Y(-\sum_{\sigma \in \Delta(1)} D_\sigma).$$

Hence we may assume that Y is complete.

Next, subdividing Δ if necessary, there is an H-equivariant birational map between complete toric varieties $g : Y' \to Y$ such that the class group is torsion-free and Y' is projective [Oda, (2.17)]. Let W be the H-stable open subvariety of Y obtained by removing the union of all the H-stable closed subvarieties of codimension grater than or equal to two (W is the toric variety corresponding to the one skelton of Δ). It is easy to see that $g|_{g^{-1}(W)} : g^{-1}(W) \to W$ is an isomorphism ($g^{-1}(W)$ also corresponds to the one-skelton of Δ). If the corollary is true for Y', then it is true also for its open subset $g^{-1}(W)$ as above, and then the assertion is also true for Y, because W is large in Y.

Thus we are in the situation of Proposition 16.1. Then as in Proposition 15.33, it suffices to show that $\omega_X \cong \mathcal{O}_X(-\sum_{\sigma\in\Delta(1)}\sigma)$. But this is trivial, since B is a polynomial ring with the variables x_σ. $\square$

Lemma 16.3. *Let $S = \operatorname{Spec} k$ with k a field, and G be an affine S-group scheme. Let $(B, \mathfrak{m})$ be a G-local G-algebra such that $k \to B/\mathfrak{m}$ is bijective. Let F be a B-finite B-projective (G, B)-module such that $F/\mathfrak{m}F$ is a projective G-module. Then F is a projective (G, B)-module, and $F \cong B\otimes_k(F/\mathfrak{m}F)$.*

Proof. By assumption, the canonical map $\pi : F \to F/\mathfrak{m}F$ has a G-linear splitting $i : F/\mathfrak{m}F \to F$. We define $\nu : B \otimes_k (F/\mathfrak{m}F) \to F$ by $\nu(b \otimes \alpha) = b \cdot i(\alpha)$. This is (G, B)-linear, and is surjective by G-Nakayama's lemma, Lemma 14.17. As F is assumed to be B-projective, ν has a B-linear splitting. So $K = \operatorname{Ker} \nu$ is a B-finite (G, B)-module. As can be seen easily, we have $K/\mathfrak{m}K = 0$, and hence $K = 0$ by G-Nakayama's lemma again.

For a (G, B)-module M, we have

$$\operatorname{Hom}_{G,B}(B \otimes_k (F/\mathfrak{m}F), M) \cong \operatorname{Hom}_G(F/\mathfrak{m}F, M).$$

So $\operatorname{Hom}_{G,B}(B \otimes_k (F/\mathfrak{m}F), ?)$ is an exact functor, and $F \cong B \otimes_k (F/\mathfrak{m}F)$ is (G, B)-projective. $\square$

Proposition 16.4 (cf. [Tho, Theorem 1], [Bru2, section 3])**.** *Let the notation be as in* Corollary 16.2. *Assume that k is a perfect field of characteristic $p > 0$. Then Y is of graded finite F-representation type by some rank-one reflexive sheaves $\mathcal{M}_1, \dots, \mathcal{M}_u$ on Y, with respect to the action of H.*

Proof. As in the proof of Corollary 16.2, we may assume that Y is complete. As before, let $g : Y' \to Y$ be a birational map between complete toric varieties such that Y' is projective and $\operatorname{Cl}(Y')$ is torsion-free. Assume that Y' is of graded finite F-representation type by rank-one reflexive $({}^{e_0}H, \mathcal{O}_{Y'})$-modules $\mathcal{M}_1, \dots, \mathcal{M}_u$. Then for each $e \geq 1$, we can write $F_*^e(\mathcal{O}_{{}^eY'}) = \bigoplus_j \mathcal{N}_j$ as $({}^eH, \mathcal{O}_{Y'})$-modules such that for each j, there exists some $l(j)$ such that $\mathcal{N}_j \cong \mathcal{M}_{l(j)}$ as $\mathcal{O}_{Y'}$-modules. Then $F_*^e(\mathcal{O}_{{}^eY}) \cong \bigoplus_j g_*\mathcal{N}_j$ as $({}^eH, \mathcal{O}_Y)$-modules, since $g_*\mathcal{O}_{Y'} = \mathcal{O}_Y$. So each $g_*\mathcal{N}_j$ is rank-one reflexive. Moreover, $g_*\mathcal{N}_j \cong g_*\mathcal{M}_{l(j)}$ as $\mathcal{O}_Y$-modules. So wasting $\mathcal{M}_l$ which does not appear in the expression at all, if any, we have that $g_*\mathcal{M}_1, \dots, g_*\mathcal{M}_u$ are rank-one reflexive $({}^{e_0}H, \mathcal{O}_Y)$-modules, and Y is of graded finite F-representation type by $g_*\mathcal{M}_1, \dots, g_*\mathcal{M}_u$. Hence we may replace Y by Y', and we are in the situation of Proposition 16.1.

By Corollary 12.8, it suffices to show that there exist some $e_0 \geq 0$ and finitely many rank-one B-free $({}^{e_0}H \times N, B)$-modules such that $({}^eB)^{eN_e}$ is a direct sum of copies of these modules as (N, B)-modules, where B is the Cox ring of Y.

