THE FIRST EULER CHARACTERISTICS VERSUS THE HOMOLOGICAL DEGREES*

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1. Introduction

The purpose of our paper is to study the relationship between the first Euler characteristics and the homological degrees of modules. We also investigate the first Hilbert coefficients of parameters in connection with the homological torsions of modules.

To state the problems and the results of our paper, first of all, let us fix some of our terminology. Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Let M be a finitely generated A-module with $s = \dim_A M$. For simplicity, throughout this paper, we assume that A is $\mathfrak{m}$–adically complete and the residue class field $A/\mathfrak{m}$ of A is infinite.

For each $j \in \mathbb{Z}$ we set

$$M_j = \mathrm{Hom}_A(\mathrm{H}^j_{\mathfrak{m}}(M), E),$$

where $E = \mathrm{E}_A(A/\mathfrak{m})$ denotes the injective envelope of $A/\mathfrak{m}$ and $\mathrm{H}^j_{\mathfrak{m}}(M)$ the j-th local cohomology module of M with respect to the maximal ideal $\mathfrak{m}$. Then M_j is a finitely generated A-module with $\dim_A M_j \le j$ for all $j \in \mathbb{Z}$ by the local duality theorem. Let I be a fixed $\mathfrak{m}$-primary ideal in A and let $\ell_A(N)$ denote, for an A-module N, the length of N. Then there exist integers $\{\mathrm{e}^i_I(M)\}_{0 \le i \le s}$ such that

$$\ell_A(M/I^{n+1}M) = \mathrm{e}^0_I(M)\binom{n+s}{s} - \mathrm{e}^1_I(M)\binom{n+s-1}{s-1} + \cdots + (-1)^s \mathrm{e}^s_I(M)$$

for all $n \gg 0$. We call $\mathrm{e}^i_I(M)$ the i-th Hilbert coefficient of M with respect to I and especially call the leading coefficient $\mathrm{e}^0_I(M)$ (> 0) the multiplicity of M with respect to I.

The homological degree $\mathrm{hdeg}_I(M)$ of M with respect to I is inductively defined in the following way, according to the dimension $s = \dim_A M$ of M.

Definition 1.1. ([12]) For each finitely generated A-module M with $s = \dim_A M$, we set

$$\mathrm{hdeg}_I(M) = \begin{cases} \ell_A(M) & \text{if } s \le 0, \\ \\ \mathrm{e}^0_I(M) + \sum_{j=0}^{s-1} \binom{s-1}{j} \mathrm{hdeg}_I(M_j) & \text{if } s > 0 \end{cases}$$

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and call it the homological degree of M with respect to I.

In this paper we need also the notion of homological torsions of modules.

Definition 1.2. Let M be a finitely generated A–module with $s = \dim_A M \geq 2$. We set

$$\mathrm{T}^i_I(M) = \sum_{j=1}^{s-i} \binom{s-i-1}{j-1} \mathrm{hdeg}_I(M_j)$$

for each $1 \leq i \leq s-1$ and call them the homological torsions of M with respect to I.

Notice that the homological degrees $\mathrm{hdeg}_I(M)$ and torsions $\mathrm{T}^i_I(M)$ of M with respect to I depend only on the integral closure of I.

In this paper we study the first Euler characteristic of modules relative to parameters in connection with homological degrees. Let $Q = (a_1, a_2, \ldots, a_s)$ be a parameter ideal of M. We denote by $\mathrm{H}_i(Q; M)$ $(i \in \mathbb{Z})$ the i–th homology module of the Koszul complex $\mathrm{K}_\bullet(Q; M)$ generated by the system $a_1, a_2, \ldots, a_s$ of parameters for M. We set

$$\chi_1(Q; M) = \sum_{i \geq 1} (-1)^{i-1} \ell_A(\mathrm{H}_i(Q; M))$$

and call it the first *Euler characteristic* of M relative to Q; hence

$$\chi_1(Q; M) = \ell_A(M/QM) - \mathrm{e}^0_Q(M) \geq 0$$

by a classical result of Serre (see [1], [9]).

