On modules of linear type*

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1 Introduction

Let R be a Noetherian ring. For an R-module N, we denote by $\mathcal{S}(N)$ the symmetric algebra of N. Let m and n be positive integers such that $1 \leq m \leq n$. We denote by $\mathrm{Mat}(m, n\,; R)$ the set of $m \times n$ matrices with entries in R. Let $A = (a_{ij}) \in \mathrm{Mat}(m, n\,; R)$. We set

$$M = \mathrm{Coker}\,(R^m \xrightarrow{{}^tA} R^n)\,.$$

Let $S = \mathcal{S}(R^n)$ and $x_1, x_2, \ldots, x_n$ be the standard free basis of R^n. Then we have

$$S = R[x_1, x_2, \ldots, x_n]\,,$$

which is a polynomial ring. For any $i = 1, \ldots, m$, we set

$$f_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \in S\,.$$

As is well known, we have $\mathcal{S}(M) = S/(f_1, f_2, \ldots, f_m)S$. In [1], after proving that $\mathrm{grade}\,(f_1, f_2, \ldots, f_m)S = m$ if and only if $\mathrm{grade}\,\mathrm{I}_i(A) \geq m - i + 1$ for any $i = 1, \ldots, m$, Avramov gave a condition for $\mathrm{I}_m(A)$ to be an ideal of linear type in the case where $n = m + 1$ (See [3] for elementary proofs for those facts). Let us notice that if $n = m + 1$, the cokernel of the homomorphism $R^m \to R^n$ defined by ${}^tA$ is isomorphic to $\mathrm{I}_m(A)$ by the theorem of Hilbert-Burch. The purpose of this report is to generalize Avramov's result. Without assuming $n = m + 1$, we will give a condition for the R-torsion part of $\mathcal{S}(M)$, which is denoted by $\mathrm{T}_R(\mathcal{S}(M))$, to be vanished. The main theorem can be stated as follows.

Theorem 1.1 *The following conditions are equivalent.*

(1) $\mathrm{grade}\,\mathrm{I}_i(A) \geq m - i + 2$ *for any* $i = 1, \ldots, m$.

(2) M *has rank* $n - m$, $\mathrm{T}_R(\mathcal{S}(M)) = 0$ *and* $\mathrm{grade}\,(f_1, f_2, \ldots, f_m)S = m$.

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In order to explain the meaning of the condition (2) of the theorem above, let us recall the definitions of the rank and the Rees algebra of a module. Let r be a non-negative integer and Q be the total quotient ring of R. We say that an R-module N has rank r if $Q \otimes_R N \cong Q^r$, which is equivalent to saying that $N_\mathfrak{p} \cong R_\mathfrak{p}^r$ for any $\mathfrak{p} \in \operatorname{Ass} R$ (cf. [2, 1.4.3]). If an R-module N has rank r and torsion free, there exist a finitely generated free R-module F and an embedding $\sigma : N \hookrightarrow F$ (cf. [2, 1.4.18]). Then we see that the kernel of $\mathcal{S}(\sigma) : \mathcal{S}(N) \to \mathcal{S}(F)$ coincides with $\mathrm{T}_R(\mathcal{S}(N))$ (cf. [7, p.613]), and so $\operatorname{Im} \mathcal{S}(\sigma) \cong \mathcal{S}(N)/\mathrm{T}_R(\mathcal{S}(N))$ as R-algebras. This means that, up to isomorphisms of R-algebras, $\operatorname{Im} \mathcal{S}(\sigma)$ is independent of the choice of F and σ. So, the Rees algebra of N is defined to be $\mathcal{S}(N)/\mathrm{T}_R(\mathcal{S}(N))$, which is denoted by $\mathcal{R}(N)$. We say that N is a module of linear type if $\mathrm{T}_R(\mathcal{S}(N)) = 0$, that is $\mathcal{S}(N) \cong \mathcal{R}(N)$ as R-algebras. Therefore, if the condition (2) of 1.1 is satisfied, we have $\mathcal{R}(M) \cong S/(f_1, f_2, \ldots, f_m)S$ and the Koszul complex of $f_1, f_2, \ldots, f_m$ gives a S-free resolution of $\mathcal{R}(M)$.

2 Preliminaries

In this section, we summarize preliminary results we need to prove Theorem 1.1.

