研究集会

第30回可換環論シンポジウム

(The 30th Symposium on Commutative Ring Theory in Japan)

2008年11月18日－21日

於：国民宿舎虹の松原ホテル

明治大学科学技術研究所重点研究（Ａ）（代表：後藤四郎）
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科学研究費若手研究（Ｂ）（代表：高木俊輔）

序

これは第３０回可換環論シンポジウムの報告集です。このシンポジウムは、以下の今年度の研究費（括弧内は研究代表者）の援助で行われました。

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「特異点の可換環論 -- blow-up代数の環構造解析」
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（宮崎誓）
・科学研究費若手研究(B)
「正標数の手法を用いた随伴イデアル層の研究」
（高木俊輔）

シンポジウムは２００８年１１月１８日から１１月２１日にかけて佐賀県唐津市の国民宿舎虹の松原ホテルにおいて開催され、海外からの招聘講演者としてのSijong Kwak（韓国）、Gennady Lyubeznik（米国）、Irena Swanson（米国）の３氏を含め、約７０名の研究者・大学(院)生の参加があり、２７の講演と活発な議論が行われました。貴重な研究費から開催費用をご援助して下さった方々に、興味深いご講演をして下さった方々に、そして全ての参加者の方々に、ここにあらためて御礼を申し上げます。

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The 30th Conference on Commutative Algebra, Japan

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Log canonical thresholds of binomial ideals

Takafumi Shibuta (Kyushu) and Shunsuke Takagi (Kyushu)

In this article, we discuss how to compute log canonical thresholds of binomial ideals.

Let k be a field of characteristic zero and $\mathfrak{a}$ be an ideal of the polynomial ring $k[x_1, \dots, x_n]$ over k. A *log resolution* of $\mathfrak{a}$ is a proper birational morphism $\pi : \widetilde{X} \to X := \mathbb{A}_k^n$ with $\widetilde{X}$ a nonsingular variety such that $\mathfrak{a}\mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-F)$ is invertible and $\mathrm{Exc}(\pi) \cup \mathrm{Supp}(F)$ is a simple normal crossing divisor.

Definition 1. Let $t > 0$ be a real number and fix a log resolution $\pi : \widetilde{X} \to X = \mathbb{A}_k^n$ with $\mathfrak{a}\mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-F)$. The *multiplier ideal* $\mathcal{J}(\mathfrak{a}^t)$ of $\mathfrak{a}$ with exponent t is

$$\mathcal{J}(\mathfrak{a}^t) = \pi_* \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}/X} - \lfloor tF \rfloor) \subseteq k[x_1, \dots, x_n],$$

where $K_{\widetilde{X}/X}$ is the relative canonical divisor of π. This definition is independent of the choice of the log resolution π.

The reader is referred to [4] for basic properties of multiplier ideals.

Definition 2. Assume that $\mathfrak{a}$ is contained in the maximal ideal $(x_1, \dots, x_n)$. The *log canonical threshold* of $\mathfrak{a}$ at the origin $0 \in \mathbb{A}_k^n$ is

$$\mathrm{lct}_0(\mathfrak{a}) = \sup\{t \in \mathbb{R}_{\geq 0} \mid \mathcal{J}(\mathfrak{a}^t)_0 = k[x_1, \dots, x_n]_{(x_1, \dots, x_n)}\}$$

(when $\mathfrak{a}$ is not contained in $(x_1, \dots, x_n)$, we put $\mathrm{lct}_0(\mathfrak{a}) = \infty$). The log canonical threshold $\mathrm{lct}_0(\mathfrak{a})$ is a rational number.

The log canonical threshold plays a very important role in birational geometry, and it has been related to various other points of view on singularities.

Remark 3. (1) The log canonical threshold $\mathrm{lct}_0(\mathfrak{a})$ is related to the behavior of symbolic powers of the ideal $\mathfrak{a}$. Assume for simplicity that $\mathfrak{a}$ is a prime ideal of height $h \geq 1$. Then, by an argument similar to that of Ein-Lazarsfeld-Smith [2], one can prove the following: if $\mathrm{lct}_0(\mathfrak{a}) > s$ for some integer s, then for every integer $r \geq 1$,

$$\mathfrak{a}^{(hr-s)} \subseteq \mathfrak{a}^r.$$

(2) Budur-Mustaţă-Saito [1] proved that the log canonical threshold $\mathrm{lct}_0(\mathfrak{a})$ coincides with the smallest root of the generalized Bernstein-Sato polynomial $b_{f,0}(s)$, where $f := (f_1, \ldots, f_r)$ is a system of generators for $\mathfrak{a}$. The reader is refereed to [1] for the definition of generalized Bernstein-Sato polynomials.

Since log canonical thresholds are defined via resolution of singularities, it is difficult to compute them in general. When the ideal $\mathfrak{a}$ is a monomial ideal, there exists a combinatorial description of the multiplier ideal $\mathcal{J}(\mathfrak{a}^t)$ due to Howald [3].

Proposition 4 ([3]). *Let $\mathfrak{a}$ be a monomial ideal of $k[x_1, \ldots, x_n]$ and $P(\mathfrak{a}) \subseteq \mathbb{R}^d$ be the Newton polyhedron of $\mathfrak{a}$. Then for every real number $t > 0$,*

$$\mathcal{J}(\mathfrak{a}^t) := (x^{\mathbf{m}} \mid \mathbf{m} + \mathbf{1} \in \mathrm{Int}(t \cdot P(\mathfrak{a})) \cap \mathbb{N}^n),$$

where $\mathbf{1} := (1, \ldots, 1) \in \mathbb{N}^n$. In particular, if $\mathfrak{a} = (x^{\mathbf{m}_1}, \ldots, x^{\mathbf{m}_r})$, then

$$\begin{aligned}\mathrm{lct}_0(\mathfrak{a}) &= \sup\{t \in \mathbb{R}_{\geq 0} \mid \mathbf{1} \in t \cdot P(\mathfrak{a})\} \\ &= \max\left\{\sum_{i=1}^{r} \lambda_i \,\middle|\, \sum_{i=1}^{r} \mathbf{m}_i \lambda_i \leq \mathbf{1},\ \lambda_i \in \mathbb{Q}_{\geq 0}\right\}.\end{aligned}$$

Motivated by Howald's result, we will compute the log canonical threshold $\mathrm{lct}_0(\mathfrak{a})$ by linear programming when $\mathfrak{a}$ is a binomial ideal. We start with the following lemma, which is proved using characteristic p methods such as F-pure thresholds (see [6] for the definition of F-pure thresholds).

Lemma 5. *Let k be a field of characteristic zero and $\mathfrak{a} = (f_1, \ldots, f_r)$ be an ideal of $k[x_1, \ldots, x_n]$ generated by binomials $f_i = x^{\mathbf{a}_i} - \gamma_i x^{\mathbf{b}_i}$, where $\mathbf{a}_i = (a_{i1}, \ldots, a_{in}), \mathbf{b}_i = (b_{i1}, \ldots, b_{in}) \in \mathbb{Z}^n_{\geq 0} \setminus \{\mathbf{0}\}$ and $\gamma_i \in k^*$ for all $i = 1, \ldots, r$. We assume that the ideal $\mathfrak{a}$ contains no monomials. Put*

$$A := \begin{pmatrix} a_{11} & \cdots & a_{r1} & b_{11} & \cdots & b_{r1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{rn} & b_{1n} & \cdots & b_{rn} \\ 1 & & 0 & 1 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & 1 & 0 & & 1 \end{pmatrix},$$

and consider the following linear programming problem:

$$\text{Maximize: } \sum_{i=1}^{r}(\mu_i + \nu_i)$$
$$\text{Subject to: } A\ (\mu_1, \dots, \mu_r, \nu_1, \dots, \nu_r)^{\mathrm{T}} \leq \mathbf{1},\ \mu_i, \nu_i \in \mathbb{Q}_{\geq 0}.$$

Suppose that there exists an optimal solution (μ, ν) *such that* $A\ (\mu, \nu)^{\mathrm{T}} \neq A\ (\mu', \nu')^{\mathrm{T}}$ *for all other optimal solutions* $(\mu', \nu') \neq (\mu, \nu)$*. Then the log canonical threshold* $\mathrm{lct}_0(\mathfrak{a})$ *is equal to the optimal value* $\sum_{i=1}^{r}(\mu_i + \nu_i)$*.*

As a corollary of Lemma 5, we have the following theorem.

Theorem 6. *Let* $\mathfrak{a} = (f_1, \dots, f_r)$ *be an ideal of* $k[x_1, \dots, x_n]$ *generated by binomials* $f_i = x^{\mathbf{a}_i} - \gamma_i x^{\mathbf{b}_i}$*, where* $\mathbf{a}_i, \mathbf{b}_i \in \mathbb{Z}^n_{\geq 0} \setminus \{\mathbf{0}\}$ *and* $\gamma_i \in k^*$ *for all* $i = 1, \dots, r$*. Suppose that the ideal* $\mathfrak{a}$ *contains no monomials and, in addition, that one of the following conditions is satisfied:*

(1) $f_1, \dots, f_r$ *form a regular sequence for* $k[x_1, \dots, x_n]$*,*

(2) $f_1, \dots, f_r$ *form a canonical system of generators of the defining ideal of a monomial curve in* $\mathbb{A}^3_k$ *(in this case,* $r \leq 3$*).*

Then the log canonical threshold $\mathrm{lct}_0(\mathfrak{a})$ *is equal to*

$$\max\left\{ \sum_{i=1}^{r}(\mu_i + \nu_i) \,\middle|\, \sum_{i=1}^{r}(\mathbf{a}_i\mu_i + \mathbf{b}_i\nu_i) \leq \mathbf{1},\ \mu_i + \nu_i \leq 1,\ \mu_i, \nu_i \in \mathbb{Q}_{\geq 0} \right\}.$$

Proof. (1) If $f_1, \dots, f_r$ form a regular sequence, then it is well-known that the matrix A in Lemma 5 has full rank. In this case, the assumption in Lemma 5 is clearly satisfied. Thus, we obtain the assertion.

(2) The proof relies on a case-by-case argument, so we omit the proof. □

Example 7. (1) Let $\mathfrak{a} = (x_1^3 - x_2x_3, x_2^2 - x_1x_3, x_3^2 - x_1^2x_2)$ be the defining ideal of the monomial curve $k[t^3, t^4, t^5]$ in $\mathbb{A}^3_k$. Then

$$(\mu_1, \mu_2, \mu_3, \nu_1, \nu_2, \nu_3) = (1/9, 1/3, 0, 1/3, 2/3, 0)$$

is an optimal solution of the linear programming problem in Theorem 6. Thus, $\mathrm{lct}_0(\mathfrak{a}) = 1/9 + 1/3 + 0 + 1/3 + 2/3 + 0 = 13/9$.

(2) Let $\mathfrak{a} = (x_1^4 - x_2x_3^2, x_2^4 - x_1^3x_3, x_3^3 - x_1x_2^3)$ be the defining ideal of the monomial curve $k[t^9, t^{10}, t^{13}]$ in $\mathbb{A}_k^3$. Then

$$(\mu_1, \mu_2, \mu_3, \nu_1, \nu_2, \nu_3) = (5/24, 0, 0, 1/2, 0, 1/6)$$

is an optimal solution of the linear programming problem in Theorem 6. Thus, $\mathrm{lct}_0(\mathfrak{a}) = 5/24 + 0 + 0 + 1/2 + 0 + 1/6 = 7/8$.

(3) Let $\mathfrak{a} = (x_1^3 - x_4^2, x_2^2 - x_1x_4, x_3^2 - x_2x_4)$ be the defining ideal of the monomial curve $k[t^8, t^{10}, t^{11}, t^{12}]$ in $\mathbb{A}_k^4$. Then

$$(\mu_1, \mu_2, \mu_3, \nu_1, \nu_2, \nu_3) = (1/9, 1/3, 1/2, 0, 2/3, 1/3)$$

is an optimal solution of the linear programming problem in Theorem 6. Thus, $\mathrm{lct}_0(\mathfrak{a}) = 1/9 + 1/3 + 1/2 + 0 + 2/3 + 1/3 = 35/18$.

(4) Let $\mathfrak{a} = (x_1^5 - x_2x_4^2, x_2^7 - x_3^4x_4, x_3^3 - x_1x_4^2, x_4^7 - x_1^3x_2^6x_3^2, x_1^4x_2^6 - x_3x_4^5)$ be the defining ideal of the monomial curve $k[t^{53}, t^{63}, t^{85}, t^{101}]$ in $\mathbb{A}_k^4$. Then $\mathfrak{a}$ does not satisfy the assumption in Theorem 6, but we can still apply Lemma 5 to this situation. It is easy to check that

$$(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5) = (1/5, 1/14, 1/3, 0, 0, 1/2, 0, 0, 0, 0)$$

is an optimal solution of the linear programming problem in Lemma 5 and, in addtion, this solution satisfies the assumption in Lemma 5. Thus, $\mathrm{lct}_0(\mathfrak{a}) = 1/5+1/14+1/3+0+0+1/2+0+0+0+0 = 116/105$. By Remark 3 (1), we see that $\mathfrak{a}^{(5)} \subseteq \mathfrak{a}^2$, $\mathfrak{a}^{(8)} \subseteq \mathfrak{a}^3$ and so on.

Question 8. Let $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$ be a binomial ideal which contains no monomials and let $f_1, \ldots, f_r$ be a system of minimal binomial generators for $\mathfrak{a}$. Then do $f_1, \ldots, f_r$ satisfy the assumption in Lemma 5? We don't know any counterexample for the moment.

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An algorithm for computing multiplier ideals

Takafumi Shibuta (Kyushu)

In this article, we give an algorithm for computing multiplier ideals using Gröbner bases in Weyl algebras.

Let X be the affine space $\mathbb{C}^n$ with the coordinate system $x = (x_1, \ldots, x_n)$ and the coordinate ring $\mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$. Let $\mathfrak{a} = \langle f_1, \cdots, f_r \rangle \subseteq \mathbb{C}[x]$ be an ideal. Suppose that $\pi : \widetilde{X} \to X$ is a log resolution of $\mathfrak{a}$, that is, π is a proper birational morphism, $\widetilde{X}$ is smooth and $\pi^{-1}V(\mathfrak{a}) = F$ is a divisor with simple normal crossing support. Then the multiplier ideal of $\mathfrak{a}$ with exponent $c \in \mathbb{R}_{>0}$ is defined by

$$\mathcal{J}(\mathfrak{a}^c) = \mathcal{J}(c \cdot \mathfrak{a}) = \pi_* \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}/X} - \lfloor cF \rfloor) \subseteq \mathcal{O}_X,$$

where $K_{\widetilde{X}/X}$ is the relative canonical divisor of π. Since multiplier ideals are defined via log resolutions, it is difficult to compute them in general. In this paper, we will give an algorithm for computing multiplier ideals using the theory of D-modules. Budur–Mustaţă–Saito introduced generalized Bernstein-Sato polynomials (or b-function) of arbitrary varieties in [2] and proved relation between generalized Bernstein-Sato polynomials and multiplier ideals using the theory of the V-filtration of Kashiwara and Malgralge. We modify the definition of Budur–Mustaţă–Saito's Bernstein-Sato polynomials to determine a system of generators of the multiplier ideals of a given ideal.

Let $Y = X \times \mathbb{C}^r$ be the affine space $\mathbb{C}^{n+r}$ with the coordinate system $(x,t) = (x_1, \ldots, x_n, t_1, \ldots, t_r)$. Then $X = X \times \{0\} = V(t_1, \ldots, t_r) = \mathbb{C}^n$ is a liner subspace of Y with the defining ideal $I_X = \langle t_1, \ldots, t_r \rangle$. We denote by $\partial_x = (\partial_{x_1}, \ldots, \partial_{x_n})$ and $\partial_t = (\partial_{t_1}, \ldots, \partial_{t_r})$ the partial differential operators $\partial_{x_i} = \frac{\partial}{\partial x_i}$ and $\partial_{t_j} = \frac{\partial}{\partial t_j}$. We denote the rings of differential operators of X

and Y by

$$\begin{aligned} D_X &= \mathbb{C}\langle x, \partial_x\rangle = \mathbb{C}\langle x_1, \dots, x_n, \partial_{x_1}, \dots, \partial_{x_n}\rangle, \\ D_Y &= \mathbb{C}\langle x, t, \partial_x, \partial_t\rangle = \mathbb{C}\langle x_1, \dots, x_n, t_1, \dots, t_r, \partial_{x_1}, \dots, \partial_{x_n}, \partial_{t_1}, \dots, \partial_{t_r}\rangle. \end{aligned}$$

We use the notation $x^{\mu_1} = \prod_{i=1}^n x_i^{\mu_{1i}}$, $t^{\mu_2} = \prod_{j=1}^r t_j^{\nu_{1i}}$, $\partial_x{}^{\nu_1} = \prod_{i=1}^n \partial_{x_i}^{\nu_{1i}}$, and $\partial_t{}^{\nu_2} = \prod_{j=1}^d \partial_{t_j}^{\nu_{2j}}$ for $\mu_1 = (\mu_{11}, \dots, \mu_{1r})$, $\nu_1 = (\nu_{11}, \dots, \nu_{1r}) \in \mathbb{Z}_{\geq 0}^n$ and $\mu_2 = (\mu_{21}, \dots, \mu_{2n})$, $\nu_2 = (\nu_{21}, \dots, \nu_{2n}) \in \mathbb{Z}_{\geq 0}^r$. We define the decreasing filtration $V^m D_Y$ called *V-filtration* of D_Y along $X \times \{0\}$ as following:

$$V^m D_Y = \{ \sum_{|\mu_2| - |\nu_2| \geq m} a_{\mu_1\mu_2\nu_1\nu_2} x^{\mu_1} t^{\mu_2} \partial_x{}^{\nu_1} \partial_t{}^{\nu_2} \mid a_{\mu_1\mu_2\nu_1\nu_2} \in \mathbb{C}\}.$$

Definition 1. *The V-filtration along $X \times \{0\}$ on a finitely generated left D_Y-module M is an exhaustive decreasing filtration $\{V^\alpha M\}_\alpha$, such that:*
(i) *$V^\alpha M$ are finitely generated $V^0 D_Y$-submodules of M.*
(ii) *$\{V^\alpha M\}_\alpha$ is indexed left-continuously and discretely by rational numbers, that is, $V^\alpha M = \bigcap_{\alpha' < \alpha} V^{\alpha'} M$, and every interval contains only finitely many α with $\mathrm{Gr}_V^\alpha M \neq 0$, and these α must be rational. Here $\mathrm{Gr}_V^\alpha M := V^\alpha M / (\bigcup_{\alpha' > \alpha} V^{\alpha'} M)$.*
(iii) *$(V^i D_Y)(V^\alpha M) \subset V^{\alpha+i} M$ for any $i \in \mathbb{Z}$, $\alpha \in \mathbb{Q}$.*
(iv) *$(V^i D_Y)(V^\alpha M) = V^{\alpha+i} M$ for any $i > 0$ if $\alpha \gg 0$.*
(v) *the action of $\sigma + \alpha$ is nilpotent on $\mathrm{Gr}_V^\alpha M$.*

Let $\iota : X \to Y$ be the graph embedding $x \mapsto (x, f_1(x), \dots, f_r(x))$ of $f = (f_1, \dots, f_r)$, and $M_f = \iota_+ \mathbb{C}[x]$, where ι_+ denotes the direct image for left D-modules. There is a natural isomorphism $M_f \cong \mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[\partial_{t_1}, \dots, \partial_{t_r}]$ (see [1]), and the action of $\mathbb{C}[x]$ and $\partial_{t_1}, \dots, \partial_{t_r}$ on M is given by the canonical one, and the action of a vector field ξ on X and t_j are given by

$$\begin{aligned} \xi(g \otimes \partial_t{}^\nu) &= \xi g \otimes \partial_t{}^\nu - \sum_j (\xi f_j) g \otimes \partial_{t_j} \partial_t{}^\nu, \\ t_j(g \otimes \partial_t{}^\nu) &= f_j g \otimes \partial_t{}^\nu - \nu_j g \otimes \partial_t{}^{\nu - 1_j}. \end{aligned}$$

where 1_j is the element of $\mathbb{Z}^r$ whose i-th component is 1 if $i = j$ and 0 otherwise.

Definition 2. *Let M be a D_Y-module with the V-filtration. For $u \in M$, the Bernstein-Sato polynomial $b_u(s)$ of u is the monic minimal polynomial of the action of σ on $V^0 D_Y u / V^1 D_Y u$.*

The existence of $b_u(s)$ is equivalent to the finiteness of the induced filtration V on $(V^0 D_Y)u/(V^1 D_Y)u$, and that follows from that of the properties of V-filtrations in Definition 1.

Definition 3 ([2]). *For $g \in \mathbb{C}[x]$, we define $b_{f,g}(s)$ to be $b_{1\otimes g}(s)$ the Bernstein-Sato polynomial of $1 \otimes g \in M_f$.*

Theorem 4 ([2]). *We will denote by V the filtration on $\mathbb{C}[x] \cong \mathbb{C}[x] \otimes 1$ induced by the V-filtration on $\iota_+\mathbb{C}[x]$. Then $\mathcal{J}(\mathfrak{a}^c) = V^{c+\varepsilon}\mathbb{C}[x]$ and $V^\alpha \mathbb{C}[x] = \mathcal{J}(\mathfrak{a}^{\alpha-\varepsilon})$ for any $\alpha \in \mathbb{Q}$ and $0 < \epsilon \ll 1$. Therefore for a given rational number $c \geq 0$,*

$$\mathcal{J}(\mathfrak{a}^c) = \{g \in \mathbb{C}[x] \mid c < c' \quad \text{if} \quad b_{f,g}(-c') = 0\}.$$

In particular, the log canonical threshold $\mathrm{lct}(\mathfrak{a})$ *of $\mathfrak{a} = \langle f_1, \ldots, f_r \rangle$ is the minimal root of $b_f(-s)$.*

To obtain an algorithm that gives the generators of multiplier ideals, we modify the definition of Budur–Mustaţă–Saito's Bernstein-Sato polynomial. We set $\delta = 1 \otimes 1 \in M_f = \iota_+\mathbb{C}[x]$ and $\overline{M}_f^{(m)} = (V^0 D_Y)\delta/(V^m D_Y)\delta$. The induced filtration V on the $\overline{M}_f^{(m)}$ is finite by the definition of the V-filtration (Definition 1) as in the case of $m = 1$. For $g \in \mathbb{C}[x]$, we denote by $\overline{g \otimes 1}$ the image of $g \otimes 1 = g\delta$ in $\overline{M}_f^{(m)}$.

Definition 5. *We define $b_{f,g}^{(m)}(s)$ to be the monic minimal polynomial of the action of σ on $(V^0 D_Y)\overline{g \otimes 1} \subset \overline{M}_f^{(m)}$.*

The existence of $b_{f,g}^{(m)}(s)$ follows from the finiteness of the filtration V on $\overline{M}_f^{(m)}$ and the rationality of its roots follows from the rationality of the V-filtration.

Theorem 6. *For a given rational number $c < m + \mathrm{lct}(\mathfrak{a})$,*

$$\mathcal{J}(\mathfrak{a}^c) = \{g \in \mathbb{C}[x] \mid c < c' \quad \text{if} \quad b_{f,g}^{(m)}(-c') = 0\}.$$

In particular, the log canonical threshold $\mathrm{lct}(\mathfrak{a})$ *of* $\mathfrak{a} = \langle f_1, \ldots, f_r \rangle$ *is the minimal root of* $b_{f,1}^{(m)}(-s)$.

Lemma 7.

$$\mathrm{Ann}_{D_Y} \prod_i f_i^{s_i} = \langle t_i - f_i \mid 1 \le i \le r \rangle + \langle \partial_{x_j} + \sum_{i=1}^{r} \partial_{x_j}(f_i)\partial_{t_i} \mid 1 \le j \le n \rangle.$$

Theorem 8 (Algorithm for multiplier ideals). *Let*

$$I_f = \langle t_i u_1 - f_i \mid 1 \le i \le r \rangle + \langle u_1 \partial_{x_j} + \sum_{i=1}^{r} \partial_{x_j}(f_i)\partial_{t_i} \mid 1 \le j \le d \rangle + \langle u_1 u_2 - 1 \rangle$$

be a left ideal of $D_Y[u_1, u_2]$. *Then compute the following ideals;*

1. $I_{f,1} = I_f \cap D_Y$,

2. $J_f(m) = D_Y[s](I_{f,1} + \mathfrak{a}^m + \langle s - \sigma \rangle) \cap \mathbb{C}[x, s]$.

