The automorphism group of a UFD over the kernel of a locally nilpotent derivation

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1 Introduction

Let A be an integral domain containing $\mathbf{Q}$, and δ a nonzero *locally nilpotent derivation* of A, i.e., a derivation of A such that, for each $a \in A$, there exists $l \geq 1$ satisfying $\delta^l(a) = 0$. We denote by $\mathrm{Aut}(A/A^\delta)$ the automorphism group of the A^δ-algebra A, and by $\mathrm{LND}(A/A^\delta)$ the set of locally nilpotent A^δ-derivations of A. For each $D \in \mathrm{LND}(A/A^\delta)$, the *exponential automorphism* $\exp D \in \mathrm{Aut}(A/A^\delta)$ is defined by

$$(\exp D)(a) = \sum_{l=0}^{\infty} \frac{D^l(a)}{l!}$$

for $a \in A$. Then, $\mathcal{N}_\delta := \{\exp D \mid D \in \mathrm{LND}(A/A^\delta)\}$ forms a normal subgroup of $\mathrm{Aut}(A/A^\delta)$ (cf. Proposition 2.1 (ii)). In this report, we discuss the structure of the quotient group

$$\mathrm{Aut}(A/A^\delta)/\mathcal{N}_\delta. \tag{1.1}$$

We call $z \in A$ a *slice* of the extension A/A^δ if $A = A^\delta[z]$. If this is the case, A is the polynomial ring in z over A^δ. Hence, we have $A^\times = (A^\delta)^\times$ and

$$\mathrm{Aut}(A/A^\delta) = \{\psi_{a,b} \mid a \in A^\times, b \in A^\delta\}, \quad \mathrm{LND}(A/A^\delta) = \{b(d/dz) \mid b \in A^\delta\}, \tag{1.2}$$

where $\psi_{a,b} \in \mathrm{Aut}(A/A^\delta)$ is such that $\psi_{a,b}(z) = az + b$. Since $\exp b(d/dz) = \psi_{1,b}$ for each $b \in A^\delta$, we see that (1.1) is isomorphic to $A^\times$ in this case. The aim of this research is to study the quotient group (1.1) when A/A^δ has no slice.

2 Key results

First, we recall some basics on locally nilpotent derivations. For each $a \in A \setminus \{0\}$, we define the δ-*degree* of a by

$$\deg_\delta(a) := \max\{l \in \mathbf{Z}_{\geq 0} \mid \delta^l(a) \neq 0\}.$$

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We call $z \in A \setminus \{0\}$ a *local slice* of δ if $\deg_\delta(z) = 1$, that is, $\delta(z)$ belongs to $A^\delta \setminus \{0\}$. If z is a slice of A/A^δ, then we have $\delta = \delta(z)(d/dz)$, and so z is a local slice of δ by (1.2). The ideal $\mathrm{pl}(\delta) := A^\delta \cap \delta(A)$ of A^δ is called the *plinth ideal* of δ. Since δ is nonzero and locally nilpotent, we have $\mathrm{pl}(\delta) \neq \{0\}$. Hence, there always exists a local slice. We define

$$\Gamma_\delta := \{a \in Q(A^\delta) \mid a\,\mathrm{pl}(\delta) = \mathrm{pl}(\delta)\}.$$

Then, Γ_δ is a subgroup of $Q(A^\delta)^\times$. Since $A^\times = (A^\delta)^\times$ (cf. [4, Corollary 1.3.36]), we see that $A^\times$ is contained in Γ_δ.

In the notation above, the following proposition holds.