By Lemma 16.3, we have that ${}^eB \cong B \otimes ({}^eB/\mathfrak{m}^e B) \cong B \otimes {}^e(B/\mathfrak{m}^{[p^e]})$ as $({}^eG, B)$-modules, where $\mathfrak{m}^{[p^e]} = \mathfrak{m}^{(e)}B$. We identify a ${}^eG$-module with a $p^{-e}W$-graded k-vector space. Then we have that ${}^e(B/\mathfrak{m}^{[p^e]})$ is the sum of one-dimensional representations

$$ {}^e(B/\mathfrak{m}^{[p^e]}) = \bigoplus_{(\alpha_\sigma)\in \operatorname{Map}(\Delta(1),[0,1)\cap p^{-e}\mathbb{Z})} k \cdot x^{\sum_\sigma \alpha_\sigma \sigma} \cong \bigoplus_{(\alpha_\sigma)} k(-\sum_\sigma \alpha_\sigma D_\sigma). $$

So

$$ {}^eB \cong \bigoplus_{(\alpha_\sigma)\in \operatorname{Map}(\Delta(1),[0,1)\cap p^{-e}\mathbb{Z})} B(-\sum_\sigma \alpha_\sigma D_\sigma). $$

Hence $({}^eB)^{eN_e} \cong \bigoplus_{(\alpha_\sigma)} B(-\sum_\sigma \alpha_\sigma D_\sigma)$, where the sum is taken over $(\alpha_\sigma) \in \operatorname{Map}(\Delta(1), [0,1) \cap p^{-e}\mathbb{Z})$ such that $-\sum_\sigma \alpha_\sigma [D_\sigma] \in p^{-e}\operatorname{Cl}(Y)$ lies in $\operatorname{Cl}(Y)$. Let $\pi : \mathbb{R}^{n+s} = \operatorname{Map}(\Delta(1), \mathbb{R}) \to \operatorname{Cl}(Y)_\mathbb{R}$ be the map given by $\pi(\alpha_\sigma) = \sum_\sigma \alpha_\sigma [D_\sigma]$. Then $\pi([0,1]^{n+s}) \cap \operatorname{Cl}(Y)$ is compact and discrete, and hence is finite. So we can find some e_0 and rank-one free summands $M_1, \ldots, M_u$ of ${}^eB^{eN_e}$ for $e \leq e_0$ (e may vary) such that any other rank-one free summand of ${}^eB^{eN_e}$ for any e is (N, B)-isomorphic to some M_l. This is what we wanted to prove. □

The following is well-known.

Proposition 16.5. *Let the notation be as in* Corollary 16.2. *Assume that k is a perfect field of characteristic $p > 0$. Then Y is globally F-regular.*

Proof. By Lemma 13.11 and Corollary 13.8, we may assume that Y is projective and the class group of Y is torsion-free. As the torus $N = \operatorname{Spec} k \operatorname{Cl}(Y)$ is smooth linearly reductive and the polynomial ring B is strongly F-regular, Y is globaly F-regular by Theorem 13.14 and Proposition 16.1. □

Corollary 16.6. *An affine normal semigroup ring over a field of characteristic $p > 0$ is strongly F-regular. In particular, it is Cohen–Macaulay.*

Proof. Let A be an affine normal semigroup ring over k. By [Has6, (3.17)], we may assume that k is algebraically closed. Then by Proposition 16.5, the associated affine toric variety $\operatorname{Spec} A$ is globally F-regular. That is, A is strongly F-regular. □

17. Surjectively graded rings

(17.1) As we have seen in the last section, we can construct a rational almost principal bundle from a multisection ring over a normal quasi-projective variety over a field. However, given a finitely generated multigraded algebra B over a field k, it seems that it is not so easy to tell if B is a multisection ring. But this is relatively easy for the case that B is surjectively graded.

(17.2) Let $\Lambda = \mathbb{Z}^s$, and $G = \operatorname{Spec} Z\Lambda$, the split s-torus over $\mathbb{Z}$. Let B be an Λ-graded ring. Let Σ be a subsemigroup (submonoid) of $\Lambda_{\mathbb{R}} = \mathbb{R}^s$. We say that B is Σ-*surjectively graded* if for $\lambda, \lambda' \in \Sigma \cap \Lambda$, the product $B_\lambda \otimes_{\mathbb{Z}} B_{\lambda'} \to B_{\lambda+\lambda'}$ is surjective. By definition, B is Σ-surjectively graded if and only if it is $\Sigma \cap \Lambda$-surjectively graded.

The definition is a variant of [Has3, (3.5)]. For a Λ-graded domain B, $\Sigma(B) := \{\lambda \in \Lambda \mid B_\lambda \neq 0\}$ is a subsemigroup of Λ. We say that B is a surjectively graded domain if B is a $\Sigma(B)$-surjectively graded domain.

Lemma 17.3. $\operatorname{Spec} B \to \operatorname{Spec} B_0$ *is a principal G-bundle if and only if B is $\Lambda_{\mathbb{R}}$-surjectively graded.*

Proof. This is Lemma 15.7. $\square$

Lemma 17.4. *Let B be a Σ-surjectively graded ring, and S a multiplicatively closed subset of B consisting of homogeneous elements. Set $|S| = \{|s| \mid s \in S\}$ be the submonoid of Λ of the degrees of the elements of S. Assume that $|S| \subset \Sigma$. Then the localization B_S is $\Sigma - |S|$-surjectively graded, where*

$$\Sigma - |S| = \{\mu - |s| \mid \mu \in \Sigma,\ s \in S\}.$$

Proof. Take $\nu_i = \mu_i - |s_i| \in (\Sigma - |S|) \cap \Lambda$ for $i = 1, 2$, where $\mu_i \in \Sigma \cap \Lambda$ and $s_i \in S$. Take $c = bs^{-1} \in (B_S)_{\nu_1+\nu_2}$, where $s \in S$ and $b \in B_{\nu_1+\nu_2+|s|}$. Then $bs_1s_2 \in B_{\mu_1+\mu_2+|s|}$. As $B_{\mu_1} \otimes_{\mathbb{Z}} B_{\mu_2+|s|} \to B_{\mu_1+\mu_2+|s|}$ is surjective, we can write $bs_1s_2 = \sum_i u_iv_i$ with $u_i \in B_{\mu_1}$ and $v_i \in B_{\mu_2+|s|}$. Then $c = \sum_i (u_is_1^{-1})(v_is^{-1}s_2^{-1})$, and $u_is_1^{-1} \in (B_S)_{\nu_1}$ and $v_is^{-1}s_2^{-1} \in (B_S)_{\nu_2}$. So $(B_S)_{\nu_1} \otimes_{\mathbb{Z}} (B_S)_{\nu_2} \to (B_S)_{\nu_1+\nu_2}$ is surjective, and B_S is $\Sigma - |S|$-surjectively graded. $\square$

(17.5) Let Σ be a rational convex polyhedral cone in $\Lambda_{\mathbb{R}}$ with $\Sigma - \Sigma = \Lambda_{\mathbb{R}}$. Let $\lambda \in \Sigma^\circ \cap \Lambda$, where Σ° is the interior of Σ. Then we have $\Sigma - \mathbb{Z}_{\geq 0}\lambda = \Lambda_{\mathbb{R}}$.