In [3] the authors and co-workers gave, for parameter ideals Q for M, an upper bound

$$\chi_1(Q; M) \leq \mathrm{hdeg}_Q(M) - \mathrm{e}^0_Q(M)$$

of $\chi_1(Q; M)$ (Proposition 3.1). It seems natural to ask what happens on the parameters Q for M, when the equality

$$\chi_1(Q; M) = \mathrm{hdeg}_Q(M) - \mathrm{e}^0_Q(M)$$

is attained. The first main result of this paper answers this question and is stated as follows, where the sequence $a_1, a_2, \cdots, a_d$ is said to be a d-sequence on M, if the equality

$$[(a_1, a_2, \cdots, a_{i-1})M :_M a_i a_j] = [(a_1, a_2, \cdots, a_{i-1})M :_M a_j]$$

holds true for all $1 \leq i \leq j \leq d$ ([6]).

Theorem 1.3. *Let M be a finitely generated A-module with $d = \dim_A M$ and let Q be a parameter ideal of A. Then the following conditions are equivalent.*

(1) $\chi_1(Q; M) = \mathrm{hdeg}_Q(M) - \mathrm{e}^0_Q(M)$.

(2) *The following two conditions are satisfied.*

(*a*)

$$(-1)^i \mathrm{e}^i_Q(M) = \begin{cases} \mathrm{T}^i_Q(M) & \text{if } 1 \leq i \leq d-1, \\ \ell_A(\mathrm{H}^0_{\mathfrak{m}}(M)) & \text{if } i = d \end{cases}$$

for all $1 \leq i \leq d$.

(*b*)

$$\ell_A(M/Q^{n+1}M) = \sum_{i=0}^{d} (-1)^i \mathrm{e}^i_Q(M) \binom{n+d-i}{d-i}$$

for all $n \geq 0$.

When this is the case, we have the following:

(i) *There exist elements* $a_1, a_2, \cdots, a_d \in A$ *such that* $Q = (a_1, a_2, \cdots, a_d)$ *and* $a_1, a_2, \cdots, a_d$ *forms a d-sequence on* M,

(ii) $QM \cap \mathrm{H}^0_{\mathfrak{m}}(M) = (0)$, *and* $Q\mathrm{H}^i_{\mathfrak{m}}(M) = (0)$ *for all* $1 \leq i \leq d-2$.

Our next purpose is to investigate the relationship between the first Hilbert coefficients and the homological torsions for modules. In [3] the authors and co-workers gave the lower bound

$$\mathrm{e}^1_Q(M) \geq -\mathrm{T}^1_Q(M)$$

of the first Hilbert coefficient $\mathrm{e}^1_Q(M)$ in terms of the homological torsion $\mathrm{T}^1_Q(M)$ (Proposition 4.1). Here we notice that the inequality $0 \geq \mathrm{e}^1_Q(M)$ holds true for every parameter ideals Q of M ([7, Theorem 3.5]) and that M is a Cohen-Macaulay A-module once $0 = \mathrm{e}^1_Q(M)$ for some parameter ideal Q, provided M is unmixed (see [2]). Remember that M is said to be unmixed, if $\dim A/\mathfrak{p} = \dim_A M$ for all $\mathfrak{p} \in \mathrm{Ass}_A M$ (since A is assumed to be $\mathfrak{m}$-adically complete). It seems now natural to ask what happens on the parameters Q of M which satisfy the equality $\mathrm{e}^1_Q(M) = -\mathrm{T}^1_Q(M)$. The second main result of this paper answers the question and is stated as follows (Theorem 4.2).

Theorem 1.4. *Let* M *be a finitely generated* A*-module with* $d = \dim_A M \geq 2$ *and suppose that* M *is unmixed. Let* Q *be a parameter ideal of* A. *Then the following conditions are equivalent.*

(1) $\chi_1(Q; M) = \mathrm{hdeg}_Q(M) - \mathrm{e}^0_Q(M)$.

(2) $\mathrm{e}^1_Q(M) = -\mathrm{T}^1_Q(M)$.

When this is the case, we have the following:

(i) $(-1)^i \mathrm{e}^i_Q(M) = \mathrm{T}^i_Q(M)$ *for* $2 \leq i \leq d-1$ *and* $\mathrm{e}^d_Q(M) = 0$,

(ii) $\ell_A(M/Q^{n+1}M) = \sum_{i=0}^{d} (-1)^i \mathrm{e}^i_Q(M) \binom{n+d-i}{d-i}$ *for all* $n \geq 0$,

(iii) *there exist elements* $a_1, a_2, \cdots, a_d \in A$ *such that* $Q = (a_1, a_2, \cdots, a_d)$ *and* $a_1, a_2, \cdots, a_d$ *forms a d-sequence on* M, *and*

(iv) $Q\mathrm{H}^i_{\mathfrak{m}}(M) = (0)$ *for all* $1 \leq i \leq d-2$.