Lemma 2.1 *Let N be a finitely generated torsion-free R-module having a rank. Then* $\mathrm{T}_{R_\mathfrak{p}}(N_\mathfrak{p}) \cong \mathrm{T}_R(N)_\mathfrak{p}$ *for any* $\mathfrak{p} \in \operatorname{Spec} R$.

Theorem 1.1 will be proved by induction on m. The next result plays a key role in the argument of induction.

Lemma 2.2 *Let $m \geq 2$ and $\mathfrak{p} \in \operatorname{Spec} R$. We assume $\mathrm{I}_1(A) \not\subseteq \mathfrak{p}$. Then there exists $B = (b_{ij}) \in \operatorname{Mat}(m-1, n-1\,; R_\mathfrak{p})$ satisfying the following conditions.*

(a) $\mathrm{I}_i(B) = \mathrm{I}_{i+1}(A)_\mathfrak{p}$ *for any* $i \in \mathbb{Z}$.

(b) *Setting* $S' = R_\mathfrak{p}[x_1, \ldots, x_{n-1}]$ *and* $g_i = b_{i1}x_1 + \cdots + b_{i,n-1} \in S'$ *for* $i = 1, \ldots, m-1$, *we have*
$$\operatorname{grade}(f_1, \ldots, f_{m-1}, f_m)S_\mathfrak{p} = 1 + \operatorname{grade}(g_1, \ldots, g_{m-1})S'.$$

(c) *Setting* $N = \operatorname{Coker}(R_\mathfrak{p}^{m-1} \xrightarrow{{}^tB} R_\mathfrak{p}^{n-1})$, *we have* $M_\mathfrak{p} \cong N$ *as* $R_\mathfrak{p}$*-modules.*

Let us denote the Koszul complex of $f_1, f_2, \ldots, f_m$ with respect to S by

$$0 \longrightarrow C_m \xrightarrow{d_m} C_{m-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{d_1} C_0 \longrightarrow 0\,.$$

Let $u_1, u_2, \ldots, u_m$ be the R-free basis of C_1 such that $d_1(u_i) = f_i$ for $i = 1, \ldots, m$. Then

$$d_r(u_{i_1} \wedge u_{i_2} \wedge \cdots \wedge u_{i_r}) = \sum_{p=1}^{r} (-1)^{p-1} f_{i_p} u_{i_1} \wedge \cdots \wedge \widehat{u_{i_p}} \wedge \cdots \wedge u_{i_r}$$

if $1 \leq r \leq m$ and $1 \leq i_1 < i_2 < \cdots < i_r \leq m$. We regard S as a graded ring by setting $\deg x_j = 1$ for all $j = 1, \ldots, n$. Moreover, we regard $C_\bullet$ as a graded complex by setting

$\deg u_i = 1$ for all $i = 1, \dots, m$. Then, taking the homogeneous component of $C_\bullet$ of degree m, we get a complex

$$0 \longrightarrow [C_m]_m \xrightarrow{[d_m]_m} [C_{m-1}]_m \longrightarrow \cdots \longrightarrow [C_1]_m \xrightarrow{[d_1]_m} [C_0]_m \longrightarrow 0$$

of finitely generated free R-modules, where $[d_r]_m$ denotes the restriction of d_r to $[C_r]_m$ for $r = 1, \dots, m$. Let us notice that $[C_m]_m$ is rank 1 and is generated by $u_1 \wedge u_2 \wedge \cdots \wedge u_m$. Furthermore, as an R-free basis of $[C_{m-1}]_m$, we can take

$$\{ x_j \check{u}_i \mid 1 \le i \le m \, , \, 1 \le j \le n \} ,$$

where $\check{u}_i = u_1 \wedge \cdots \wedge \widehat{u_i} \wedge \cdots \wedge u_m$ for $i = 1, \dots m$. Because

$$\begin{aligned} \partial_m(u_1 \wedge u_2 \wedge \cdots \wedge u_m) &= \sum_{i=1}^{m} (-1)^{i-1} f_i \check{u}_i \\ &= \sum_{i=1}^{m} (-1)^{i-1} (\sum_{j=1}^{n} a_{ij} x_j) \check{u}_i \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} (-1)^{i-1} a_{ij} \cdot x_j \check{u}_i \, , \end{aligned}$$

we get the following result.