Proposition 2.1. (i) *For each $\phi \in \mathrm{Aut}(A/A^\delta)$ and a local slice $z \in A$ of δ, there exist $u_\phi \in \Gamma_\delta$ and $b \in A^\delta$ such that $\phi(z) = u_\phi z + b$. Moreover, u_ϕ is defined only from ϕ, and does not depend on the choice of the local slice z.*
(ii) $\theta : \mathrm{Aut}(A/A^\delta) \ni \phi \mapsto u_\phi \in \Gamma_\delta$ *is a homomorphism of groups with* $\ker\theta = \mathcal{N}_\delta$.
(iii) *For each* $\phi \in \mathrm{Aut}(A/A^\delta) \setminus \mathcal{N}_\delta$, *we have* $\mathrm{ord}(\phi) = \mathrm{ord}(\theta(\phi))$.
(iv) *If* $\mathrm{LND}(A/A^\delta) = \{a\delta_0 \mid a \in A^\delta\}$ *for some* $\delta_0 \in \mathrm{LND}(A/A^\delta)$, *then* $\mathrm{Im}\,\theta$ *is contained in* $A^\times$.

By Proposition 2.1 (ii), we know that $\mathrm{Aut}(A/A^\delta)/\mathcal{N}_\delta$ is isomorphic to $\mathrm{Im}\,\theta$, and hence is an abelian group. We note that every element of $\mathcal{N}_\delta \setminus \{\mathrm{id}_A\}$ has infinite order.

As for Γ_δ, we have the following result.

Proposition 2.2. *We have $\Gamma_\delta = A^\times$ if one of the following conditions holds.*
(a) $\mathrm{pl}(\delta)$ *is a principal ideal.*
(b) *A is normal and* $\mathrm{pl}(\delta)$ *is finitely generated.*
(c) *A satisfies the Ascending Chain Condition for principal ideals, and there exist a finite number of prime elements $p_1, \ldots, p_l$ of A such that Γ_δ is contained in $A^\delta_{p_1\cdots p_l}$.*
(d) *A satisfies the Ascending Chain Condition for principal ideals, and there exists a local slice $z \in A$ of δ such that $\delta(z)$ is a product of prime elements of A.*
(e) *A is a UFD.*

Now, we define

$$\mathrm{ord}(A/A^\delta) := \begin{cases} \min\{\deg_\delta a \mid a \in A \setminus A^\delta[z]\} & \text{if } \mathrm{pl}(\delta) \text{ is a principal ideal} \\ 1 & \text{otherwise,} \end{cases}$$

where $z \in A$ is such that $\mathrm{pl}(\delta) = \delta(z)A^\delta$. Since $\delta(z)A^\delta = \delta(w)A^\delta$ implies $z = \alpha w + \beta$ for some $\alpha \in (A^\delta)^\times$ and $\beta \in A^\delta$, we see that the definition of $\mathrm{ord}(A/A^\delta)$ does not depend on the choice of z. By definition, A/A^δ has a slice if $\mathrm{ord}(A/A^\delta) = \infty$. Conversely, if A/A^δ has a slice z, then $A = A^\delta[z]$, and $\mathrm{pl}(\delta) = \delta(z)A^\delta$, since $\delta(A^\delta[z]) \subset \delta(z)A$ and $\delta(z) \in A^\delta$. Hence, we have $\mathrm{ord}(A/A^\delta) = \infty$.

Proposition 2.3. *Assume that A/A^δ has no slice. If* $\mathrm{pl}(\delta)$ *contains the product of a finite number of prime elements of A, then $(\mathrm{Im}\,\theta)_{\mathrm{tor}}$ is a finite cyclic group of order at most* $\mathrm{ord}(A/A^\delta)$.

Here, we define $M_{\rm tor} := \{a \in M \mid \mathrm{ord}(a) < \infty\}$ for each group M.

In the case of UFD, $\mathrm{Im}\,\theta$ is a subgroup of $A^\times$ by Proposition 2.2 (e). Since $\mathrm{pl}(\delta)$ contains the product of a finite number of prime elements of A, (i) of the following theorem is a consequence of Proposition 2.3.

