# Arithmetical rank of Gorenstein squarefree monomial ideals of height three [1] [2]

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## 1. Introduction

Let $S$ be a polynomial ring over a field $K$ and $I$ a squarefree monomial ideal of $S$. The *arithmetical rank* of $I$, denoted by $\operatorname{ara} I$, is defined as the minimum number $u$ of elements $q_1, \ldots, q_u \in S$ such that $\sqrt{(q_1, \ldots, q_u)} = \sqrt{I} (= I)$. When this is the case, we say that $q_1, \ldots, q_u$ *generate* $I$ *up to radical.* By the result of Lyubeznik [13], we have the following inequalities:

$$\operatorname{height} I \leq \operatorname{pd} S/I \leq \operatorname{ara} I,$$

where $\operatorname{pd} S/I$ is the projective dimension of $S/I$ (over $S$). If $\operatorname{ara} I = \operatorname{height} I$ holds, then $I$ is said to be a *set-theoretic complete intersection.* By the inequalities, it is natural to ask which ideal $I$ satisfies $\operatorname{ara} I = \operatorname{pd} S/I$ or which (Cohen–Macaulay) ideal $I$ is a set-theoretic complete intersection. Many authors have studied this problem and proved $\operatorname{ara} I = \operatorname{pd} S/I$ for some ideals $I$, see e.g., [2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 14]. However, counterexamples for the equality were also found; see [15, 11], though the projective dimensions of those are depend on the characteristic of the base field $K$.

Among the above references, we note [7] and [15]. In [7], the first author proved that $\operatorname{ara} I = \operatorname{pd} S/I$ holds (and thus, $I$ is a set-theoretic complete intersection) for a Cohen–Macaulay squarefree monomial ideal $I$ of height 2. On the other hand, in [15], Yan found a counterexample for the equality among Cohen–Macaulay squarefree monomial ideals of height 3: let $\Delta$ be the triangulation of the real projective plane with 6 vertices. Then the Stanley–Reisner ideal $I_\Delta$ is of height 3, $\operatorname{pd} S/I_\Delta$ is 3 if $\operatorname{char} K \neq 2$; 4 if $\operatorname{char} K = 2$. Yan [15] proved that $\operatorname{ara} I_\Delta = 4$ for any characteristic $K$.

Then it is natural to ask whether the equality holds for a Gorenstein squarefree monomial ideal of height 3. The following theorem is the main result of this article.

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**Theorem 1.1.** *Let $I \subset S$ be a Gorenstein squarefree monomial ideal of height 3. Then $I$ is a set-theoretic complete intersection. That is,* $\operatorname{ara} I = \operatorname{pd} S/I = \operatorname{height} I = 3$.

*Remark* 1.2. It follows that any Gorenstein monomial ideal is a set-theoretic complete intersection since the radical of a Gorenstein monomial ideal is Gorenstein.

In order to prove Theorem 1.1, we must construct 3 elements which generate the ideal up to radical. We will explain the construction by an example instead of complete construction.

## 2. Gorenstein squarefree monomial ideals of height three

Bruns and Herzog [5] proved that a Gorenstein squarefree monomial ideal of height 3 is essentially $I_r$ (see below). In this section, we recall their result.

Let $r \geq 1$ be an integer and $I_r$ the ideal of $K[x_1, \ldots, x_{2r+1}]$ generated by $2r+1$ monomials $u_1, \ldots, u_{2r+1}$:

$$u_i = x_i x_{i+1} \cdots x_{i+r-1}, \qquad i = 1, 2, \ldots, 2r+1,$$

where we consider $x_j$ as $x_{j-(2r+1)}$ if $j > 2r+1$.

*Remark* 2.1. $I_r$ is the Stanley–Reisner ideal of the boundary complex of cyclic polytope $C(2r+1, 2r-2)$.

Before stating the result by Bruns and Herzog [5], we define a terminology. Let $I$ be a squarefree monomial ideal of $S = K[x_1, \ldots, x_n]$. Let $x_{i1}, x_{i2}$ be new variables. Set $S' = K[x_1, \ldots, x_{i-1}, x_{i1}, x_{i2}, x_{i+1}, \ldots, x_n]$. Then by substitution $x_i \mapsto x_{i1}x_{i2}$ for each monomial generator of $I$, we obtain the new ideal $J \subset S'$. We call this transformation a 1-*vertex inflation*.