Lemma 2.3 $\mathrm{I}_1([d_m]_m) = \mathrm{I}_1(A)$.

The following fact can be regarded as the core of Theorem 1.1.

Lemma 2.4 (cf. [6, Proposition]) *The following conditions are equivalent.*

(1) $\operatorname{grade} \mathrm{I}_m(A) \ge 2$.

(2) *M is torsion-free and has rank $n - m$.*

When this is the case, $\operatorname{pd}_R M \le 1$ *and M can be embedded into a finitely generated free R-module.*

3 Proof of the main theorem

In this section we prove Theorem 1.1.

Proof of (1) $\Rightarrow$ (2).

As $\operatorname{grade} \mathrm{I}_m(A) \ge 2$ by (1), it follows that M is torsion-free and has rank $n - m$ by 2.3. Moreover, we get $\operatorname{grade} (f_1, f_2, \dots, f_m)S = m$ by [1, Proposition 1]. Let us prove $\mathrm{T}_R(\mathcal{S}(M)) = 0$ by induction on m.

First, we consider the case where $m = 1$. Then $\mathcal{S}(M) = S/f_1S$, where $f_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$. Suppose $\mathrm{T}_R(\mathcal{S}(M)) \ne 0$. Then there exists $P \in \operatorname{Ass}_S \mathrm{T}_R(\mathcal{S}(M))$. Because $\mathrm{T}_R(\mathcal{S}(M))$ is an S-submodule of S/f_1S and $\operatorname{grade} f_1S = 1$, we have $\operatorname{depth} S_P =$

1. We set $\mathfrak{p} = P \cap R$. Then $\operatorname{grade} \mathfrak{p} \leq 1$, and so $\mathrm{I}_1(A) \not\subseteq \mathfrak{p}$ as $\operatorname{grade} \mathrm{I}_1(A) \geq 2$. Hence, replacing the columns of A if necessary, we may assume $a_{11} \notin \mathfrak{p}$. Then we have

$$\mathcal{S}(M_{\mathfrak{p}}) = \mathcal{S}(M)_{\mathfrak{p}} = \frac{R_{\mathfrak{p}}[x_1, x_2, \ldots, x_n]}{(x_1 + (a_{12}/a_{11})x_2 + \cdots + (a_{1n}/a_{11})x_n)} \cong R_{\mathfrak{p}}[x_2, \ldots, x_n] .$$

which means $\mathrm{T}_{R_{\mathfrak{p}}}(\mathcal{S}(M_{\mathfrak{p}})) = 0$, and so $\mathrm{T}_R(\mathcal{S}(M))_{\mathfrak{p}} = 0$ by 2.1. Therefore it follows that $\mathrm{T}_R(\mathcal{S}(M))_P = 0$, which contradicts to $P \in \operatorname{Ass}_S \mathrm{T}_R(\mathcal{S}(M))$. Thus we see $\mathrm{T}_R(\mathcal{S}(M)) = 0$ in the case where $m = 1$.

Next, we assume $m \geq 2$ and the required implication is true for matrices having $m - 1$ rows. Suppose $\mathrm{T}_R(\mathcal{S}(M)) \neq 0$. Then there exists $P \in \operatorname{Ass}_R \mathrm{T}_R(\mathcal{S}(M))$. Because $\mathrm{T}_R(\mathcal{S}(M))$ is an S-submodule of $S/(f_1, f_2, \ldots, f_m)S$ and $\operatorname{grade}(f_1, f_2, \ldots, f_m)S = m$, we have $\operatorname{depth} S_P = m$. We set $\mathfrak{p} = P \cap R$. Then $\operatorname{grade} \mathfrak{p} \leq m$, and so $\mathrm{I}_{\mathrm{I}_1(A)}() \not\subseteq \mathfrak{p}$ as $\operatorname{grade} 1A \geq m + 1$. Hence, there exists $B = (b_{ij}) \in \operatorname{Mat}(m-1, n-1; R_{\mathfrak{p}})$ satisfying the conditions (a), (b) and (c) of 2.2. By (a), for any $i = 1, \ldots, m-1$, we have

$$\operatorname{grade} \mathrm{I}_i(B) = \operatorname{grade} \mathrm{I}_{i+1}(A)_{\mathfrak{p}} \geq \operatorname{grade} \mathrm{I}_{i+1}(A) \geq m - (i+1) + 2 = (m-1) - i + 2 .$$