Let B be a Σ-surjectively graded ring. Let $J(\lambda)$ be the ideal of B generated by B_λ. Set $X = \operatorname{Spec} B$, and $U = X \setminus V(J(\lambda))$.

Lemma 17.6 (cf. [Has3, (3.8)]). *There is a principal G-bundle $\rho : U \to Y$. If* $\operatorname{ht} J(\lambda) \geq 2$, *then*

$$X \xleftarrow{i} U \xrightarrow{\rho} Y \xrightarrow{\mathrm{id}_Y} Y \tag{37}$$

is a rational almost principal G-bundle, where $i : U \to X$ is the inclusion. We have $Y = \operatorname{Proj} \bigoplus_{n\geq 0} B_{n\lambda} t^n$. U is independent of the choice of $\lambda \in \Sigma^\circ \cap \Lambda$, and hence Y is also independent of λ.

Proof. For $\mu \in \Lambda$, let $B(\mu)$ be the rank-one B-free (G, B)-module given by $B(\mu)_\nu = B_{\mu+\nu}$. The corresponding G-linearized invertible sheaf on X is denoted by $\mathcal{O}(\mu)$. Let C be the section ring

$$C = \Gamma_{\geq 0}(X; \mathcal{O}(\lambda)) = \bigoplus_{n\geq 0} \Gamma(X, \mathcal{O}(n\lambda))t^n = \bigoplus_{n\geq 0} B(n\lambda)t^n.$$

Let D be the ring of invariants C^G. That is, $D = \bigoplus_{n\geq 0} B_{n\lambda} t^n$. Then we have a sequence of morphisms

$$U \xrightarrow{\iota} \operatorname{Proj} C \setminus V_+(D_+C) \xrightarrow{\psi} \operatorname{Proj} D,$$

where D_+C is the ideal of C generated by $D_+ = \bigoplus_{n>0} B_{n\lambda} t^n$. It is easy to see that ι is an isomorphism (recall that $X = \operatorname{Proj} C$). On the other hand, it is easy to see that the map ψ induced by the graded homomorphism of graded rings $D \to C$ is an algebraic quotient by G. For $a \in B_{n\lambda} \setminus 0$ for $n \geq 1$, $B[a^{-1}]$ is $\Lambda_{\mathbb{R}}$-surjectively graded. By Lemma 17.3, ψ is a principal G-bundle. So letting $\rho = \psi\iota$, we are done.

As $B_\lambda^{\otimes n} \to B_{n\lambda}$ is surjective, $J(\lambda)^n = J(n\lambda)$. If μ is another element of $\Sigma^\circ \cap \Lambda$, then $n\lambda - \mu \in \Sigma$ for sufficiently large n. So $B_\mu \otimes_{\mathbb{Z}} B_{n\lambda-\mu} \to B_{n\lambda}$ is surjective, and $J(\mu) \supset J(\lambda)^n$. Hence $\sqrt{J(\mu)} \supset \sqrt{J(\lambda)}$. Similarly, $\sqrt{J(\mu)} \subset \sqrt{J(\lambda)}$ is also true, and the definition of U is independent of λ. As the principal bundle is a categorical quotient and hence is unique, Y is also independent of the choice of λ. □

(17.7) Let the assumption be as in Lemma 17.6. Let $\mathcal{L}(\mu)$ be the invertible sheaf on Y corresponding to $\mathcal{O}(\mu)$. Namely, $\mathcal{L}(\mu) = \rho_*(\mathcal{O}_U(\mu))^G$. Then we have $\rho^*(\mathcal{L}(\mu)) \cong \mathcal{O}_U(\mu)$. Note that ρ is affine, and $\rho_*\mathcal{O}_U$ is a graded $\mathcal{O}_Y$-algebra: $\rho_*\mathcal{O}_U = \bigoplus_{\mu\in\Lambda} \mathcal{A}_\mu t^\mu$. As $\mathcal{L}(\mu) = \rho_*\mathcal{O}_U(\mu)^G = (\bigoplus_\nu \mathcal{A}_{\mu+\nu} t^\nu)^G = \mathcal{A}_\mu$, we have that $U = \underline{\operatorname{Spec}}_Y \bigoplus_\mu \mathcal{L}(\mu)t^\mu$. Hence

Lemma 17.8 (cf. [Has3, (4.4)])**.** *If B is a Krull domain, then U and Y are locally Krull and integral. If B is Noetherian and (S_2), then U and Y are Noetherian and (S_2). In both cases, B is isomorphic to the multisection ring $R(Y; \mathcal{L}_1, \ldots, \mathcal{L}_s) = \bigoplus_{\mu \in \Lambda} \Gamma(Y, \mathcal{L}(\mu))$.*

Proof. Assume that B is a Krull domain. Being locally Krull and integral is inherited by a nonempty open subset, and U is locally Krull and integral. Then by Theorem 10.13, Y is also locally Krull, and clearly integral. As U is large, $\mathcal{O}_X \to i_*\mathcal{O}_U$ is an isomorphism by [Has9, (5.28)], and B is isomorphic to $R(Y; \mathcal{L}_1, \ldots, \mathcal{L}_s)$.

