NILPOTENCY OF FROBENIUS AND DIVISOR CLASS GROUPS

VASUDEVAN SRINIVAS AND SHUNSUKE TAKAGI

In this note, we will briefly summarize our results on two-dimensional F-nilpotent rings. See [7] for the details. All rings are excellent in this note.

Let R be a ring of prime characteristic p and $F : R \to R$ the Frobenius map which sends $x \in R$ to $x^p \in R$. If $(R, \mathfrak{m})$ is local, then the Frobenius map F induces a p-linear map $H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R)$ for each i, which we denote by the same letter F. The e-th iteration of F is denoted by F^e. Also, we denote by R° the set of elements of R which are not in any minimal prime ideal.

Definition 1. Let $(R, \mathfrak{m})$ be a d-dimensional reduced local ring of characteristic $p > 0$.

(i) We say that R is *F-injective* if $F : H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R)$ is injective for all i.
(ii) We say that R is *F-rational* if R is Cohen-Macaulay and if for any $c \in R^\circ$, there exists $e \in \mathbb{N}$ such that $cF^e : H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R)$ is injective.

Remark 2. F-rationality implies F-injectivity.

The *tight closure* $0^*_{H^d_{\mathfrak{m}}(R)}$ of the zero submodule in $H^d_{\mathfrak{m}}(R)$ is the submodule of $H^d_{\mathfrak{m}}(R)$ consisting of all elements $z \in H^d_{\mathfrak{m}}(R)$ for which there exists $c \in R^\circ$ such that $cF^e(z) = 0$ for all large $e \in \mathbb{N}$. When R is analytically irreducible, $0^*_{H^d_{\mathfrak{m}}(R)}$ is the unique maximal proper R-submodule of $H^d_{\mathfrak{m}}(R)$ stable under the Frobenius action F (see [6]). It follows from the definition of F-rational rings that R is F-rational if and only if R is Cohen-Macaulay and $0^*_{H^d_{\mathfrak{m}}(R)} = 0$.

Definition 3. Let $(R, \mathfrak{m})$ be a d-dimensional reduced local ring of characteristic $p > 0$. We say that R is *F-nilpotent*[1] if the natural Frobenius actions F on $H^0_{\mathfrak{m}}(R), \dots, H^{d-1}_{\mathfrak{m}}(R)$, $0^*_{H^d_{\mathfrak{m}}(R)}$ are all nilpotent, that is, there exists $e \in \mathbb{N}$ such that $F^e(H^0_{\mathfrak{m}}(R)) = \cdots = F^e(H^{d-1}_{\mathfrak{m}}(R)) = F^e(0^*_{H^d_{\mathfrak{m}}(R)}) = 0$.

Remark 4. (i) When a (not necessarily finitely generated) R-module M has a Frobenius action F, we denote $M_{\mathrm{nil}} := \{z \in M \mid F^e(z) = 0 \text{ for some } e \in \mathbb{N}\}$. By Hartshorne–Speiser–Lyubeznik Theorem, the definition of F-nilpotency is equivalent to saying that $H^i_{\mathfrak{m}}(R)_{\mathrm{nil}} = H^i_{\mathfrak{m}}(R)$ for all $i \le d-1$ and $(0^*_{H^d_{\mathfrak{m}}(R)})_{\mathrm{nil}} = 0^*_{H^d_{\mathfrak{m}}(R)}$.
(ii) R is F-rational if and only if R is F-injective and F-nilpotent.

This paper is an announcement of our result and the detailed version will be submitted to somewhere.

[1]Blickle and Bondu [2] called such rings "rings close to F-rational".

Example 5. Let k be a perfect field of characteristic $p > 0$.

(1) $k[[x,y,z]]/(x^2+y^3+z^7)$ is F-nilpotent but not F-injective.
(2) $k[[x,y,z]]/(x^2+y^3+z^7+xyz)$ is not F-nilpotent but F-injective.
(3) ([1, Example 5.28]) $k[[x,y,z]]/(x^4+y^4+z^4)$ is F-nilpotent if and only if $p \equiv 3 \bmod 4$.

Using reduction from characteristic zero to positive characteristic, we can define the notion of F-singularities in characteristic zero.

Definition 6. Let $R = k[X_1,\ldots,X_n]/(f_1,\ldots,f_r)$ be a ring of finite type over a field k of characteristic zero. Let A be a $\mathbb{Z}$-subalgebra of k generated by the coefficients of the f_i, and put $R_A = A[X_1,\ldots,X_n]/(f_1,\ldots,f_r)$. Then $R_A \otimes_A k \cong R$. By the generic freeness, after possibly localizing A at a single element, we may assume that R_A is flat over A. We refer to R_A as a *model* of R.