Theorem 2.4. *Assume that A is a UFD. Then, the following assertions hold.*
(i) *If A/A^δ has no slice, then $(\mathrm{Im}\,\theta)_{\rm tor}$ is a finite cyclic group of order at most $\mathrm{ord}(A/A^\delta)$.*
(ii) *If A/A^δ has no slice, and if $\zeta^i - 1$ belongs to $A^\times$ for any $i \geq 1$ and $\zeta \in A^\times \setminus (A^\times)_{\rm tor}$, then we have $\mathrm{Im}\,\theta = (\mathrm{Im}\,\theta)_{\rm tor}$.*
(iii) *If ζ is an element of $\mathrm{Im}\,\theta \setminus (\mathrm{Im}\,\theta)_{\rm tor}$, then $A_{\underline{\zeta}}/A_{\underline{\zeta}}^\delta$ has a slice, where δ is the unique extension of δ to $A_{\underline{\zeta}} := A[\{1/(\zeta^i - 1) \mid i \geq 1\}]$.*

Thanks to (i) and (ii) of Theorem 2.4, we obtain the following theorem.

Theorem 2.5. *Assume that A is a UFD such that $A^\times \cup \{0\}$ is a field. If A/A^δ has no slice, then $\mathrm{Aut}(A/A^\delta)/\mathcal{N}_\delta$ is isomorphic to a finite cyclic subgroup of $A^\times$ of order at most $\mathrm{ord}(A/A^\delta)$.*

In the situation of Theorem 2.5, each element of $\mathrm{Aut}(A/A^\delta) \setminus \mathcal{N}_\delta$ has finite order by Proposition 2.1 (iii).

3 Polynomial ring

We are especially interested in the case where A is the polynomial ring $k[\mathbf{x}] := k[x_1, \ldots, x_n]$ over a field k of characteristic zero. Even in the case of $n = 3$, the structure of the automorphism group $\mathrm{Aut}_k\, k[\mathbf{x}]$ of this k-algebra remains mysterious. Since $k[\mathbf{x}]$ is a UFD with $k[\mathbf{x}]^\times \cup \{0\} = k$, the assumption of Theorem 2.5 is satisfied. Hence, if $k[\mathbf{x}]/k[\mathbf{x}]^\delta$ has no slice, then $\mathrm{Aut}(k[\mathbf{x}]/k[\mathbf{x}]^\delta)/\mathcal{N}_\delta$ is isomorphic to a finite cyclic subgroup of $k^\times$.

For each $f = \sum_a u_a \mathbf{x}^a \in k[\mathbf{x}]$, we define $\mathrm{supp}(f) := \{a \mid u_a \neq 0\}$, where $u_a \in k$ and $\mathbf{x}^a = x_1^{a_1} \cdots x_n^{a_n}$ for each $a = (a_1, \ldots, a_n)$. We define M_δ to be the $\mathbf{Z}$-submodule of $\mathbf{Z}^n$ generated by

$$\bigcup_{f \in k[\mathbf{x}]^\delta} \mathrm{supp}(f).$$

We mention that, for any given δ, the generators of M_δ can be computed by means of a standard technique for locally nilpotent derivations. In fact, we can construct $f_1, \ldots, f_n, g \in k[\mathbf{x}]^\delta \setminus \{0\}$ satisfying $k[\mathbf{x}]^\delta \subset k[f_1, \ldots, f_n, g^{-1}]$. Then, M_δ is generated by $\mathrm{supp}(f_1) \cup \cdots \cup \mathrm{supp}(f_n) \cup \mathrm{supp}(g)$.

(ii) of the following theorem is a consequence of Theorem 2.5.