**Theorem 2.2** (Bruns and Herzog [5])**.** *Let $I_r$ be the ideal defined above.*

1. *$I_r$ is a Gorenstein squarefree monomial ideal of height* 3*.*
2. *Any Gorenstein squarefree monomial ideal of height* 3 *is obtained from $I_r$ for some $r$ by a series of* 1*-vertex inflations.*

By Theorem 2.2, if we prove that $I_r$ is a set-theoretic complete intersection, then Theorem 1.1 follows.

Next we modify $I_r$ by renumbering variables. Let $r_o$ be the largest odd integer with $r_o \leq r$ and $r_e$ the largest even integer with $r_e \leq r$. Let us consider the following $2r+1$ variables:

(2.1)

$$x_1, x_3, \ldots, x_{r_o},\ x_{-r_e}, x_{-(r_e-2)}, \ldots, x_{-2},\ x_0,\ x_2, \ldots, x_{r_e-2}, x_{r_e},\ x_{-r_o}, \ldots, x_{-3}, x_{-1}.$$

Let $S_r$ be the polynomial ring over $K$ in the above variables. Recall that $I_r$ is generated by the $2r+1$ products of continuous $r$ variables. Thus we may assume that the order of variables are as in (2.1). Then $I_r \subset S_r$ is generated by the following $2r+1$ monomials:

$$\begin{array}{lll} n_{+r}^{(0)},\ n_{-r}^{(0)}, & n_{+r}^{(s)},\ n_{-r}^{(s)}, & s = 1, 3, \ldots, r_o - 2, \\ m^{(r-1)}, & m_{+r}^{(t)},\ m_{-r}^{(t)}, & t = 0, 2, \ldots, r_e - 2, \end{array}$$

where

$$n_{+r}^{(0)} := x_r x_{-(r-1)} \cdots x_{\pm 3} x_{\mp 2} x_{\pm 1}, \qquad n_{-r}^{(0)} := x_{-r} x_{r-1} \cdots x_{\mp 3} x_{\pm 2} x_{\mp 1},$$

and where for an odd integer $s$,

$$\begin{cases} m_{+r}^{(s)} := x_r x_{-(r-1)} \cdots x_{\mp(s+3)} x_{\pm(s+2)} \cdot x_{\pm s} x_{\pm(s-2)} \cdots x_{\pm 1}, \\ m_{-r}^{(s)} := x_{-r} x_{r-1} \cdots x_{\pm(s+3)} x_{\mp(s+2)} \cdot x_{\mp s} x_{\mp(s-2)} \cdots x_{\mp 1}, \\ m^{(s)} := x_s x_{s-2} \cdots x_1 x_{-1} \cdots x_{-(s-2)} x_{-s}, \\ n_{+r}^{(s)} := \sqrt{m_{+r}^{(s)} m^{(s)}}, \qquad n_{-r}^{(s)} := \sqrt{m_{-r}^{(s)} m^{(s)}}, \end{cases}$$

and where for an even integer $t$,

$$\begin{cases} m_{+r}^{(t)} := x_r x_{-(r-1)} \cdots x_{\pm(t+3)} x_{\mp(t+2)} \cdot x_t x_{t-2} \cdots x_2 x_0 x_{-2} \cdots x_{-(t-2)} x_{-t}, \\ m_{-r}^{(t)} := x_{-r} x_{r-1} \cdots x_{\mp(t+3)} x_{\pm(t+2)} \cdot x_t x_{t-2} \cdots x_2 x_0 x_{-2} \cdots x_{-(t-2)} x_{-t}, \\ m^{(t)} := x_t x_{t-2} \cdots x_2 x_0 x_{-2} \cdots x_{-(t-2)} x_{-t}. \end{cases}$$

**Example 2.3.** $I_4$ is generated by the following 9 monomials:

$$\begin{array}{llll} x_4x_{-3}x_2x_{-1}, & x_{-4}x_3x_{-2}x_1, & x_4x_{-3}x_2x_0, & x_{-4}x_3x_{-2}x_0, \\ x_4x_{-3}\cdot x_1x_{-1}, & x_{-4}x_3\cdot x_1x_{-1}, & x_4\cdot x_2x_0x_{-2}, & x_{-4}\cdot x_2x_0x_{-2}, \\ x_3x_1x_{-1}x_{-3}. & & & \end{array}$$