Then the followings hold:

(i) $b_{f,g}^{(m)}(s)$ *is the generator of* $(J_f(m) : g) \cap \mathbb{C}[s]$.

(ii) *Let* $J_f(m) = \cap_{i=1}^{\ell} \mathfrak{q}_i$ *be a primary decomposition of* $J_f(m)$. *Then, for* $1 \le i \le \ell$, *there exists* $c(i)$ *a root of* $b_{f,1}^{(m)}(-s)$ *such that the generator of* $\mathfrak{q}_i \cap \mathbb{C}[s]$ *is some power of* $s + c(i)$, *and*

$$\{c(i) \mid 1 \le i \le \ell\} = \{c' \mid b_{f,1}^{(m)}(-c') = 0\}.$$

(iii) *For* $c < \mathrm{lct}(\mathfrak{a}) + m$,

$$\mathcal{J}(\mathfrak{a}^c) = \bigcap_{i \in \{j \mid c(j) \le c\}} \mathfrak{q}_i \cap \mathbb{C}[x].$$

The computations were made using Kan/sm1 [6] and Risa/Asir [4].

Example 9. Let $\mathfrak{a} \subset \mathbb{C}[x_1, x_2, x_3]$ be the defining ideal of the space monomial curve $\mathrm{Spec}\,\mathbb{C}[T^3, T^4, T^5] \subset \mathbb{C}^3$ with a system of generators $f = (x_1^3 - x_2 x_3, x_2^2 - x_1 x_3, x_3^2 - x_1^2 x_2)$. Then

$$b_f(s) = \left(s + \frac{13}{9}\right)\left(s + \frac{3}{2}\right)\left(s + \frac{14}{9}\right)\left(s + \frac{16}{9}\right)\left(s + \frac{17}{9}\right)(s+2)^2\left(s + \frac{19}{9}\right)\left(s + \frac{20}{9}\right),$$

and

$$
\mathcal{J}(\mathfrak{a}^c) = \begin{cases} \mathbb{C}[x_1, x_2, x_3] & 0 \leq c < \frac{13}{9}, \\ \langle x_1, x_2, x_3 \rangle & \frac{13}{9} \leq c < \frac{16}{9}, \\ \langle x_1^2, x_2, x_3 \rangle & \frac{16}{9} \leq c < \frac{17}{9}, \\ \langle x_1^2, x_1x_2, x_2^2, x_3 \rangle & \frac{17}{9} \leq c < 2. \\ \mathfrak{a} & 2 \leq c < \frac{22}{9}. \end{cases}
$$

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A simple proof of Nowicki's conjecture on the kernel of an elementary derivation

Shigeru Kuroda

1 Introduction

Let $A[\mathbf{x}] = A[x_1, \ldots, x_n]$ be the polynomial ring in n variables over an integral domain A for $n \in \mathbf{N}$, and D an A-derivation of $A[\mathbf{x}]$, i.e., an A-linear map $D : A[\mathbf{x}] \to A[\mathbf{x}]$ satisfying $D(fg) = D(f)g + fD(g)$ for each $f, g \in A[\mathbf{x}]$. We say that D is *elementary* if $D(x_i)$ belongs to A for each i. Then, the kernel $\ker D$ of D is an A-subalgebra of $A[\mathbf{x}]$ containing

$$L_{i,j}^{D} := D(x_j)x_i - D(x_i)x_j \quad \text{for each } i, j \in \{1, \ldots, n\}.$$

In general, it is difficult to determine the structure of $\ker D$. The problem of finite generation of $\ker D$ is a special case of the Fourteenth Problem of Hilbert if $A = k[\mathbf{y}] := k[y_1, \ldots, y_m]$ is a polynomial ring over a field k. In 1990, Roberts [13] gave a new counterexample to the Fourteenth Problem of Hilbert by a construction different from that of Nagata [11]. Roberts' counterexample is obtained as the kernel of an elementary derivation (see [6] and [9] for generalizations of Roberts' counterexample). On the other hand, Weitzenböck's theorem says that $\ker D$ is always finitely generated if k is of characteristic zero and D is *linear*, i.e., each $D(x_i)$ is a linear form in $y_1, \ldots, y_m$ over k. We mention that Kurano [7, Proposition 3.1] found a finite set of generators of $\ker D$ for a certain non-linear elementary derivation D (see also [2] and [4] for affirmative results).

Now, assume that k is of characteristic zero and $m = n$. Consider the elementary derivation Δ of $k[\mathbf{y}][\mathbf{x}]$ defined by

$$\Delta(x_i) = y_i \quad (i = 1, \ldots, n).$$

Weitzenböck's theorem says that $\ker \Delta$ is finitely generated, but it says nothing about the finite set of generators. Nowicki [12, Conjecture 6.9.10] conjectured the following:

2000 *Mathematics Subject Classification.* Primary 13N15; Secondary 13E15, 13A50.

Key words and phrases. Derivations, The Fourteenth Problem of Hilbert.

Partly supported by the Grant-in-Aid for Young Scientists (Start-up) 19840041, The Ministry of Education, Culture, Sports, Science and Technology, Japan.

Conjecture (Nowicki) *The $k[\mathbf{y}]$-algebra $\ker\Delta$ is generated by $L^{\Delta}_{i,j} = y_j x_i - y_i x_j$ for $1 \le i < j \le n$.*

Recently, Khoury [5] solved the conjecture in the affirmative by calculating a Gröbner basis for certain ideal. Khoury's Gröbner basis consists of several families of polynomials, and he had to check many cases to show that all the S-polynomials are reduced to zero.

The aim of this article is to give a simple new proof of Nowicki's conjecture by a method similar to that used in the proof of Kurano [7, Proposition 3.1].

We remark that the result on the structure of $\ker\Delta$ implies a more general result as follows. For each A-domain B and an elementary A-derivation D of $A[\mathbf{x}]$, the B-derivation $D_B := \mathrm{id}_B \otimes D$ of $B \otimes_A A[\mathbf{x}] = B[x_1, \dots, x_n]$ is elementary. Moreover, if B is flat over A, then $\ker D_B = B \otimes_A \ker D$. Therefore, the result on $\ker\Delta$ implies the following theorem.

Theorem 1 *Let A be an integral domain containing a field k of characteristic zero, and let D be an elementary A-derivation of $A[\mathbf{x}]$ such that A is flat over $k[D(x_1), \dots, D(x_n)]$ and $D(x_1), \dots, D(x_n)$ are algebraically independent over k. Then, $\ker D$ is generated by $L^{D}_{i,j}$ for $1 \le i < j \le n$ over A.*

Actually, D induces an elementary R-derivation D' of $R[x_1, \dots, x_n]$, for which $\ker D = A \otimes_R \ker D'$, where $R = k[D(x_1), \dots, D(x_n)] \simeq k[\mathbf{y}]$.

We note that Khoury [5, Theorem 1.1] showed that $\ker D$ is generated by $L^{D}_{i,j}$ for $1 \le i < j \le n$ over $k[\mathbf{y}]$ for the $k[\mathbf{y}]$-derivation D of $k[\mathbf{y}][\mathbf{x}]$ defined by $D(x_i) = y_i^{t_i}$ with $t_i \in \mathrm{N}$ for $i = 1, \dots, n$. In this case, $y_1^{t_1}, \dots, y_n^{t_n}$ are algebraically independent over k, and $k[\mathbf{y}]$ is free over $k[y_1^{t_1}, \dots, y_n^{t_n}]$.

2 Idea of the proof

We only explain the idea of the proof in this report. The complete proof can be found in [8], which will be published in Tokyo Journal of Mathematics.

First, we recall a useful lemma on initial algebras. Let $A[T]$ be the polynomial ring in a variable T over a k-algebra A. For each $f = a_0 T^m + a_1 T^{m-1} + \cdots + a_m \in A[T] \setminus \{0\}$ with $a_0 \ne 0$, we define $\mathrm{in}_T f = a_0 T^m$. For a k-subalgebra B of $A[T]$, we define the *initial algebra* $\mathrm{in}_T B$ to be the k-subalgebra of $A[T]$ generated by $\mathrm{in}_T f$ for $f \in B \setminus \{0\}$.

The following lemma is well-known.

Lemma 2 *If $\mathrm{in}_T B = k[\mathrm{in}_T g_1, \dots, \mathrm{in}_T g_r]$ for $g_1, \dots, g_r \in B$, then $B = k[g_1, \dots, g_r]$.*

The conjecture is proved by induction on n. The assertion is clear when $n = 1$. Assume that $n \ge 2$, and let S_l be the set of $L_{i,j} := x_j y_i - x_i y_j$ for $1 \le i < j \le l$ for each $1 \le l \le n$. By the assumption on induction, it easily follows that

$$k[\mathbf{y}][x_1, \dots, x_{n-1}] \cap \ker\Delta = k[\mathbf{y}][S_{n-1}]. \qquad (1)$$

We claim that Lemma 2 and the following proposition imply that the k-algebra $\ker\Delta$ is generated by $S := \{y_i \mid i = 1, \dots, n\} \cup S_n$, and therefore that the conjecture is true.

Proposition 3 *The initial algebra* $\mathrm{in}_{x_n} \ker \Delta$ *is equal to* $k[\mathbf{y}][S_{n-1}][y_1x_n, \ldots, y_{n-1}x_n]$.

In fact, $k[\mathbf{y}][S_{n-1}][y_1x_n, \ldots, y_{n-1}x_n]$ is generated by $\mathrm{in}_{x_n} f$ for $f \in S$, since $\mathrm{in}_{x_n} y_i = y_i$ for $i = 1, \ldots, n$, $\mathrm{in}_{x_n} L_{i,j} = L_{i,j}$ for $1 \le i < j \le n-1$, and $\mathrm{in}_{x_n} L_{i,n} = y_ix_n$ for $i = 1, \ldots, n-1$.

To show Proposition 3, take any $\Phi \in \ker \Delta \setminus \{0\}$, and write

$$\Phi = \phi_0 x_n^m + \phi_1 x_n^{m-1} + \cdots + \phi_m \quad (\phi_0, \ldots, \phi_m \in k[\mathbf{y}][x_1, \ldots, x_{n-1}],\ \phi_0 \neq 0),$$

where m is a nonnegative integer. Then, we get

$$\begin{gathered} 0 = \Delta(\Phi) = \Delta(\phi_0)x_n^m + m\phi_0 y_n x_n^{m-1} + \Delta(\Phi - \phi_0 x_n^m), \\ \deg_{x_n}(m\phi_0 y_n x_n^{m-1} + \Delta(\Phi - \phi_0 x_n^m)) \le m-1, \end{gathered}$$

which imply $\Delta(\phi_0) = 0$. Hence, ϕ_0 belongs to $k[\mathbf{y}][S_{n-1}]$ by (1). Thus, we can express

$$\phi_0 = \sum_{\substack{\mathbf{a} = (a_1, \ldots, a_n) \\ \mathbf{u} = (u_{i,j})_{1 \le i < j \le n-1}}} c_{\mathbf{a},\mathbf{u}} y_1^{a_1} \cdots y_n^{a_n} \prod_{1 \le i < j \le n-1} L_{i,j}^{u_{i,j}}, \tag{2}$$

where $c_{\mathbf{a},\mathbf{u}} \in k$ for each $\mathbf{a}$ and $\mathbf{u}$.

The following is a key lemma.

Lemma 4 *In* (2), *we have* $\sum_{i=1}^{n-1} a_i \ge m$ *for each* $\mathbf{a} = (a_1, \ldots, a_n)$ *with* $c_{\mathbf{a},\mathbf{u}} \neq 0$.

We can deduce Proposition 3 from this lemma as follows. Assuming Lemma 4, there exists $0 \le a_i' \le a_i$ for $i = 1, \ldots, n-1$ such that $\sum_{i=1}^{n-1} a_i' = m$ for each $\mathbf{a}$ with $c_{\mathbf{a}} \neq 0$. Then, we can write

$$\mathrm{in}_{x_n} \Phi = \phi_0 x_n^m = \sum_{\mathbf{a},\mathbf{u}} c_{\mathbf{a},\mathbf{u}} y_1^{a_1 - a_1'} \cdots y_{n-1}^{a_{n-1} - a_{n-1}'} (y_1x_n)^{a_1'} \cdots (y_{n-1}x_n)^{a_{n-1}'} y_n^{a_n} \prod_{1 \le i < j \le n-1} L_{i,j}^{u_{i,j}}.$$

This proves that $\mathrm{in}_{x_n} \Phi$ belongs to $k[\mathbf{y}][S_{n-1}][y_1x_n, \ldots, y_{n-1}x_n]$. Thus, $\mathrm{in}_{x_n} \ker \Delta$ is contained in $k[\mathbf{y}][S_{n-1}][y_1x_n, \ldots, y_{n-1}x_n]$. The reverse inclusion is obvious.

Lemma 4 is proved by a method similar to the method used in the proof of Kurano [7, Proposition 3.1].

Note Drensky–Makar-Limanov [1] also gave a simple proof of Nowicki's conjecture. Very recently, Professor Mitsuyasu Hashimoto (Nagoya University) informed the author that Goto-Hayasaka-Kurano-Nakamura [3, Theorem 3.2] and Miyazaki [10, Theorem 3.7] also gave results which imply that Nowicki's conjecture is true. Actually, $\ker \Delta$ is equal to the invariant subring for the G_a-action on $k[\mathbf{y}][\mathbf{x}]$ defined by $y_i \mapsto y_i$ and $x_i \mapsto x_i + ty_i$ for $i = 1, \ldots, n$ for each $t \in G_a$. On the other hand, Goto-Hayasaka-Kurano-Nakamura and Miyazaki determined sets of generators for certain invariant rings in which $\ker \Delta$ appears.

The author would like to thank Professor Hashimoto for the information.

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ON SOME NUMERICAL INVARIANTS OF LOCAL RINGS AND PROJECTIVE SCHEMES

GENNADY LYUBEZNIK

1. Introduction

Let $(A, \mathfrak{m}, k)$ be a commutative Noetherian local ring containing a field. Let $\phi : R \to A$ be a surjection from a regular local ring R containing a field and let $I = \ker\phi$. Let $H^i_I(R)$ denote the ith local cohomology module of R with support in I. Let

$$\lambda_{i,j}(A) = \dim_k(\mathrm{Ext}^i_R(k, H^{n-j}_I(R)))$$

where $n = \dim R$. These integers were introduced in our old paper [9] where it has been shown that $\lambda_{i,j}(A)$ are finite and depend only on A, i.e. they are independent of R and ϕ.

The above definition of $\lambda_{i.j}(A)$ is valid only if A admits a surjection from a regular local ring containing a field in which case it is not hard to show that $\lambda_{i,j}(A) = \lambda_{i,j}(\hat{A})$ where $\hat{A}$ is the completion of A with respect to $\mathfrak{m}$. A complete local ring containing a field always admits a surjection from a regular local ring containing a field. Hence even if A does not admit a surjection from a regular local ring, one can set $\lambda_{i,j}(A) \stackrel{\text{def}}{=} \lambda_{i,j}(\hat{A})$. In this way $\lambda_{i,j}(A)$ becomes defined for every local ring containing a field.

Some of the properties of these integers proven in our paper [9] are the following:

• If $\lambda_{i,j}(A) \neq 0$, then $0 \leq i, j \leq d$ where $d = \dim A$.

•• $\lambda_{d,d} \neq 0$.

If A is regular, the integers $\lambda_{i,j}(A)$ are trivial in the sense that $\lambda_{i,j}(A) = 1$ and all other $\lambda_{i,j}(A)$ vanish [9]. Thus $\lambda_{i,j}(A)$ are measures of the singularity of A.

Part of the reason these integers are so interesting is that they are defined algebraically yet exhibit some striking and mysterious connections to topology. For example, if A is the local ring at an isolated singular point x of a complex analytic variety X, then $\lambda_{i,j}(A)$ can be completely expressed in terms of classical singular (not Zarisky!) topology of X around x [3].

2. The integer $\lambda_{d,d}(A)$

The "top" integer $\lambda_{d,d}(A)$ has attracted special attention. In our original paper [9] we asked if this integer is always equal to 1. Counterexamples in

NSF support is gratefully acknowledged.

dimension $d = 2$ have been given independently by K.-I. Kawasaki [7] and U. Walther [12]. Another important result of K.-I. Kawasaki on the way to a complete understanding of $\lambda_{d,d}(A)$ says that if A is Cohen-Macaulay, then $\lambda_{d,d}(A) = 1$ [8].

In our survey [10] we had stated a conjecture about what $\lambda_{d,d}(A)$ is equal to in general. In addition to the above-mentioned results of Kawasaki and Walther the conjecture was motivated by a result of Yanagawa [13, 3.16] on $\lambda_{d,d}(A)$ in the case that A is a quotient of a regular local ring by a monomial ideal. This conjecture has then been proven by us in characteristic $p > 0$ [11] and later in general by our student W. Zhang in a beautiful and completely characteristic-free paper [14].

The Hochster-Huneke graph Γ_B of a commutative Noetherian local ring B is defined as follows. The vertices of Γ_B are the top-dimensional minimal primes of B and two distinct vertices P and Q are joined by an edge if and only if the height of the ideal $P + Q$ equals 1. W. Zhang's result [14] is the following.

Theorem 2.1. *$\lambda_{d,d}(A)$ equals the number of the connected components of the Hochster-Huneke graph Γ_B where $B = \widehat{A^{\mathrm{sh}}}$ is the completion of the strict Henselization of A.*

This theorem completely describes $\lambda_{d,d}(A)$ in topological terms. This underscores the tantalizing and mysterious connection between $\lambda_{i,j}(A)$ and topology.

In the case $d = 2$ this result was essentially proven by K.-I. Kawasaki [7] and U. Walther [12]. W. Zhang's proof for $d > 2$ proceeds by induction on d, the case $d = 2$ being known. Namely, let $a \in A$ be an element outside all the minimal primes of A and let $\bar{A} = A/aA$. It is easy to see that $\bar{B} = B/aB$ is the completion of the strict Henselization of $\bar{A}$. The induction step basically consists in proving two things, both of which are given short and beautiful proofs in [14]: (i) $\lambda_{d,d}(A) = \lambda_{d-1,d-1}(\bar{A})$, and (ii) Γ_B and $\Gamma_{\bar{B}}$ have the same number of connected components.

The theorem is stated in terms of Γ_B rather than Γ_A because the faithful flatness of B over A implies that $\lambda_{d,d}(B) = \lambda_{d,d}(A)$ and because B has two important properties that A need not have, namely, B is complete and has a separably closed residue field. In general there may be several minimal primes of B over every minimal prime of A and consequently Γ_A may have fewer connected components than Γ_B. In fact each of the two operations, completing and separably closing the residue field, may increase the number of connected components of the Hochster-Huneke graph of the ring in question. Equality between $\lambda_{d,d}(A)$ and the number of the connected components of Γ_A holds if A is complete and has a separably closed residue field, but in general it need not hold.

3. The projective case

In [10, p. 133] we asked the following question:

Let Y be a projective scheme over a field k and let A be the local ring at the vertex of the affine cone over Y in some embedding of Y into $\mathbb{P}^n_k$. Is it true that $\lambda_{i,j}(A)$ depends only on Y, i and j, but not on the embedding?

A positive answer to this question would produce a new set of numerical invariants of projective varieties.

In a beautiful and very recent preprint W. Zhang [16] gave a positive answer to this question in the characteristic $p > 0$ case. We have no doubt that the answer is positive in characteristic 0 as well, but this remains to be proven.

In the rest of this note we sketch some of the ideas of W. Zhang's proof. We assume for the rest of this note that k is a field of characteristic $p > 0$. Let $Y \subset \mathbb{P}^n_k$ be a projective embedding of Y, let R be the homogeneous coordinate ring of $\mathbb{P}^n_k$ and let $I \subset R$ be the defining ideal of Y. Let

$$\mathcal{M}^{i,j} = \mathrm{Ext}_R^{n+1-i}(\mathrm{Ext}_R^{n+1-j}(R/I, R), R).$$

This R-module is naturally graded and $\mathcal{M}^{i,j}_l$, the degree-l piece of $\mathcal{M}^{i,j}$, is a finite-dimensional k-vector space for every l. W. Zhang proved the following.

Theorem 3.1. *$\mathcal{M}^{i,j}_0$ (i.e. the degree zero piece of $\mathcal{M}^{i,j}$) is independent of the embedding.*

This is proven by showing that

$$\mathcal{M}^{i,j}_0 \cong \mathrm{Ext}^{1-i}_{\mathcal{O}_Y}(\underline{\mathrm{Ext}}^{1-j}_{\mathcal{O}_Y}(\mathcal{O}_Y, \omega^\bullet_Y), \omega^\bullet_Y)$$

where $\omega^\bullet_Y = f^!(\mathcal{O}_{\mathrm{Spec}k})$ is a dualising complex of Y (here $f : Y \to \mathrm{Spec}k$ is the structure morphism of Y) and $\underline{\mathrm{Ext}}$ denotes sheaf Ext (we use the same notation as in [4]). Since the right-hand side does not involve any embedding, the theorem follows. Both the statement and the proof of this theorem are completely characteristic-free.

Next we recall the definition of the Frobenius morphism F from the category of R-modules to itself. Let R' be the additive group of R regarded as an R-bimodule as follows. Left multiplication by elements of R is the usual one while $r'r = r^p r'$ for all $r \in R$ and $r' \in R'$. For an R-module M one sets

$$F(M) = R' \otimes_R M$$

$$F(\phi : M \to N) = (R' \otimes_R M \overset{r'\otimes x \to r'\otimes\phi(x)}{\longrightarrow} R' \otimes_R N).$$

$F(M)$ is viewed as an R-module via multipication by elements of R on the left.

Since R is a regular ring, R' is flat over R, hence the functor F is exact [6]. Another elementary property of F is that $F(R/I) \cong R/I^{[p]}$ where $I^{[p]}$ is the ideal generated by the p-th powers of the elements of I; in particular, $F(R) \cong R$. Exactness of F implies that F commutes with $\mathrm{Ext}^i_R(-, R)$, i.e. $F(\mathrm{Ext}^i_R(M, R)) \cong \mathrm{Ext}^i_R(F(M), R)$. All of this together implies that

$$F(\mathcal{M}^{i,j}) \cong \mathrm{Ext}_R^{n+1-i}(\mathrm{Ext}_R^{n+1-j}(R/I^{[p]}, R), R).$$

The natural surjection $R/I^{[p]} \to R/I$ is degree-preserving. Since the functor $\mathrm{Ext}_R^{n+1-i}(\mathrm{Ext}_R^{n+1-j}(-,R),R)$ is covariant, this surjection induces a degree preserving map of graded R-modules

$$\phi : F(\mathcal{M}^{i,j}) \cong R' \otimes_R \mathcal{M}^{i,j} \to \mathcal{M}^{i,j}.$$

Identifying $1 \otimes_R \mathcal{M}^{i,j}$ with $\mathcal{M}^{i,j}$ we get an induced map $\mathfrak{f} : \mathcal{M}^{i,j} \to \mathcal{M}^{i,j}$ that satisfies the following properties: (i) $\mathfrak{f}(rx) = r^p\mathfrak{f}(x)$ for all $r \in R$ and $x \in \mathcal{M}^{i,j}$, and (ii) $\deg\mathfrak{f}(x) = p\deg x$. In particular we get a map in degree zero

$$\mathfrak{f}_0 : \mathcal{M}_0^{i,j} \to \mathcal{M}_0^{i,j}$$

satisfying $\mathfrak{f}(cx) = c^p\mathfrak{f}(x)$ for every $c \in k$ and every $x \in \mathcal{M}_0^{i,j}$.

From now on we assume that the field k is separably closed, in which case $\mathfrak{f}_0(\mathcal{M}_0^{i,j})$ is a k-vector subspace of $\mathcal{M}_0^{i,j}$. This does not involve any loss of generality for separably closing the field does not affect $\lambda_{i,j}(A)$. The resulting descending chain of k-vector spaces

$$\mathcal{M}_0^{i,j} \supset \mathfrak{f}_0(\mathcal{M}_0^{i,j}) \supset \mathfrak{f}_0^2(\mathcal{M}_0^{i,j}) \supset \dots$$

stabilizes because $\mathcal{M}_0^{i,j}$ is finite-dimensional. We denote this stable space $(\mathcal{M}_0^{i,j})_s$. The next step in W. Zhang's proof is the following.