Theorem 3.1. (i) *If $\mathrm{rank}\, M_\delta < n$, then we have $\delta = f\partial/\partial x_i$ for some $1 \leq i \leq n$ and $f \in k[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$. Hence, $k[\mathbf{x}]/k[\mathbf{x}]^\delta$ has a slice, and $\mathbf{Z}^n/M_\delta \simeq \mathbf{Z}$.*
(ii) *Assume that $d := \#(\mathbf{Z}^n/M_\delta)$ is finite. Then, $\mathbf{Z}^n/M_\delta$ is a cyclic group. If k contains a primitive d-th root of unity, then $\mathrm{Aut}(k[\mathbf{x}]/k[\mathbf{x}]^\delta) \setminus \mathcal{N}_\delta$ contains an element of order d.*

For example, let δ be the locally nilpotent derivation of $k[\mathbf{x}]$ for $n = 3$ defined by $\delta(x_1) = 0$, $\delta(x_2) = x_1$ and $\delta(x_3) = -2x_2$. Then, we have $k[\mathbf{x}]^\delta = k[x_1, x_1x_3 + x_2^2]$. In this case, M_δ is generated by

$$\mathrm{supp}(x_1) \cup \mathrm{supp}(x_1x_3 + x_2^2) = \{(1,0,0),(1,0,1),(0,2,0)\}.$$

Hence, we have $\mathbf{Z}^3/M_\delta \simeq \mathbf{Z}/2\mathbf{Z}$. The automorphism of $k[\mathbf{x}]$ defined by $x_2 \mapsto -x_2$ and $x_i \mapsto x_i$ for $i = 1, 3$ belongs to $\mathrm{Aut}(k[\mathbf{x}]/k[\mathbf{x}]^\delta) \setminus \mathcal{N}_\delta$.

The *rank* $\mathrm{rank}(\delta)$ of δ is by definition the minimal number $0 \leq r \leq n$ for which there exist $\phi \in \mathrm{Aut}_k\, k[\mathbf{x}]$ and $f_1, \ldots, f_r \in k[\mathbf{x}]$ such that

$$\phi \circ \delta \circ \phi^{-1} = f_1 \frac{\partial}{\partial x_1} + \cdots + f_r \frac{\partial}{\partial x_r}.$$

Due to Rentschler [15], the extension $k[\mathbf{x}]/k[\mathbf{x}]^\delta$ always has a slice if $n = 2$. In the case of $n = 3$, there always exist $f_1, f_2 \in A^\delta$ such that $A^\delta = k[f_1, f_2]$ by Miyanishi [12]. This means that $\mathrm{rank}(\delta) = 1$ if A/A^δ has a slice. Thus, $\mathrm{rank}(\delta) \geq 2$ implies that $k[\mathbf{x}]/k[\mathbf{x}]^\delta$ has no slice when $n = 3$. Using Asanuma [2] (see also [6]), we can prove that $k[\mathbf{x}]/k[\mathbf{x}]^\delta$ has no slice if $n \geq 3$ and $\mathrm{rank}(\delta) = 2$. Therefore, we have the following corollary to Theorem 2.5.

Corollary 3.2. *Assume that $n = 3$ and $\mathrm{rank}(\delta) \geq 2$, or $n \geq 3$ and $\mathrm{rank}(\delta) = 2$. Then, $\mathrm{Aut}(k[\mathbf{x}]/k[\mathbf{x}]^\delta)/\mathcal{N}_\delta$ is isomorphic to a finite cyclic subgroup of $k^\times$ of order at most $\mathrm{ord}(k[\mathbf{x}]/k[\mathbf{x}]^\delta)$.*

We mention that $\mathrm{pl}(\delta)$ is a principal ideal if $n = 3$ by Daigle-Kaliman [3, Theorem 1]. The following theorem is a consequence of Theorem 5.3 stated later.

Theorem 3.3. *Assume that $n = 3$ and let δ be a locally nilpotent derivation of $k[\mathbf{x}]$ with $\mathrm{rank}(\delta) = 3$. Then, we have $M_\delta = \mathbf{Z}^3$.*

A k-derivation D of $k[\mathbf{x}]$ is said to be *triangular* if $D(x_i)$ belongs to $k[x_1, \ldots, x_{i-1}]$ for $i = 1, \ldots, n$. It is easy to see that D is locally nilpotent if D is triangular. We say that D is *triangularizable* if $\phi \circ D \circ \phi^{-1}$ is triangular for some $\phi \in \mathrm{Aut}_k\, k[\mathbf{x}]$. Since every triangular k-derivation of $k[\mathbf{x}]$ has rank at most $n - 1$, the same holds for every triangularizable k-derivation of $k[\mathbf{x}]$.