We now briefly explain how this paper is organized. In Section 2 we will summarize, for the later use in this paper, some auxiliary results on the homological degrees and torsions. We shall prove Theorem 1.3 in Section 3 (Theorem 3.2). In Section 3 we will explore an example of parameter ideals which satisfy the equality in Theorem 1.3 (1). Theorem 1.4 will be proven in Section 4 (Theorem 4.2). Unless M is unmixed, the implication (2) $\Rightarrow$ (1) in Theorem 1.4 does not hold true in general. We will show in Section 4 an example of parameter ideals Q in a two-dimensional mixed local ring A such that $\mathrm{e}^1_Q(A) = -\mathrm{T}^1_Q(A)$ but $\chi_1(Q; A) < \mathrm{hdeg}_Q(A) - \mathrm{e}^0_Q(A)$.

2. Preliminaries

In this section we summarize some basic properties of homological degrees and torsions of modules, which we need throughout this paper. We begin with the following.

Fact 2.1. Let M and M' are finitely generated A-modules. Let I be an $\mathfrak{m}$-primary ideal in A. Then $0 \leq \mathrm{hdeg}_I(M) \in \mathbb{Z}$. We furthermore have the following.

(1) $\mathrm{hdeg}_I(M) = 0$ if and only if $M = (0)$.

(2) If $M \cong M'$, then $\mathrm{hdeg}_I(M) = \mathrm{hdeg}_I(M')$.

(3) $\mathrm{hdeg}_I(M)$ depends only on the integral closure of I.
(4) If M is a generalized Cohen-Macaulay A-module, then
$$\mathrm{hdeg}_I(M) - \mathrm{e}_I^0(M) = \mathbb{I}(M)$$
and
$$\ell_A(M/QM) - \mathrm{e}_Q^0(M) \leq \mathbb{I}(M)$$
for all parameter ideals Q for M ([10]), where $\mathbb{I}(M) = \sum_{j=0}^{s-1} \binom{s-1}{j} \ell_A(\mathrm{H}_{\mathfrak{m}}^j(M))$ denotes the Stückrad-Vogel invariant of M.

The following result plays a key role in the analysis of homological degree.

Lemma 2.2. ([12, Proposition 3.18]) *Let $0 \to X \to Y \to Z \to 0$ be an exact sequence of finitely generated A-modules. Then the following assertions hold true.*
(1) *If $\ell_A(Z) < \infty$, then* $\mathrm{hdeg}_I(Y) \leq \mathrm{hdeg}_I(X) + \mathrm{hdeg}_I(Z)$.
(2) *If $\ell_A(X) < \infty$, then* $\mathrm{hdeg}_I(Y) = \mathrm{hdeg}_I(X) + \mathrm{hdeg}_I(Z)$.

Let $R = \mathrm{R}(I) = A[It] \subseteq A[t]$ be the Rees algebra of I (here t denotes an indeterminate over A) and let $f : I \to R,\ a \mapsto at$ be the identification of I with $R_1 = It$. Set
$$\mathrm{Proj}\, R = \{\mathfrak{p} \mid \mathfrak{p} \text{ is a graded prime ideal of } R \text{ such that } \mathfrak{p} \not\supseteq R_+\}.$$
We then have the following.

Lemma 2.3. ([11, Theorem 2.13]) *Let M be a finitely generated A-module. Then there exists a finite subset $\mathcal{F} \subseteq \mathrm{Proj}\, R$ such that*
(1) *every $a \in I \setminus \bigcup_{\mathfrak{p} \in \mathcal{F}} [f^{-1}(\mathfrak{p}) + \mathfrak{m}I]$ is superficial for M with respect to I and*
(2) $\mathrm{hdeg}_I(M/aM) \leq \mathrm{hdeg}_I(M)$ *for each $a \in I \setminus \bigcup_{\mathfrak{p} \in \mathcal{F}} [f^{-1}(\mathfrak{p}) + \mathfrak{m}I]$.*