Theorem 3.2. $\lambda_{i,j}(A) = \dim_k(\mathcal{M}_0^{i,j})_s$.

Since $\mathcal{M}_0^{i,j}$ is independent of the embedding and $(\mathcal{M}_0^{i,j})_s$ is a subspace of $\mathcal{M}_0^{i,j}$, this theorem implies that there is an upper bound on $\lambda_{i,j}(A)$ over all the projective embeddings of Y, i.e. the set of possible values of $\lambda_{i,j}(A)$, for a fixed Y, is finite. This was the main result of W. Zhang's earlier preprint [15]. In his very recent preprint [16] W. Zhang has finally proved the following

Theorem 3.3. *The map* $\mathfrak{f}_0 : \mathcal{M}_0^{i,j} \to \mathcal{M}_0^{i,j}$ *is independent of the embedding.*

Theorems 3.1, 3.2 and 3.3 immediately imply

Main Theorem. *If the characteristic of the field* k *is* $p > 0$, *then* $\lambda_{i,j}(A)$ *depends only on* Y, i *and* j *but not on the embedding.*

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Cohomology groups of semigroup rings and toric varieties

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1 Introduction

I will explain the relation between Yanagawa's result on the local cohomologies of squarefree modules and Fujino's cohomology vanishing theorems of differential modules on projective toric varieties and toric polyhedra.

Let $M \simeq \mathbf{Z}^r$ ($r \geq 0$) and $\mathcal{S} \subset M$ be a finitely generated additive subsemigroup with $0 \in \mathcal{S}$. We assume $\mathcal{S} + (-\mathcal{S}) = M$. For a field k of any characteristic, the semigroup ring $S = k[\mathcal{S}]$ is defined. We denote the k-basis of S by $\{\mathbf{e}(m) \ ; \ m \in \mathcal{S}\}$. This is a k-subalgebra of the group ring $k[M]$ with the basis $\{\mathbf{e}(m) \ ; \ m \in M\}$. For a subset $A \subset M$, we denote by $\langle A \rangle_k$ the vector space with the basis $\{\mathbf{e}(m) \ ; \ m \in A\}$.

We investigate this ring combinatorially by using the associated cone. Let $M_{\mathbf{R}} = M \otimes_{\mathbf{Z}} \mathbf{R} \simeq \mathbf{R}^r$ and $C(\mathcal{S}) \subset M_{\mathbf{R}}$ the closed convex cone generated by $\mathcal{S}$. Then $C(\mathcal{S})$ is a rational polyhedral cone of dimension r. We denote by π the dual cone of $C(\mathcal{S})$ in the dual space $N_{\mathbf{R}}$ of $M_{\mathbf{R}}$. The normalization of S is $k[M \cap C(\mathcal{S})]$. In particular, S is normal if and only if $\mathcal{S} = M \cap C(\mathcal{S})$ $(= M \cap \pi^{\vee})$. The definition of squarefree modules on a normal S is given in the next section.

If $C(\mathcal{S})$ is strongly convex, i.e., if $C(\mathcal{S}) \cap (-C(\mathcal{S})) = \{0\}$, then the vector subspace $\mathfrak{m} = \langle \mathcal{S} \setminus \{0\} \rangle_k$ is the M-homogeneous maximal ideal of S. If E is an M-graded S module, then each local cohomology group $H^i_{\mathfrak{m}}(E)$ is an M-graded S-module.

Theorem 1.1 (Yanagawa[Y]) *Assume that $\mathcal{S} = C(\mathcal{S}) \cap M$ and $C(\mathcal{S})$ is strongly convex. Let E be a finitely generated M-graded S-module. If E is squarefree, then each local cohomology group $H^i_{\mathfrak{m}}(E)$ is the Matlis dual of a squarefree module. In particular $H^i_{\mathfrak{m}}(E)(m) = 0$ if $m \notin -\mathcal{S}$.*

On the other hand, Fujino proved the following theorem by his method of multiplication maps which is analogous to that of Frobenius morphisms.

Theorem 1.2 (Fujino[F1]) *Let X be a projective toric variety, L an ample line bundle, B a reduced torus invariant Weil divisor and i a nonnegative integer. Then*

$$H^j(X, \widetilde{\Omega}^i_X(\log B) \otimes L) = 0$$

for all $j > 0$.

This theorem is generalized for the modules of differentials on a projective toric polyhedron [F2]. Here toric polyhedron is a torus action invariant subvariety of a toric variety defined by a squarefree ideal.

Yanagawa used the description of the local cohomology groups by Burns and Herzog [BH] for the proof of his theorem. There is a similar description of the cohomology groups of coherent sheaves on a projective toric variety (cf. [I3]). Then we can understand the relation between these two theorems.

2 Squarefree modules

Let π be a strongly convex rational polyhedral cone of $N_{\mathbf{R}}$. The set of faces of π is denoted by $F(\pi)$. For each integer $0 \leq i \leq r$, we set $F(\pi)(i) = \{\sigma \in F(\pi) \; ; \dim \sigma = i\}$. We denote by S_π the normal semigroup ring $k[M \cap \pi^\vee]$.

Let E be an M-graded S_π-module. For each m in M, we denote by $E(m)$ the homogeneous part of degree m. If m is in M and m' is in $M \cap \pi^\vee$, the multiplication of $\mathrm{e}(m)$ defines a k-linear map $\mu_E(m, m') : E(m) \to E(m+m')$.

Definition 2.1 An M-graded S_π-module E is said to be *squarefree* if the following conditions are satisfied.
(1) $E(m) = 0$ if $m \notin M \cap \pi^\vee$.
(2) $\mu_E(m, m')$ is an isomorphism if $m \in M \cap \pi^\vee$ and $\pi \cap m^\perp = \pi \cap (m+m')^\perp$.

Note that for an element m in $M \cap \pi^\vee$, $\sigma = \pi \cap m^\perp$ is a face of π and m is in the relative interior of the face $\pi^\vee \cap \sigma^\perp$ of $\pi^\vee$. If m_1, m_2 are in the relative interior of a face of $\pi^\vee$, then $m_1 + m_2$ is also in the relative interior. Hence if E is squarefree, then both $E(m_1)$ and $E(m_2)$ are isomorphic to $E(m_1 + m_2)$. This implies that there exists a k-vector space $E(\sigma)$ for each $\sigma \in F(\pi)$ such that $E(m)$ is identified with $E(\sigma)\mathrm{e}(m)$ for all m in $M \cap \mathrm{rel.int}(\pi^\vee \cap \sigma^\perp)$. If σ and τ are in $F(\pi)$ and $\sigma \prec \tau$, then for m, m' with $m \in M \cap \mathrm{rel.int}(\pi^\vee \cap \tau^\perp) \subset M \cap \pi^\vee \cap \sigma^\perp$ and $m' \in M \cap \mathrm{rel.int}(\pi^\vee \cap \sigma^\perp)$, we have $m + m' \in M \cap \mathrm{rel.int}(\pi^\vee \cap \sigma^\perp)$. Hence the multiplication of $\mathrm{e}(m')$ induces a k-linear map $f_E(\sigma/\tau) : E(\tau) \to E(\sigma)$, which does not depend on the choice of m, m'. Namely, we have a contravariant functor f_E from $F(\pi)$ to k-vector spaces defined by $f_E(\sigma) = E(\sigma)$. Conversely, if a contravariant functor f from $F(\pi)$ to k-vector spaces is given, then we define a squarefree M-graded S_π-module $\mathbf{E}_f$ by

$$\mathbf{E}_f = \bigoplus_{m \in M \cap \pi^\vee} f(\pi \cap m^\perp)\mathrm{e}(m) \, .$$

The multiplication map $\mathrm{e}(m') : \mathbf{E}_f(m) \to \mathbf{E}_f(m+m')$ for the above m, m' is defined by $f(\sigma/\tau) : f(\tau) \to f(\sigma)$.

The following proposition is proved easily (cf. [Y]).

Proposition 2.2 *Let E be a squarefree M-graded S_π-module.*
(1) E is finitely generated if and only if the dimension of $E(\sigma)$ is finite for every σ in $F(\pi)$.

(2) *E is a free S_π-module if and only if $f_E(\sigma/\pi)$ is an isomorphism for every σ in $F(\pi)$. In this case, E is isomorphic to $S_\pi \otimes_k E(\pi)$.*
(3) *E is a quotient of a squarefree free S_π-module if and only if $f_E(\sigma/\pi)$ is surjective for every σ in $F(\pi)$. In this case, E is a quotient of $S_\pi \otimes_k E(\pi)$.*
(4) *E is an S_π-submodule of a squarefree free S_π-module if and only if $f_E(\mathbf{0}/\sigma)$ is injective for every σ in $F(\pi)$. In this case, E is an S_π-submodule of $S_\pi \otimes_k E(\mathbf{0})$.*

Example 2.3 *We denote by $k_{F(\pi)}$ or simply k the constant functor defined by $k(\sigma) = k$. Then the squarefree module $\mathbf{E}_k$ is equal to S_π. For a star closed subset Φ of $F(\pi)$, the ideal $I(\Phi) = \bigcap_{\sigma\in\Phi} P(\sigma)$ is a squarefree module which corresponds to the functor*

$$G(\sigma) = \begin{cases} 0 & \text{if } \sigma \in \Phi \\ k & \text{if } \sigma \in F(\pi) \setminus \Phi \,. \end{cases}$$

Here a subset Φ of $F(\pi)$ is said to be star closed if $\sigma \in \Phi$ and $\sigma \prec \tau \prec \pi$ imply $\tau \in F(\pi)$.

Example 2.4 *Let V be a k-vector space of finite dimension. Suppose that a subspace $A(\gamma) \subset V$ is given for every $\gamma \in F(\pi)(1)$. We define the functor A by*

$$A(\sigma) = \bigcap_{\gamma\in F(\sigma)(1)} A(\gamma)$$

for every $\sigma \in F(\pi)$. For $\sigma, \tau \in F(\pi)$ with $\sigma \prec \tau$, the morphism $A(\sigma/\tau)$ is defined to be the inclusion map $A(\tau) \hookrightarrow A(\sigma)$. Then the squarefree module $\mathbf{E}_A$ is a reflective submodule of $S_\pi \otimes_k V$.

When $\operatorname{char} k = 0$, the modules of differentials on an affine toric variety defined in Danilov's paper [D] belong to Example 2.4. Let $V = M_k = M \otimes_{\mathbf{Z}} k$ and $\Omega(\sigma) = M[\sigma]_k$ for every σ in $F(\pi)$ where $M[\sigma] = M \cap \sigma^\perp$. Then $\mathbf{E}_\Omega$ is equal to Danilov's sheaf $\widetilde{\Omega}^1_{X(\pi)}$ of 1-forms on the affine toric variety $X(\pi) = \operatorname{Spec} S_\pi$. More generally, for any p with $0 \le p \le r$, the contravariant functor Ω^p defined by $\Omega^p(\sigma) = \bigwedge^p M[\sigma]_k$ defines a squarefree module $\mathbf{E}_{\Omega^p}$ which is equal to Danilov's sheaf $\widetilde{\Omega}^p_{X(\pi)}$ of p-forms on $X(\pi)$.

Let ρ be a face of π. If m is an element of M with $\pi \cap m^\perp = \rho$, then the localization $S_\pi[\mathrm{e}(m)^{-1}]$ is equal to $S_\rho = k[M \cap \rho^\vee]$.

Proposition 2.5 *Let E be a squarefree M-graded S_π-module and ρ a face of π. Then $E_\rho = E \otimes_{S_\pi} S_\rho$ is a squarefree S_ρ-modules. The corresponding contravariant functor f_{E_ρ} from $F(\rho)$ to k-vector spaces is equal to the restriction $f_E|F(\rho)$.*

Let Σ be a fan of $N_{\mathbf{R}}$ and $Z = Z(\Sigma)$ the associated toric variety. For a T_N-equivariant $\mathcal{O}_Z$-module $\mathcal{E}$ and for an element σ in Σ, the $\mathcal{O}_Z(X(\sigma))$-module $\mathcal{E}(X(\sigma))$ has M-grading corresponding to the T_N-action. We call $\mathcal{E}$ a squarefree sheaf if $\mathcal{E}(X(\sigma))$ is a squarefree S_σ-module for every σ in Σ. A quasicoherent squarefree $\mathcal{O}_Z$-module $\mathcal{E}$ corresponds to a contravariant functor from Σ to k-vector spaces. We denote the functor by $f_{\mathcal{E}}$.

Let $P \subset M_{\mathbf{R}}$ be an integral convex polytope of dimension r. The set of cones

$$\Delta(P) = \{(P - x)^\vee \,;\, x \in P\}$$

is a projective fan of $N_{\mathbf{R}}$, and the associated projective toric variety $Z(P)$ has the tautological line bundle $\mathcal{O}_{Z(P)}(1)$ such that

$$H^0(Z(P), \mathcal{O}_{Z(P)}(1)) = \langle M \cap P \rangle_k .$$

Set $\widetilde{M} = M \oplus \mathbf{Z}$ and let $\widetilde{N}$ be its dual $\mathbf{Z}$-module. Denote by $C(P)$ the closed convex cone generated by $P \times \{1\} \subset \widetilde{M}_{\mathbf{R}} = \widetilde{M} \otimes_{\mathbf{Z}} \mathbf{R}$ and ω the dual cone in $\widetilde{N}_{\mathbf{R}} = \widetilde{N} \otimes_{\mathbf{Z}} \mathbf{R}$. Then $Z(P) = \operatorname{Proj} S_\omega$ for $S_\omega = k[\widetilde{M} \cap \omega^\vee]$, where the degree of the monomial $\mathbf{e}((m,d))$ is defined to be d for every (m,d) in $\widetilde{M} = M \oplus \mathbf{Z}$. There exists a natural bijection between $\Delta(P)$ and $F(\omega) \setminus \{\omega\}$. Namely, if $\sigma = (P - x)^\vee$, the corresponding face $\tilde{\sigma}$ of ω is defined by $\omega \cap (x,1)^\perp$. The projection $\widetilde{N}_{\mathbf{R}} = N_{\mathbf{R}} \oplus \mathbf{R} \longrightarrow N_{\mathbf{R}}$ induces bijection $\tilde{\sigma} \longrightarrow \sigma$ for every σ. For a finitely generated graded S_ω-module E, we denote by $\mathcal{E}$ the coherent $\mathcal{O}_{Z(P)}$-module $E^\sim$. For the tautological ample line bundle $\mathcal{O}_{Z(P)}(1)$, we denote $\mathcal{E}(d) = \mathcal{E} \otimes \mathcal{O}_{Z(P)}(1)^{\otimes d}$.

For a contravariant functor f from $\Delta(P)$ to k-vector spaces, the functor $\tilde{f}$ from $F(\omega)$ is defined by $\tilde{f}(\tilde{\sigma}) = f(\sigma)$ for every σ in $\Delta(P)$ while $\tilde{f}(\omega)$ is defined to be the projective limit of $\{f(\sigma)\ ;\ \sigma \in \Delta(P)\}$.

Proposition 2.6 *Let $\mathcal{E}$ be a finitely generated squarefree $\mathcal{O}_{Z(P)}$-module, and E the squarefree $\widetilde{M}$-graded S_ω-module associated to the contravariant functor $\tilde{f}_{\mathcal{E}}$. Then $\mathcal{E}$ is isomorphic to the associated sheaf $E^\sim$.*

For every σ in $\Delta(P)$, there exists an exact sequence

$$0 \longrightarrow M[\sigma] \longrightarrow \widetilde{M}[\tilde{\sigma}] \longrightarrow \mathbf{Z} \longrightarrow 0 ,$$

which induces the exact squence

$$0 \longrightarrow \bigwedge^p M[\sigma] \longrightarrow \bigwedge^p \widetilde{M}[\tilde{\sigma}] \longrightarrow \bigwedge^{p-1} M[\sigma] \longrightarrow 0 \qquad (1)$$

for every integer $0 \le p \le r$. Let F_P^p be the functor from $\Delta(P)$ to k-vector spaces defined by $F_P^p(\sigma) = \Omega^p(\tilde{\sigma})$ for every σ in $\Delta(P)$. If $\operatorname{char} k = 0$, the sequence

$$0 \longrightarrow \Omega^p(\sigma) \longrightarrow F_P^p(\tilde{\sigma}) \longrightarrow \Omega^{p-1}(\sigma) \longrightarrow 0$$

induced by (1) is exact. Hence we get an exact squence

$$0 \longrightarrow \tilde{\Omega}^p_{Z(P)} \longrightarrow \mathbf{E}_{F_P^p} \longrightarrow \tilde{\Omega}^{p-1}_{Z(P)} \longrightarrow 0$$

of squarefree modules on $Z(P)$ for every integer $0 \le p \le r$. Well-known exact sequence

$$0 \longrightarrow \Omega^1_{\mathbf{P}^n} \longrightarrow \mathcal{O}_{\mathbf{P}^n}(-1)^{\oplus n+1} \longrightarrow \mathcal{O}_{\mathbf{P}^n} \longrightarrow 0$$

on a projective space is a special case. Here $\mathcal{O}_{\mathbf{P}^n}(-1)^{\oplus n+1}$ can be understand as a squarefree module $\bigoplus_{i=0}^n \mathcal{O}_{\mathbf{P}^n} dx_i$ on $\mathbf{P}^n$ with respect to the homogeneous coordinates $[x_0, x_1, \cdots, x_n]$.

A subset $\Phi \subset F(\omega)$ is said to be locally star closed if $\sigma, \rho \in \Phi$ and $\sigma \prec \tau \prec \rho$ imply $\tau \in \Phi$. For a locally star closed subset $\Phi \subset F(\omega)$ and a contravariant functor G from $F(\omega)$ to an abelian category $\mathcal{A}$, the complex $C^\bullet(\Phi, G)$ in $\mathcal{A}$ is defined as follow.

We define $\Phi(i) = \{\sigma \in \Phi \; ; \dim \sigma = i\}$ for every $i \in \mathbf{Z}$. We set

$$C^i(\Phi, G) = \bigoplus_{\sigma \in \Phi(r-i)} G(\sigma) \otimes \mathbf{Z}(\sigma)^*$$

for every i, where $\mathbf{Z}(\sigma)^*$ is the dual of a free $\mathbf{Z}$-module of rank one whose generators are the orientations of σ. Namely, $\mathbf{Z}(\sigma)^* = \mathbf{Z}e_\sigma^*$ if we take an orientation e_σ. The homomorphism $d^i : C^i(\Phi, G) \to C^{i+1}(\Phi, G)$ is defind as follows. The component of d^i for $\tau \in \Phi(r-i)$ and $\sigma \in \Phi(r-i-1)$ is $G(\sigma/\tau) \otimes q^*_{\sigma/\tau}$, where $q^*_{\sigma/\tau}$ is the isomorphism $\mathbf{Z}(\tau)^* \to \mathbf{Z}(\sigma)^*$ defined by the incidence of σ and τ if $\sigma \prec \tau$ and the zero map otherwise. Note that $C^i(\Phi, G) = 0$ if $\Phi(r-i) = \emptyset$. In particular, $C^i(\Phi, G) = 0$ if $i \notin [0, r]$.

For a finitely generated graded S-module E, the local cohomology group $H^i_{\mathfrak{m}}(E)$ is equal to that of the complex $C^\bullet(F(\omega), E_*)$ by [BH], where E_* is the contravariant functor defined by $E_*(\sigma) = E \otimes_{S_\omega} S_\sigma$.

We set $F(\omega)' = F(\omega) \setminus \{\omega\}$. For the cohomology groups of the coherent sheaf $\mathcal{E}(d)$, we get the following theorem.

Theorem 2.7 *For any $0 \le i \le r$, we have*

$$\bigoplus_{d \in \mathbf{Z}} H^i(Z(P), \mathcal{E}(d)) = H^{i+1}(C^\bullet(F(\omega)', E_*)) \,.$$

Since the cohomology groups of $C^\bullet(F(\omega), E_*)$ and $C^\bullet(F(\omega)', E_*)$ are equal in degree greater than one, we get the following corollary.

Corollary 2.8 *$H^i_{\mathfrak{m}}(E)$ is a graded module. For any $2 \le i \le r+1$ and $d \in \mathbf{Z}$, we have*

$$H^i_{\mathfrak{m}}(E)_d = H^i(C^\bullet(F(\omega)', E_*))_d = H^{i-1}(Z(P), \mathcal{E}(d)) \,.$$

We also have the following.

Corollary 2.9 *There exists an exact sequence of k-vector spaces*

$$0 \longrightarrow H^0_{\mathfrak{m}}(E)_d \longrightarrow E_d \longrightarrow H^0(Z(P), \mathcal{E}(d)) \longrightarrow H^1_{\mathfrak{m}}(E)_d \longrightarrow 0 \,.$$

If E is a squarefree S_ω-module, then $H^i_{\mathfrak{m}}(E)$ is an $\widetilde{M}$-graded S_ω-module. Its m-component is zero if m is outside $\widetilde{M} \cap (-\omega)$ by Yanagawa's theorem. This implies the vanishing of $H^i(Z(P), \mathcal{E}(d))$ for $i > 0$. This shows the relation of Yanagawa's theory and Fujino's vanishing theorem of differential modules on projective toric varieties. We can discuss the case of toric polyhedra similarly.

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G-prime and G-primary G-ideals on G-schemes

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1 Introduction

This report is a preliminary version, and a more detailed final version will be published elsewhere.

Let A be a $\mathbb{Z}^n$-graded ring, I a prime ideal (resp. radical ideal, primary ideal) of A, and I^* the homogeneous ideal generated by the all homogeneous elements of I. Then it is well-known that I^* is again a prime ideal (resp. radical ideal, primary ideal). In particular, if P is a prime ideal of A, then the local ring A_{P^*} makes sense. In particular, the following theorem makes sense.

Theorem 1.1. Let $M = \mathbb{Z}^n$, A be an M-graded noetherian ring, and P a prime ideal of A. If A_{P^*} is Cohen–Macaulay (resp. Gorenstein, complete intersection, regular), then so is A_P.

This theorem was conjectured by Nagata [8] for the case that $n = 1$ for the Cohen–Macaulay property, and solved by Hochster–Ratliff [5], Matijevic–Roberts [7], Matijevic [6], Aoyama–Goto [1], and Avramov–Achiles [2], affirmatively.

If M is a finitely generated abelian group with torsion elements and A is M-graded, then even if P is a prime ideal, P^* may not be a prime. However, a homogeneous ideal of the form P^* has some special interest. For homogeneous ideals I and J, if $IJ \subset P^*$, then either $I \subset P^*$ or $J \subset P^*$. Our start of this research is to consider a substitute of A_{P^*} in this context.

More generally, let S be a scheme, G an S-group scheme, and X a noetherian G-scheme, where a G-scheme means an S scheme on which G acts. We assume that the second projection $p_2 : G \times X \to X$ is flat of finite type. Under these settings, we define a G-prime, G-primary, and G-radical G-ideals. As we will see, these are natural generalization of prime, primary, and radical ideals, respectively. We study some important properties of G-stable closed subschemes defined by G-primary ideals. Moreover, we generalize Theorem 1.1.

Utilizing this research, we can remove the assumption that G is smooth with connected fibers from the talk of Ohtani [9] given at the 29th Symposium on Commutative Algebra in Japan. This will be discussed elsewhere.

2 G-prime ideals

Let S, G, and X be as in the introduction.

Definition 2.1 (Mumford). A G-linearized $\mathcal{O}_X$-module (an equivariant $(G, \mathcal{O}_X)$-module) is a pair $(\mathcal{M}, \Phi)$ such that $\mathcal{M}$ is an $\mathcal{O}_X$-module, and $\Phi : a^*\mathcal{M} \to p_2^*\mathcal{M}$ is an isomorphism of $\mathcal{O}_{G\times X}$-modules such that

$$(\mu \times 1_X)^*\Phi : (\mu \times 1_X)^* a^*\mathcal{M} \to (\mu \times 1_X)^* p_2^*\mathcal{M}$$

agrees with

$$\begin{aligned}(\mu \times 1_X)^* a^*\mathcal{M} &\xrightarrow{\cong} (1_G \times a)^* a^*\mathcal{M} \xrightarrow{\Phi} (1_G \times a)^* p_2^*\mathcal{M} \\ &\xrightarrow{\cong} p_{23}^* a^*\mathcal{M} \xrightarrow{\Phi} p_{23}^* p_2^*\mathcal{M} \xrightarrow{\cong} (\mu \times 1_X)^* p_2^*\mathcal{M},\end{aligned}$$

where $p_{23} : G \times G \times X \to G \times X$ is the projection. In this case, we sometimes say that $\mathcal{M}$ is a G-linearized $\mathcal{O}_X$-module with Φ its structure map.