Next, assume that B is Noetherian and (S_2). This property is inherited by the open subset U, and then descends to Y. Again, as U is large, $\mathcal{O}_X \to i_*\mathcal{O}_U$ is an isomorphism by Lemma 7.31, and we have $B \cong R(Y; \mathcal{L}_1, \ldots, \mathcal{L}_s)$. $\square$

Proposition 17.9. *Let k be a field, and let $B = \bigoplus_{n \geq 0} B_n$ be a standard graded algebra, that is, $B = k[B_1]$ with $\dim_k B_1 < \infty$. Assume moreover that $\dim B \geq 2$. Then we have*

1 *Letting $U = X \setminus 0$ and $Y = \operatorname{Proj} B$, (37) is a rational almost principal $\mathbb{G}_m$-bundle, where 0 is the origin of X.*

2 *$\omega_Y \cong \tilde{\omega}_B$ and $\omega_B \cong \bigoplus_{n \in \mathbb{Z}} \Gamma(Y, \omega_Y(n)) t^n$, where $\tilde{(?)}$ denotes the sheaf on Y associated with a graded module.*

3 *Let $d > 1$. Let $B_{d\mathbb{Z}} = \bigoplus_{n \geq 0} B_{nd}$ be the Veronese subring. Then $(\omega_B)_{d\mathbb{Z}} \cong \omega_{B_{d\mathbb{Z}}}$ as $\mathbb{Z}$-graded modules. If, moreover, B has a graded full 2-canonical module M, then $\omega_B \cong (B \otimes_{B_{d\mathbb{Z}}} \omega_{B_{d\mathbb{Z}}})^{\vee\vee}$, where $(?)^\vee = \operatorname{Hom}_B(?, M)$.*

4 *(cf. Goto–Watanabe [GW, (3.2.1)]) For $r \in \mathbb{Z}$, the following are equivalent.*

- **a** *B is quasi-Gorenstein of a-invariant rd (that is, $\omega_B \cong B(rd)$).*
- **b** *$\operatorname{depth} B_{\mathfrak{m}} \geq 2$, and $B_{d\mathbb{Z}}$ is quasi-Gorenstein of a-invariant rd (that is, $\omega_{B_{d\mathbb{Z}}} \cong B_{d\mathbb{Z}}(rd)$),*

where $\mathfrak{m}$ is the irrelevant ideal B_+ of B. In particular, B is qusi-Gorenstein and its a-invariant is divisible by d if and only if $\operatorname{depth} B_{\mathfrak{m}} \geq 2$ and the Veronese subring $B_{d\mathbb{Z}}$ is quasi-Gorenstein.

5 *Assume that B is normal. Then* $0 \to \mathbb{Z} \xrightarrow{\beta} \mathrm{Cl}(Y) \xrightarrow{\gamma} \mathrm{Cl}(X) \to 0$ *is exact, where* $\beta(1) = \mathcal{O}(1)$*, and* $\gamma(\mathcal{M}) = \bigoplus_{n\in\mathbb{Z}} \Gamma(Y, \mathcal{M}(n))$*.*

6 *Let* $d > 1$*, and set* $X' = \operatorname{Spec} B_{d\mathbb{Z}}$*. If B is normal, then*

$$0 \to \mathbb{Z}/d\mathbb{Z} \xrightarrow{\bar{\beta}} \mathrm{Cl}(X') \xrightarrow{\bar{\gamma}} \mathrm{Cl}(X) \to 0$$

is exact.

Proof. Let $s = 1$ and $\Lambda = \mathbb{Z}$, and $\lambda = 1$. Set $S = Y_0 = \operatorname{Spec} k$, $G := \operatorname{Spec} k\Lambda$, and $Y_0 := \operatorname{Spec} k$. Then $J(1)$ is the irrelevant ideal B_+ by assumption, and $\operatorname{ht} J(1) \geq 2$, since $\dim B \geq 2$. Thus **1** follows from Lemma 17.6.

Note that Θ_{G,Y_0} is G-trivial, since G is an abelian group, see Remark 11.21. So $\omega_Y \cong (\rho_* i^* \omega_X)^G$ by Theorem 11.18. The right-hand side agrees with $\tilde{\omega}_B$ by definition. On the other hand, by Theorem 11.18, $\omega_X \cong i_* \rho^* \omega_Y$, and hence $\omega_B \cong \bigoplus_{n\in\mathbb{Z}} \Gamma(Y, \omega_Y(n)) t^n$. So **2** has been proved.

Set $\Lambda_d = d\Lambda = \mathbb{Z}d$, and $H := \operatorname{Spec} \mathbb{Z}\Lambda_d$. Let $f : G \to H$ be the canonical homomorphism induced by $\Lambda_d \hookrightarrow \Lambda$, and $N := \operatorname{Ker} f = \operatorname{Spec} \mathbb{Z}(\Lambda/\Lambda_d) = \mu_d$. As G acts freely on U, N acts on U freely. So the canonical map $\theta : X \to X'$ corresponding to $B_{d\mathbb{Z}} = B^N \to B$ is a G-enriched almost principal N-bundle. As N is linearly reductive, **3** follows from Corollary 11.22.

4. **a⇒b**. As B is quasi-Gorenstein, it satisfies (S_2). As $\dim B_{\mathfrak{m}} \geq 2$ by assumption, we have that $\operatorname{depth} B_{\mathfrak{m}} \geq 2$. Letting $\mathcal{L} = \mathcal{O}_{X'}(rd)$ in Theorem 14.24, **5**, $\omega_{B_{d\mathbb{Z}}} \cong B_{d\mathbb{Z}}(rd)$ follows.

b⇒a. Let $U = X \setminus 0$, $X = \operatorname{Spec} B_{d\mathbb{Z}}$ as above, and $\theta : X \to X'$ the canonical map. Let $U' = \theta(U)$, and $\upsilon : U \to U'$ be the restriction of θ. As υ is a principal N-bundle by Lemma 15.36, it is flat with Cohen–Macaulay fibers. As U' is quasi-Gorenstein, U satisfies the (S_2) condition. As $U = X \setminus 0$ and $\operatorname{depth} B_{\mathfrak{m}} \geq 2$, X satisfies the (S_2) condition. Now the result follows from Theorem 14.24, **5**.