**Example 2.4.** $I_5$ is generated by the following 11 monomials:

$$\begin{array}{llll} x_5x_{-4}x_3x_{-2}x_1, & x_{-5}x_4x_{-3}x_2x_{-1}, & x_5x_{-4}x_3x_{-2}x_0, & x_{-5}x_4x_{-3}x_2x_0, \\ x_5x_{-4}x_3\cdot x_1x_{-1}, & x_{-5}x_4x_{-3}\cdot x_1x_{-1}, & x_5x_{-4}\cdot x_2x_0x_{-2}, & x_{-5}x_4\cdot x_2x_0x_{-2}, \\ x_5\cdot x_3x_1x_{-1}x_{-3}, & x_{-5}\cdot x_3x_1x_{-1}x_{-3}, & x_4x_2x_0x_{-2}x_{-4}. & \end{array}$$

## 3. Key lemmas and 3 elements which generate $I_r$ up to radical

In this section, we explain the idea of the proof of Theorem 1.1.

The cases $r = 1, 2$ are easy.

**Example 3.1.** Since $I_1 = (x_0, x_{-1}, x_1)$, there is nothing to prove for the case $r = 1$.

Let us consider the case $r = 2$. $I_2$ is generated by the following 5 monomials:

$$x_2x_{-1},\ x_{-2}x_1,\ x_1x_{-1},\ x_2x_0,\ x_{-2}x_0.$$

Actually, $I_2$ is the Stanley–Reisner ideal of 5-cycle. This ideal is known to be a set-theoretic complete intersection; see e.g., [2, 4]. For example, following 3 elements generate $I_2$ up to radical:

$$x_1x_{-1},\ x_2x_{-1} + x_{-2}x_0,\ x_{-2}x_1 + x_2x_0.$$

In what follows, we assume $r \geq 3$. We divide the minimal monomial generators of $I_r$ by the divisibility by $x_0$. We denote by $J_r$, the ideal of $S_r$ generated by the minimal monomial generators of $I_r$ which are not divisible by $x_0$. Let $J'_r$ be the ideal of $S_{r+1}$ obtained from $J_r$ by substitutions $x_k \mapsto x_{k+1}$ and $x_{-k} \mapsto x_{-(k+1)}$ $(k = 1, 2, \ldots, r)$.

**Lemma 3.2.** *Let $r \geq 3$ be an integer. Then $I_r = J_r + x_0 J'_{r-1}$.*

We first construct 2 elements which generate $x_0 J_r$ up to radical. Set

$$\begin{cases} g_{1r}^{(1)} := x_0((m_{+r}^{(r_o-2)})^{r+3} - (m_{-r}^{(r_o-2)})^{r+3})((m_{+r}^{(r_o-4)})^{r+3} - (m_{-r}^{(r_o-4)})^{r+3}) \\ \qquad\qquad \cdots ((m_{+r}^{(1)})^{r+3} - (m_{-r}^{(1)})^{r+3})((n_{+r}^{(0)})^{r+3} - (n_{-r}^{(0)})^{r+3}), \\ g_{2r}^{(1)} := x_1 x_{-1}, \end{cases}$$

and for $s = 3, 5, \ldots, r_o$,

$$\begin{cases} g_{1r}^{(s)} := x_0(g_{2r}^{(s-2)})^{r+3}((m_{+r}^{(r_o-2)})^{r+3} - (m_{-r}^{(r_o-2)})^{r+3})((m_{+r}^{(r_o-4)})^{r+3} - (m_{-r}^{(r_o-4)})^{r+3}) \\ \qquad\qquad \cdots ((m_{+r}^{(s-2)})^{r+3} - (m_{-r}^{(s-2)})^{r+3}), \\ g_{2r}^{(s)} := g_{2r}^{(s-2)} x_s x_{-s} + g_{1r}^{(s-2)}. \end{cases}$$

Put $g_{1r} := g_{1r}^{(r_o)}$, $g_{2r} := g_{2r}^{(r_o)}$.

**Proposition 3.3.** *$x_0 J_r$ is generated by $x_0 g_{1r}, x_0 g_{2r}$ up to radical. Moreover, $g_{1r}, g_{2r} - m^{(r_o)} \in x_0 (J_r)^{r+3}$.*

*Remark* 3.4. If we remove $x_0$ on the construction $g_{1r}^{(s)}$, we obtain two elements which generate $J_r$ up to radical. (We may also omit the power $r+3$ in each $g_{1r}^{(s)}, g_{2r}^{(s)}$.) Combining this with Lemma 3.2, we have $\operatorname{ara} I_r \leq 4$.