Definition 2.2. A morphism $\varphi : (\mathcal{M}, \Phi) \to (\mathcal{N}, \Psi)$ of G-linearized $\mathcal{O}_X$-modules is a morphism $\varphi : \mathcal{M} \to \mathcal{N}$ such that $\Psi \circ (a^*\varphi) = (p_2^*\varphi) \circ \Phi$.

Thus we have a category of G-linearized $\mathcal{O}_X$-modules in a natural way.

Definition 2.3. Let $(\mathcal{M}, \Phi)$ be a G-linearized $\mathcal{O}_X$-module. We say that $\mathcal{N}$ is an equivariant $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$ if $\mathcal{N}$ is an $\mathcal{O}_X$-submodule of $\mathcal{M}$, and $\Phi(a^*\mathcal{N}) = p_2^*\mathcal{N}$ (note that a and p_2 are flat). If, moreover, $\mathcal{M} = \mathcal{O}_X$, then we say that $\mathcal{N}$ is a *G-ideal* of $\mathcal{O}_X$.

If $\mathcal{N}$ is an equivariant $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$, then $(\mathcal{N}, \Phi|_{\mathcal{N}})$ is a G-linearized $\mathcal{O}_X$-module, and the inclusion $\mathcal{N} \hookrightarrow \mathcal{M}$ is a morphism of G-linearized $\mathcal{O}_X$-modules. Conversely, if $\varphi : \mathcal{N} \to \mathcal{M}$ is a morphism of G-linearized $\mathcal{O}_X$-modules, then the image of φ is an equivariant $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$.

The following is [4, Corollary 12.8, Lemma 12.12].

Theorem 2.4. The category $\mathrm{Qch}(G, X)$ of quasi-coherent G-linearized $\mathcal{O}_X$-modules is a locally noetherian abelian category, and $(\mathcal{M}, \Phi)$ is a noetherian object of $\mathrm{Qch}(G, X)$ if and only if $\mathcal{M}$ is coherent. The forgetful functor $F_X : \mathrm{Qch}(G, X) \to \mathrm{Qch}(X)$ given by $(\mathcal{M}, \Phi) \mapsto \mathcal{M}$ is faithful exact, and admits a right adjoint.

(Quasi-) coherent G-linearized $\mathcal{O}_X$-modules are closed under various ring-theoretic operations.

Lemma 2.5. Let $\mathcal{M}$, $\mathcal{N}$, $\mathcal{L}$ be in $\mathrm{Qch}(G, X)$, $\mathcal{I}$ be a G-ideal, and $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, and $\mathcal{M}_\lambda$ be quasi-coherent equivariant $(G, \mathcal{O}_X)$-submodules of $\mathcal{M}$. Let $\mathcal{L}$ and $\mathcal{M}_3$ be coherent. Then the following modules have structures of quasi-coherent G-linearized $\mathcal{O}_X$-modules: $\underline{\mathrm{Tor}}_i^{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$, $\underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(\mathcal{L}, \mathcal{M})$, $\underline{H}^i_{\mathcal{I}}(\mathcal{M}) \cong \varinjlim \underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, \mathcal{M})$, the Fitting ideal $\underline{\mathrm{Fitt}}_j(\mathcal{L})$, $\mathcal{M}_1 \cap \mathcal{M}_2$, $\sum_\lambda \mathcal{M}_\lambda$, $\mathcal{I}\mathcal{M}_1$, $\mathcal{M}_1 : \mathcal{M}_3$, and $\mathcal{M}_1 : \mathcal{I}$.

Let $\mathcal{M}$ be in $\mathrm{Qch}(G, X)$, and $\mathfrak{m}$ be an $\mathcal{O}_X$-submodule of $\mathcal{M}$. The sum of all quasi-coherent equivariant $(G, \mathcal{O}_X)$-submodules of $\mathcal{M}$ contained in $\mathfrak{m}$ is denoted by $\mathfrak{m}^*$. $\mathfrak{m}^*$ is the largest quasi-coherent equivariant $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$ contained in $\mathfrak{m}$.

Let $Y = V(\mathfrak{a})$ be a closed subscheme of X. Then $Y^* := V(\mathfrak{a}^*)$ is the smallest G-stable closed subscheme of X containing Y.

From now on, all ideals and G-ideals are required to be coherent. All modules and G-linearized modules are required to be quasi-coherent.

Lemma 2.6. Let $\mathcal{M}$ be in $\mathrm{Qch}(G, X)$, $\mathfrak{m}$, $\mathfrak{n}$, and $\mathfrak{m}_\lambda$ be $\mathcal{O}_X$-submodules of $\mathcal{M}$, and $\mathcal{N}$ be a coherent equivariant $(G, \mathcal{O}_X)$-submodule of $\mathcal{M}$. Let $\mathcal{I}$ be a G-ideal of $\mathcal{O}_X$. Then we have: **1)** $(\bigcap_\lambda \mathfrak{m}_\lambda^*)^* = (\bigcap_\lambda \mathfrak{m}_\lambda)^*$; **2)** $\mathfrak{m}^* \cap \mathfrak{n}^* = (\mathfrak{m} \cap \mathfrak{n})^*$; **3)** $(\mathfrak{m} : \mathcal{N})^* = \mathfrak{m}^* : \mathcal{N}$; **4)** $(\mathfrak{m} : \mathcal{I})^* = \mathfrak{m}^* : \mathcal{I}$.

3 G-prime and G-radical G-ideals

Lemma 3.1. Let $\mathcal{P}$ be a G-ideal of $\mathcal{O}_X$. Then the following are equivalent.

- There exists some ideal $\mathfrak{p}$ of $\mathcal{O}_X$ such that $\mathfrak{p}$ is prime (i.e., $V(\mathfrak{p})$ is integral) and $\mathfrak{p}^* = \mathcal{P}$.
- $\mathcal{P} \neq \mathcal{O}_X$, and if $\mathcal{I}$ and $\mathcal{J}$ are G-ideals of $\mathcal{O}_X$ and $\mathcal{I}\mathcal{J} \subset \mathcal{P}$, then $\mathcal{I} \subset \mathcal{P}$ or $\mathcal{J} \subset \mathcal{P}$.

Definition 3.2. If the equivalent conditions in the lemma are satisfied, we say that $\mathcal{P}$ is a *G-prime* G-ideal.

Definition 3.3. Let $\mathcal{I}$ be a G-ideal of $\mathcal{O}_X$. Then $V_G(\mathcal{I})$ denotes the set of G-prime ideals containing $\mathcal{I}$. We set $\sqrt[G]{\mathcal{I}} := (\bigcap_{\mathcal{P}\in V_G(\mathcal{I})} \mathcal{P})^*$, and call $\sqrt[G]{\mathcal{I}}$ the *G-radical* of $\mathcal{I}$.

Lemma 3.4. Let $\mathcal{I}$, $\mathcal{J}$, and $\mathcal{P}$ be G-ideals of $\mathcal{O}_X$. Then we have: **1)** $\mathcal{I} \subset \sqrt[G]{\mathcal{I}} \subset \sqrt{\mathcal{I}}$, $\sqrt[G]{\mathcal{I}} = \sqrt{\mathcal{I}}^*$. **2)** If $\mathcal{I} \supset \mathcal{J}$, then $\sqrt[G]{\mathcal{I}} \supset \sqrt[G]{\mathcal{J}}$. **3)** $\sqrt[G]{\mathcal{I}\mathcal{J}} = \sqrt[G]{\mathcal{I}\cap\mathcal{J}} = \sqrt[G]{\mathcal{I}} \cap \sqrt[G]{\mathcal{J}}$. **4)** $\sqrt[G]{\sqrt[G]{\mathcal{I}}} = \sqrt[G]{\mathcal{I}}$. **5)** If $\mathcal{P}$ is a G-prime, then $\sqrt[G]{\mathcal{P}} = \mathcal{P}$.

Lemma 3.5. Let $\mathcal{I}$ be a G-ideal of $\mathcal{O}_X$. Then the following are equivalent. **1)** $\mathcal{I} = \sqrt[G]{\mathcal{I}}$; **2)** $\mathcal{I}$ is the intersection of finitely many G-prime G-ideals; **3)** There exists some ideal $\mathfrak{a}$ of $\mathcal{O}_X$ such that $\mathfrak{a}$ is radical (i.e., $V(\mathfrak{a})$ is reduced), and $\mathfrak{a}^* = \mathcal{I}$.

If the equivalent conditions in the lemma are satisfied, then we say that $\mathcal{I}$ is *G-radical.* A G-prime G-ideal is G-radical.

4 G-primary submodules

From now on, until the end of this report, let $\mathcal{M}$ be a coherent G-linearized $\mathcal{O}_X$-module, and $\mathcal{N}$ its coherent equivariant $(G, \mathcal{O}_X)$-submodule.

Definition 4.1. We say that $\mathcal{N}$ is *G-primary* if $\mathcal{N} \neq \mathcal{M}$, and for any coherent equivariant $(G, \mathcal{O}_X)$-submodule $\mathcal{L}$ of $\mathcal{M}$, either $\mathcal{N} : \mathcal{L} = \mathcal{O}_X$ or $\mathcal{N} : \mathcal{L} \subset \sqrt[G]{\mathcal{N} : \mathcal{M}}$ holds.

If $\mathcal{N}$ is G-primary, then $\mathcal{P} = \sqrt[G]{\mathcal{N} : \mathcal{M}}$ is G-prime. In this case, we say that $\mathcal{N}$ is $\mathcal{P}$-G-primary.

Lemma 4.2. For a prime ideal $\mathfrak{p}$ of $\mathcal{O}_X$, $\mathfrak{p}^*$ is G-prime. For a radical ideal $\mathfrak{a}$ of $\mathcal{O}_X$, $\mathfrak{a}^*$ is G-radical. If $\mathfrak{n}$ is a $\mathfrak{p}$-primary $\mathcal{O}_X$-submodule of $\mathcal{M}$, then $\mathfrak{n}^*$ is a $\mathfrak{p}^*$-G-primary submodule of $\mathcal{M}$. For a G-primary submodule $\mathcal{N}$ of $\mathcal{M}$, there exists some primary $\mathcal{O}_X$-submodule $\mathfrak{n}$ of $\mathcal{M}$ such that $\mathfrak{n}^* = \mathcal{N}$.

An expression

$$\mathcal{N} = \mathcal{M}_1 \cap \cdots \cap \mathcal{M}_r$$

is called a *G-primary decomposition* if this equation holds, and each $\mathcal{M}_i$ is a G-primary submodule of $\mathcal{M}$. We say that the decomposition is *minimal* if $\mathcal{N} \neq \bigcap_{j \neq i} \mathcal{M}_j$ for any i, and $\sqrt[G]{\mathcal{M}_i : \mathcal{M}}$ is distinct.

Proposition 4.3. $\mathcal{N}$ has a minimal G-primary decomposition.

Proof (sketch). Let

$$\mathcal{N} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_r$$

be a usual primary decomposition. Then

$$\mathcal{N} = \mathcal{N}^* = (\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_r)^* = \mathfrak{m}_1^* \cap \cdots \cap \mathfrak{m}_r^*$$

is a G-primary decomposition. We can make it minimal, as usual. □

Theorem 4.4. The set

$$\mathrm{Ass}_G(\mathcal{M}/\mathcal{N}) = \{ \sqrt[G]{\mathcal{M}_i : \mathcal{M}} \mid i = 1, \ldots, r\}$$

is independent of the choice of minimal G-primary decomposition

$$\mathcal{N} = \mathcal{M}_1 \cap \cdots \cap \mathcal{M}_r,$$

and depends only on $\mathcal{M}/\mathcal{N}$.

We call an element of $\mathrm{Ass}_G(\mathcal{M}/\mathcal{N})$ a *G-associated G-prime.* The set of minimal elements of $\mathrm{Ass}_G(\mathcal{M}/\mathcal{N})$ is denoted by $\mathrm{Min}_G(\mathcal{M}/\mathcal{N})$, and its element is called a *G-minimal G-prime.* An element of the set $\mathrm{Ass}_G(\mathcal{M}/\mathcal{N}) \setminus \mathrm{Min}_G(\mathcal{M}/\mathcal{N})$ is called a *G-embedded G-prime.*

Theorem 4.5. Let

$$\mathcal{N} = \mathcal{M}_1 \cap \cdots \cap \mathcal{M}_r$$

be a minimal G-primary decomposition and

$$\mathcal{M}_i = \mathfrak{m}_{i,1} \cap \cdots \cap \mathfrak{m}_{i,s_i}$$

a minimal primary decomposition. Then

$$\mathcal{N} = \bigcap_{i=1}^{r} (\mathfrak{m}_{i,1} \cap \cdots \cap \mathfrak{m}_{i,s_i})$$

is a minimal primary decomposition.

Proposition 4.6. A G-primary submodule $\mathcal{N}$ of $\mathcal{M}$ does not have an embedded prime. For each minimal prime $\mathfrak{p}$ of $\mathcal{M}/\mathcal{N}$, we have $\mathfrak{p}^* = \sqrt[G]{\mathcal{N} : \mathcal{M}}$.

Corollary 4.7. We have

$$\mathrm{Ass}(\mathcal{M}/\mathcal{N}) = \coprod_{i=1}^{s} \mathrm{Ass}(\mathcal{M}/\mathcal{M}_i) = \coprod_{\mathcal{P} \in \mathrm{Ass}_G(\mathcal{M}/\mathcal{N})} \mathrm{Ass}(\mathcal{O}_X/\mathcal{P})$$

and

$$\mathrm{Ass}_G(\mathcal{M}/\mathcal{N}) = \{\mathfrak{p}^* \mid \mathfrak{p} \in \mathrm{Ass}(\mathcal{M}/\mathcal{N})\}$$

Corollary 4.8. $\mathrm{Ass}(\mathcal{M}/\mathcal{N}) = \mathrm{Min}(\mathcal{M}/\mathcal{N})$ if and only if $\mathrm{Ass}_G(\mathcal{M}/\mathcal{N}) = \mathrm{Min}_G(\mathcal{M}/\mathcal{N})$.

5 Smooth group schemes and Group schemes with connected fibers

For some groups, the notion of G-prime G-ideal agrees with that of G-ideal which is a prime ideal.

Lemma 5.1. Assume that G is S-smooth. If $\mathfrak{a}$ is a radical ideal of $\mathcal{O}_X$, then $\mathfrak{a}^*$ is also radical. In particular, any G-radical G-ideal is radical.

Corollary 5.2. Assume that G is S-smooth. If $\mathcal{I}$ is a G-ideal of $\mathcal{O}_X$, then $\sqrt{\mathcal{I}} = \sqrt[G]{\mathcal{I}}$. In particular, $\sqrt{\mathcal{I}}$ is a G-radical G-ideal.

Lemma 5.3. Assume that $G \to S$ has connected fibers. If $\mathfrak{q}$ is a primary ideal of $\mathcal{O}_X$, then $\mathfrak{q}^*$ is also primary. In particular, a G-primary G-ideal is primary.

Corollary 5.4. Assume that $G \to S$ has connected fibers. If $\mathcal{I}$ is a G-ideal, then a minimal G-primary decomposition of $\mathcal{I}$ is also a minimal primary decomposition.

Corollary 5.5. Assume that $G \to S$ is smooth with connected fibers. If $\mathfrak{p}$ is a prime, then $\mathfrak{p}^*$ is also a prime. Any G-prime G-ideal is a prime. For a G-ideal $\mathcal{I}$ of $\mathcal{O}_X$, any associated prime of $\mathcal{I}$ is a G-prime G-ideal.

6 G-stable closed subschemes defined by G-primary G-ideals

Theorem 6.1. Let 0 be G-primary in $\mathcal{O}_X$. Then the dimension of the fiber of $p_2 : G \times X \to X$ is constant.

Theorem 6.2. Let 0 be G-primary in $\mathcal{O}_X$. If X has an affine open covering $(\operatorname{Spec} A_i)$ such that each A_i is **Hilbert, universally catenary, and for any minimal prime P of A_i, the heights of maximal ideals of A_i/P are the same** (for example, X is of finite type over a field or $\mathbb{Z}$). Then the dimensions of the irreducible components of X are the same.

Remark 6.3. There is an example of $G = X$ such that the dimensions of the irreducible components are different. The **bold face** assumptions are necessary. The **bold face** property is preserved by of-finite-type extensions.

The following is a generalization of Theorem 1.1.

Theorem 6.4. Let $y \in X$ and $Y = \bar{y}$. Let η be the generic point of an irreducible component of Y^*. Then: **1)** $\dim \mathcal{O}_{X,y} \geq \dim \mathcal{O}_{X,\eta}$. **2)** If $\mathcal{M}_\eta$ is maximal Cohen–Macaulay (resp. of finite injective dimension, projective dimension m, dim − depth $= n$, torsionless, reflexive, G-dimension g), then so is $\mathcal{M}_y$. **3)** If $\mathcal{O}_{X,\eta}$ is a complete intersection, then so is $\mathcal{O}_{X,y}$. **4)** If G is smooth and $\mathcal{O}_{X,\eta}$ is regular, then $\mathcal{O}_{X,y}$ is regular. **5)** Assume that G is smooth and X is a locally excellent $\mathbb{F}_p$-scheme. If $\mathcal{O}_{X,\eta}$ is weakly F-regular (resp. F-regular, F-rational), then so is $\mathcal{O}_{X,y}$.

Some special cases of Theorem 6.4 was proved by the author [3], and the author and M. Ohtani (unpublished).

Consider the case $S = \operatorname{Spec} \mathbb{Z}$, $G = \mathbb{G}_m^n$, and $X = \operatorname{Spec} A$ is affine. Then A is a $\mathbb{Z}^n$-graded ring.

Corollary 6.5. Let A be a locally excellent $\mathbb{Z}^n$-graded $\mathbb{F}_p$-algebra. Let P be a prime ideal of A, and let P^* be the prime ideal generated by homogeneous elements of P. If A_{P^*} is weakly F-regular (resp. F-regular, F-rational), then so is A_P.

Corollary 6.6. Let Y be a G-stable closed subscheme of X defined by a G-primary G-ideal. If η and ζ are generic points of irreducible components of Y, then $\dim \mathcal{O}_{X,\eta} = \dim \mathcal{O}_{X,\zeta}$.

X is said to be *G-artinian* if every G-prime of $\mathcal{O}_X$ is a G-minimal G-prime of 0.

Corollary 6.7. A G-artinian G-scheme is Cohen–Macaulay.

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永田雅宜先生の業績の一部
(a part of Works of Professor Masayoshi Nagata)

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永田雅宜先生 (February 9, 1927–August 27, 2008) の業績は適任者が解説されるでしょうから，ここでは永田先生の最初の論文 [97] が完備局所環の構造定理 (the structure theorem of complete local rings) であることを注意し，永田先生が提起された可換環論の問いについてその後の展開を述べるにとどめます.

Question 1. Is there a local domain whose completion has embedded primes?

よく知られているように，D. Ferrand–M. Raynaud によって 2-次元 local domain で完備化が embedded prime を持つものが構成された.

Question 2. Is there a normal catenary local domain which is not universally catenary?

L. J. Ratliff Jr. の詳しい研究の後，C. Rotthaus, T. Ogoma, R. C. Heitmann 等によりそのような例が構成された.

Question 3. Is there a Nagata local domain whose singular locus is not closed?

これも M. Brodmann–C. Rotthaus により regular locus が open でない例が構成された.

MathSciNet等で調べると，永田先生は140編以上の論文・著書・論説等を執筆されています.（ご協力下さった，蔵野和彦氏，橋本光靖氏に感謝します.）

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Goto numbers of parameter ideals

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Let $(R, \mathfrak{m})$ be a Noetherian local ring. The **Goto number** $g(Q)$ of a parameter ideal Q is defined as the largest integer g such that $Q : \mathfrak{m}^g$ is integral over Q. By $\mathcal{G}(R)$ we denote the set of all Goto integers in R.

This work started from the group work at the workshop "Integral closure, multiplier ideals, and cores", at the American Institute of Mathematics (AIM) in December 2006. Shiro Goto presented the background, motivation, and some intriguing open questions. A motivating result for the group work at AIM was:

Theorem 1. (Corso, Huneke, Vasconcelos [2], Corso, Polini [3], Corso, Polini, Vasconcelos [4], Goto [5]) *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of positive dimension. Let Q be a parameter ideal in R and let $I = Q : \mathfrak{m}$. Then the following are equivalent:*

1. $I^2 \neq QI$.

2. *The integral closure of Q is Q.*

3. *R is a regular local ring and $\mu(\mathfrak{m}/Q) \leq 1$.*

Consequently, if $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring that is not regular, then $I^2 = QI$. If $\dim R > 1$, it follows that the Rees algebra $R[It]$ is a Cohen-Macaulay ring, and even without the assumption that $\dim R > 1$, the fact that $I^2 = QI$ implies that the associated graded ring $gr_I(R) = R[It]/IR[It]$ and the fiber ring $R[It]/\mathfrak{m}R[It]$ are both Cohen-Macaulay.

Goto and Sakurai [8, 9] explored the Buchsbaumness properties of the associated rings of $I = Q : \mathfrak{m}$ if $(R, \mathfrak{m})$ is a Buchsbaum local ring. Namely, if $I^2 = QI$, then by Yamagishi [12, 13] and by Goto–Nishida [7], the Rees algebra, the associated graded ring, and the fiber ring of I are all Buchsbaum rings with certain specific graded local cohomology modules. Goto and Sakurai proved many instances of when this is the case. Namely, they proved that the equality $I^2 = QI$ holds if $e(R) = 2$ and the depth of R is positive, giving many specific conditions for sufficiency of the Buchsbaumness of the associated graded, fiber, and Rees rings. They proved more generally for Buchsbaum rings of dimension at least 2 that $I^2 = QI$ holds for infinitely many parameter ideals Q.

In [6], Goto, Matsuoka, and Takahashi explored the Cohen-Macaulayness of the associated graded and fiber rings and of Rees algebras for quasi-socle ideals $I = Q : \mathfrak{m}^2$. They proved that if $(R, \mathfrak{m})$ is a Gorenstein local ring of positive dimension and multiplicity at least 3, then $\mathfrak{m}^2 I = \mathfrak{m}^2 Q$ and $I^3 = QI^2$, the associated graded ring and the fiber ring of I are both Cohen–Macaulay rings, and the Rees algebra of I is Cohen–Macaulay if $\dim R \geq 3$. They also showed that wild behavior can occur in general. For example, if R is the numerical semigroup ring $k[[t^4, t^7, t^9]]$, where k is a field and t an indeterminate over k, then R is a one-dimensional non-Gorenstein Cohen–Macaulay local ring such that with $Q = (t^s)$, the associated graded ring of $I = Q : \mathfrak{m}^2$ is Cohen–Macaulay if and only if $s = 4, 8, 9$, and it is Buchsbaum ring if and only if $s \neq 7$. The fiber ring is always Buchsbaum and it is Cohen–Macaulay if and only if $s = 4, 9$.

In all examples above, $Q : \mathfrak{m}$ and $Q : \mathfrak{m}^2$ are integral over Q. This paper is about understanding the largest g such that $Q : \mathfrak{m}^g$ is integral over Q, namely about the Goto number of Q.

Even the Goto numbers in regular rings have interesting behavior: If R is one-dimensional, $g(Q) = 0$ for all Q; if R is two-dimensional, $g(Q)$ equals the order of Q minus 1 [10, Theorem 2.2]; but in higher-dimensional regular rings, $g(Q)$ can be arbitrarily larger than the order of Q. In fact, $g((x_1^e, x_2^n, \ldots, x_d^n)) = (d-2)(n-1) + e - 1$, if $x_1, \ldots, x_d$ is a regular system of parameters, and $e \leq n$ are positive integers. More general results have been proved recently by Goto, Kimura, Matsuoka, and Phuong.

We study in detail the behavior of Goto numbers in non-regular one-dimensional Noetherian domains.