5, 6. As we assume that B is normal, $B = \bigoplus_{n\in\mathbb{Z}} \Gamma(Y, \mathcal{O}_Y(n))$ by Lemma 17.8. So **5** follows from Proposition 15.32. **6** follows from Lemma 15.37. □

Example 17.10. Let $B = k[x, y]$ with $\deg x = \deg y = 1$. Then $X = \mathbb{A}^2$, $U = \mathbb{A}^2 \setminus 0$, and $Y = \mathbb{P}^1$. The category of locally free sheaves on $\mathbb{P}^1$ is $\mathrm{Ref}(Y)$, which is equivalent to $\mathrm{Ref}(\mathbb{G}_m, X) \cong \mathrm{Ref}(\mathbb{G}_m, B)$. As $\dim B = 2$, a reflexive $(\mathbb{G}_m, B)$-module is nothing but a graded finite free B-module. A graded finite free B-module is a direct sum of copies of $B(n)$, $n \in \mathbb{Z}$, and the

Krull–Schmidt theorem holds. Hence a locally free sheaf on $\mathbb{P}^1$ is a direct sum of copies of $\mathcal{O}(n)$, $n \in \mathbb{Z}$, and the Krull–Schmidt theorem holds. This is a well-known theorem of Grothendieck, see [HazM, (4.1)].

Example 17.11. Let $B = k[x, y]$ with $\deg x = 1$ and $\deg y = -1$, and $X = \operatorname{Spec} B = \mathbb{A}^2$. As $\mathbb{G}_m$ acts freely on $B[x^{-1}]$ and on $B[y^{-1}]$, we have that G acts freely on $X \setminus 0$. In particular, for $n \geq 1$, the subgroup scheme $N = \mu_{n+1}$ acts freely on $X \setminus 0$. In particular, $\varphi : X = \operatorname{Spec} B \to \operatorname{Spec} B^N = Y$ is an almost principal N-bundle. So the class group of $B^N = k[x^{n+1}, xy, y^{n+1}]$ is $\mathcal{X}(N) = \mathbb{Z}/(n+1)\mathbb{Z}$ [Wat, Proposition 4].

Lemma 17.12. *Let k be a perfect field, G a finite k-group scheme acting on a k-scheme X. Assume that there is a separated G-invariant morphism $\varphi : X \to Y$. Then the action of G on X is free if and only if the actions of G_{red} and G° on X are free.*

Proof. The only if part is trivial. We prove the if part. We may assume that k is algebraically closed. Assume that the actions of G_{red} and G° are free, but the action of G is not free. Then take $x \in X \setminus U$, where U is the free locus. Then the stabilizer G_x is nontrivial by Nakayama's lemma. As $(G_x)_{\mathrm{red}} \subset G_x \cap (G_{\mathrm{red}} \otimes_k \kappa(x)) = (G_{\mathrm{red}})_x = e$, G_x is contained in $G_x \cap (G \otimes_k \kappa(x))^\circ = (G^\circ)_x = e$, and G_x is trivial. A contradiction. $\square$

Lemma 17.13. *Let k be a perfect field, G a finite k-group scheme acting on a k-scheme X. Let $\varphi : X \to Y$ be an algebraic quotient by G°. Assume that there is a separated G_{red}-invariant morphism $\psi : Y \to Z$. Then $\mathcal{S}_{G_{\mathrm{red}},X} = \mathcal{S}_{G_{\mathrm{red}},Y} \times_Y X$. The action of G_{red} on X is free if and only if its action on Y is free.*

Proof. We may assume that the characteristic of k is $p > 0$. By Lemma 1.8, $\mathcal{S}_{G_{\mathrm{red}},X} \subset \mathcal{S}_{G_{\mathrm{red}},Y} \times_Y X$. As both of them are finite over X, to prove the equality, it suffices to show that $\mathcal{S}_{G_{\mathrm{red}},x} = \mathcal{S}_{G_{\mathrm{red}},y} \times_y x$ for each point x of X, by Nakayama's lemma, where $y = \varphi(x)$. Let $a_x : G_{\mathrm{red}} \times x \to X$ and $a_y : G_{\mathrm{red}} \times y \to Y$ be the actions. Then

$$\mathcal{S}_{G_{\mathrm{red}},y} \times_y x = a_y^{-1}(y) \times_y x = (1_{G_{\mathrm{red}}} \times \varphi)^{-1} a_y^{-1}(y) = a_x^{-1}(\varphi^{-1}(y)).$$

As $G_{\mathrm{red}} \times x$ is étale over $\kappa(x)$, it is reduced, and hence any morphism from $G_{\mathrm{red}} \times x$ to $\varphi^{-1}(y)$ factors through $(\varphi^{-1}(y))_{\mathrm{red}}$. As G° is infinitesimal, φ is purely inseparable. So $(\varphi^{-1}(y))_{\mathrm{red}} = x$, and

$$\mathcal{S}_{G_{\mathrm{red}},y} \times_y x = a_x^{-1}((\varphi^{-1}(y))_{\mathrm{red}}) = a_x^{-1}(x) = \mathcal{S}_{G_{\mathrm{red}},x}.$$

Thus $\mathcal{S}_{G_{\mathrm{red}},X} = \mathcal{S}_{G_{\mathrm{red}},Y} \times_Y X$. If the action of G_{red} on Y is free, then $\mathcal{S}_{G_{\mathrm{red}},Y}$ is trivial, and its base change $\mathcal{S}_{G_{\mathrm{red}},X}$ is also trivial, and the action on X is also free. Conversely, assume that the action on X is free and $\mathcal{S}_{G_{\mathrm{red}},X}$ is trivial. Let $\mathcal{C}$ be the cokernel of $\mathcal{O}_Y \to \phi^Y_* \mathcal{O}_{\mathcal{S}_{G_{\mathrm{red}},Y}}$, where $\phi^Y : \mathcal{S}_{G_{\mathrm{red}},Y} \to Y$ is the structure map. Then

$$\varphi_* \mathcal{O}_X \to \varphi_* \phi^X_* \mathcal{O}_{\mathcal{S}_{G_{\mathrm{red}},X}} \to \varphi_* \varphi^* \mathcal{C} \to 0$$

is exact. By assumption, $\varphi_*\varphi^*\mathcal{C} = 0$. As φ is finite surjective, $\mathcal{C} = 0$ by Nakayama's lemma. This shows that the action of G_{red} on Y is free. $\square$