On the proof of Proposition 3.3, the following result, which is essentially due to Schmitt and Vogel [14, Lemma p. 249], is useful.

**Lemma 3.5.** *Let $R$ be a commutative ring with unitary and $I$ an ideal of $R$. Suppose that $a, b \in R$ satisfy $ab \in \sqrt{I}$. Then $a, b \in \sqrt{I + (a+b)}$.*

*Proof.* Put $J = I + (a+b)$. Since $a^2 = a(a+b) - ab$ and $ab \in \sqrt{I} \subset \sqrt{J}$, we have $a^2 \in \sqrt{J}$. Hence $a \in \sqrt{J}$. □

Instead of proving Proposition 3.3, we see the case where $r = 4$.

**Example 3.6.** When $r = 4$, the construction is done by 2 steps:

$$\begin{cases} g_{14}^{(1)} = x_0((x_4 x_{-3} x_{-1})^7 - (x_{-4} x_3 x_1)^7)((x_4 x_{-3} x_2 x_{-1})^7 - (x_{-4} x_3 x_{-2} x_1)^7), \\ g_{24}^{(1)} = x_1 x_{-1}, \end{cases}$$

$$\begin{cases} g_{14} = g_{14}^{(3)} = x_0(g_{24}^{(1)})^7((x_4 x_{-3} x_{-1})^7 - (x_{-4} x_3 x_1)^7), \\ g_{24} = g_{24}^{(3)} = x_3 x_1 x_{-1} x_{-3} + g_{14}^{(1)}. \end{cases}$$

It is easily to see that the product of two summands of $g_{24}$ is in $\sqrt{(g_{14})}$. Then we have $x_3 x_1 x_{-1} x_{-3}, g_{14}^{(1)} \in \sqrt{(g_{14}, g_{24})}$ by Lemma 3.5. Since the product of 2 terms in each bracket of $g_{14}, g_{14}^{(1)}$ are divisible by $x_3 x_1 x_{-1} x_{-3}$, we conclude that $x_0 g_{14}, x_0 g_{24}$ generate $x_0 J_4$ up to radical by repeated use of Lemma 3.5.

Now we return to the ideal $I_r$ and explain the construction of 3 elements $q_{0r}, q_{1r}, q_{2r}$ which generate $I_r$ up to radical.

Set

$$q_{0r} := \begin{cases} n_{+r}^{(0)} - n_{-r}^{(0)}, & \text{if } r \text{ is odd,} \\ n_{-r}^{(0)} - n_{+r}^{(0)}, & \text{if } r \text{ is even.} \end{cases}$$

The construction of $q_{1r}, q_{2r}$ is done inductively. Let $h_{1r}, h_{2r}$ be elements obtained from $x_0 g_{1\,r-1}, x_0 g_{2\,r-1}$ respectively, by substitutions $x_k \mapsto x_{k+1}$; $x_{-k} \mapsto x_{-(k+1)}$ $(k = 1, 2, \ldots, r-1)$, which is the same ones we used to obtain $J'_{r-1}$ from $J_{r-1}$.

Starting with $q_{0r}, h_{1r}, h_{2r}$, we construct $q_{i_t r}^{(r_o - t)}$ for $t = 0, 2, 4, \ldots, r_o - 1$, where $i_t$ is 1 if $t$ is a multiple of 4; otherwise 2. For $t = 2, 4, \ldots, r_o - 1$, we set

$$M_{r_o - t} := \begin{cases} x_{r_o - t+2} m_{+r}^{(r_o - t)} - x_{-(r_o - t+2)} m_{-r}^{(r_o - t)}, & \text{if } r \text{ is odd,} \\ x_{r_o - t+2} m_{-r}^{(r_o - t)} - x_{-(r_o - t+2)} m_{+r}^{(r_o - t)}, & \text{if } r \text{ is even.} \end{cases}$$

Put $Q_{r_o - t} := (q_{0r}, q_{i_{t-2} r}^{(r_o - t+2)}, q_{i_t r}^{(r_o - t)})$ $(t = 0, 2, \ldots, r_0 - 1)$, where $q_{i_{-2} r}^{(r_o+2)} := h_{2r}$. We will construct $q_{i_t r}^{(r_o - t)}$ so that $q_{i_t r}^{(r_o - t)}$ and $Q_{r_o - t}$ satisfy the following lemmas:

**Lemma 3.7.** *For $t = 0, 2, \ldots, r_o - 1$,*

$$\begin{aligned} & q_{i_t r}^{(r_o - t)} - M_{r_o - 2} M_{r_o - 4} \cdots M_{r_o - t} m^{(r_o - t)} \\ & \in (m^{(r_o)}, M_{r_o - 2} m^{(r_o - 2)}, M_{r_o - 4} m^{(r_o - 4)}, \ldots, M_{r_o - t+4} m^{(r_o - t+4)}) m^{(r_o - t)} \\ & \quad + x_0 (J'_{r-1})^{r - (t/2)}. \end{aligned}$$

**Lemma 3.8.** $x_0 J'_{r-1} \subset \sqrt{Q_{r_o}}$ *and* $m^{(r_o)}, n_{+r}^{(0)}, n_{-r}^{(0)} \in \sqrt{Q_{r_o}}$.

**Lemma 3.9.** *For $t = 2, 4, \ldots, r_o - 1$, $Q_{r_o - t+2} \subset \sqrt{Q_{r_o - t}}$ and $n_{+r}^{(r_o - t)}, n_{-r}^{(r_o - t)} \in \sqrt{Q_{r_o - t}}$. In particular,*

(1) $x_0 J'_{r-1} + (m^{(r_o)}, n_{+r}^{(0)}, n_{-r}^{(0)}) \subset \sqrt{Q_{r_o - t}}$.

(2) $n_{+r}^{(r_o - 2)}, n_{-r}^{(r_o - 2)}, n_{+r}^{(r_o - 4)}, n_{-r}^{(r_o - 4)}, \ldots, n_{+r}^{(r_o - t)}, n_{-r}^{(r_o - t)} \in \sqrt{Q_{r_o - t}}$.

By Lemma 3.9, we can conclude that $q_{0r}, q_{i_{r_o - 3} r}^{(3)}, q_{i_{r_o - 1} r}^{(1)}$, which are generators of $Q_1$, generate $I_r$ up to radical.

The key idea of the construction is the following lemma which based on Barile's idea [1] (see also [3, 4, 7]).

**Lemma 3.10.** *Let $R$ be a commutative ring with unitary and $I$ an ideal of $R$. Take elements $q_1, q_2 \in I$ and $p_1, p_2 \in R$. Suppose $q_1, q_2 \in (p_1, p_2)$ :*

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = A \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \tag{3.1}$$

*where $A$ is $2 \times 2$ matrix whose entries are in $R$. Then* $(\det A) p_1, (\det A) p_2 \in I$.

*Proof.* Multiply each side of (3.1) by the cofactor matrix of $A$ from left. □

We show the construction when $r = 5$.

**Example 3.11.** In order to construct 3 elements $q_{05}, q_{15}, q_{25}$ which generate $I_5$ up to radical, we need 3 steps. The starting 3 elements are

$$\begin{aligned} q_{05} &= x_5x_{-4}x_3x_{-2}x_1 - x_{-5}x_4x_{-3}x_2x_{-1}, \\ h_{15} &\in x_0(J_4')^7, \\ h_{25} &= x_4x_2x_0x_{-2}x_{-4} + \eta, \end{aligned}$$

where $\eta \in x_0^2(J_4')^7$.

**(Step 1)** We first construct $q_{15}^{(5)}$. Since $q_{05}, h_{25} \in (x_{-4}x_{-2}, x_4x_2)$, we can write

$$\begin{pmatrix} q_{05} \\ h_{25} \end{pmatrix} = A_1^{(5)} \begin{pmatrix} x_{-4}x_{-2} \\ x_4x_2 \end{pmatrix},$$

where

$$A_1^{(5)} = \begin{pmatrix} x_5x_3x_1 & * \\ x_0\eta_-^{(51)} & x_0x_{-2}x_{-4} + x_0\eta_+^{(51)} \end{pmatrix},$$