The following fact about parameter ideals in general makes it easier to study parameter ideals in one-dimensional Noetherian domains (not necessarily regular): If Q_1 and Q_2 are ideals such that Q_2 is not contained in any minimal prime, and if e is a positive integer such that $Q_1 : \mathfrak{m}^e$ is not integral over Q_1, then $Q_1 Q_2 : \mathfrak{m}^e$, is not integral over $Q_1 Q_2$. Thus in dimension 1, if Q_1, Q_2 are parameter ideals, then $g(Q_1 Q_2) \leq \min\{g(Q_1), g(Q_2)\}$. A consequence is that in dimension 1, there exists a positive integer n such that all parameter ideals contained in $\mathfrak{m}^n$ have the same Goto number, and this number is the minimal possible Goto number. Namely, the set of all Goto numbers is a non-empty subset of non-negative integers. This set has a minimum, which means that there exists a parameter ideal Q_0 in R such that for all parameter ideals Q, $g(Q_0) \leq g(Q)$. Let n be a positive integer such that $\mathfrak{m}^n \subseteq Q_0$. Then for all parameter ideals Q in $\mathfrak{m}^n$, $Q \subseteq Q_0$, and so by the previous part, $g(Q) \leq g(Q_0) \leq g(Q)$, whence equality holds.

A more specific lower bound for Goto numbers is given by the following theorem:

Theorem 2. *[10, Theorem 3.4] Let $(R, \mathfrak{m})$ be a one-dimensional Noetherian local reduced ring such that $\overline{R}$ is module-finite over R. Let $C = R :_R \overline{R}$ be the conductor of R in $\overline{R}$, and let $x \in \mathfrak{m}$ and $y \in C$ generate parameter ideals. Then for each positive integer n, the Goto number $g(x^n y) = g(xy)$. Thus for all parameter ideals $Q = qR \subseteq xC = \overline{xC}$, we have $g(Q) = g(xy)$. Furthermore, this is the minimal possible Goto number of a parameter ideal in R.*

Goto numbers are one measure of the size of the integral closure of a parameter ideal. This size is perhaps more aptly measured by $\ell(\overline{Q}/Q)$. The following theorem describes when the set of all such lengths is bounded:

Theorem 3. *[10, Theorem 3.5] Let* $(R, \mathfrak{m})$ *be a one-dimensional Noetherian local ring with* $\mathfrak{m}$*-adic completion* $\widehat{R}$*. Then the set* $\{\ell_R(\overline{Q}/Q) \mid Q$ *is a parameter ideal of* $R\}$ *is finite if and only if* $\ell_{\widehat{R}}(\sqrt{0\widehat{R}})$ *is finite.*

The conditions in the theorem above imply that $\mathcal{G}(R)$ is finite, and we suspect that the converse holds as well. The famous example by Nagata, of a Noetherian local domain of dimension 1 whose integral closure has nilpotents (and since it is Cohen–Macaulay, $\sqrt{0\widehat{R}}$ does not have finite length), is as follows: Let $A = k^p[[X]][k]$, where k is a field of characteristic $p > 0$ such that $[k : k^p] = \infty$, and let

$$R = \frac{A[Y]}{(Y^p - \sum_{i\geq 1} b_i^p X^{ip})},$$

where $\{b_i\}_{i=1}^{\infty}$ are elements of k that are p-independent over k^p. Here the set $\mathcal{G}(R)$ of Goto numbers of parameter ideals of R is infinite. It suffices to prove that the completion $\widehat{R}$ of R has this property. But $\widehat{R}$ is isomorphic to $S = k[[X, Z]]/(Z^p)$. We prove below that under these conditions, $\mathcal{G}(R)$ is infinite.

Theorem 4. *[10, Theorem 3.8] Let* $(R, \mathfrak{m})$ *be a one-dimensional Noetherian local ring. If there exists a nonzero principal ideal* yR *such that* R/yR *is one-dimensional and Cohen-Macaulay and* $(0) : y$ *is contained in the nilradical, then the set* $\mathcal{G}(R)$ *is infinite.*

Proof The assumption that R/yR is one-dimensional and Cohen-Macaulay implies that each $P \in AssR/yR$ is a minimal prime of R. Let

$$x \in \mathfrak{m} \setminus \bigcup_{P \in AssR/yR} P.$$

If R has minimal primes other than those in $AssR/yR$, choose x also to be in each of these other minimal primes of R. For each positive integer n, let $Q_n := (y + x^n)R$. Notice that Q_n is a parameter ideal of R. Checking integral closure modulo minimal primes, we see that $(y, x^n)R + \mathfrak{n} \subseteq \overline{Q_n}$, where $\mathfrak{n}$ is the nilradical of R. We prove that $g(Q_n) \geq n$. Let $r \in (Q_n : \mathfrak{m}^n)$. Then $r \in (Q_n : x^n)$, so $rx^n = a(y + x^n)$, for some $a \in R$. Hence $(r - a)x^n = ay$, so $r - a \in (yR : x^n)$. Since x^n is regular on R/yR, we have $r - a = by$, for some $b \in R$. It follows that $x^n by = ay$, so $(x^n b - a)y = 0$ and $x^n b - a \in (0) : y \subseteq \mathfrak{n}$. Therefore $a = x^n b + c$, where $c \in \mathfrak{n}$. Hence $r = bx^n + by + c \in \overline{Q_n}$. We conclude that $g(Q_n) \geq n$, and therefore that $\mathcal{G}(R)$ is infinite.

In particular, if $(R, \mathfrak{m})$ is a one-dimensional Cohen-Macaulay local ring such that $\mathfrak{m}$ is minimally 2-generated, then $\mathcal{G}(R)$ is finite if and only if the $\mathfrak{m}$-adic completion $\widehat{R}$ of R is reduced, and this holds if and only if $\overline{R}$ is module-finite over R.

A possibly more general case of when the set of Goto numbers is finite is given by [10, Proposition 3.11]:

Proposition 1. *Let $(R, \mathfrak{m})$ be a one-dimensional Noetherian local ring, and let x, y be elements of R such that for all n, $y + x^n$ is a parameter. Assume that for all n, $(y) : x^n \subseteq \overline{(y + x^n)}$ and $(x^n) : y \subseteq \overline{(y + x^n)}$. Then $\mathcal{G}(R)$ is infinite.*

It is likely that $\mathcal{G}(R)$ being finite is equivalent to $\overline{R}$ being module-finite over R in general (for one-dimensional Noetherian local domains R). In search of a counterexample, we tried to modify Nagata's example, always either confirming the suspicion of equivalence, or producing rings in which it is too hard to compute Goto numbers or the integral closure.

More is known about Goto numbers in numerical semigroup rings. Such rings have module-finite integral closure, thus the set of Goto numbers is a finite set. In the sequel, S denotes a numerical semigroup, minimally generated by $a_1, \ldots, a_d$, with $a_1 < \cdots < a_d$. The Frobenius number of S is the largest integer f that is not contained in S. The corresponding numerical semigroup ring is $R = k[[t^s : s \in S]]$.

Theorem 5. *[10, Theorem 4.1] Let R be the numerical semigroup ring. Then*

$$g(t^{f+a_1+1}) = \min\{g(Q) \mid Q \text{ is a parameter ideal of } R\}.$$

Moreover, for all $e \geq f + a_1 + 1$, we have $g(t^e) = g(t^{f+a_1+1})$.

The lower bound for e given in theorem above is sharp: if $G = \langle 9, 19 \rangle$, then $f = 143$, $a_1 = 9$, $f + a_1 + 1 = 153$, and $g(x^{152}) = 9 > \min\{g(x^{a_i}) : i = 1, \ldots, d\} = 8$.

For parameter ideals (t^{a_i}), the following are upper bounds on their Goto numbers [10, Propositions 5.1, 5.3]: $g(t^{a_1}) \leq \lceil \frac{f+a_1+1}{a_2} \rceil - 1$, and for $j > 1$, $g(t^{a_1}) \leq \lfloor \frac{f+a_j-b_j}{a_1} \rfloor$, where b_j is the largest integer in S that is strictly smaller than a_j.

Below is a small list of computed Goto numbers for a few numerical semigroups. The generators of the semigroups are in the first column, column 2 contains the expected bounds from the previous paragraph for $g(t^{a_j})$, column 3 contains the Goto numbers $g(t^{a_j})$, column 4 contains the Frobenius number f of the semigroup, and the last column lists all elements of S that are smaller than f:

generators of G	$\left\lceil \frac{f+a_1+1}{a_2} \right\rceil - 1$, $\left\lfloor \frac{a_2-b_2+f}{a_1} \right\rfloor, \ldots,$ $\left\lfloor \frac{a_d-b_d+f}{a_1} \right\rfloor$	$g(x^{a_1}), \ldots,$ $g(x^{a_d})$	f	$s \in S, 0 < s < f$
3, 4, 5	1, 1, 1	1, 1, 1	2	
3, 5, 7	1, 2, 1	1, 2, 1	4	3
3, 7, 8	1, 2, 2	1, 2, 2	5	3
3, 7, 11	1, 3, 3	1, 3, 2	8	3, 6, 7
3, 8, 10	1, 3, 2	1, 3, 2	7	3, 6
3, 8, 13	1, 4, 3	1, 4, 2	10	3, 6, 8, 9
3, 10, 11	1, 3, 3	1, 3, 3	8	3, 6
3, 10, 14	1, 4, 4	1, 4, 3	11	3, 6, 9, 10
3, 10, 17	1, 5, 5	1, 5, 3	14	3, 6, 9, 10, 12, 13

3, 11, 13	1, 4, 3	1, 4, 3	10	3, 6, 9
3, 11, 16	1, 5, 4	1, 5, 3	13	3, 6, 9, 11, 12
3, 11, 19	1, 6, 5	1, 6, 3	16	3, 6, 9, 11, 12, 14, 15
4, 5, 6	2, 2, 2	2, 2, 2	7	4, 5, 6
4, 5, 7	2, 1, 2	2, 1, 2	6	4, 5
4, 5, 11	2, 2, 2	2, 2, 2	7	4, 5
4, 6, 7	2, 2, 2	2, 2, 2	9	4, 6, 7, 8
4, 6, 9	2, 3, 3	2, 2, 3	11	4, 6, 8, 9, 10
4, 6, 11	2, 3, 3	2, 2, 3	13	4, 6, 8, 10, 11, 12
4, 6, 13	3, 4, 4	2, 2, 4	15	4, 6, 8, 10, 12, 13, 14
4, 7, 9	2, 3, 2	2, 2, 2	10	4, 7, 8, 9
4, 7, 10	2, 4, 3	2, 4, 3	13	4, 7, 8, 10, 11, 12
4, 7, 13	2, 3, 2	2, 3, 2	10	4, 7, 8
4, 7, 17	2, 4, 3	2, 4, 2	13	4, 7, 8, 11, 12
4, 9, 10	2, 4, 4	2, 4, 3	15	4, 8, 9, 10, 12, 13, 14
4, 9, 11	2, 3, 4	2, 2, 3	14	4, 8, 9, 11, 12, 13
4, 9, 14	2, 5, 5	2, 5, 4	19	4, 8, 9, 12, 13, 14, 16, 17, 18
4, 9, 15	2, 3, 4	2, 3, 3	14	4, 8, 9, 12, 13
4, 9, 19	2, 4, 4	2, 4, 3	15	4, 8, 9, 12, 13
4, 10, 11	2, 4, 4	2, 3, 3	17	4, 8, 10, 11, 12, 14, 15, 16
4, 10, 13	2, 5, 5	2, 3, 4	19	4, 8, 10, 12, 13, 14, 16, 17, 18
4, 10, 15	2, 5, 5	2, 3, 4	21	4, 8, 10, 12, 14, 15, 16, 18, 19, 20
4, 10, 17	2, 6, 6	2, 3, 5	23	4, 8, 10, 12, 14, 16, 17, 18, 20, 21, 22
5, 6, 7	2, 2, 2	2, 2, 2	9	5, 6, 7
5, 6, 8	2, 2, 2	2, 2, 2	9	5, 6, 8
5, 6, 9	3, 2, 3	3, 2, 3	13	5, 6, 9, 10, 11, 12
5, 6, 13	3, 3, 3	2, 3, 3	14	5, 6, 10, 11, 12, 13
5, 6, 14	3, 2, 3	3, 2, 3	13	5, 6, 10, 11, 12
5, 6, 19	3, 3, 3	3, 3, 3	14	5, 6, 10, 11, 12
5, 7, 8	2, 2, 2	2, 2, 2	11	5, 7, 8, 10
5, 7, 9	2, 3, 3	2, 3, 3	13	5, 7, 9, 10, 12
5, 7, 11	2, 3, 2	2, 3, 2	13	5, 7, 10, 11, 12
5, 7, 13	3, 3, 3	3, 2, 3	16	5, 7, 10, 12, 13, 14, 15
5, 8, 9	2, 3, 2	2, 3, 2	12	5, 8, 9, 10
5, 8, 11	2, 4, 3	2, 4, 3	17	5, 8, 10, 11, 13, 15, 16
5, 11, 14	2, 4, 5	2, 3, 3	23	5, 10, 11, 14, 15, 16, 19, 20, 21, 22
6, 7, 8	3, 3, 3	3, 3, 3	17	6, 7, 8, 12, 13, 14, 15, 16
6, 7, 9	3, 3, 3	3, 3, 3	17	6, 7, 9, 12, 13, 14, 15, 16
6, 7, 10	3, 2, 3	3, 2, 3	15	6, 7, 10, 12, 13, 14
6, 7, 11	3, 2, 3	3, 2, 3	16	6, 7, 11, 12, 13, 14

6, 10, 15	3, 5, 5	3, 4, 4	29	6, 10, 12, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28
7, 8, 9	3, 3, 3	3, 3, 3	20	7, 8, 9, 14, 15, 16, 17, 18
7, 8, 10	3, 2, 3	3, 2, 3	19	7, 8, 10, 14, 15, 16, 17, 18
7, 8, 11	3, 3, 3	3, 3, 3	20	7, 8, 11, 14, 15, 16, 18, 19
7, 9, 10	3, 3, 3	3, 3, 3	22	7, 9, 10, 14, 16, 17, 18, 19, 20, 21
7, 9, 11	3, 4, 4	3, 4, 4	26	7, 9, 11, 14, 16, 18, 20, 21, 22, 23, 25
7, 9, 12	4, 4, 4	4, 3, 4	29	7, 9, 12, 14, 16, 18, 19, 21, 23, 24, 25, 26, 27, 28
7, 9, 13	3, 3, 4	3, 3, 4	24	7, 9, 13, 14, 16, 18, 20, 21, 22, 23
7, 9, 15	3, 4, 3	3, 4, 3	26	7, 9, 14, 15, 16, 18, 21, 22, 23, 24, 25
7, 9, 17	4, 4, 4	4, 3, 4	29	7, 9, 14, 16, 17, 18, 21, 23, 24, 25, 26, 27, 28
7, 9, 19	4, 4, 4	4, 3, 4	31	7, 9, 14, 16, 18, 19, 21, 23, 25, 26, 27, 28, 30
7, 9, 20	4, 5, 5	3, 5, 5	33	7, 9, 14, 16, 18, 20, 21, 23, 25, 27, 28, 29, 30, 32
7, 9, 22	4, 5, 4	3, 5, 4	33	7, 9, 14, 16, 18, 21, 22, 23, 25, 27, 28, 29, 30, 31, 32
7, 9, 24	4, 4, 4	4, 4, 4	29	7, 9, 14, 16, 18, 21, 23, 24, 25, 27, 28
7, 9, 26	5, 5, 5	5, 3, 5	38	7, 9, 14, 16, 18, 21, 23, 25, 26, 27, 28, 30, 32, 33, 34, 35, 36, 37
7, 9, 29	5, 6, 5	3, 6, 5	40	7, 9, 14, 16, 18, 21, 23, 25, 27, 28, 29, 30, 32, 34, 35, 36, 37, 38, 39
7, 9, 31	4, 5, 4	4, 5, 4	33	7, 9, 14, 16, 18, 21, 23, 25, 27, 28, 30, 31, 32
7, 9, 33	5, 5, 5	5, 4, 5	38	7, 9, 14, 16, 18, 21, 23, 25, 27, 28, 30, 32, 33, 34, 35, 36, 37
7, 10, 11	3, 4, 3	3, 3, 3	26	7, 10, 11, 14, 17, 18, 20, 21, 22, 24, 25
7, 10, 12	3, 4, 3	3, 4, 3	25	7, 10, 12, 14, 17, 19, 20, 21, 22, 24
7, 10, 13	3, 5, 5	3, 5, 5	32	7, 10, 13, 14, 17, 20, 21, 23, 24, 26, 27, 28, 30, 31
7, 10, 15	4, 5, 4	4, 4, 4	33	7, 10, 14, 15, 17, 20, 21, 22, 24, 25, 27, 28, 29, 30, 31, 32
7, 10, 16	3, 4, 4	3, 4, 4	29	7, 10, 14, 16, 17, 20, 21, 23, 24, 26, 27, 28
7, 10, 18	4, 5, 4	4, 3, 4	33	7, 10, 14, 17, 18, 20, 21, 24, 25, 27, 28, 30, 31, 32

$7, 10, 19$	$3, 5, 4$	$3, 5, 4$	32	7, 10, 14, 17, 19, 20, 21, 24, 26, 27, 28, 29, 30, 31
$8, 9, 30$	$5, 4, 5$	$5, 4, 5$	37	8, 9, 16, 17, 18, 24, 25, 26, 27, 30, 32, 33, 34, 35, 36
$8, 9, 39$	$6, 5, 6$	$6, 4, 6$	46	8, 9, 16, 17, 18, 24, 25, 26, 27, 32, 33, 34, 35, 36, 39, 40, 41, 42, 43, 44, 45
$9, 10, 13$	$4, 3, 4$	$4, 3, 4$	34	9, 10, 13, 18, 19, 20, 22, 23, 26, 27, 28, 29, 30, 31, 32, 33
$9, 19, 21$	$4, 8, 8$	$4, 8, 6$	71	9, 18, 19, 21, 27, 28, 30, 36, 37, 38, 39, 40, 42, 45, 46, 47, 48, 49, 51, 54, 55, 56, 57, 58, 59, 60, 61, 63, 64, 65, 66, 67, 68, 69, 70
$4, 6, 9, 11$	$1, 2, 2, 2$	$1, 1, 2, 1$	7	4, 6
$4, 6, 11, 13$	$2, 2, 2, 2$	$1, 1, 2, 1$	9	4, 6, 8
$4, 6, 13, 15$	$2, 3, 3, 3$	$1, 1, 3, 1$	11	4, 6, 8, 10
$4, 7, 9, 10$	$1, 2, 1, 1$	$1, 2, 1, 1$	6	4
$4, 7, 10, 13$	$1, 3, 2, 2$	$1, 3, 2, 1$	9	4, 7, 8
$4, 9, 10, 11$	$1, 2, 2, 2$	$1, 2, 2, 2$	7	4
$4, 9, 10, 15$	$1, 3, 3, 3$	$1, 3, 2, 2$	11	4, 8, 9, 10
$4, 9, 11, 14$	$1, 2, 3, 2$	$1, 2, 2, 2$	10	4, 8, 9
$4, 9, 14, 15$	$1, 3, 3, 3$	$1, 3, 3, 2$	11	4, 8, 9
$4, 10, 11, 13$	$1, 2, 2, 2$	$1, 2, 2, 2$	9	4, 8
$4, 10, 11, 17$	$1, 3, 3, 3$	$1, 2, 2, 2$	13	4, 8, 10, 11, 12
$4, 10, 13, 15$	$1, 3, 3, 3$	$1, 2, 3, 2$	11	4, 8, 10
$5, 14, 16, 21$	$2, 6, 5, 5$	$2, 4, 3, 2$	27	5, 10, 14, 15, 16, 19, 20, 21, 24, 25, 26
$7, 9, 19, 20$	$4, 4, 4, 4$	$3, 3, 4, 3$	31	7, 9, 14, 16, 18, 19, 20, 21, 23, 25, 26, 27, 28, 29, 30
$7, 9, 19, 22$	$3, 3, 3, 3$	$3, 2, 3, 3$	24	7, 9, 14, 16, 18, 19, 21, 22, 23
$7, 10, 12, 13$	$2, 3, 2, 2$	$2, 3, 2, 2$	18	7, 10, 12, 13, 14, 17
$7, 10, 12, 13, 16$	$2, 3, 2, 2, 2$	$2, 3, 2, 2, 2$	18	7, 10, 12, 13, 14, 16, 17
$7, 10, 12, 16$	$3, 4, 3, 3$	$3, 4, 3, 3$	25	7, 10, 12, 14, 16, 17, 19, 20, 21, 22, 23, 24

Observe that it need not be that the maximum of the values in the second column in the previous table is the same as the maximum of the values in the third column. See $R = k[[x^4, x^7, x^9]]$: in this case, the associated graded ring of R is even Cohen-Macaulay, so the failure of the Cohen-Macaulay property is not a reason for this phenomenon.

For two-generated numerical semigroups, Goto numbers are more predictable: $g(t^{f+a_1+1}) = g(t^{a_1}) = a_1 - 1 \leq g(t^{a_2}) = a_2 - 1 - \lfloor \frac{a_2-1}{a_1} \rfloor$ [10, Theorems 5.5 and 5.10].

The minimum possibly Goto number has many other characterizations:

Proposition 2. *[10, Proposition 5.6] Let t be the maximum integer such that for all $\alpha \in \{1, 2, \ldots, a_1\}$, $\mathfrak{m}^t \not\subseteq t^\alpha R$ (R-module containment). Then $t = g(t^{f+a_1+1})$.*

Proposition 3. *[10, Proposition 5.7] For each $\alpha \in \{1, \ldots, a_1\}$, find elements $\beta \in S$ such that $\beta - \alpha \notin S$. Among all such β, fix one for which t^β has the largest $\mathfrak{m}$-adic order. As α varies, let t' be the smallest of these orders. Then $t' = g(t^{f+a_1+1})$.*

The following characterization is due to Bryant [1]:

Proposition 4. *[1, Proposition 2.11] The minimum Goto number among all Goto numbers of parameter ideals in a numerical semigroup ring $k[[t^{a_1}, \ldots, t^{a_d}]]$ with $a_1 < a_2 < \cdots < a_d$, is $\min\{g(t^{a_1}), g(t^{f+a_1})\}$.*

Bryant [1] further studied under what semigroup conditions the associated graded ring of $\mathfrak{m}$ is Cohen–Macaulay or Gorenstein.

The maximum possible Goto number of any parameter ideal in a numerical semigroup ring is at most $\lceil \frac{f}{a_1} \rceil$, as proved by Bryant (and appears in [10, Theorem 4.7]). However, it is a much harder problem to understand this issue. For one thing, for any parameter ideal of the form (t^s + higher order terms), its Goto number is at least $g(t^s)$, and strict inequality can happen in many cases. For example, if $S = \langle 5, 11 \rangle$, then $f = 39$, and $g(t^{40}) = 4 < g(t^{40} + t^{44}) = 5 = g(t^{44})$. Note that t^{40} and $t^{40} + t^{44}$ are in the conductor C of R. An example that plays out differently is with $S = \langle 5, 6, 9 \rangle$, where $g(t^6) = 2$, $g(t^9) = 3$, but $g(t^6 + t^9) = 2$ (since $(t^5 + t^{14})\mathfrak{m}^3 \subseteq (t^6 + t^9)$).

Another paper of interest is Shen's [11].

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THE STRUCTURE OF SALLY MODULES OF RANK ONE – THE BUCHSBAUM CASE –

SHIRO GOTO AND KAZUHO OZEKI

1. Introduction

Throughout this paper let A denote a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Let I be an $\mathfrak{m}$-primary ideal in A and suppose that our ideal I contains a parameter ideal $Q = (a_1, a_2, \cdots, a_d)$ of A as a reduction, that is $Q \subseteq I$ and the equality $I^{n+1} = QI^n$ holds true for some (and hence for any) integer $n \gg 0$. Let $\ell_A(M)$ denote, for an A-module M, the length of M. We then have integers $\{\mathrm{e}_i(I)\}_{0 \le i \le d}$ such that the equality

$$\ell_A(A/I^{n+1}) = \mathrm{e}_0(I)\binom{n+d}{d} - \mathrm{e}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d \mathrm{e}_d(I)$$

holds true for all integers $n \gg 0$, which we call the Hilbert coefficients of A with respect to I. Let

$$R = \mathcal{R}(I) := A[It] \text{ and } T = \mathcal{R}(Q) := A[Qt] \subseteq A[t]$$

denote, respectively, the Rees algebras of I and Q, where t stands for an indeterminate over A. Let

$$R' = \mathcal{R}'(I) := A[It, t^{-1}] \text{ and } G = \mathrm{G}(I) := R'/t^{-1}R' \cong \bigoplus_{n \ge 0} I^n/I^{n+1}.$$

Following Vasconcelos [16], we then define

$$S = \mathrm{S}_Q(I) := IR/IT \cong \bigoplus_{n \ge 1} I^{n+1}/Q^n I$$

and call it the Sally module of I with respect to Q. Here we notice that S is a finitely generated graded T-module and $\mathfrak{m}^\ell \cdot S = (0)$ for some integer $\ell \gg 0$, since R is a module finite extension of the graded ring T and $\mathfrak{m} = \sqrt{I}$, so that $\dim_T \mathrm{S}_Q(I) \le d$.