We say that R is of *F-rational type* (resp. *F-nilpotent type*) if there exists a model R_A of R over a finitely generated $\mathbb{Z}$-subalgebra $A \subseteq k$ and a dense open subset $S \subseteq \operatorname{Spec} A$ such that $R_\mu := R_A \otimes_A A/\mu$ is F-rational (resp. F-nilpotent) for all closed points $\mu \in S$.

Example 7. By Example 5, $\mathbb{C}[x,y,z]/(x^2+y^3+z^7)$ is of F-nilpotent type, but $\mathbb{C}[x,y,z]/(x^4+y^4+z^4)$ is not.

As the name suggests, F-rational rings correspond to rational singularities.

Theorem 8 ([3], [5], [6])**.** *Let $(R,\mathfrak{m})$ be a normal local ring essentially of finite type over an a field of characteristic zero. R is of F-rational type if and only if* $\operatorname{Spec} R$ *has only rational singularities, that is, for every (some) resolution of singularities $\pi : Y \to X = \operatorname{Spec} R$, $R^i\pi_*\mathcal{O}_Y = 0$ for all $i \geq 1$.*

We obtain a characterization of two-dimensional rings of F-nilpotent type in terms of dual graphs of resolutions of singularities.

Theorem 9. *Let $(R,\mathfrak{m})$ be a two-dimensional normal local ring essentially of finite type over an algebraically closed field of characteristic zero. Let $\pi : Y \to X = \operatorname{Spec} R$ be a resolution of singularities such that the exceptional locus E of π is a simple normal crossing divisor and $\pi|_{Y\setminus E} : Y \setminus E \to X \setminus \{\mathfrak{m}\}$ is an isomorphism. Then R is of F-nilpotent type if and only if E is a tree of smooth rational curves.*

A combination of a result of Lipman [4] with Theorem 8 gives a characterization of two-dimensional local rings of F-rational type in terms of divisor class groups.

Theorem 10 (cf. [4, Theorem 17.4])**.** *Let $(R,\mathfrak{m})$ be a two-dimensional normal local ring essentially of finite type over an algebraically closed field of characteristic zero. Let $\widehat{R}$ be the $\mathfrak{m}$-adic completion of R. Then R is of F-rational type if and only if the divisor class group $\operatorname{Cl}(\widehat{R})$ is finite.*

As a corollary of Theorem 9, we give a similar characterization of two-dimensional local rings of F-nilpotent type.

Theorem 11. *Let $(R, \mathfrak{m})$ be a two-dimensional normal local ring essentially of finite type over an algebraically closed field of characteristic zero. Let $\widehat{R}$ be the $\mathfrak{m}$-adic completion of R. Then R is of F-nilpotent type if and only if the divisor class group $\mathrm{Cl}(\widehat{R})$ does not contain the torsion group $\mathbb{Q}/\mathbb{Z}$.*

For example, the divisor class group of $\mathbb{C}[[x, y, z]]/(x^2 + y^3 + z^7)$ does not contain $\mathbb{Q}/\mathbb{Z}$, whereas that of $\mathbb{C}[[x, y, z]]/(x^2 + y^3 + z^7 + xyz)$ contain $\mathbb{Q}/\mathbb{Z}$.

References

[1] M. Blickle, The intersection homology D-module in positive characteristic, Univ. of Michigan Dissertation, 2001.

[2] M. Blickle and R. Bondu, Local cohomology multiplicities in terms of étale cohomology, Ann. Inst. Fourier (Grenoble) **55** (2005), no. 7, 2239–2256.

[3] N. Hara, A characterization of rational singularities in terms of injectivity of Frobenius maps, Amer. J. Math. **120** (1998), no. 5, 981–996.

[4] J. Lipman, Rational singularities, with applications to algebraic surfaces and unique factorization, Inst. Hautes Études Sci. Publ. Math., No. 36, 1969, 195–279.

[5] V. B. Mehta and V. Srinivas, A characterization of rational singularities, Asian J. Math. **1** (1997), no. 2, 249–271.

[6] K. E. Smith, F-rational rings have rational singularities, Amer. J. Math. **119** (1997), 159–180.

[7] V. Srinivas and S. Takagi, On the nilpotency of Frobenius acting on local cohomology modules, in preparation.

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Colaba, Mumbai 400005, India
E-mail address: `Srinivas@math.tifr.res.in`

Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan
E-mail address: `stakagi@ms.u-tokyo.ac.jp`