Therefore, setting

$$N = \operatorname{Coker}(R_{\mathfrak{p}}^{m-1} \xrightarrow{{}^tB} R_{\mathfrak{p}}^{n-1}) ,$$

we get $\mathrm{T}_{R_{\mathfrak{p}}}(\mathcal{S}(N)) = 0$ by the hypothesis of induction. Because $\mathcal{S}(M_{\mathfrak{p}}) \cong \mathcal{S}(N)$ by (c), we have $\mathrm{T}_{R_{\mathfrak{p}}}(\mathcal{S}(M_{\mathfrak{p}})) = 0$, and so $\mathrm{T}_R(M)_{\mathfrak{p}} = 0$ by 2.1. This means $\mathrm{T}_R(\mathcal{S}(M))_P = 0$, which contradicts to $P \in \operatorname{Ass}_S \mathrm{T}_R(\mathcal{S}(M))$. Thus we see $\mathrm{T}_R(\mathcal{S}(M)) = 0$.

Proof of (2) $\Rightarrow$ (1).

Let us consider the Koszul complex

$$0 \longrightarrow C_m \xrightarrow{d_m} C_{m-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{d_1} C_0 \longrightarrow 0$$

described in Section 2. This is graded and acyclic as $\operatorname{grade}(f_1, f_2, \ldots, f_m)S = m$. Moreover, we have

$$\operatorname{Coker} d_1 = S/(f_1, f_2, \ldots, f_m)S = \mathcal{S}(M) .$$

The condition $\mathrm{T}_R(\mathcal{S}(M)) = 0$ implies that M is torsion-free over R, and so by [2, 1.4.18] there exist a finitely generated free R-module F and an embedding $\sigma : M \hookrightarrow F$. Then the induced homomorphism $\mathcal{S}(\sigma) : \mathcal{S}(M) \longrightarrow \mathcal{S}(F)$ is injective since $\operatorname{Ker} \mathcal{S}(\sigma) = \mathrm{T}_R(\mathcal{S}(M)) = 0$. Thus we get a graded acyclic complex

$$0 \longrightarrow C_m \xrightarrow{d_m} C_{m-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{d_1} C_0 \longrightarrow \mathcal{S}(F) \longrightarrow 0 .$$

Taking its homogeneous component of degree m, we get an acyclic complex

$$0 \longrightarrow [C_m]_m \xrightarrow{[d_m]_m} [C_{m-1}]_m \longrightarrow \cdots \longrightarrow [C_1]_m \xrightarrow{[d_1]_m} [C_0]_m \longrightarrow [\mathcal{S}(F]_m) \longrightarrow 0 ,$$

of finitely generated free R-modules. Let us notice that $\operatorname{rank}_R [C_m]_m = 1$ and $\mathrm{I}_1([d_m]_m) = \mathrm{I}_1(A)$ by 2.3. Hence we get $\operatorname{grade} \mathrm{I}_1(A) \geq m + 1$ by [2, 1.4.13].

In the rest of this proof, we show that the condition (1) holds by induction on m. If $m = 1$, it is certainly true by the observation stated above. So, let us consider the case where $m \geq 2$. We suppose that $\operatorname{grade} \mathrm{I}_j(A) \leq m-j+1$ for some j with $2 \leq j \leq m$. Then there exists $\mathfrak{p} \in \operatorname{Spec} R$ such that $\mathrm{I}_j(A) \subseteq \mathfrak{p}$ and $\operatorname{depth} R_\mathfrak{p} \leq m-j+1$. As $\operatorname{grade} \mathrm{I}_1(A) \geq m+1$, we have $\mathrm{I}_1(A) \not\subseteq \mathfrak{p}$, and so there exists $B = (b_{ij}) \in \operatorname{Mat}(m-1, n-1\,; R_\mathfrak{p})$ satisfying the conditions (a), (b) and (c) of 2.2. We set

$$N = \operatorname{Coker}(R_\mathfrak{p}^{m-1} \xrightarrow{{}^tB} R_\mathfrak{p}^{n-1}).$$