In [16] Vasconcelos gave an elegant review, in terms of his *Sally* module, of the works [11, 12, 13] of Judith Sally about the structure of $\mathfrak{m}$-primary ideals I with interaction to the structure of the graded rings G and the Hilbert coefficients $e_i(I)$'s of I. Let us recall a part of his work.

As is well-known, in the case where A is a Cohen-Macaulay local ring, we have the inequality

$$\mathrm{e}_1(I) \ge \mathrm{e}_0(I) - \ell_A(A/I)$$

([9]), and Craig Huneke [7] showed that the equality $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I)$ holds true if and only if $I^2 = QI$. When this is the case, the associated graded ring $G = \mathrm{G}(I)$ and the fiber cone $\mathrm{F}(I) = \bigoplus_{n \ge 0} I^n/\mathfrak{m}I^n$ of I are both Cohen-Macaulay, and the Rees algebra R of I is also a Cohen-Macaulay ring, provided $d \geq 2$. Thus, the ideals I with $e_1(I) = e_0(I) - \ell_A(A/I)$ enjoy very nice properties.

Sally [13] firstly investigated the second border, that is the ideals I satisfying the equality

$$e_1(I) = e_0(I) - \ell_A(A/I) + 1$$

and gave several important results. Among them, one finds the characterization of ideals I with $e_1(I) = e_0(I) - \ell_A(A/I) + 1$ and $e_2(I) \neq 0$.

She says, however, nothing about the case where $e_2(I) = 0$ and it seems natural to ask what happens, when $e_2(I) = 0$, on the ideals I which satisfy the equality $e_1(I) = e_0(I) - \ell_A(A/I) + 1$. This long standing question has motivated the researches [3, 4], where the authors and Koji Nishida gave the following structure theorem of Sally modules of $\mathfrak{m}$-primary ideals I with $e_1(I) = e_0(I) - \ell_A(A/I) + 1$. To state it, let

$$B = T/\mathfrak{m}T \cong k[X_1, X_2, \cdots, X_d],$$

which is the polynomial ring with d indeterminates over the residue class field $k = A/\mathfrak{m}$ of A.

Theorem 1.1 ([4, Theorem 1.2]). *Suppose that $(A, \mathfrak{m})$ is a Cohen-Macaulay local ring. Then the following three conditions are equivalent to each other.*

(1) $e_1(I) = e_0(I) - \ell_A(A/I) + 1$.

(2) $\mathfrak{m}{\cdot}S = (0)$ *and* $\operatorname{rank}_B S = 1$.

(3) $S \cong (X_1, X_2, \cdots, X_c)B$ *as graded T-modules for some $0 < c \leq d$, where $\{X_i\}_{1 \leq i \leq c}$ are linearly independent linear forms of the polynomial ring B.*

When this is the case, $c = \ell_A(I^2/QI)$ and $I^3 = QI^2$, and the following assertions hold true.

(i) $\operatorname{depth} G \geq d - c$ *and* $\operatorname{depth}_T S = d - c + 1$.

(ii) $\operatorname{depth} G = d - c$, *if $c \geq 2$.*

(iii) *Suppose $c < d$. Then*

$$\ell_A(A/I^{n+1}) = e_0(I)\binom{n+d}{d} - e_1(I)\binom{n+d-1}{d-1} + \binom{n+d-(c+1)}{d-(c+1)}$$

for all $n \geq 0$. Hence

$$e_i(I) = \begin{cases} 0 & \text{if } i \neq c+1, \\ (-1)^{c+1} & \text{if } i = c+1 \end{cases}$$

for $2 \leq i \leq d$.

(iv) *Suppose $c = d$. Then*

$$\ell_A(A/I^{n+1}) = e_0(I)\binom{n+d}{d} - e_1(I)\binom{n+d-1}{d-1}$$

for all $n \geq 1$. Hence $e_i(I) = 0$ for $2 \leq i \leq d$.

In the case where $(A, \mathfrak{m})$ is a Buchsbaum local ring, Alberto Corso [1] also investigated the Sally module $S_Q(\mathfrak{m})$ of the maximal ideal $\mathfrak{m}$ and gave several inspiring results about the dimension of $S_Q(\mathfrak{m})$ together with an important description of the Hilbert function of $\mathfrak{m}$ in terms of the Sally module $\mathrm{S}_Q(\mathfrak{m})$. These results of [1, 3, 4] are, however, only known ones about the structure of Sally modules.

The present research aims, being inspired by Corso [1], at a systematic approach towards further developments of the theory of Sally modules $\mathrm{S}_Q(I)$ of $\mathfrak{m}$-primary ideals I in *not-necessarily* Cohen-Macaulay local rings $(A, \mathfrak{m})$, in order to answer the natural questions of what is a possible equality corresponding to the equality

$$\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 1$$

in the Cohen-Macaulay case and of what kind of properties the Sally modules $\mathrm{S}_Q(I)$ and the ideals I enjoy, provided the equality holds true.

To sate the results of the present paper, let us consider the following four conditions:

(C_0) The sequence $a_1, a_2, \cdots, a_d$ is a d-sequence in A in the sense of Huneke [6].

(C_1) The sequence $a_1, a_2, \cdots, a_d$ is a d^+-sequence in A, that is for all integers $n_1, n_2, \cdots, n_d \geq 1$ the sequence $a_1^{n_1}, a_2^{n_2}, \cdots, a_d^{n_d}$ forms a d-sequence in any order.

(C_2) $(a_1, a_2, \cdots, \check{a}_i, \cdots, a_d) :_A a_i \subseteq I$ for all $1 \leq i \leq d$.

(C_3) $\operatorname{depth} A > 0$.

These conditions $(\mathrm{C}_0), (\mathrm{C}_1), (\mathrm{C}_2)$, and (C_3) are naturally satisfied, when A is a Cohen-Macaulay local ring. Condition (C_1) (resp. condition (C_2)) is always satisfied, if A is a Buchsbaum local ring (resp. $I = \mathfrak{m}$). Here we notice that condition (C_1) is equivalent to saying that our local ring A is a generalized Cohen-Macaulay ring, that is all the local cohomology modules $\mathrm{H}^i_\mathfrak{m}(A)$ $(i \neq d)$ of A with respect to the maximal ideal $\mathfrak{m}$ are finitely generated and the parameter ideal Q is standard, that is the equality

$$\ell_A(A/Q) - \mathrm{e}_0(Q) = \sum_{i=0}^{d-1} \binom{d-1}{i} \cdot \ell_A(\mathrm{H}^i_\mathfrak{m}(A))$$

holds true. Hence condition (C_1) is independent of the choice of a minimal system $\{a_i\}_{1 \leq i \leq d}$ of generators of the parameter ideal Q. We note here that condition (C_2) is also independent of the choice of a minimal system $\{a_i\}_{1 \leq i \leq d}$ of generators of Q.

Although some parts of the results which we shall refer to in this section still hold true under milder assumptions $(\mathrm{C}_0), (\mathrm{C}_2)$, and (C_3), or under the assumption that $I = \mathfrak{m}$ only, for the sake of simplicity of the statement let us now assume that conditions $(\mathrm{C}_1), (\mathrm{C}_2)$, and (C_3) are satisfied. We then have the inequality

$$\mathrm{e}_1(I) \geq \mathrm{e}_0(I) + \mathrm{e}_1(Q) - \ell_A(A/I),$$

and the equality $\mathrm{e}_1(I) = \mathrm{e}_0(I) + \mathrm{e}_1(Q) - \ell_A(A/I)$ holds true if and only if $I^2 = QI$ (Corollary 2.5). When this is the case, we have

$$\mathrm{H}^i_M(G) = [\mathrm{H}^i_M(G)]_{1-i} \cong \mathrm{H}^i_\mathfrak{m}(A)$$

for all $0 \leq i < d$ and the a-invariant

$$\mathrm{a}(G) = \max\{n \in \mathbb{Z} \mid [\mathrm{H}^d_M(G)]_n \neq (0)\}$$

of G is at most $1 - d$, where $G = \mathrm{G}(I)$ and $M = G_+$, and the ring G is a Buchsbaum ring if so is A (cf. [2]). Thus the ideals I again enjoy very nice properties, if $\mathrm{e}_1(I) =$

$\mathrm{e}_0(I) + \mathrm{e}_1(Q) - \ell_A(A/I)$. The next target is, of course, the case where the equality

$$\mathrm{e}_1(I) = \mathrm{e}_0(I) + \mathrm{e}_1(Q) - \ell_A(A/I) + 1$$

holds true, which leads us to the main result Theorem 1.2 of this paper.

The following Theorem 1.2 completely generalizes Theorem 1.1 given in the case where A is a Cohen-Macaulay local ring, because $\mathrm{e}_i(Q) = 0$ for all $1 \leq i \leq d$. We notice that, thanks to condition (C_1), the Hilbert coefficients $\mathrm{e}_i(Q)$ of Q are given by the formula

$$(-1)^i \mathrm{e}_i(Q) = \begin{cases} \mathrm{e}_0(Q) & \text{if } i = 0, \\ \ell_A(\mathrm{H}^0_{\mathfrak{m}}(A)) & \text{if } i = d, \\ \sum_{j=1}^{d-i} \binom{d-i-1}{j-1} \ell_A(\mathrm{H}^j_{\mathfrak{m}}(A)) & \text{if } 1 \leq i \leq d-1 \end{cases}$$

and one has the equality $\ell_A(A/Q^{n+1}) = \sum_{i=0}^{d} (-1)^i \mathrm{e}_i(Q) \binom{n+d-i}{d-i}$ for all $n \geq 0$ ([14, Korollar 3.2]), so that $\{\mathrm{e}_i(Q)\}_{1 \leq i \leq d}$ are independent of the choice of the reduction Q of I and so, are invariants of A.

Theorem 1.2. *Suppose conditions* $(\mathrm{C}_1), (\mathrm{C}_2)$, *and* (C_3) *are satisfied. Then the following three conditions are equivalent to each other.*

(1) $\mathrm{e}_1(I) = \mathrm{e}_0(I) + \mathrm{e}_1(Q) - \ell_A(A/I) + 1$.

(2) $\mathfrak{m} \cdot S = (0)$ *and* $\mathrm{rank}_B\, S = 1$.

(3) $S \cong (X_1, X_2, \cdots, X_c)B$ *as graded* T*-modules, where* $1 \leq c \leq d$ *and* $\{X_i\}_{1 \leq i \leq c}$ *are linearly independent linear forms of the polynomial ring* B.

When this is the case, we get $c = \ell_A(I^2/QI)$ *and* $I^3 = QI^2$, *and the following assertions also hold true.*

(i) $\mathrm{depth}_T\, S = d - c + 1$.

(ii) *Suppose* $c < d$. *Then*

$$\begin{aligned} \ell_A(A/I^{n+1}) &= \mathrm{e}_0(I)\binom{n+d}{d} - \mathrm{e}_1(I)\binom{n+d-1}{d-1} + \binom{n+d-(c+1)}{d-(c+1)} \\ &+ \sum_{i=2}^{d} (-1)^i \{\mathrm{e}_{i-1}(Q) + \mathrm{e}_i(Q)\} \binom{n+d-i}{d-i} \end{aligned}$$

for all $n \geq 0$. *Hence*

$$\mathrm{e}_i(I) = \begin{cases} \mathrm{e}_{i-1}(Q) + \mathrm{e}_i(Q) & \textit{if } i \neq c+1, \\ \mathrm{e}_{i-1}(Q) + \mathrm{e}_i(Q) + (-1)^{c+1} & \textit{if } i = c+1 \end{cases}$$

for $2 \leq i \leq d$.

(iii) *Suppose* $c = d$. *Then*

$$\begin{aligned} \ell_A(A/I^{n+1}) &= \mathrm{e}_0(I)\binom{n+d}{d} - \mathrm{e}_1(I)\binom{n+d-1}{d-1} \\ &+ \sum_{i=2}^{d} (-1)^i \{\mathrm{e}_{i-1}(Q) + \mathrm{e}_i(Q)\} \binom{n+d-i}{d-i} \end{aligned}$$

for all $n \geq 1$. *Hence* $\mathrm{e}_i(I) = \mathrm{e}_{i-1}(Q) + \mathrm{e}_i(Q)$ *for* $2 \leq i \leq d$.

In contrast with the case where A is a Cohen-Macaulay local ring, in general one cannot control depth G in terms of the integer c, since the Rees algebra $T = \mathcal{R}(Q)$ of Q is not necessarily a Cohen-Macaulay ring.

We are now in a position to briefly explain how we organize this paper.

We shall discuss outline of proof Theorem 1.2 in Section 3. We will summarize in Section 2 some auxiliary results on Sally modules for the later use in this paper. We will introduce the result that $S \cong \mathfrak{a}$ as graded T-modules for some graded ideal $\mathfrak{a}$ of B, once the equality $\mathrm{e}_1(I) = \mathrm{e}_0(I) + \mathrm{e}_1(Q) - \ell_A(A/I) + 1$ holds true (Theorem 2.6), which is one of the keys for our proof of Theorem 1.2, similarly as in the Cohen-Macaulay case. In Section 3 we shall also discuss consequences of Theorem 1.2.

In what follows, unless otherwise specified, let $(A, \mathfrak{m})$ be a Noetherian local ring with $d = \dim A > 0$. Let I be an $\mathfrak{m}$-primary ideal in A and let $S = \mathrm{S}_Q(I)$ be the Sally module of I with respect to a minimal reduction $Q = (a_1, a_2, \cdots, a_d)$ of I. We put $R = A[It]$, $T = A[Qt]$, $R' = A[It, t^{-1}]$, $G = R'/t^{-1}R'$, and $B = T/\mathfrak{m}T$. We denote by $\mathrm{H}^i_{\mathfrak{m}}(*)$ $(i \in \mathbb{Z})$ the i-th local cohomology functor of A with respect to $\mathfrak{m}$. Let $\mathfrak{M} = \mathfrak{m}T + T_+$ be the unique graded maximal ideal in T.

Let $\tilde{I} = \bigcup_{n \geq 1}[I^{n+1} :_A I^n] = \bigcup_{n \geq 1}[I^{n+1} :_A (a_1^n, a_2^n, \cdots, a_d^n)]$ denote the Ratliff-Rush closure of I, which is the largest $\mathfrak{m}$-primary ideal in A such that $I \subseteq \tilde{I}$ and $\mathrm{e}_i(\tilde{I}) = \mathrm{e}_i(I)$ for all $0 \leq i \leq d$ (cf. [10]).

2. Preliminary steps for the proof

The purpose of this section is to summarize some auxiliary results on Sally modules, which we need throughout this paper. Let us begin with the following.

Lemma 2.1. *The following assertions hold true.*

(1) $\mathfrak{m}^\ell S = (0)$ *for integers* $\ell \gg 0$. *Hence* $\dim_T S \leq d$.

(2) *The homogeneous components* $\{S_n\}_{n \in \mathbb{Z}}$ *of the graded* T*-module* S *are given by*

$$S_n \cong \begin{cases} (0) & \text{if } n \leq 0, \\ I^{n+1}/Q^n I & \text{if } n \geq 1. \end{cases}$$

(3) $S = (0)$ *if and only if* $I^2 = QI$.

(4) *Suppose that* $S \neq (0)$ *and put* $V = S/\mathfrak{M}S$. *Let* V_n $(n \in \mathbb{Z})$ *denote the homogeneous component of the finite-dimensional graded* $T/\mathfrak{M}$*-space* V *with degree* n *and put* $\Lambda = \{n \in \mathbb{Z} \mid V_n \neq (0)\}$. *Let* $q = \max \Lambda$. *Then we have* $\Lambda = \{1, 2, \cdots, q\}$ *and* $\mathrm{r}_Q(I) = q + 1$, *where*

$$\mathrm{r}_Q(I) = \min\{n \in \mathbb{Z} \mid I^{n+1} = QI^n\}$$

stands for the reduction number of I *with respect to* Q.

(5) $S = TS_1$ *if and only if* $I^3 = QI^2$.

Proof. See [3, Lemma 2.1]. □

In the following Lemma 2.2 Serre's condition (S_2) on T plays a crucial role. This condition is automatically satisfied, once both conditions (C_1) and (C_3) are satisfied (cf. [15, Theorem 6.2]).

Lemma 2.2. *Suppose that conditions* (C_0), (C_2), *and* (C_3) *are satisfied. Assume that the ring* T *satisfies Serre's condition* (S_2). *Then* $\mathrm{Ass}_T S \subseteq \{\mathfrak{m}T\}$, *whence* $\dim_T S = d$ *if* $S \neq (0)$.

Proof. See [5, Lemma 2.3]. □

Proposition 2.3. *Suppose that conditions* (C_0) *and* (C_2) *are satisfied. Then*

$$\begin{aligned}\ell_A(A/I^{n+1}) &= \mathrm{e}_0(I)\binom{n+d}{d} - \{\mathrm{e}_0(I) + \mathrm{e}_1(Q) - \ell_A(A/I)\}\binom{n+d-1}{d-1} \\ &+ \sum_{i=2}^{d}(-1)^i\{\mathrm{e}_{i-1}(Q) + \mathrm{e}_i(Q)\}\binom{n+d-i}{d-i} - \ell_A(S_n)\end{aligned}$$

for all $n \gg 0$.

Proof. See [5, Proposition 2.4]. □

We put $\mathfrak{p} = \mathfrak{m}T$. Then since $\mathrm{Ass}_T S \subseteq \{\mathfrak{p}\}$ by Lemma 2.2, the following result is proven exactly in the same way as in the proof of [3, Proposition 2.2 (3)].

Proposition 2.4. *Suppose that conditions* (C_0) *and* (C_2) *are satisfied. Then*

$$\mathrm{e}_1(I) = \mathrm{e}_0(I) + \mathrm{e}_1(Q) - \ell_A(A/I) + \ell_{T_\mathfrak{p}}(S_\mathfrak{p}),$$

whence $\mathrm{e}_1(I) \geq \mathrm{e}_0(I) + \mathrm{e}_1(Q) - \ell_A(A/I)$.

Combining Lemmas 2.1 (3), 2.2, and Proposition 2.3 with Proposition 2.4, we get the following, which is more or less a finer version of [1].

Corollary 2.5. *Suppose that conditions* (C_1), (C_2), *and* (C_3) *are satisfied. Then* $\mathrm{e}_1(I) = \mathrm{e}_0(I) + \mathrm{e}_1(Q) - \ell_A(A/I) + \ell_{T_\mathfrak{p}}(S_\mathfrak{p})$. *The equality* $\mathrm{e}_1(I) = \mathrm{e}_0(I) + \mathrm{e}_1(Q) - \ell_A(A/I)$ *holds true if and only if* $I^2 = QI$. *When this is the case, one has* $\mathrm{e}_i(I) = \mathrm{e}_{i-1}(Q) + \mathrm{e}_i(Q)$ *for* $2 \leq i \leq d$.

Proof. Since $\mathrm{Ass}_T S \subseteq \{\mathfrak{p}\}$ by Lemma 2.2, we have $S_\mathfrak{p} = (0)$ if and only if $S = (0)$, that is equivalent to saying that $I^2 = QI$ by Lemma 2.1. When this is the case, by Proposition 2.3 we readily get $\mathrm{e}_i(I) = \mathrm{e}_{i-1}(Q) + \mathrm{e}_i(Q)$ for all $2 \leq i \leq d$. □

The following result is one of the keys for our proof of Theorem 1.2.

Theorem 2.6. *Suppose that conditions* (C_1), (C_2), *and* (C_3) *are satisfied. Then the following three conditions are equivalent to each other.*

(1) $\mathrm{e}_1(I) = \mathrm{e}_0(I) + \mathrm{e}_1(Q) - \ell_A(A/I) + 1$.
(2) $\mathfrak{m}S = (0)$ *and* $\mathrm{rank}_B S = 1$.
(3) *There exists a non-zero graded ideal* $\mathfrak{a}$ *of* B *such that* $S \cong \mathfrak{a}$ *as graded* T*-modules.*

To prove Theorem 2.6, we need the following estimation of $\mathrm{e}_2(I)$, which is a generalization of Narita's theorem [8] given in the case where A is a Cohen-Macaulay local ring.

Theorem 2.7. *Suppose that $d \geq 2$ and that condition* (C_1) *is satisfied. Then*

$$e_2(I) \geq e_1(Q) + e_2(Q).$$

Proof. See [5, Theorem 2.10]. □

3. Outline of proof of the main theorem

We notice that the equivalence of conditions (1) and (2) in Theorem 1.2 follows from Theorem 2.6. The implication (3) ⇒ (2) is clear. Therefore we have only to show the implication (1) ⇒ (3) together with the last assertions in Theorem 1.2. Suppose that condition (1) in Theorem 1.2 is satisfied. Then, thanks to Theorem 2.6, $\mathfrak{m}S = (0)$ and we get an isomorphism

$$\varphi : S \to \mathfrak{a}$$

of graded B-modules, where $\mathfrak{a}$ is a graded ideal in B. Notice that once we are able to prove that $I^3 = QI^2$, since $S = BS_1$ by Lemma 2.1 (5), the ideal $\mathfrak{a}$ is generated by lineally independent linear forms $\{X_i\}_{1 \leq i \leq c}$ of B with $c = \ell_A(I^2/QI)$ (recall that $\mathfrak{a}_1 \cong S_1 \cong I^2/QI$; see Lemma 2.1 (2)). When this is the case, (1) ⇒ (3) and the last assertions in Theorem 1.2 follow. Hence our Theorem 1.2 has been proven modulo the following.

Theorem 3.1. *Let $W = H^0_{\mathfrak{m}}(A)$ and assume that conditions* (C_1) *and* (C_2) *are satisfied. Suppose that $e_1(I) = e_0(I) + e_1(Q) - \ell_A(A/I) + 1$. Then $I^3 \subseteq QI^2 + W$.*

Proof. See [5, Theorem 1.3]. □

We lastly discuss consequences of Theorem 1.2.

Thanks to Theorem 1.2, we get the following generalization of Sally's theorem ([13, 16]), which corresponds to the case where $c = 1$ in Theorem 1.2. We denote by $B(-1)$ the graded B-module whose grading is given by $[B(-1)]_n = B_{n-1}$ for all $n \in \mathbb{Z}$.

Corollary 3.2. *Suppose that $d \geq 2$ and that conditions* (C_1), (C_2), *and* (C_3) *are satisfied. Then the following three conditions are equivalent to each other.*

(1) $S \cong B(-1)$ *as graded T-modules.*
(2) $e_1(I) = e_0(I) + e_1(Q) - \ell_A(A/I) + 1$ *and* $e_2(I) \neq e_1(Q) + e_2(Q)$.
(3) $I^3 = QI^2$ *and* $\ell_A(I^2/QI) = 1$.

When this is the case, we have

$$\begin{aligned} \ell_A(A/I^{n+1}) &= e_0(I)\binom{n+d}{d} - e_1(I)\binom{n+d-1}{d-1} \\ &+ \sum_{i=2}^{d}\{e_{i-1}(Q) + e_i(Q)\}\binom{n+d-i}{d-i} + \binom{n+d-2}{d-2} \end{aligned}$$

for all $n \geq 0$, and the following assertions hold true.

(*a*) $\operatorname{depth}_T S = d$.
(*b*) $e_2(I) = e_1(Q) + e_2(Q) + 1$.
(*c*) $e_i(I) = e_{i-1}(Q) + e_i(Q)$ *for* $3 \leq i \leq d$.

The characterization of the case where $c = d$ in Theorem 1.2 is the following.