Lemma 2.4. *Let M be a finitely generated A–module with $s = \dim_A M \geq 3$ and I an $\mathfrak{m}$-primary ideal of A. Then, for each $1 \leq i \leq s-2$, there exists a finite subset $\mathcal{F} \subseteq \mathrm{Proj}\, R$ such that every $a \in I \setminus \bigcup_{\mathfrak{p} \in \mathcal{F}} [f^{-1}(\mathfrak{p}) + \mathfrak{m}I]$ is superficial for M with respect to I, satisfying the inequality*
$$\mathrm{T}_I^i(M/aM) \leq \mathrm{T}_I^i(M).$$

3. Relation between the first Euler characteristics and the homological degrees

In this section we study the relation between the first Euler characteristics and the homological degrees. The following inequality is due to [3].

Proposition 3.1. ([3, Theorem 7.2]) *Let M be a finitely generated A-module with $d = \dim_A M$. Then*
$$\chi_1(Q; M) \leq \mathrm{hdeg}_Q(M) - \mathrm{e}_Q^0(M)$$
for every parameter ideal Q of A.

Proof. Suppose $d = 1$. Then M is a generalized Cohen-Macaulay A-module and hence
$$\chi_1(Q; M) = \ell_A(M/QM) - \mathrm{e}_Q^0(M) \leq \ell_A(\mathrm{H}_{\mathfrak{m}}^0(M)) = \mathrm{hdeg}_Q(M_0) = \mathrm{hdeg}_Q(M) - \mathrm{e}_Q^0(M)$$
by Fact 2.1 (4). Assume that $d \geq 2$ and that our assertion holds true for $d-1$. We choose an element $a \in Q \backslash \mathfrak{m}Q$ so that a is superficial for M with respect to Q and

$\mathrm{hdeg}_Q(M/aM) \le \mathrm{hdeg}_Q(M)$ (Lemma 2.3). Then, setting $\overline{M} = M/aM$, by the hypothesis of induction on d we get

$$\chi_1(Q; M) = \chi_1(\overline{Q}; \overline{M}) \le \mathrm{hdeg}_Q(\overline{M}) - \mathrm{e}_Q^0(\overline{M}) \le \mathrm{hdeg}_Q(M) - \mathrm{e}_Q^0(M),$$

as wanted. $\square$

It seems natural to ask what happens on the parameters Q for M, when $\chi_1(Q; M) = \mathrm{hdeg}_Q(M) - \mathrm{e}_Q^0(M)$. The following theorem answers the question, which is the main result of this section.

Theorem 3.2. *Let M be a finitely generated A-module with $d = \dim_A M$. Let Q be a parameter ideal of A. Then the following conditions are equivalent.*

(1) $\chi_1(Q; M) = \mathrm{hdeg}_Q(M) - \mathrm{e}_Q^0(M)$.

(2) *The following conditions are satisfied.*

(a)

$$(-1)^i \mathrm{e}_Q^i(M) = \begin{cases} \mathrm{T}_Q^i(M) & \textit{if } 1 \le i \le d-1, \\ \ell_A(\mathrm{H}_{\mathfrak{m}}^0(M)) & \textit{if } i = d \end{cases}$$

for all $1 \le i \le d$ *and*

(b)

$$\ell_A(M/Q^{n+1}M) = \sum_{i=0}^{d} (-1)^i \mathrm{e}_Q^i(M) \binom{n+d-i}{d-i}$$

for all $n \ge 0$.

When this is the case, we have the following:

(i) *There exist elements $a_1, a_2, \cdots, a_d \in A$ such that $Q = (a_1, a_2, \cdots, a_d)$ and $a_1, a_2, \cdots, a_d$ forms a d-sequence on M.*

(ii) *$QM \cap \mathrm{H}_{\mathfrak{m}}^0(M) = (0)$ and $Q\mathrm{H}_{\mathfrak{m}}^i(M) = (0)$ for all $1 \le i \le d-2$.*

The following result shows that Theorem 3.2 (i) holds true, once $\chi_1(Q; M) = \mathrm{hdeg}_Q(M) - \mathrm{e}_Q^0(M)$.