Proposition 17.14. *Let k be a field, $n \geq 1$, and G be a linearly reductive finite subgroup scheme of SL_n. Then the canonical action of G on $k[x_1, \ldots, x_n]$ is small. The action of G on $k[[x_1, \ldots, x_n]]$ is also small.*

Proof. We prove that the action of G on $k[x_1, \ldots, x_n]$ is small. We may assume that k is algebraically closed of characteristic $p > 0$, and $n \geq 2$. It suffices to show that the actions of G° and G_{red} are small by Lemma 17.12. As $G_{\mathrm{red}} \subset SL_n$, G_{red} does not have a diagonalizable pseudoreflection. As G_{red} is linearly reductive, G_{red} does not have a transvection (that is, a pseudoreflection $g \in GL_n$ such that $1 - g$ is nilpotent) by Maschke's theorem (as the Jordan normal form shows, the order of a transvection in characteristic p is p). Thus G_{red} does not have a pseudoreflection of any kind, and the action of G_{red} is small.

So we may assume that G is infinitesimal. Let $B = k[x_1, \ldots, x_n]$. As G is also linearly reductive, G is diagonalizable [Swe2]. So $\bigoplus_i kx_i$ is a direct sum of one-dimensional G-modules. Changing variables, we may assume that $G \subset T \cap SL_n$, where T is the subgroup of GL_n consisting of the invertible diagonal matricies. Considering the action of $T \cap SL_n$ is to consider a $\mathbb{Z}^n/(\alpha)$-grading, where $\alpha = (1, 1, \ldots, 1)$. So when we invert $y_i = (x_1 \cdots x_n)/x_i$, then the action of the torus $T \cap SL_n$ on $B[y_i^{-1}]$ is free for $1 \leq i \leq n$, and hence the action of G is also free. Thus it suffices to show that the ideal $I = (y_1, \ldots, y_n)$ of B is height two. By definition, I is the Stanley–Reisner ideal (the defining ideal of the Stanley–Reisner ring, see [Stan, (II.1.1)]) of the $(n-3)$-skelton of the $(n-1)$-simplex. So $\dim B/I = n - 2$, and $\operatorname{ht} I = 2$.

The last assertion follows from the first assertion and Lemma 14.2. $\square$

Remark 17.15. Let k be a field, $V = k^n$, and $\tilde{G} = GL_n = GL(V)$. We define

$$\mathrm{PR}(\tilde{G}) = \{g \in \tilde{G} \mid \operatorname{rank}(1_V - g) \leq 1\}.$$

It is a closed subscheme of $\tilde{G}$. For a closed subscheme F of $\tilde{G}$, we define that $\mathrm{PR}(F) = F \cap \mathrm{PR}(\tilde{G})$. We say that a finite subgroup scheme G of $\tilde{G}$ does not have a pseudoreflection if $\mathrm{PR}(G) = \{e\} = \operatorname{Spec} k$, scheme theoretically. On the other hand, we say that G is small if the action of G on V is small.

By Example 14.5, if G is étale, then G is small if and only if G does not have a pseudoreflection. However, in general, a small subgroup G of $\tilde{G}$ may have a pseudoreflection. For example, if $p = 2$ and G is the subgroup scheme of SL_2 of type (A_1), then it is easy to see that $G = \mathrm{PR}(G)$.

The author does not have an appropriate way to connect the smallness of the action and the non-existence of pseudoreflections for non-reduced finite group schemes.

(17.16) Let k be an algebraically closed field, and N be a nontrivial finite linearly reductive k-subgroup scheme of SL_2. Such N is classified with Dynkin diagrams of type ADE [Has10]. Let $H = \mathbb{G}_m$, which acts on $B = k[x, y]$ by $\deg x = \deg y = 1$. Then $G = H \times N$ acts on B in a natural way. By Proposition 17.14, the action of N on $X = \operatorname{Spec} B$ is small.

The following is well-known for the case that N is étale, see [LeW, Chapter 6].

Theorem 17.17. *Let k, $N \subset SL_2$, H, G, $B = k[x, y]$, and X be as above (N may not be reduced). Set $A = B^N$. Let $\hat{A}$ and $\hat{B}$ respectively be the completion of A and B with respect to the irrelevant ideal. Let $\varphi : X = \operatorname{Spec} B \to Y = \operatorname{Spec} A$ be the canonical algebraic quotient, and $\hat{\varphi} : \hat{X} = \operatorname{Spec} \hat{B} \to \hat{Y} = \operatorname{Spec} \hat{A}$ be its completion. Then*

1 *The free locus of the action of N on X (resp. $\hat{X}$) is $X \setminus 0$ (resp. $\hat{X} \setminus 0$). In particular, φ and $\hat{\varphi}$ are G-enriched almost principal N-bundles.*

2 *A is strongly F-regular Gorenstein of the a-invariant -2.*

3 *The category of $\hat{B}$-finite $\hat{B}$-free $(N, \hat{B})$-modules and the category of maximal Cohen–Macaulay $\hat{A}$-modules are equivalent. The Cohen–Macaulay ring $\hat{A}$ has finite representation type, and any maximal Cohen–Macaulay module of $\hat{A}$ is isomorphic to $\hat{M}_V := (\hat{B} \otimes_k V)^N$ for some finite dimensional N-module V. $\hat{M}_V$ is indecomposable if and only if V is simple. $\hat{M}_V \cong \hat{M}_{V'}$ if and only if $V \cong V'$. An isomorphism class of simple modules of N corresponds to a vertex of the corresponding extended Dynkin diagram.*

4 *The category of B-finite B-free (G, B)-modules (that is, graded (N, B)-modules) is equivalent to the category of maximal Cohen–Macaulay (H, A)-modules (that is, graded maximal Cohen–Macaulay A-modules). Any graded maximal Cohen–Macaulay A-module is isomorphic to $M_V = (B \otimes_k V)^N$, where V is a finite dimensional G-module. M_V is indecomposable if and only if V is simple. So A is of finite representation type in the graded sense (see* [LeW, Chapter 15]*). $M_V \cong M_{V'}$ if and only if $V \cong V'$.*