and $\eta_-^{(51)}, \eta_+^{(51)} \in x_0(J_4')^6$. Therefore

$$\det A_1^{(5)} - x_5x_3x_1 \cdot x_0x_{-2}x_{-4} \in x_0(J_4')^6.$$

Then since $q_{05}, \det A_1^{(5)} \in (x_{-4}x_{-2}, x_4x_2)$, we can write

$$\begin{pmatrix} q_{05} \\ \det A_1^{(5)} \end{pmatrix} = A_2^{(5)} \begin{pmatrix} x_{-4}x_{-2} \\ x_4x_2 \end{pmatrix},$$

where

$$A_2^{(5)} = \begin{pmatrix} * & -x_{-5}x_{-3}x_{-1} \\ x_0x_5x_3x_1 + x_0\eta_-^{(52)} & x_0\eta_+^{(52)} \end{pmatrix},$$

and $\eta_-^{(52)}, \eta_+^{(52)} \in x_0(J_4')^5$. We set

$$q_{15}^{(5)} := \frac{\det A_2^{(5)}}{x_0} + h_{15}.$$

Note that $q_{15}^{(5)} = x_5x_3x_1x_{-1}x_{-3}x_{-5} + \eta^{(5)}$, where $\eta^{(5)} \in x_0(J_4')^5$. Therefore $q_{15}^{(5)}$ satisfies Lemma 3.7 with $t = 0$. We show that $Q_5 = (q_{05}, h_{25}, q_{15}^{(5)})$ satisfies Lemma 3.8.

By Lemma 3.10, we have

$$\det A_2^{(5)} x_4x_2,\ \det A_2^{(5)} x_{-4}x_{-2} \in \sqrt{(q_{05}, h_{25})}$$

Therefore the product of two terms of $q_{15}^{(5)}$ is in $\sqrt{(q_{05}, h_{25})}$. Thus each term of $q_{15}^{(5)}$ is in $\sqrt{Q_5}$ by Lemma 3.5. In particular, $h_{15}, h_{25} \in \sqrt{Q_5}$. Since $h_{15}$ and $h_{25}$ generate $x_0J_4'$ up to radical, we have $x_0J_4' \subset \sqrt{Q_5}$. Then $x_5x_3x_1x_{-1}x_{-3}x_{-5} \in \sqrt{Q_5}$ also follows. Moreover, by $q_{05} \in Q_5$ and Lemma 3.5, we have

$$x_5x_{-4}x_3x_{-2}x_1,\ x_{-5}x_4x_{-3}x_2x_{-1} \in \sqrt{Q_5}.$$

**(Step 2)** Next we construct $q_{25}^{(3)}$. Since $q_{05}, q_{15}^{(5)} \in (x_{-4}x_{-2}, x_{-5})$, we can write

$$\begin{pmatrix} q_{05} \\ q_{15}^{(5)} \end{pmatrix} = A_+^{(3)} \begin{pmatrix} x_{-4}x_{-2} \\ x_{-5} \end{pmatrix},$$

where

$$A_+^{(3)} = \begin{pmatrix} x_5x_3x_1 & * \\ \eta_+^{(31)} & x_5 \cdot x_3x_1x_{-1}x_{-3} + \eta_+^{(32)} \end{pmatrix},$$

and $\eta_+^{(31)}, \eta_+^{(32)} \in x_0(J_4')^4$. Similarly, since $q_{05}, q_{15}^{(5)} \in (x_4x_2, x_5)$, we can write

$$\begin{pmatrix} q_{05} \\ q_{15}^{(5)} \end{pmatrix} = A_-^{(3)} \begin{pmatrix} x_4x_2 \\ x_5 \end{pmatrix},$$

where

$$A_-^{(3)} = \begin{pmatrix} -x_{-5}x_{-3}x_{-1} & * \\ \eta_-^{(31)} & x_{-5} \cdot x_3x_1x_{-1}x_{-3} + \eta_-^{(32)} \end{pmatrix},$$

and $\eta_-^{(31)}, \eta_-^{(32)} \in x_0(J_4')^4$. Then

$$\det A_+^{(3)} + \det A_-^{(3)} = (x_5^2x_3x_1 - x_{-5}^2x_{-3}x_{-1})x_3x_1x_{-1}x_{-3} + \eta^{(3)},$$

where $\eta^{(3)} \in x_0(J_4')^4$. We set

$$q_{25}^{(3)} := \det A_+^{(3)} + \det A_-^{(3)} + (h_{25})^7.$$

It is easy to see that $q_{25}^{(3)}$ satisfies Lemma 3.7 with $t = 2$. We show that $Q_3 = (q_{05}, q_{15}^{(5)}, q_{25}^{(3)})$ satisfies Lemma 3.9 with $t = 2$.