Corollary 3.3. *Suppose that $d \geq 2$ and that conditions* (C_1), (C_2), *and* (C_3) *are satisfied. Then the following three conditions are equivalent to each other.*

(1) $S \cong B_+$ *as graded T-modules and* $\operatorname{depth} G = 0$.

(2) $e_1(I) = e_0(I) + e_1(Q) - \ell_A(A/I) + 1$, $e_i(I) = e_{i-1}(Q) + e_i(Q)$ *for* $2 \leq i \leq d$, *and* $\operatorname{depth} G = 0$.

(3) $\widetilde{I}^2 = Q\widetilde{I}$ *and* $\ell_A(\widetilde{I}/I) = 1$.

Here $\tilde{I} = \bigcup_{n \geq 1} [I^{n+1} :_A I^n]$ *denotes the Ratliff-Rush closure of I.*

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A remark on the regularity of a system of parameters of a local ring of mixed characteristic

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Throughout this article, $(R, \mathfrak{m})$ will denote a local Noetherian ring. An R-algebra B is said to be a *(balanced) big Cohen-Macaulay R-algebra* if some (every) system of parameters of R is a regular sequence on B. Then it is well known that any local ring containing a field has a big Cohen-Macaulay algebra. However, for a local ring $(R, \mathfrak{m})$ of mixed characteristic, the existence of such an algebra is known only in the case when $\dim R \leq 3$ and the final settlement in all characteristics has been resisted for more than 30 years. By this reason, it is expected that one might be able to find, at least, a weaker version of big Cohen-Macaulay algebras, which suffices to prove certain homological conjectures.

Recently, P. Roberts suggested in [6], [7] a new approach based upon Fontaine's theory for constructing almost Cohen-Macaulay algebras in mixed characteristic. In this report, we prove a result on the regularity of a certain system of parameters of a local ring of mixed characteristic. After stating the main theorem, we will make a brief comment on the proof of the theorem with its relation to almost Cohen-Macaulay algebras. We prove the following theorem:

Theorem 1. *Let $(R, \mathfrak{m})$ be a complete local domain of mixed characteristic $p > 0$ and let $p, x_2, \ldots, x_d$ be a system of parameters for R. Then there exists an R-algebra B such that:*

(1) $(p, x_2, \ldots, x_d)B \neq B$,

(2) $x_2, \ldots, x_d$ *forms a regular sequence on B/pB, and*

(3) p *is not a nilpotent element of B.*

Before starting the proof, let us make a couple of comments. First off, the method we use in the proof of the theorem is nothing new, and it has been one of the commonly used methods since M. Hochster succeeded in proving the existence of big Cohen-Macaulay modules to establish the homological conjectures in the equicharacteristic case. However, we want to emphasize that the main idea used to produce the required algebra B has been extended to the construction of almost Cohen-Macaulay algebras in mixed characteristic (see [8]).

Second, we recall a recent work of G. Dietz [1] on the study of algebras over local rings of positive characteristic that can be mapped to big Cohen-Macaulay algebras. Especially, he proved that such an algebra can be modified into an absolutely integrally closed, $\mathfrak{m}$-adically complete and separated quasilocal big Cohen-Macaulay algebra domain. He also developed a great deal of machinery to study intrinsic properties of such algebras over local rings of positive characteristic and his idea plays a role in our proof. Now we recall the definition of Hochster's algebra modification [4].

Definition 2 (algebra modifications). Let $x_1, \ldots, x_{k+1}$ be a sequence of a local ring $(R, \mathfrak{m})$ and for an R-algebra T, let $t_1, \ldots, t_{k+1}$ be a sequence of T such that $x_{k+1}t_{k+1} = \sum_{i=1}^{k} x_i t_i$. Let $X_1, \ldots, X_k$ be a set of indeterminates over T. Then we say that

$$T' = \frac{T[X_1, \ldots, X_k]}{(t_{k+1} - \sum_{i=1}^{k} x_i X_i)}$$

is an *algebra modification of* T. We define a *sequence of algebra modifications*:

$$T = T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_s \longrightarrow \cdots$$

such that every T_{i+1} is an algebra modification of T_i.

Let $\mathcal{F}$ denote a nonempty family of systems of parameters of R. Then as a universal object, we define $\mathrm{Mod}(T/R)$ as a polynomial algebra over T (quite possibly in infinitely many variables) modulo an ideal generated by elements as stated in the definition of algebra modification that are associated to all possible relations on T with respect to systems of parameters in $\mathcal{F}$. Inductively, we put $\mathrm{Mod}_0(T/R) := T$, $\mathrm{Mod}_{k+1}(T/R) := \mathrm{Mod}(\mathrm{Mod}_k(T/R)/R)$, and $\mathrm{Mod}_\infty(T/R) := \varinjlim_{k \in \mathbb{N}} \mathrm{Mod}_k(T/R)$. It turns out that $\mathrm{Mod}_\infty(T/R)$ is a possibly improper big Cohen-Macaulay R-algebra.

The *absolute integral closure* of an integral domain A is defined as the integral closure of A in an algebraic closure of the field of fractions of A and denote it by A^+. We recall the following brilliant result:

Theorem 3 (Hochster,Huneke [3]; Huneke,Lyubeznik [5]). *Let $(R, \mathfrak{m})$ be a local Noetherian domain of characteristic $p > 0$. Assume one of the following conditions:*

(1) *R is an excellent local domain.*

(2) *R is a homomorphic image of a Gorenstein local ring.*

Then R^+ is a balanced big Cohen-Macaulay R-algebra.

Definition 4. Let T be an algebra over a local ring $(R, \mathfrak{m})$ and let $0 \le t < \dim(R)$ be fixed. Then we say that a sequence of algebra modifications with respect to $\mathcal{F}$ is *of type* $\ge t$ if every T_{i+1} is a modification of T_i with respect to a relation of type at least t.

The proof of the following proposition follows directly from the definition. We will apply the proposition to the case where the family $\mathcal{F}$ consists of a single system of parameters of a mixed characteristic local ring, say $p, x_2, \ldots, x_d$.

Proposition 5. *Suppose that $x_1, \ldots, x_d$ is a system of parameters of R and S is an R-algebra. Let $0 \le t < \dim(R)$ be any fixed integer. Then the following conditions are equivalent:*

(1) *There exists an S-algebra B such that $1 \notin (x_1, \ldots, x_d)B$ and $x_{t+1}, \ldots, x_d$ forms a regular sequence on $B/(x_1, \ldots, x_t)B$.*

(2) *Suppose that*

$$S = T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_s$$

is any finite sequence of algebra modifications of S of type $\ge t$ with respect to $x_1, \ldots, x_d$. Then we have $1 \notin (x_1, \ldots, x_d)T_s$.

Proof of Theorem 1. We begin with a sequence of algebra modifications of relations of type ≥ 1 with respect to $p, x_2, \ldots, x_d$ and we prove the theorem by contradiction. Suppose that, as in Proposition 5, there is a sequence of algebra modifications of type ≥ 1 with respect to $p, x_2, \ldots, x_d$:

$$\mathcal{T} : R = T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_s$$

such that $1 \in (p, x_1, \ldots, x_d)T_s$. We remark that R/pR is a complete local ring of characteristic $p > 0$ in which $x_2, \ldots, x_d$ forms a system of parameters. We keep the same notation for a system of parameters of R/pR.

After dividing the sequence $\mathcal{T}$ out by p, we show that the induced sequence maps to a sequence of modifications of R/pR. We let

$$T_{i+1} = \frac{T_i[X_1^{(i)}, \ldots, X_k^{(i)}]}{(s_{k+1}^{(i)} - \sum_{j=1}^{k} x_j X_j^{(i)})}, \quad s_{k+1}^{(i)} \in T_i.$$

Then we have

$$T_{i+1} \equiv \frac{T_i[X_1^{(i)}, \ldots, X_k^{(i)}]}{(s_{k+1}^{(i)} - \sum_{j=2}^{k} x_j X_j^{(i)})} \ (\mathrm{mod} p).$$

We set

$$\overline{T}_{i+1} := \frac{T_{i+1}}{(p, X_1^{(0)}, \ldots, X_1^{(i)})T_{i+1}}.$$

Now it follows that a new sequence:

$$\overline{\mathcal{T}} : R/pR = \overline{T}_0 \longrightarrow \overline{T}_1 \longrightarrow \cdots \longrightarrow \overline{T}_s$$

is a sequence of modifications of R/pR with respect to $x_2, \ldots, x_d$, satisfying $1 \in (x_2, \ldots, x_d)\overline{T}_s$. Let Q be a fixed minimal prime ideal of R^+ over pR^+, where R^+ is the absolute integral closure of R. Since R^+/Q is the absolute integral closure of $R/(Q \cap R)$, a complete local domain of characteristic $p > 0$, it follows that $x_2, \ldots, x_d$ is a regular sequence on R^+/Q, as mentioned previously. By replacing the sequence $\overline{T}$ with $\overline{T} \otimes (R^+/Q)$, we have a sequence of bad modifications of R^+/Q. Then a standard method (see [2] for more detail) provides us the following commutative diagram:

$$\begin{array}{ccccccc}
R^+/Q & = & R^+/Q & = & \cdots & = & R^+/Q \\
\uparrow & & \uparrow & & & & \uparrow \\
R^+/Q & \longrightarrow & \overline{T}_1 & \longrightarrow & \cdots & \longrightarrow & \overline{T}_s
\end{array}$$

in which the first vertical arrow is the identity map, and so we get a contradiction $(x_2, \ldots, x_d)R^+/Q = R^+/Q$. Hence we have proved (1) and (2).

In order to prove (3), we need to construct a commutative diagram similar to the one as above. Let $S := R[\frac{1}{px_2 \cdots x_d}]$, the localization of R at $px_2 \cdots x_d$. Then $p, x_2, \ldots, x_d$ truely forms an improper regular sequence on S, and for any sequence of algebra modifications $R \to T_1 \to \cdots \to T_s$ of R, we get the commutative diagram:

$$\begin{array}{ccccccc}
S & = & S & = & \cdots & = & S \\
\uparrow & & \uparrow & & & & \uparrow \\
R & \longrightarrow & T_1 & \longrightarrow & \cdots & \longrightarrow & T_s
\end{array}$$

in which $R \to S$ is the natural inclusion. If we assume p is nilpotent in T_s, then it implies that p is nilpotent in S. But this is a contradiction. As the required algebra B can be constructed as a large direct limit of various sequences of algebra modifications of R of type ≥ 1, we conclude that p is not nilpotent in B as well, which proves (3). □

Question 6. Let $(R, \mathfrak{m})$ be a complete local domain of arbitrary characteristic. Then is R^+ an almost Cohen-Macaulay R-algebra in the sense of P. Roberts?

Finally, I would like to acknowledge a simple remark due to Prof. Y. Yoshino that has led to simplifications and improvements of the main result in [8].

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COEFFICIENT IDEAL OF IDEALS GENERATED BY MONOMIALS

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ABSTRACT. In a commutative Noetherian ring R, the *coefficient ideal* of I relative to J the largest ideal $\mathfrak{b}$ for which $I\mathfrak{b} = J\mathfrak{b}$ when I is integral over J. In this paper, we compute $\mathfrak{a}(I, J)$ by the *socle sequence* when $R = k[X_1, \ldots, X_d]$, a polynomial ring over a field k and I, J are ideals generated by monomials. Also we will show that our computation is closely related to the reduction number of I.

1. INTORODUCTION

Let R be a Noetherian ring and $J \subset I$ be ideals in R. If I is integral over J, there exists an integer r such that $I^{r+1} = I^r J$ or $I \cdot I^r = J \cdot I^r$. Conversely, if $I\mathfrak{a} = J\mathfrak{a}$ for some ideal $\mathfrak{a}$ which contains a non zero divisor, then I is integral over J. If $\mathfrak{a}$, $\mathfrak{b}$ are ideals in R satisfying $I\mathfrak{a} = J\mathfrak{a}$ and $I\mathfrak{b} = J\mathfrak{b}$, then we can easily see that $I(\mathfrak{a}+\mathfrak{b}) = J(\mathfrak{a}+\mathfrak{b})$. So, there is the largest ideal $\mathfrak{a}(I, J)$ such that $I\mathfrak{a}(I, J) = J\mathfrak{a}(I, J)$. We call this $\mathfrak{a}(I, J)$ the *coefficient ideal* of I with respect to J.

This notion was defined by I. M. Aberbach and C.Huneke [AH] and they also showed a theoretical algorithm to compute $\mathfrak{a}(I, J)$ when R/I is an Artinian ring. But this algorithm is too complicated to compute $\mathfrak{a}(I, J)$ actually.

In this paper, we treat the case when $R = k[X_1, \ldots, X_d]$, a polynomial ring over a field k, $J = (X_1^{a_1}, \ldots, X_d^{a_d})$ and I is a monomial ideal containing J. We give an algorithm to compute $\mathfrak{a}(I, J)$ by giving the socle of $\mathfrak{a}(I, J)$ since $\mathfrak{a}(I, J)$ is also generated by monomials and is determined by giving its socle.

To give the socle of $\mathfrak{a}(I, J)$, we introduce the notion of "socle sequence" and "step" of the elements of the sequence. Then the maximum number of steps gives the reduction number $r_J(I)$.

2. BASIC PROPERTIES

In this paper, let R be a commutative Noetherian ring, $J \subset I$ be two ideals of R. We always assume I is integral over J and $I \neq J$. By $\overline{J}$ we denote the integral closure of J.

Definition 2.1. The *coefficient ideal* of I with respect to J is

$$\mathfrak{a}(I, J) = \max\{\mathfrak{a} \mid \mathfrak{a} \quad \text{is an ideal with} \quad \mathfrak{a}I = \mathfrak{a}J\}.$$

Obviously, $\mathfrak{a}(I, J) \subset J : I$. Let us list some basic properties of $\mathfrak{a}(I, J)$.

Proposition 2.2. *Let I, I', J, J' be ideals of R and $I, I' \subset \overline{J}$. Then*

$$J \subset I \subset I' \Longrightarrow \mathfrak{a}(I, J) \supseteq \mathfrak{a}(I', J),$$
$$J \subset J' \subset I \Longrightarrow \mathfrak{a}(I, J) \subseteq \mathfrak{a}(I, J').$$

We will recall a theoretical algorithm to compute $\mathfrak{a}(I, J)$ for $\mathfrak{m}$ primary ideals I, J due to I. M. Aberbach and C. Huneke.

Definition 2.3. Let $J \subset I$ be ideals, and $I \subset \overline{J}$. The *reduction number* of I with respect to J, $r_J(I)$, is the least integer $r = r_J(I)$ such that $I^{r+1} = I^r J$.

Proposition 2.4. *(cf.* [AH], [Hun]*) Let $J \subset I$ be ideals of R with $I \subset \overline{J}$. We define the sequence of ideals $\{\mathfrak{a}_n\}$ as follows.*
(i) $\mathfrak{a}_1 = J : I$.
(ii) If $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ is defined, then $\mathfrak{a}_{n+1} = J\mathfrak{a}_n : I$.
Then we can assert the following.
(1) If $r = r_J(I)$, $I^r \subseteq \mathfrak{a}(I, J)$,
(2) $\mathfrak{a}_{n+1} \subseteq \mathfrak{a}_n$ and $\mathfrak{a}(I, J) \subseteq \mathfrak{a}_n$ for all n,
(3) If R/I is Artinian, then the sequence $\mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \cdots$ stabilizes and $\mathfrak{a}(I, J) = \mathfrak{a}_n$ for $n \gg 0$.

Corollary 2.5. *Let $R \to R'$ be a finite flat local extension of Noetherian rings and if $J \subset I$ are ideals in R primary to some maximal ideal $\mathfrak{m}$ with $I \subset \overline{J}$, then $\mathfrak{a}(IR', JR') = \mathfrak{a}(I, J)R'$.*

Proof. Since the colon is preserved under flat ring extension, our assertion is clear from the construction in Proposition 2.4. □

We recall some properties of ideals of $k[X_1, \dots, X_d]$ generated by monomials. First, we fix some notation. For $\underline{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}^d$ (we denote $\mathbb{N} = \{0, 1, 2, \dots\}$), we write $\underline{X}^{\underline{m}} = X_1^{m_1} X_2^{m_2} \cdots X_d^{m_d}$. For $\underline{m} = (m_1, m_2, \dots, m_d), \underline{n} = (n_1, n_2, \dots, n_d) \in \mathbb{N}^d$, we define operations,

$$\underline{m} \pm \underline{n} = (m_1 \pm n_1, m_2 \pm n_2, \dots, m_d \pm n_d),$$
$$a\underline{m} = (am_1, am_2, \dots, am_d) \text{ for } a \in \mathbb{N}.$$

Also, we write $\underline{m} \geq \underline{n}$ when $m_i \geq n_i$ for all $i = 1, 2, \dots, d$.

In the following, $R = k[X_1, \dots, X_d], \mathfrak{m} = (X_1, \dots, X_d)$ and I, J are $\mathfrak{m}$ primary ideals and we always assume that I, J are generated by monomials.

Remark 2.6. Note that $I : \mathfrak{m}$ is also generated by monomials. By Proposition 2.4, $\mathfrak{a}(I, J)$ is a monomial ideal too.

The socle of an $\mathfrak{m}$ primary ideal plays very important role in our computation.

Definition 2.7. Let I be an $\mathfrak{m}$ primary ideal generated by monomials. We denote by $\mu(I)$ the number of minimal generators of I.

If $(I : \mathfrak{m})/I$ is minimally generated by $\underline{X}^{\underline{m}_1}, \dots, \underline{X}^{\underline{m}_s}$, we define the *socle* of I

$$\mathrm{Soc}(I) = \{\underline{m}_1, \dots, \underline{m}_s\}.$$

Conversely, if $S = \{\underline{m}_1, \dots, \underline{m}_s\}$, then we denote by I_S the unique $\mathfrak{m}$ primary ideal generated by monomials, whose socle is S (cf. the next proposition).

Proposition 2.8. *Let I be an $\mathfrak{m}$ primary ideal generated by monomials.*

(1) *Let $R = k[X, Y]$ and $I = (X^{k_0}, X^{k_1}Y^{l_1}, \dots, X^{k_{s-1}}Y^{l_{s-1}}, Y^{l_s})$ with $k_0 > k_1 > \cdots > k_{s-1} > 0$ and $0 < l_1 < \cdots < l_s$. Then*

$$\mathrm{Soc}(I) = \{(k_0 - 1, l_1 - 1), (k_1 - 1, l_2 - 1), \dots, (k_{s-1}, l_s - 1)\}.$$

(2) *Conversely, if* $S = \{(k'_1, l'_1), \ldots, (k'_s, l'_s)\}$ *with* $k'_1 > \ldots > k'_s$ *and* $l'_1 < \ldots < l'_s$, *then*

$$I_S = (X^{k'_1+1}, X^{k'_2+1}Y^{l'_1+1}, \ldots, Y^{l'_s+1}).$$

In particular, $\mu(I) = \sharp\,\mathrm{Soc}(I) + 1$.

(3) *Let* $R = k[X_1, \ldots, X_d]$ *and* $S = \{\underline{m}_1, \ldots, \underline{m}_s\}$ *where* $\underline{m}_i = (m_{i_1}, m_{i_2}, \ldots, m_{i_d})$. *Then the monomial* $\mathfrak{m}$ *primary ideal* I_S *with* $\mathrm{Soc}(I_S) = S$ *is given by the following.*

$$I_S = \bigcap_{i=1}^{s} \left(X_1^{m_{i_1}+1}, X_2^{m_{i_2}+1}, \ldots, X_d^{m_{i_d}+1} \right).$$

3. Calculations of $\mathfrak{a}(I, J)$ and socle sequence

Let us calculate the coefficient ideal of ideals generated by monomials. Let $R = k[X_1, \ldots, X_d]$ be a polynomial ring over a field k with maximal ideal $\mathfrak{m} = (X_1, \ldots, X_d)$ and put $J = (X_1^{a_1}, \ldots, X_d^{a_d}) \subset I = (J, X_1^{s_1}X_2^{s_2}\cdots X_d^{s_d})$ $(0 \le s_i < a_i)$. Write $F = X_1^{s_1}X_2^{s_2}\cdots X_d^{s_d}$. In this section, we treat only this case and in the next section we treat the case when $I = (J, F_1, \ldots, F_n)$. Since I must be integral over J, we assume

$$\frac{s_1}{a_1} + \cdots + \frac{s_d}{a_d} \ge 1.$$

Remark 3.1. Let $(a_i, s_i) = d_i$, $a'_i = a_i/d_i$ and $s'_i = s_i/d_i$ $(i = 1, 2, \ldots, d)$. Let $R' = k[Y_1, \ldots, Y_d]$ and consider R' as a subring of R by putting $Y_i = X_i^{d_i}$. Define $J' = (Y_1^{a'_1}, \ldots, Y_d^{a'_d}), F' = Y_1^{s'_1}Y_2^{s'_2}\cdots Y_d^{s'_d}, I' = (J', F')R'$. Then since R is a free R' module, we have $\mathfrak{a}(I, J) = \mathfrak{a}(I', J')$ and thus we can reduce to the case $(a_i, s_i) = 1$.

Proposition 3.2. *Let* $J = (X_1^{a_1}, \ldots, X_d^{a_d}) \subset I = (J, F)$ *with* $F = X_1^{s_1}X_2^{s_2}\cdots X_d^{s_d}$. *We put* $w = a_1 + \cdots + a_d - (s_1 + \cdots + s_d)$.

(1) If we assume $a_i \le s_1 + \cdots + s_d$ *for all* $i = 1, 2, \ldots, d$, *then* $\mathfrak{a}(I, J) \supset \mathfrak{m}^{w-d+1}$ *and* $\mathfrak{a}(I, J) \not\supset \mathfrak{m}^{w-d}$.

(2) Let $\underline{m} = (m_1, \ldots, m_d)$ *with* $\sum_{i=1}^{d} \frac{m_i}{a_i} > \sum_{i=1}^{d} \frac{a_i - s_i - 1}{a_i}$, *then* $X^{\underline{m}} \in \mathfrak{a}(I, J)$.

Proof. (1) Since $\mathfrak{m}^{w-d+1}F \subset \mathfrak{m}^{\sum(a_i-1)+1}$, we can easily reduce that $\mathfrak{m}^{w-d+1}F \subset J$ and hence $\mathfrak{m}^{w-d+1}I = \mathfrak{m}^{w-d+1}J$. This shows that $\mathfrak{m}^{w-d+1} \subset \mathfrak{a}(I, J)$.

Next, put $M = X_1^{a_1-1-s_1}\cdots X_d^{a_d-1-s_d} \in \mathfrak{m}^{w-d}$. Then $FM = X_1^{a_1-1}\cdots X_d^{a_d-1} \notin J$. Hence $\mathfrak{a}(I, J) \not\supset \mathfrak{m}^{w-d}$.

(2) Assume $FX^{\underline{m}} \notin J$. Then we must have $\sum_{i=1}^{d} \frac{m_i + s_i}{a_i} \le \sum_{i=1}^{d} \frac{a_i - 1}{a_i}$ since the socle of R/J is generated by $X_1^{a_1-1}\cdots X_d^{a_d-1}$. Hence if $\sum_{i=1}^{d} \frac{m_i}{a_i} > \sum_{i=1}^{d} \frac{a_i - s_i - 1}{a_i}$, we have $FX^{\underline{m}} = X_i^{a_i}X^{\underline{m}'}$ for some i and $\underline{m}' = (m'_1, \ldots, m'_d)$. Since $\sum_{i=1}^{d} \frac{s_i}{a_i} \ge 1$, $\sum_{i=1}^{d} \frac{m'_i}{a_i} \ge \sum_{i=1}^{d} \frac{m_i}{a_i}$, we can repeat this process. This means that $X^{\underline{m}} \in \mathfrak{a}(I, J)$. □

Since $\mathfrak{a}(I, J)$ is generated by monomials, we can express $\mathfrak{a}(I, J)$ by giving its socle. Now, we compute $\mathrm{Soc}(\mathfrak{a}(I, J))$ by "socle sequence".

Construction 3.3. Construct the sequence $S = (\underline{m}_\sigma) \subset \mathbb{N}^d$ inductively as follows. We call this sequence *the socle sequence of I with respect to J.*

(1) Put $\underline{m}_\emptyset = (a_1 - s_1 - 1, \ldots, a_d - s_d - 1)$. We say $\underline{m}_\emptyset$ is the element of step 1 of the socle sequence and denote $\mathrm{Step}(m_\emptyset) = 1$.