Then by (c) we have $N \cong M_\mathfrak{p}$ as $R_\mathfrak{p}$-modules, so N has rank $n-m$ and $\mathcal{S}(N) \cong \mathcal{S}(M_\mathfrak{p})$ as $R_\mathfrak{p}$-algebras. Because $\mathrm{T}_{R_\mathfrak{p}}(\mathcal{S}(M_\mathfrak{p})) = \mathrm{T}_R(\mathcal{S}(M))_\mathfrak{p} = 0$ by 2.1, it follows that $\mathrm{T}_{R_\mathfrak{p}}(\mathcal{S}(N)) = 0$. Moreover, setting $S' = R_\mathfrak{p}[x_1, \ldots, x_{n-1}]$ and $g_i = b_{i1}x_1 + \cdots + b_{i,n-1}x_{n-1} \in S'$ for $i = 1, \ldots, m-1$, we get

$$\operatorname{grade}(g_1, \ldots, g_{m-1})S' = \operatorname{grade}(f_1, \ldots, f_{m-1}, f_m)S_\mathfrak{p} - 1 = m-1$$

by (b). Therefore the hypothesis of induction implies

$$\operatorname{grade} \mathrm{I}_{j-1}(A) \geq (m-1) - (j-1) + 2 = m-j+2.$$

Then, as $\mathrm{I}_{j-1}(B) = \mathrm{I}_j(A)_\mathfrak{p}$ by (a), we get $\operatorname{grade} \mathrm{I}_j(A)_\mathfrak{p} \geq m-j+2$, which contradicts to $\operatorname{depth} R_\mathfrak{p} \leq m-j+1$. Thus we see $\operatorname{grade} \mathrm{I}_i(A) \geq m-i+2$ for any $i = 1, 2, \ldots, m$ and the proof is complete.

4 Example

In this section, we give examples of matrices satisfying the condition (1) of Theorem 1.1.

Example 4.1 *Let m and d be positive integers such that $m < d$. Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring and $x_1, x_2, \ldots, x_n$ be elements of R generating an $\mathfrak{m}$-primary ideal. We take a family $\{\alpha_{ij}\}_{1 \leq i \leq m,\, 1 \leq j \leq n}$ of positive integers, and set*

$$a_{ij} = \begin{cases} x_{i+j-1}^{\alpha_{ij}} & \text{if } i+j \leq n+1 \\[2ex] x_{i+j-n-1}^{\alpha_{ij}} & \text{if } i+j > n+1 \end{cases}$$

and $A = (a_{ij}) \in \operatorname{Mat}(m, n\,; R)$. Then we have $\operatorname{grade} \mathrm{I}_i(A) \geq m-i+2$ for $1 \leq \forall i \leq m$.

If $\alpha_{ij} = 1$ for any i and j, the matrix A stated above looks like

$$\begin{pmatrix} x_1 & x_2 & \cdots & & x_m & \cdots & x_n \\ x_2 & & & x_m & & x_n & x_1 \\ \vdots & & \iddots & & \iddots & \iddots & \vdots \\ & x_m & & x_n & x_1 & & x_{m-2} \\ x_m & \cdots & x_n & x_1 & \cdots & x_{m-2} & x_{m-1} \end{pmatrix}.$$

However, we can take any power at each entries.

References

[1] L. Avramov, *Complete intersections and symmetric algebras*, J. Algebra, **73** (1981), 248–263.

[2] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Stud. Adv. Math. **39**, Cambridge University Press, 1997.

[3] K. Fukumuro, *On the symmetric and Rees algebras of certain determinantal ideals*, to appear in Tokyo J. Math., arXiv:1306.0993.

[4] K. Fukumuro, T. Inagawa and K. Nishida, *On a transform of an acyclic complex of length* 3, J. Algebra, **384** (2013), 84–109.

[5] T. Inagawa, $*$-*transforms of acyclic complexes*, Preprint (2013), arXiv:1307.1500.

[6] D. Katz, *Torsion-free modules and syzygies*, Mathematica Pannonica, **5**/1 (1994), 7–13.

[7] A. Simis, B. Ulrich and W. Vasconcelos, *Rees algebras of modules*, Proc. London Math. Soc. (3), **87** (2003), 610–646