Proposition 3.3. *Let M be a finitely generated A-module with $d = \dim_A M$ and Q a parameter ideal of A. Let $a_1 \in Q \backslash \mathfrak{m}Q$ be a superficial element for M with respect to Q such that* $\mathrm{hdeg}_Q(M/a_1M) \le \mathrm{hdeg}_Q(M)$. *Assume that*

$$\chi_1(Q; M) = \mathrm{hdeg}_Q(M) - \mathrm{e}_Q^0(M).$$

Then there exist elements $a_2, a_3, \cdots, a_d \in A$ such that $Q = (a_1, a_2, \cdots, a_d)$ and $a_1, a_2, \cdots, a_d$ forms a d-sequence on M.

The following result plays a key role in our proof of Theorem 3.2.

Proposition 3.4 (cf. [5, Proposition 3.4])**.** *Let M be a finitely generated A-module with $d = \dim_A M$. Let $Q = (a_1, a_2, \cdots, a_d)$ be a parameter ideal of A and assume that $a_1, a_2, \cdots, a_d$ forms a d-sequence on M. Then we have the following, where $Q_i = (a_1, a_2, \cdots, a_i)$ for each $0 \le i \le d$.*

(1) $\mathrm{e}_Q^0(M) = \ell_A(M/QM) - \ell_A([Q_{d-1}M :_M a_d]/Q_{d-1}M)$.

(2) $(-1)^i \mathrm{e}_Q^i(M) = \ell_A(\mathrm{H}_{\mathfrak{m}}^0(M/Q_{d-i}M)) - \ell_A(\mathrm{H}_{\mathfrak{m}}^0(M/Q_{d-i-1}M))$ *for* $1 \le i \le d-1$ *and* $(-1)^d \mathrm{e}_Q^d(M) = \ell_A(\mathrm{H}_{\mathfrak{m}}^0(M))$.

(3) $\ell_A(M/Q^{n+1}M) = \sum_{i=0}^{d}(-1)^i \mathrm{e}_Q^i(M)\binom{n+d-i}{d-i}$ *for all* $n \geq 0$.

We are now in a position to prove Theorem 3.2. We have only to show the implication (1) $\Rightarrow$ (2).

Proof of Theorem 3.2. (1) $\Rightarrow$ (2) Since the last assertion (i) follows from Proposition 3.3, we have assertion (b) by Proposition 3.4. It is now enough to show that assertion (a) holds true. We proceed by induction on d. Thanks to [7, Proposition 3.1] we have $\mathrm{e}_Q^1(M) = -\ell_A(\mathrm{H}_{\mathfrak{m}}^0(M))$ if $d = 1$.

We may assume that $d \geq 2$ and that our assertion holds true for $d-1$. Choose an element $a \in Q \setminus \mathfrak{m}Q$ so that a is superficial for M and M_j with respect to Q and $\mathrm{hdeg}_Q(M_j/aM_j) \leq \mathrm{hdeg}_Q(M_j)$ for all $1 \leq j \leq d-1$ (Lemma 2.3). We set $\overline{M} = M/aM$ and $\overline{Q} = Q/(a)$. Consider the long exact sequence

$$\mathrm{H}_{\mathfrak{m}}^j(M) \xrightarrow{a} \mathrm{H}_{\mathfrak{m}}^j(M) \to \mathrm{H}_{\mathfrak{m}}^j(\overline{M}) \to \mathrm{H}_{\mathfrak{m}}^{j+1}(M) \xrightarrow{a} \mathrm{H}_{\mathfrak{m}}^{j+1}(M)$$

of local cohomology modules induced from the exact sequence

$$0 \to (0):_M a \to M \xrightarrow{a} M \to \overline{M} \to 0.$$

Then, taking the Matlis dual of the above long exact sequence, we get exact sequences

$$0 \to M_{j+1}/aM_{j+1} \to \overline{M}_j \to (0):_{M_j} a \to 0$$

and embeddings

$$0 \to (0):_{M_j} a \to M_j$$

for all $0 \leq j \leq d-2$. Consequently, because $\ell_A((0):_{M_j} a) < \infty$, by Lemma 2.2 we have

$$\begin{aligned}\mathrm{hdeg}_I(\overline{M}_j) &\leq \mathrm{hdeg}_I([(0):_{M_j} a]) + \mathrm{hdeg}_I(M_{j+1}/aM_{j+1}) \\ &\leq \mathrm{hdeg}_I(M_j) + \mathrm{hdeg}_I(M_{j+1})\end{aligned}$$