The following theorem is proved by using Theorem 4.4 stated later.

Theorem 3.4. *Assume that $n \geq 3$ and $\mathrm{rank}(\delta) = 2$. If $\mathrm{Aut}(k[\mathbf{x}]/k[\mathbf{x}]^\delta) \neq \mathcal{N}_\delta$, then δ is triangularizable.*

Now, let R be a UFD, and $R[\mathbf{x}] = R[x_1, x_2]$ the polynomial ring in two variables over R. We discuss a triangular R-derivation of $R[\mathbf{x}]$ of a special form. Let $p(z) = \sum_{i \geq 0} b_i z^i \in R[z]$ be a polynomial in one variable over R, and $a \in R \setminus (R^\times \cup \{0\})$ such that a and $p(z) - b_0$ have no non-unit common factor. We define a triangular R-derivation D of $R[x_1, x_2]$ by

$$D = a \frac{\partial}{\partial x_1} - p'(x_1) \frac{\partial}{\partial x_2}, \tag{3.1}$$

where $p'(z)$ is the derivative of $p(z)$. Then, the R-algebra $R[\mathbf{x}]^D$ is generated by $f := ax_2 + p(x_1)$, and the extension $R[\mathbf{x}]/R[\mathbf{x}]^D$ has a slice if and only if

(†) the image of b_i in R/aR is a unit if $i = 1$, and nilpotent if $i \geq 2$.
We note that the image of $b \in R$ in R/aR is nilpotent if and only if b is divisible by $\sqrt{a}$, where $\sqrt{a} \in R$ is such that $\sqrt{a}R$ is the radical of aR.

In the notation and assumption above, the following theorem holds.

Theorem 3.5. *Let $A := R[x_1, x_2]$ and $\delta := D$ be as above. When R contains a primitive d-th root $\zeta \in R^\times$ of unity with $d \geq 2$, the following conditions are equivalent:*
(1) $\mathrm{Aut}(A/A^\delta)/\mathcal{N}_\delta$ *contains an element of order d.*
(2) *$p(z)$ belongs to $R[(z + q(p(z)))^d] + aR[z]$ for some $q(z) \in \sqrt{a}R[z]z + R$.*

If this is the case, we can define $\phi \in \mathrm{Aut}(A/A^\delta)$ with $\theta(\phi) = \zeta$ by

$$\phi(x_1) = \zeta x_1 + (\zeta - 1)q(f), \quad \phi(x_2) = x_2 + \frac{p(x_1) - \phi(p(x_1))}{a}. \tag{3.2}$$

4 Linearization Problem

The following problem is a difficult problem with very little progress.

Problem 4.1 (Linearization Problem)**.** Let $\phi \in \mathrm{Aut}_{\mathbf{C}}\,\mathbf{C}[\mathbf{x}]$ be such that $\phi^d = \mathrm{id}_{\mathbf{C}[\mathbf{x}]}$ for some $d \geq 2$. Does it follow that ϕ is linearizable.

Note that ϕ is linearizable if and only if there exist $\psi \in \mathrm{Aut}_{\mathbf{C}}\,\mathbf{C}[\mathbf{x}]$ and $\alpha_1, \ldots, \alpha_n \in \mathbf{C}^\times$ such that $(\psi^{-1} \circ \phi \circ \psi)(x_i) = \alpha_i x_i$ for $i = 1, \ldots, n$.

Due to Kambayashi [7], the answer is affirmative if $n = 2$. The problem remains open for $n \geq 3$. Quite recently, the author proved the following.