By construction and Lemmas 3.10 and 3.5, we have

$$\det A_+^{(3)} + \det A_-^{(3)},\ h_{25} \in \sqrt{Q_3}.$$

Then $Q_5 \subset \sqrt{Q_3}$ follows. In particular, $x_0J_4' \subset \sqrt{Q_3}$. It then follows that

$$(x_5^2x_3x_1 - x_{-5}^2x_{-3}x_{-1})x_3x_1x_{-1}x_{-3} \in \sqrt{Q_3}.$$

Since $x_5x_3x_1x_{-1}x_{-3}x_{-5} \in \sqrt{Q_5} \subset \sqrt{Q_3}$, we also have

$$x_5 \cdot x_3x_1x_{-1}x_{-3},\ x_{-5} \cdot x_3x_1x_{-1}x_{-3} \in \sqrt{Q_3}$$

by Lemma 3.5, as desired.

**(Step 3)** Finally we construct $q_{15}^{(1)}$. Since $q_{05}, q_{25}^{(3)} \in (x_{-2}, x_{-3})$, we can write

$$\begin{pmatrix} q_{05} \\ q_{25}^{(3)} \end{pmatrix} = A_+^{(1)} \begin{pmatrix} x_{-2} \\ x_{-3} \end{pmatrix},$$

where

$$A_+^{(3)} = \begin{pmatrix} x_5x_{-4}x_3x_1 & * \\ \eta_+^{(11)} & (x_5^2x_3x_1 - x_{-5}^2x_{-3}x_{-1})x_3x_1x_{-1} + \eta_+^{(12)} \end{pmatrix},$$

and $\eta_+^{(11)}, \eta_+^{(12)} \in x_0(J_4')^3$. Similarly, since $q_{05}, q_{15}^{(5)} \in (x_2, x_3)$, we can write

$$\begin{pmatrix} q_{05} \\ q_{25}^{(3)} \end{pmatrix} = A_-^{(1)} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix},$$

where

$$A_-^{(1)} = \begin{pmatrix} -x_{-5}x_4x_{-3}x_{-1} & * \\ \eta_-^{(11)} & (x_5^2x_3x_1 - x_{-5}^2x_{-3}x_{-1})x_{-3}x_1x_{-1} + \eta_-^{(12)} \end{pmatrix},$$

and $\eta_-^{(11)}, \eta_-^{(12)} \in x_0(J_4')^3$. Then

$\det A_+^{(1)} + \det A_-^{(1)} = (x_5^2 x_3 x_1 - x_{-5}^2 x_{-3} x_{-1})(x_5 x_{-4} x_3^2 x_1 - x_{-5} x_4 x_{-3}^2 x_{-1}) x_1 x_{-1} + \eta^{(1)}$,

where $\eta^{(1)} \in x_0(J_4')^3$. We set

$$q_{15}^{(1)} := \det A_+^{(1)} + \det A_-^{(1)} + (q_{15}^{(5)})^2.$$

It is easy to see that $q_{15}^{(1)}$ satisfies Lemma 3.7 with $t = 4$. We show that $Q_1 = (q_{05}, q_{25}^{(3)}, q_{15}^{(1)})$ satisfies Lemma 3.9 with $t = 4$.

By construction and Lemmas 3.10 and 3.5, we have

$$\det A_+^{(1)} + \det A_-^{(1)},\ q_{15}^{(5)} \in \sqrt{Q_1}.$$

Then $Q_3 \subset \sqrt{Q_1}$ follows. In particular, $x_0 J_4' \subset \sqrt{Q_1}$. It then follows that

$$(x_5^2 x_3 x_1 - x_{-5}^2 x_{-3} x_{-1})(x_5 x_{-4} x_3^2 x_1 - x_{-5} x_4 x_{-3}^2 x_{-1}) x_1 x_{-1} \in \sqrt{Q_1}.$$

Note that we also have $x_5 x_3 x_1 x_{-1} x_{-3} x_{-5} \in \sqrt{Q_1}$. Then by repeated use of Lemma 3.5, we have

$$x_5 x_{-4} x_3 \cdot x_1 x_{-1},\ x_{-5} x_4 x_{-3} \cdot x_1 x_{-1} \in \sqrt{Q_1},$$

as desired.

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