for each $0 \leq j \leq d-2$. Hence

$$\begin{aligned}\chi_1(Q;M) = \chi_1(\overline{Q};\overline{M}) &\leq \mathrm{hdeg}_Q(\overline{M}) - \mathrm{e}_Q^0(\overline{M}) \\ &= \sum_{j=0}^{d-2}\binom{d-2}{j}\mathrm{hdeg}_Q(\overline{M}_j) \\ &\leq \sum_{j=0}^{d-2}\binom{d-2}{j}\{\mathrm{hdeg}_Q([(0):_{M_j} a]) + \mathrm{hdeg}_Q(M_{j+1}/aM_{j+1})\} \\ &\leq \sum_{j=0}^{d-2}\binom{d-2}{j}\{\mathrm{hdeg}_Q(M_j) + \mathrm{hdeg}_Q(M_{j+1})\} \\ &= \sum_{j=0}^{d-1}\binom{d-1}{j}\mathrm{hdeg}_Q(M_j) \\ &= \mathrm{hdeg}_Q(M) - \mathrm{e}_Q^0(M) = \chi_1(Q;M),\end{aligned}$$

because $\chi_1(Q;M) = \chi_1(\overline{Q};\overline{M})$ and $\chi_1(\overline{Q};\overline{M}) \leq \mathrm{hdeg}_Q(\overline{M}) - \mathrm{e}_Q^0(\overline{M})$ by Proposition 3.1. Thus

$$\chi_1(\overline{Q};\overline{M}) = \mathrm{hdeg}_Q(\overline{M}) - \mathrm{e}_Q^0(\overline{M}),$$

$$\mathrm{hdeg}_Q(\overline{M}_j) = \mathrm{hdeg}_Q(M_j) + \mathrm{hdeg}_Q(M_{j+1}),$$

and $aM_j = (0)$ for all $0 \le j \le d-2$. On the other hand, since a is superficial for M with respect to Q, we have $\mathrm{e}_Q^i(M) = \mathrm{e}_Q^i(\overline{M})$ for all $0 \le i \le d-2$ and $(-1)^{d-1}\mathrm{e}_Q^{d-1}(M) = (-1)^{d-1}\mathrm{e}_Q^{d-1}(\overline{M}) - \ell_A([(0):_M a])$ ([8, (22.6)]).

Therefore the hypothesis of induction on d yields that

$$\begin{aligned}
(-1)^i\mathrm{e}_Q^i(M) = (-1)^i\mathrm{e}_Q^i(\overline{M}) &= \mathrm{T}_Q^i(\overline{M}) \\
&= \sum_{j=1}^{d-1-i} \binom{d-i-2}{j-1} \mathrm{hdeg}_Q(\overline{M}_j) \\
&= \sum_{j=1}^{d-1-i} \binom{d-i-2}{j-1} \{\mathrm{hdeg}_Q(M_j) + \mathrm{hdeg}_Q(M_{j+1})\} \\
&= \sum_{j=1}^{d-i} \binom{d-i-1}{j-1} \mathrm{hdeg}_Q(M_j) \\
&= \mathrm{T}_Q^i(M)
\end{aligned}$$

for $1 \le i \le d-2$ and that

$$\begin{aligned}
(-1)^{d-1}\mathrm{e}_Q^{d-1}(M) &= (-1)^{d-1}\mathrm{e}_Q^{d-1}(\overline{M}) - \ell_A([(0):_M a]) \\
&= \ell_A(\mathrm{H}_\mathfrak{m}^0(\overline{M})) - \ell_A(\mathrm{H}_\mathfrak{m}^0(M)) \\
&= \mathrm{hdeg}_Q(M_1) \\
&= \mathrm{T}_Q^{d-1}(M),
\end{aligned}$$

because $a\mathrm{H}_\mathfrak{m}^0(M) = (0)$ and $\ell_A(\mathrm{H}_\mathfrak{m}^0(\overline{M})) = \mathrm{hdeg}_Q(\overline{M}_0) = \mathrm{hdeg}_Q(M_0) + \mathrm{hdeg}_Q(M_1)$. Thus, as the equality $(-1)^d\mathrm{e}_Q^d(M) = \ell_A(\mathrm{H}_\mathfrak{m}^0(M))$ holds true by Proposition 3.4, assertion (a) follows, which proves the implication (1) $\Rightarrow$ (2). $\square$

We close this section with the following example of parameter ideals Q such that $\chi_1(Q; A) = \mathrm{hdeg}_Q(A) - \mathrm{e}_Q^0(A)$ but A is not a generalized Cohen-Macaulay ring.