5 *The class groups of A and $\hat{A}$ are isomorphic to the character group $\mathcal{X}(N)$. $\mathcal{X}(N)$ is $\mathbb{Z}/(n+1)\mathbb{Z}$ for type (A_n), $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ for type (D_n), $\mathbb{Z}/3\mathbb{Z}$ for type (E_6), $\mathbb{Z}/2\mathbb{Z}$ for type (E_7), and is trivial for (E_8), and is independent of the characteristic of k.*

Proof. **1** As we have seen, the free locus U of the action of N on X is large in X. As U is G-stable and large, we have that $U = X$ or $U = X \setminus \{0\}$. However, the origin is a fixed point of the action, and $0 \notin U$. The case of $\hat{X}$ is similar.

2 As N is Reynolds, A is a pure subring of B by Lemma 5.13. Hence A is strongly F-regular by [HocH, (3.1)]. As we have that $N \subset SL_2$ and linearly reductive, A is Gorenstein of a-invariant -2 by Example 14.28, **4**.

3 By **1**, the categories $\mathrm{Ref}(\hat{A})$ and $\mathrm{Ref}(N, \hat{B})$ are equivalent. As $\hat{A}$ is a two-dimensional Cohen–Macaulay local ring, a reflexive $\hat{A}$-module is nothing but a maximal Cohen–Macaulay module. As $\hat{B}$ is a two-dimensional regular local ring, any reflexive $\hat{B}$-module is free. By Lemma 16.3, such a module is of the form $\hat{B} \otimes_k V$ with V a finite dimensional N-module. $V \mapsto \hat{B} \otimes_k V$ and $F \mapsto F/\mathfrak{m}F$ is a one-to-one correspondence between the set of isomorphism classes of finite dimensional N-modules and the set of isomorphism classes of $\hat{B}$-finite $\hat{B}$-free $(N, \hat{B})$-modules, and this correspondence respects finite direct sums. So $V \mapsto \hat{M}_V$ gives a one-to-one correspondence which respects the finite direct sums.

It remains to show that the simple N-modules are in one-to-one correspondence with the vertices of the corresponding extended Dynkin diagram. First, we define the McKay graph Γ_N of $N \subset SL_2$ as in the case of usual finite groups (see [Yos, (10.3)]). It is a finite quiver defined as follows. A vertex of Γ_N is an isomorphism class of simple N-modules. We draw $n_{ij} = \dim_k \mathrm{Hom}_G(V_i, V \otimes_k V_j)$ arrows from $[V_i]$ to $[V_j]$, where $[V_i]$ and $[V_j]$ are vertices. As $V \cong V^*$, it is easy to see that $n_{ij} = n_{ji}$, and we regard Γ_N as an unoriented graph. As $N \to \mathrm{End}(V)$ is a closed immersion, $k[\mathrm{End}(V)] \to k[N]$

is surjective. So it is easy to see that any simple N-module is a direct summand of some $V^{\otimes r}$. So Γ_N must be a connected graph (if V_j is a direct summnad of $V^{\otimes r}$, then starting from the trivial module $[V_0] = [k]$, we reach $[V_j]$ along a path of the length r). Moreover, when we set $a_j = \dim_k V_j$, we have $2a_j = \dim_k(V \otimes V_j) = \sum_i n_{ij} a_i$. If $V_N = \{[V_0], \dots, [V_n]\}$ is the set of vertices, then $n \geq 1$, since N is assumed to be non-trivial. A connected finite graph with the vertex set V_N (with $\#V_N \geq 2$) with n_{ij} arrows from $[V_i]$ to $[V_j]$ with a function $[V_j] \mapsto a_j$ with the property $2a_j = \sum_i n_{ij} a_i$ is classified easily, and is one of (A_n) $(n \geq 1)$, (D_n) $(n \geq 4)$, (E_6), (E_7), or (E_8) displayed in [Yos, section 10] (the symbol $[R]$ there should be replaced by the trivial representation $[k]$ here), or the graph

(38)
$$\begin{tikzcd}
{[k]} \arrow[dr, dash] & \\
& 2 \arrow[loop right] \\
1 \arrow[ur, dash] &
\end{tikzcd},$$

which has a self arrow. N is abelian if and only if $a_j = 1$ for all j if and only Γ_N is of type (A_n). So N is of type (A_n) if and only if Γ_N is of type (A_n). If N is of type (D_n) $(n \geq 4)$, then N has the Klein group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ as a quotient. So Γ_N is not (A_{4n-9}), and $a_j = 1$ for at least four j. By dimension counting, Γ_N must be (D_n). If N is of type (E_6) (resp. (E_7), (E_8)), then $N/[N,N]$ is of order 3 (resp. 2, 1), and there are exactly three (two, one) one-dimensional representations. So it is easy to see that Γ_N is (E_6) (resp. (E_7), (E_8)). After all, (38) does not have a corresponding N.

4 is similar to **3**.

5 follows easily from the discussion in the proof of **3**. $\square$

18. Determinantal rings

Lemma 18.1. *Let S be a scheme, and G a flat quasi-compact quasi-separated S-group scheme. Let $\varphi : X \to Y$ be an almost principal G-bundle with respect to $U \subset Y$ and $V \subset X$. Assume that X is Noetherian and normal, and Y is Noetherian and satisfies Serre's condition (S_2). Then Y is normal, and $\bar{\eta} : \mathcal{O}_Y \to (\varphi_* \mathcal{O}_X)^G$ is an isomorphism.*

Proof. As $\rho : V \to U$ is fpqc and V is normal, we have that U is normal. As U_{reg} is large in U and U is large in Y, we have that Y satisfies Serre's (R_1) condition, and hence Y is normal. By Theorem 10.13, $\bar{\eta}$ is an isomorphism. $\square$

(18.2) Let k be a field, and $n \geq m \geq t \geq 2$. Let $X = \mathrm{Mat}(m, t-1) \times \mathrm{Mat}(t-1, n)$, where $\mathrm{Mat}(a, b)$ denotes the ab-dimensional affine space of the set of $a \times b$ matrices. Let $Y = Y_t(m, n)$ be the determinantal variety $\{C \in \mathrm{Mat}(m, n) \mid \operatorname{rank} C < t\}$. Let $\varphi : X \to Y$ be the map $\varphi(A, B) = AB$. Let U be the open set $Y \setminus Y_{t-1}$, and $V = \varphi^{-1}(U)$. Let $N = GL(t-1)$, and $G = GL(m) \times GL(t-1) \times GL(n)$. The proof of [Has4, (3.1)] shows the following.