Theorem 4.2. *Let R be a PID, and $\phi \in \mathrm{Aut}_R\, R[x_1, x_2]$ such that $\mathrm{ord}(\phi) = d$ for some $d \geq 1$. If R contains a primitive d-th root of unity, then ϕ is linearizable.*

This theorem immediately implies the following.

Corollary 4.3. *Let $\phi \in \mathrm{Aut}_k[x_1, x_2, x_3]$ be such that $\phi(x_3) = x_3$ and $\mathrm{ord}(\phi) = d$ for some $d \geq 1$. If k contains a primitive d-th root ζ of unity, then ϕ is linearizable as an automorphism over $k[x_3]$.*

Assume that $n \geq 3$, and let $\phi \in \mathrm{Aut}_k\, k[\mathbf{x}]$ be such that $\phi(x_i) = x_i$ for $i = 3, \ldots, n$ and $\mathrm{ord}(\phi) = d$ for some $d \geq 2$. Then, ϕ is regarded as an element of $\mathrm{Aut}_K\, K[x_1, x_2]$, where $K := k(x_3, \ldots, x_n)$. Hence, if k contains a primitive d-th root ζ of unity, then there exist $\psi \in \mathrm{Aut}_K\, K[x_1, x_2]$ and $d_1, d_2 \in \mathbf{Z}$ such that $(\psi^{-1} \circ \phi \circ \psi)(x_i) = \zeta^{d_i} x_i$ for $i = 1, 2$. In this situation, we have the following theorem.

Theorem 4.4. *If $\gcd(d, d_1) > 1$ or $\gcd(d, d_2) > 1$, then ϕ is linearizable as an automorphism over $k[x_3, \ldots, x_n]$.*

Finally, we mention a relation between Problem 4.1 and the *Cancellation Problem.*

Problem 4.5 (Cancellation Problem)**.** Let R be a $\mathbf{C}$-algebra, and $R[z]$ the polynomial ring in one variable over R. Assume $R[z]$ is $\mathbf{C}$-isomorphic to $\mathbf{C}[x_1, \ldots, x_n]$. Does it follow that R is $\mathbf{C}$-isomorphic to $\mathbf{C}[x_1, \ldots, x_{n-1}]$?

This is a famous problem in Affine Algebraic Geometry. The answer is affirmative if $n = 2$ by Abhyankar-Heinzer-Eakin [1], and if $n = 3$ by Fujita [5] and Miyanishi-Sugie [13]. The problem remains open for $n \geq 4$.

It is well known that Problem 4.1 implies Problem 4.5. More precisely, the following remark holds.

Remark 4.6. *Fix $n \in \mathbf{N}$. If there exists $d \geq 2$ such that Problem 4.1 has an affirmative answer for each $\phi \in \mathrm{Aut}_{\mathbf{C}}\, \mathbf{C}[\mathbf{x}]$ with $\mathrm{ord}(\phi) = d$, then Problem 4.5 has an affirmative answer.*

As this remark suggests, the statement of Problem 4.1 is quite strong.

5 Wang's type theorem

Wang [16] proved the following theorem.

Theorem 5.1 (Wang)**.** *Let δ be a locally nilpotent derivation of $k[x_1, x_2, x_3]$ such that $\delta^2(x_i) = 0$ for $i = 1, 2, 3$. Then, we have $\mathrm{rank}(\delta) \leq 1$.*

We proved the following theorem similar to Wang's by using the Shestakov-Umirbaev inequality [14] (cf. [8]) and some deep results on locally nilpotent derivations.

Theorem 5.2. *Let δ be a locally nilpotent derivation of $k[x_1, x_2, x_3]$ such that $\delta^2(x_1) = 0$. Then, we have $\mathrm{rank}(\delta) \leq 2$.*

As an application of Theorem 5.2, we obtain the following result.