Example 3.5. Let $\ell \ge 2$ and $m \ge 1$ be integers. Let

$$S = k[[X_i, Y_i, Z_j \mid 1 \le i \le \ell, 1 \le j \le m]]$$

be the formal power series ring with $2\ell + m$ indeterminates over an infinite field k. Let

$$A = S/(X_1, X_2, \cdots, X_\ell) \cap (Y_1, Y_2, \cdots, Y_\ell),$$

$$\mathfrak{m} = (x_i, y_i, z_j \mid 1 \le i \le \ell, 1 \le j \le m)A, \text{ and}$$

$$Q = (x_i - y_i \mid 1 \le i \le \ell)A + (z_j \mid 1 \le j \le m)A,$$

where x_i, y_i, and z_j denote the images of X_i, Y_i, and Z_j in A respectively. Then $\mathfrak{m}^2 = Q\mathfrak{m}$, whence Q is a reduction of $\mathfrak{m}$. We furthermore have the following.

1. A is an unmixed local ring with $\dim A = \ell + m$, $\mathrm{depth} A = m + 1$, and $\mathrm{H}_\mathfrak{m}^{m+1}(A)$ is not finitely generated.
2. $\ell_A(A/Q) = \ell + 1$, $\mathrm{e}_Q^0(A) = 2$, and hence $\chi_1(Q; A) = \ell - 1$.
3. $\mathrm{hdeg}_Q(A) = 2 + \binom{\ell+m-1}{m+1}$.
4. Hence $\chi_1(Q; A) = \mathrm{hdeg}_Q(A) - \mathrm{e}_Q^0(A)$, if $\ell = 2$.

4. The first Hilbert coefficients versus the homological torsions

The purpose of this section is to estimate the first Hilbert coefficients of parameters in terms of the homological torsions of modules. The following inequality is given by [3]. We indicate a brief proof for the sake of completeness.

Proposition 4.1. ([3, Theorem 6.6]) *Suppose that $d \geq 2$ and let Q be a parameter ideal of A. Then*

$$\mathrm{e}_Q^1(M) \geq -\mathrm{T}_Q^1(M)$$

for every finitely generated A-module M with $d = \dim_A M$.

Proof. We proceed by induction on d. Let $M' = M/\mathrm{H}_\mathfrak{m}^0(M)$. Then, since $\mathrm{e}_Q^1(M) = \mathrm{e}_Q^1(M')$ and $\mathrm{T}_Q^1(M) = \mathrm{T}_Q^1(M')$, to see that $\mathrm{e}_Q^1(M) \geq -\mathrm{T}_Q^1(M)$, we may assume, passing to M', that $\mathrm{depth}_A M > 0$. Suppose that $d = 2$. Choose $a \in Q \setminus \mathfrak{m}Q$ so that a is superficial for M and M_1 with respect to Q and $\mathrm{hdeg}_Q(M_1/aM_1) \leq \mathrm{hdeg}_Q(M_1)$. Set $\overline{M} = M/aM$. Then since a is M–regular, we get the exact sequence

$$0 \to \mathrm{H}_\mathfrak{m}^0(\overline{M}) \to \mathrm{H}_\mathfrak{m}^1(M) \xrightarrow{a} \mathrm{H}_\mathfrak{m}^1(M)$$

of local cohomology modules. Taking the Matlis dual, we get an isomorphism $M_1/aM_1 \cong \overline{M}_0$ and hence, because $\mathrm{e}_Q^1(\overline{M}) = -\ell_A(\mathrm{H}_\mathfrak{m}^0(\overline{M}))$ by [7, Proposition 3.1], we have

$$\begin{aligned} \mathrm{e}_Q^1(M) = \mathrm{e}_Q^1(\overline{M}) &= -\ell_A(\mathrm{H}_\mathfrak{m}^0(\overline{M})) \\ &= -\,\mathrm{hdeg}_Q(M_1/aM_1) \\ &\geq -\,\mathrm{hdeg}_Q(M_1) \\ &= -\mathrm{T}_Q^1(M). \end{aligned}$$