Theorem 18.3. *Let the notation be as above. Then φ is a G-enriched almost principal N-bundle with respect to U and V.* $\square$

Using this theorem and the fact that Y is Cohen–Macaulay [HocE], we give short proofs to some well-known results on determinantal rings.

Corollary 18.4 (de Concini–Procesi [DeCP], [Has4])**.** *Y is normal, and φ is an algebraic quotient by the action of N (as N is reductive, φ is also a categorical quotient).*

Proof. Follows immediately from Theorem 18.3, and Lemma 18.1. $\square$

Corollary 18.5 (Bruns [Bru])**.** $\mathrm{Cl}(Y) = \mathbb{Z}$.

Proof. As $\mathrm{Cl}(X) = 0$, $\mathrm{Cl}(Y) \cong H^1_{\mathrm{alg}}(G, \mathcal{O}_X^\times)$ by Theorem 11.5. By [Has9, (4.15)], we have that $\mathrm{Cl}(Y) \cong \mathcal{X}(N)$. It is well-known that $N/[N,N] \cong \mathbb{G}_m$, and $\mathcal{X}(N) \cong \mathcal{X}(\mathbb{G}_m) \cong \mathbb{Z}$. $\square$

Corollary 18.6 (Svanes [Sva])**.** *Y is Gorenstein if and only if $m = n$.*

Proof. Let $V = k^n$, $W = k^m$ and $E = k^{t-1}$ be the vector representations of GL_n, GL_m, and GL_{t-1}, respectively. Then letting $B := \mathrm{Sym}(W^* \otimes E) \otimes \mathrm{Sym}(E^* \otimes V)$ (so $X = \operatorname{Spec} B$), we have that

$$\begin{aligned}\omega_B = B \otimes_k {\textstyle\bigwedge}^{\mathrm{top}}(W^* \otimes E) \otimes_k {\textstyle\bigwedge}^{\mathrm{top}}(E^* \otimes V) \\ \cong B \otimes_k ({\textstyle\bigwedge}^{\mathrm{top}} E)^{\otimes(m-n)} \otimes_k ({\textstyle\bigwedge}^{\mathrm{top}} W)^{\otimes(1-t)} \otimes_k ({\textstyle\bigwedge}^{\mathrm{top}} V)^{\otimes(t-1)}.\end{aligned}$$

In particular, $\omega_B \cong B$ as (N, B)-modules if and only if $m = n$. If $m = n$, then by Corollary 11.19, $\omega_A \cong A$ as A-modules, and hence A is Gorenstein (note that $\Theta_{N,k}$ is trivial, since N is connected reductive, see Remark 11.21). Conversely, if A is Gorenstein, being a positively graded ring over a field, $\omega_A \cong A$ as A-modules. So $\omega_B \cong B$ as (N, B)-modules by Corollary 11.19, and hence $m = n$. $\square$

(18.7) We can do a similar discussion also on the invariant subrings under the action of symplectic groups.

Let k be a field, $t, n \in \mathbb{Z}$ with $4 \leq 2t \leq n$, and $X = \mathrm{Mat}(2t-2, n)$. Let $Y = Y_t$ be the Pfaffian subvariety of $\mathrm{Alt}(n)$, the affine space of $n \times n$ alternating matrices, defined by $2t$-Pfaffians. That is, when $C = k[x_{ij}]_{1 \leq i < j \leq n}$ is the coordinate ring of $\mathrm{Alt}(n)$ and $\Gamma = (x_{ij})$ (where $x_{ii} = 0$ and $x_{ji} = -x_{ij}$), then Y is the closed subscheme of $\mathrm{Alt}(n)$ defined by the ideal generated by all the $2t$-Pfaffians of the alternating matrix Γ. We set $J = J_{t-1} = (\delta_{i+j,t})_{1 \leq i,j < t} \in GL(t-1)$, where δ denotes Kronecker's delta. We define

$$\tilde{J} = \tilde{J}_{t-1} = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} \in GL(2t-2).$$

The symplectic group is defined as

$$Sp_{2t-2} := \{A \in GL(2t-2) \mid {}^tA\tilde{J}A = \tilde{J}\}.$$

Let $N = Sp_{2t-2}$, $V = k^n$, $E = k^{2t-2}$, and $G = GL(n) \times N$. Note that G acts on X by $(h, n) \cdot A = nAh^{-1}$.

Let $\varphi : X \to Y$ be the map given by $\varphi(C) = {}^tC\tilde{J}C$. Almost by definition, φ is N-invariant. For $C \in X$, $\varphi(C)$ has rank at most $2t-2$, and hence $2t$-Pfaffians of $\varphi(C)$ vanish, and φ is well-defined. Set $V = Y \setminus Y_{t-1}$, and $U = \varphi^{-1}(V)$. Then the discussion in [Has4, section 5] shows the following.

Theorem 18.8. *Let the notation be as above. Then φ is a G-enriched almost principal N-bundle with respect to U and V. Y is Cohen–Macaulay.*

Corollary 18.9. *Let the notation be as above.*

1 (*De Concini and Procesi* [DeCP]) *φ is an algebraic quotient, and Y is a normal variety.*

2 *As $N = [N, N]$, we have that* $\mathrm{Cl}(Y)$ *is trivial. That is, the coordinate ring of Y is a UFD* (*hence is Gorenstein*).

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