Theorem 5.3. *Assume that $n = 3$ and let δ be a locally nilpotent derivation of $k[\mathbf{x}]$ with $\mathrm{rank}(\delta) = 3$. Then, no element of $\mathrm{Aut}(k[\mathbf{x}]/k[\mathbf{x}]^{\delta}) \setminus \{1\}$ is linearizable.*

By Proposition 2.1 (iii) and Corollary 3.2, every element of $\mathrm{Aut}(\mathbf{C}[\mathbf{x}]/\mathbf{C}[\mathbf{x}]^{\delta}) \setminus \mathcal{N}_{\delta}$ has finite order if $n = \mathrm{rank}(\delta) = 3$. Therefore, if $\mathrm{Aut}(\mathbf{C}[\mathbf{x}]/\mathbf{C}[\mathbf{x}]^{\delta}) \neq \mathcal{N}_{\delta}$ for some δ, then Problem 4.1 has a negative answer by Theorem 5.3.

6 Examples

To end this report, we give some examples.

First, we construct an example in which $\mathrm{Im}\,\theta$ is an infinite group when A is a UFD. Let $R = \mathbf{Q}[t^{\pm 1}]$ be the Laurent polynomial ring in one variable over $\mathbf{Q}$ and $A = R[x_1, x_2]$. Take any $p(x_1) \in R[x_1]$ such that $\gcd(a, p'(x_1)) = 1$, and define D as in (3.1) with $a := t - 1$. Then, we have $A^D = R[f]$, where $f = ax_2 + p(x_1)$. We can define $\phi \in \mathrm{Aut}(A/A^D)$ by

$$\phi(x_1) = tx_1 \quad \text{and} \quad \phi(x_2) = x_2 + \frac{p(x_1) - p(tx_1)}{t - 1}.$$

Since x_1 is a local slice of D, we have $\theta(\phi) = t$. Therefore, $\mathrm{Im}\,\theta$ is an infinite group.

Next, we give an example in which $\mathrm{Im}\,\theta$ is not contained in $A^{\times}$. Consider the $\mathbf{Q}$-subalgebras $R := \mathbf{Q} + \mathbf{Q}[x_1^{\pm 1}, x_2]x_2$ and $A := R + \mathbf{Q}[x_1^{\pm 1}, x_2]x_3$ of the polynomial ring

$\mathbf{Q}[x_1^{\pm 1}][x_2, x_3]$ in x_2 and x_3 over the Laurent polynomial ring $\mathbf{Q}[x_1^{\pm 1}]$. It is easy to see that $A^\times = R^\times = \mathbf{Q}^\times$, and the $\mathbf{Q}$-algebra A is not finitely generated. For the locally nilpotent derivation $\delta = x_2 \partial/\partial x_3$ of A, we have

$$A^\delta = A \cap \mathbf{Q}[x_1^{\pm 1}, x_2, x_3]^\delta = A \cap \mathbf{Q}[x_1^{\pm 1}, x_2] = R, \quad \mathrm{pl}(\delta) = \mathbf{Q}[x_1^{\pm 1}, x_2]x_2.$$

Actually, $\mathrm{pl}(\delta) = A^\delta \cap \delta(A)$ is contained in $R \cap x_2 A = \mathbf{Q}[x_1^{\pm 1}, x_2]x_2$. Conversely, for each $l \in \mathbf{Z}$, the element $x_1^l x_2 = \delta(x_1^l x_3)$ of $R = A^\delta$ belongs to $\delta(A)$, and hence belongs to $\mathrm{pl}(\delta)$. Define $\phi \in \mathrm{Aut}(A/A^\delta) = \mathrm{Aut}(A/R)$ by $\phi(x_i) = x_i$ for $i = 1, 2$ and $\phi(x_3) = x_1 x_3$. Then, we have $\theta(\phi) = x_1$, since x_3 is a local slice of δ. Therefore, $\mathrm{Im}\,\theta$ is not contained in $\mathbf{Q}^\times = A^\times$.

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