Suppose that $d \geq 3$ and that our assertion holds true for $d-1$. Choose $a \in Q \setminus \mathfrak{m}Q$ so that a is superficial for M with respect to Q and $\mathrm{T}_Q^1(\overline{M}) \leq \mathrm{T}_Q^1(M)$ (Lemma 2.4). Then the hypothesis of induction on d shows

$$\mathrm{e}_Q^1(M) = \mathrm{e}_Q^1(\overline{M}) \geq -\mathrm{T}_Q^1(\overline{M}) \geq -\mathrm{T}_Q^1(M),$$

as wanted. $\square$

The first Hilbert coefficients $\mathrm{e}_Q^1(M)$ of parameter ideals are bounded below by the homological torsion $\mathrm{T}_Q^1(M)$. It is now natural to ask what happens on the parameters Q of M, once the equality $\mathrm{e}_Q^1(M) = -\mathrm{T}_Q^1(M)$ is attained. The main result of this section answers the question and is stated as follows.

Theorem 4.2. *Let M be a finitely generated A-module with $d = \dim_A M \geq 2$ and suppose that M is unmixed. Let Q be a parameter ideal of A. Then the following conditions are equivalent.*

(1) $\chi_1(Q; M) = \mathrm{hdeg}_Q(M) - \mathrm{e}_Q^0(M)$.
(2) $\mathrm{e}_Q^1(M) = -\mathrm{T}_Q^1(M)$.

When this is the case, we have the following:

(i) $(-1)^i \mathrm{e}_Q^i(M) = \mathrm{T}_Q^i(M)$ *for* $2 \leq i \leq d-1$ *and* $\mathrm{e}_Q^d(M) = 0$.
(ii) $\ell_A(M/Q^{n+1}M) = \sum_{i=0}^{d} (-1)^i \mathrm{e}_Q^i(M) \binom{n+d-i}{d-i}$ *for all* $n \geq 0$.
(iii) *There exist elements* $a_1, a_2, \cdots, a_d \in A$ *such that* $Q = (a_1, a_2, \cdots, a_d)$ *and* $a_1, a_2, \cdots, a_d$ *forms a d-sequence on M.*
(iv) $Q\mathrm{H}_\mathfrak{m}^i(M) = (0)$ *for all* $1 \leq i \leq d-2$.

To prove Theorem 4.2 we need the following.

Proposition 4.3. ([3, Theorem 2.5]) *Let M be a finitely generated A-module with $d = \dim_A M$. Suppose that M is unmixed. Then there exist a surjective homomorphism $B \to A$ of rings such that B is a Gorenstein complete local ring with $\dim B = \dim A$ and an exact sequence*

$$0 \to M \to F \to X \to 0$$

of B-modules with F finitely generated and free.

As a direct consequence we get the following.

Corollary 4.4. ([4, Lemma 3.1]) *Let M be a finitely generated A-module with $d = \dim_A M \geq 2$. If M is unmixed, then $\mathrm{H}^1_{\mathfrak{m}}(M)$ is finitely generated.*

The following example shows that the implication (2) $\Rightarrow$ (1) does not hold true in general, unless M is unmixed.

Example 4.5. Let S be a complete regular local ring with maximal ideal $\mathfrak{n}$, $\dim S = 3$, and infinite residue class field. Let $\mathfrak{n} = (X, Y, Z)$ and $\ell \geq 1$ be integers. We set

$$A = S/(X) \cap (Y^\ell, Z).$$

Let $\mathfrak{m} = (x, y, z)A$ be the maximal ideal of A and $Q = (x - y, x - z)A$, where x, y, and z denote the images of X, Y, and Z in A, respectively. Then since $\mathfrak{m}^{\ell+1} = Q\mathfrak{m}^\ell$, Q is a reduction of $\mathfrak{m}$. We furthermore have the following.

(1) A is mixed with $\dim A = 2$ and $\mathrm{depth} A = 1$,
(2) $\mathrm{e}^0_Q(A) = 1$, $\mathrm{e}^1_Q(A) = -\ell$, and $\mathrm{e}^2_Q(A) = -\binom{\ell}{2}$,
(3) $\chi_1(Q; A) = 1$, $\mathrm{hdeg}_Q(A) = \ell + 1$, and $\mathrm{T}^1_Q(A) = \ell$.
(4) Hence $\mathrm{e}^1_Q(A) = -\mathrm{T}^1_Q(A)$ but if $\ell \geq 2$, $\chi_1(Q; A) < \mathrm{hdeg}_Q(A) - \mathrm{e}^0_Q(A)$.

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