(2) Define

$$\begin{cases} \underline{u}_1 = (a_1 - s_1, -s_2, \ldots, -s_d) \\ \underline{u}_2 = (-s_1, a_2 - s_2, -s_3, \ldots, -s_d) \\ \quad \vdots \\ \underline{u}_d = (-s_1, \ldots, -s_{d-1}, a_d - s_d). \end{cases}$$

(3) Let $\sigma = (i_1, \ldots, i_h)$ be the sequence of integers with $1 \le i_j \le d$ for every j. We define $\underline{m}_\sigma = \underline{m}_\emptyset + \underline{u}_{i_1} + \ldots + \underline{u}_{i_h}$. We call σ' a subsequence of σ if $\sigma' = (i_1, \ldots, i_{h'})$ for some $h' \le h$.

(4) We call a sequence σ *permissible* if $\underline{m}_{\sigma'} \ge \mathbf{0}$ for every subsequence σ' of σ. If σ is permissible, then we define $\mathrm{Step}(\underline{m}_\sigma) = \sharp\sigma + 1$. Note that if $\sigma' = (i_1, \ldots, i_{h-1})$, then we have

$$F\underline{X}^{\underline{m}_\sigma} = X_{i_h}^{a_{i_h}} \underline{X}^{\underline{m}_{\sigma'}}.$$

(5) $S = \{\underline{m}_\sigma \mid \sigma \text{ is permissible}\}$.

Theorem 3.4. $\mathrm{Soc}(\mathfrak{a}(I,J)) = S$. *Namely* $\mathfrak{a}(I,J) = I_S$

Proof. Part 1. First we prove $\mathfrak{a}(I,J) \subset I_S$. Since $\mathrm{Soc}(I_S) = S$, we check $\underline{X}^{\underline{m}_\sigma} \notin \mathfrak{a}(I,J)$ for every $\underline{m}_\sigma \in S$ by induction on step $h = \mathrm{Step}(\underline{m}_\sigma)$. If $h = 1$, since $\underline{X}^{\underline{m}_\emptyset} \cdot F = X_1^{a_1-1} \cdots X_d^{a_d-1} \notin J$, $\underline{X}^{\underline{m}_\emptyset} \notin J : I$ and hence $\underline{X}^{\underline{m}_\emptyset} \notin \mathfrak{a}(I,J)$. Let $h > 1$ and assume $\underline{X}^{\underline{m}_\sigma} \notin \mathfrak{a}(I,J)$ for smaller values of h.

If there exists $\underline{m}_\sigma$ such that $\underline{X}^{\underline{m}_\sigma} \in \mathfrak{a}(I,J)$ then since $\underline{X}^{\underline{m}_\sigma} \cdot F \in \mathfrak{a}(I,J)I = \mathfrak{a}(I,J)J$, we must have

$$(*) \qquad \underline{X}^{\underline{m}_\sigma} \cdot F = X_k^{a_k} \underline{X}^{\underline{m}_\sigma - \underline{u}_k}$$

for some k with $\underline{X}^{\underline{m}_\sigma - \underline{u}_k} \in \mathfrak{a}(I,J)$. So it suffices to show that $\underline{m}_\sigma - \underline{u}_k \in S$ and $\mathrm{Step}(\underline{m}_\sigma - \underline{u}_k) = \mathrm{Step}(\underline{m}_\sigma) - 1$ for every k satisfying $(*)$, since $\underline{X}^{\underline{m}_\sigma - \underline{u}_k} \notin \mathfrak{a}(I,J)$ by induction hypothesis. So it suffices to prove the following Lemma. □

Lemma 3.5. *Assume that σ is permissible with $\sharp\sigma = h \ge 1$. If $\underline{m}_\sigma - \underline{u}_k \ge \mathbf{0}$ then there exists a permissible sequence τ whose last entry is k such that $\underline{m}_\tau = \underline{m}_\sigma$ and $\sharp\sigma = \sharp\tau$.*

Proof of Lemma. Use induction on $h = \mathrm{Step}(m_\sigma)$. If $\sigma = (i_1)$ then $\underline{m}_\sigma - \underline{u}_k = \underline{m}_\emptyset + (0, \ldots, a_{i_1}, \ldots, -a_k, \ldots, 0) \not\ge \mathbf{0}$ for $k \ne i_1$. Therefore we have $k = i_1$.

Assume that $h > 2$, and assume the lemma is true for smaller values of h. Suppose $\sigma = (\sigma', i_{h-1})$ where $\sigma' = (i_1, \ldots, i_{h-2})$. Put $\underline{m}_{\sigma'} = (m'_1, \ldots, m'_d)$. If $k = i_{h-1}$, we have nothing to prove. If $k \ne i_{h-1}$, since $\underline{m}_\sigma - \underline{u}_k = \underline{m}_{\sigma'} + (0, \ldots, a_{i_{h-1}}, \ldots, -a_k, \ldots, 0) \ge \mathbf{0}$, we have $m'_k \ge a_k$. This is possible only if $i_j = k$ for some j, $1 \le j \le h-2$. Now, let j be the last index with $i_j = k$. Then put $\tau = (i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_{h-1}, k)$. It is easy to show that τ is permissible by our construction. □

Proof of 3.4. Part 2. Next we prove $\mathfrak{a}(I,J) \supset I_S$. It suffices to show that for every $\underline{m}_\sigma \in S$ and for every i, $X_i\underline{X}^{\underline{m}_\sigma} \in \mathfrak{a}(I,J)$. If $\sigma = (\sigma', i_h)$ and $X_i\underline{X}^{\underline{m}_{\sigma'}} \in \mathfrak{a}(I,J)$, then $X_i\underline{X}^{\underline{m}_\sigma} \in \mathfrak{a}(I,J)$ since

$$F(X_i\underline{X}^{\underline{m}_\sigma}) = X_{i_h}^{a_{i_h}}(X_i\underline{X}^{\underline{m}_{\sigma'}}).$$

Thus by induction on $\mathrm{Step}(\underline{m}_\sigma)$, we have only to show that $X_i\underline{X}^{\underline{m}_\emptyset} \in \mathfrak{a}(I,J)$ for every i. But this is already proved in 3.2 (2). □

Remark 3.6. Let $\underline{m}_\sigma, \underline{m}_\tau \in S$. Then the following statements hold.
(1) If $\underline{m}_\sigma = \underline{m}_\tau$ then $\mathrm{Step}(\underline{m}_\sigma) = \mathrm{Step}(\underline{m}_\tau)$ and $s_1/a_1 + \cdots + s_d/a_d = 1$.
(2) There is no relation such as $\underline{m}_\sigma > \underline{m}_\tau$ for any σ, τ.

Proof. Let $\mathrm{Step}(\underline{m}_\sigma) = h$ and $\mathrm{Step}(\underline{m}_\tau) = h'$. If $\underline{m}_\sigma \geq \underline{m}_\tau$, then we have the following equation for each coefficient;

$$(\alpha_i+1)a_i - hs_i - 1 \geq (\beta_i+1)a_i - h's_i - 1 \ (h-1 = \sum_i \alpha_i, h'-1 = \sum_i \beta_i).$$

Hence $(\alpha_i - \beta_i)a_i - (h-h')s_i \geq 0$ for every i. If $h = h'$ then $\alpha_i = \beta_i$ for $i = 1, 2, \ldots, d$ and $\underline{m}_\sigma = \underline{m}_\tau$. If $h \neq h'$, there exists i such that $(\alpha_i - \beta_i)a_i - (h-h')s_i > 0$.

$$\sum_i (\alpha_i - \beta_i) = h - h' > (h-h')\sum_i \frac{s_i}{a_i}.$$

Thus we have $1 > \sum_i \frac{s_i}{a_i}$ contradicting our hypothesis. □

Remark 3.7. We can show that the number of steps of the socle sequence is always finite. Since an element of S is a lattice point in a bounded region, S is always a finite set.
(1) If $s_1/a_1 + \cdots + s_d/a_d > 1$, and if $\sigma' \in \mathbb{N}^h$ is a subsequence of $\sigma \in \mathbb{N}^{h+t}$ with $t > 0$, $\underline{m}_\sigma = (k_{1,\sigma}, \ldots, k_{d,\sigma})$, $\underline{m}_{\sigma'} = (k_{1,\sigma'}, \ldots, k_{d,\sigma'})$, then we have

$$\left(\frac{k_{1,\sigma}}{a_1} + \cdots + \frac{k_{d,\sigma}}{a_d}\right) - \left(\frac{k_{1,\sigma'}}{a_1} + \cdots + \frac{k_{d,\sigma'}}{a_d}\right) < 0.$$

Since $\left(\frac{k_{1,\sigma}}{a_1} + \cdots + \frac{k_{d,\sigma}}{a_d}\right)$ is strictly decreasing, the number of steps is bounded.

(2) In the case $s_1/a_1 + \cdots + s_d/a_d = 1$, $\left(\frac{k_{1,\sigma}}{a_1} + \cdots + \frac{k_{d,\sigma}}{a_d}\right)$ is constant for all σ which is permissible. If the number of steps is not bounded, then the socle sequence must have a loop. But if $\underline{m}_\sigma = \underline{m}_\tau$, where $\tau = (\sigma, i_1, \ldots, i_l)$, then $\underline{X}^{\underline{m}_\sigma} \in \mathfrak{a}(I,J)$, which contradicts our theorem. So the socle sequence has no loop and the maximal number of steps is finite.

We will see in Section 5 that the maximum number of steps of the socle sequence is closely related to the reduction number $r_J(I)$.

Example 3.8. Let $J = (X_1^{a_1}, \ldots, X_d^{a_d}) \subset I = (J, F)$ with $F = X_1^{s_1}X_2^{s_2}\cdots X_d^{s_d}$. If $s_i \geq \frac{a_i}{2}$ for at least two indices i, then $\underline{m}_\emptyset - u_j \not\geq \mathbf{0}$ for any j. Hence $\mathfrak{a}(I,J) = (X_1^{a_1-s_1}, \ldots, X_d^{a_d-s_d})$.

4. Calculations of $\mathfrak{a}(I, J)$ in general case

Let $R = k[X_1, \ldots, X_d]$ be a polynomial ring over a field k with maximal ideal $\mathfrak{m} = (X_1, \ldots, X_d)$ and put $J = (X_1^{a_1}, \ldots, X_d^{a_d})$. In this section, let $I = (J, F_1, F_2, \ldots, F_n)$, where each F_i is a monomial integral over J for $i = 1, 2 \ldots, n$. Also in this case, we can construct the socle sequence S and show that $\mathfrak{a}(I, J) = I_S$ as in 3.4. In the following, we treat the case $n = 2$, namely $I = (J, F, G)$, since to describe the general case, the notation will be too complicated and the argument is essentially the same.

Now, let $I = (J, F, G)$ with $F = X_1^{s_1} X_2^{s_2} \cdots X_d^{s_d}$ and $G = X_1^{t_1} X_2^{t_2} \cdots X_d^{t_d}$. Since I must be integral over J, we assume $\frac{s_1}{a_1} + \cdots + \frac{s_d}{a_d} \geq 1$ and $\frac{t_1}{a_1} + \cdots + \frac{t_d}{a_d} \geq 1$. We may assume $G \notin (J, F)$ and $F \notin (J, G)$, so that for some i, j, $s_i > t_i$ and $s_j < t_j$.

Construction 4.1. We construct the sequence $S' = (\underline{m}_\sigma) \subset \mathbb{N}^d$ inductively and call this sequence *the socle sequance of I with respect to J.*

(1) Let $\underline{m}_{\emptyset_1} = (a_1 - s_1 - 1, \ldots, a_d - s_d - 1)$ and $\underline{m}_{\emptyset_2} = (a_1 - t_1 - 1, \ldots, a_d - t_d - 1)$. We say $\underline{m}_{\emptyset_1}$ and $\underline{m}_{\emptyset_2}$ are elements of step 1 of the socle sequence.

(2) Define

$$\begin{cases} \underline{u}_1 = (a_1 - s_1, -s_2, \ldots, -s_d) \\ \underline{u}_2 = (-s_1, a_2 - s_2, -s_3, \ldots, -s_d) \\ \quad \vdots \\ \underline{u}_d = (-s_1, \ldots, -s_{d-1}, a_d - s_d) \end{cases} \qquad \begin{cases} \underline{v}_1 = (a_1 - t_1, -t_2, \ldots, -t_d) \\ \underline{v}_2 = (-t_1, a_2 - t_2, -t_3, \ldots, -t_d) \\ \quad \vdots \\ \underline{v}_d = (-t_1, \ldots, -t_{d-1}, a_d - t_d). \end{cases}$$

(3) Let $\sigma = (c; \underline{w}_{i_1}, \ldots, \underline{w}_{i_h})$ be the sequence where $c = 1$ or 2, and $\underline{w}_{i_j} = \underline{u}_{i_j}$ or $\underline{v}_{i_j}$ for every j. If $c = 1$ (resp. $c = 2$) then we define $\underline{m}_\sigma = \underline{m}_{\emptyset_1} + \underline{w}_{i_1} + \ldots + \underline{w}_{i_h}$ (resp. $\underline{m}_\sigma = \underline{m}_{\emptyset_2} + \underline{w}_{i_1} + \ldots + \underline{w}_{i_h}$). We call σ' a subsequence of σ if $\sigma' = (c; \underline{w}_{i_1}, \ldots, \underline{w}_{i_{h'}})$ for some $h' \leq h$.

(4) We call a sequence σ *permissible* if $m_{\sigma'} \geq \mathbf{0}$ for every subsequence σ' of σ. If σ is permissible, then we define $\mathrm{Step}(\underline{m}_\sigma) = \sharp\sigma + 1$. Here $\sharp\sigma = h$, if $\sigma = (c; \underline{w}_{i_1}, \ldots, \underline{w}_{i_h})$. Also, we define $\mathrm{Step}(m_{\emptyset_i}) = 1$ for $i = 1, 2$.

(5) We put $S' = \{\underline{m}_\sigma \mid \sigma \text{ is permissible}\}$ and
$S = \{\underline{m}_\sigma \mid \text{ for every } \tau \neq \sigma, \underline{m}_\tau \not\geq \underline{m}_\sigma\}$.

Theorem 4.2. *Let $I = (J, F_1, F_2, \ldots, F_n)$ and S be as above. Then $\mathfrak{a}(I, J) = I_S$.*

We treat the case $n = 2$ and follow the notation of 4.1.

Proof. Part 1. First we prove $\mathfrak{a}(I, J) \subset I_S$. Since $\mathrm{Soc}(I_S) = S$, we check $\underline{X}^{\underline{m}_\sigma} \notin \mathfrak{a}(I, J)$ for all $\underline{m}_\sigma \in S$ by induction on step h. If $h = 1$, since $\underline{X}^{\underline{m}_{\emptyset_1}} \cdot F = \underline{X}^{\underline{m}_{\emptyset_2}} \cdot G = X^{a_1 - 1} \cdots X_d^{a_d - 1} \notin J$, $\underline{X}^{\underline{m}_{\emptyset_1}}, \underline{X}^{\underline{m}_{\emptyset_2}} \notin J : I$ and hence $\notin \mathfrak{a}(I, J)$.

Let $h > 1$, and assume $\underline{X}^{\underline{m}_\sigma} \notin \mathfrak{a}(I, J)$ if $\underline{m}_\sigma \in S$ and $\mathrm{Step}(\underline{m}_\sigma) \leq h - 1$. Now assume that there exists $\underline{m}_\sigma \in S$ such that $\underline{X}^{\underline{m}_\sigma} \in \mathfrak{a}(I, J)$. As in the proof of 3.4, we have only to prove the following Lemma. □

Lemma 4.3. *Let $\sigma = (c; \underline{w}_{i_1}, \ldots, \underline{u}_{i_{h-1}})$ be permissible, $\sharp\sigma = h > 1$. If $\underline{m}_\sigma - \underline{u}_k \geq \mathbf{0}$ then there exists a permissible sequence τ whose last entry is $\underline{u}_k$ such that $\underline{m}_\tau = \underline{m}_\sigma$ and $\sharp\sigma = \sharp\tau$.*

The proof of this lemma is the same as that of 3.5 and we omit it.

Proof of 4.2. Part 2. Let us prove $\mathfrak{a}(I,J) \supset I_S$. Assume that $\mathfrak{a}(I,J) \subsetneq I_S$. Then $I_S \cdot J \subsetneq I_S \cdot I$. Hence there exists $M \in I_S$ such that $M \notin \mathfrak{a}(I,J)$. $MF \notin I_SJ$ or $MG \notin I_SJ$. Suppose that $MF \notin I_SJ$, write $M = \underline{X}^{\underline{m}}$ and $\underline{m} = (m_1, \dots, m_d)$. Then one of the following conditions hold:

(1) $m_i + (s_i - a_i) < 0$ for all $i = 1, 2, \dots, d$.

(2) $m_i + (s_i - a_i) \geq 0$ for some $i = 1, 2, \dots, d$, and $X_i^{-a_i}MF \notin I_S$ for all such i.

Let us assume (1). $m_i \leq a_i - s_i - 1$ for all $i = 1, 2, \dots, d$. Thus we can write $\underline{m} = \underline{m}_{\emptyset_1} - \boldsymbol{p}$ with $\boldsymbol{p} = (p_1, \dots, p_d) \geq \mathbf{0}$. Since $\underline{m}_{\emptyset_1} \in S = \mathrm{Soc}(I_S)$, $\underline{X}^{\underline{m}} \notin I_S$. This is a contradiction!

Next, let us assume (2) and $M' = X_i^{-a_i}MF (= \underline{X}^{\underline{m}-\underline{u}_i})$. Since $M' \notin I_S$, M' generates some element $\underline{X}^{\underline{m}_\sigma}$ where $\underline{m}_\sigma \in S = \mathrm{Soc}(I_S)$. Therefore we can write $\underline{m} - \underline{u}_i = \underline{m}_\sigma - \boldsymbol{p} \geq \mathbf{0}$ with $\boldsymbol{p} = (p_1, \dots, p_d) \geq \mathbf{0}$. Actually, $(\sigma, \underline{u}_i)$ is permissible. Hence $\underline{X}^{\underline{m}} \notin I_S$. This is a contradiction! So $\mathfrak{a}(I,J) = I_S$, the theorem is proved. □

Remark 4.4. We ask if the number of steps is finite or not in this case.

(1) If $s_1/a_1 + \cdots + s_d/a_d > 1$ and $t_1/a_1 + \cdots + t_d/a_d > 1$ then $\left(\frac{k_{1,\sigma}}{a_1} + \cdots + \frac{k_{d,\sigma}}{a_d}\right)$ is strictly decreasing sequence, so the number of steps is finite.

(2) Let us consider the case $s_1/a_1 + \cdots + s_d/a_d = 1$ or $t_1/a_1 + \cdots + t_d/a_d = 1$. In this case, there are examples when some permissible sequence contains a loop as is shown in the next example.

Example 4.5. We give an example where the number of steps of the socle sequence is not bounded. Let $R = [X, Y]$ over a field k with maximal ideal $\mathfrak{m} = (X, Y)$ and $J = (X^{10}, Y^{10}) \subset I = (J, F, G)$ with $F = X^3Y^9$, $G = X^5Y^5$. Then $\underline{m}_{\emptyset_1} = (6, 0), \underline{v}_1 = (5, -5), \underline{v}_2 = (-5, 5)$. We have a loop $\underline{m}_{\emptyset_1} + \underline{v}_1 + \underline{v}_2 = \underline{m}_{\emptyset_1}$.

5. Reduction number and number of steps

Again let $R = k[X_1, \dots, X_d]$ be a polynomial ring over a field k with maximal ideal $\mathfrak{m} = (X_1, \dots, X_d)$ and put $J = (X_1^{a_1}, \dots, X_d^{a_d}) \subset I = (J, X_1^{s_1}X_2^{s_2}\cdots X_d^{s_d}) \subset \overline{J}$ $(0 \leq s_i < a_i)$. Write $F = X_1^{s_1}X_2^{s_2}\cdots X_d^{s_d}$. In this section, we will show that the reduction number $r_J(I)$ is equal to the maximal number of steps of the socle sequence. Recall that $r_J(I) = \min\{r \in \mathbb{N} \mid I^{r+1} = JI^r\}$.

Definition 5.1. We denote by Step(S) the maximum step in the socle sequence S of I with respect to J (cf. Remark 3.7).

We begin with a property of $r_J(I)$. For a real number x, $[x]$ denotes the largest integer not larger than x.

Lemma 5.2.

$$r_J(I) = \min\left\{h \;\middle|\; \sum_{i=1}^{d}\left[\frac{(h+1)s_i}{a_i}\right] \geq h+1\right\}.$$

Proof. Since $I^{r+1} = F^{r+1}R + JI^r$, it is easy to see that

$$F^{r+1} \in JI^r \iff F^{r+1} \in J^{r+1}.$$

and if $t = \sum_{i=1}^{d} \left[\dfrac{(h+1)s_i}{a_i}\right]$, then $F^{h+1} \in J^t$ and $\notin J^{t+1}$. □

Theorem 5.3. *Let S be the socle sequence of I with respect to J. Then*

$$r_J(I) = \mathrm{Step}(S).$$

Proof. In 3.8 we have seen that $\mathrm{Step}(S) = 1$ if and only if $\sum_i \left[\dfrac{2s_i}{a_i}\right] \geq 2$. So let $r_J(I), \mathrm{Step}(S) > 1$. First we show that $r_J(I) \geq \mathrm{Step}(S)$. Let $\underline{m} = \underline{m_\emptyset} + \alpha_1\underline{u}_1 + \cdots + \alpha_d\underline{u}_d \in S$ $(\alpha_1 + \cdots + \alpha_d = t-1)$ of step $t(\leq \mathrm{Step}(S))$ ($\underline{u}_i$ is as in Construction 3.4). Then we have the following:

$$\begin{aligned} \underline{m} \geq 0 &\iff (\alpha_i + 1)a_i - ts_i - 1 \geq 0 \text{ for } i = 1, 2, \ldots, d \\ &\iff \alpha_i \geq \left[\frac{ts_i}{a_i}\right] \text{ for } i = 1, 2, \ldots, d \iff t - 1 \geq \sum_i \left[\frac{ts_i}{a_i}\right]. \end{aligned}$$

Hence $t - 1 \notin \left\{ h \;\middle|\; \sum_{i=1}^{d} \left[\dfrac{(h+1)s_i}{a_i}\right] \geq h+1 \right\}$. Therefore $r_J(I) \geq \mathrm{Step}(S)$.

Next we show that the existence of $\underline{m} \in S$ of step $r = r_J(I)$ to prove $r_J(I) \leq \mathrm{Step}(S)$. We want to show the existence of a permissible sequence σ such that $\sharp\sigma = r - 1$. We can prove the existence of σ by the following lemma. We put $n_{i,j} = \left[\dfrac{js_i}{a_i}\right]$, $r = r_J(I)$ and $\alpha_{1,r}, \ldots, \alpha_{d,r}$ are non-negative integers which satisfy $\alpha_i \geq \left[\dfrac{rs_i}{a_i}\right]$ and $\sum_{i=1}^{d} \alpha_{i,r} = r - 1$.

Lemma 5.4. *Let $n_{i,j}, \alpha_{i,r} \in \mathbb{Z}_{\geq 0}$ for $i = 1, 2, \ldots, d$ and $j = 1, 2, \ldots, r$ satisfying the following conditions.*

(1) $\sum_{i=1}^{d} n_{i,j} \leq j - 1$ *for all* j.

(2) $n_{i,j+1} = n_{i,j}$ *or* $n_{i,j} + 1$ *for all* i, j.

(3) $\alpha_{i,r} \geq n_{i,r}$ *for all* i.

(4) $\sum_{i=1}^{d} \alpha_{i,r} \leq r - 1$.

Then For all $j = 1, 2, \ldots, r-1$, there exists $(\alpha_{1,j}, \ldots, \alpha_{d,j})$ such that

(a) $\sum_{i=1}^{d} \alpha_{i,j} = j - 1$, *(b)* $\alpha_{i,j+1} \geq \alpha_{i,j}$ *for all* i, *(c)* $\alpha_{i,j} \geq n_{i,j}$.

Proof of Lemma. It sufficces to prove that there exists $(\alpha_{1,j}, \ldots, \alpha_{d,j})$ satisfying (a), (b), (c) for $j = r-1$. If $n_{i,r} = n_{i,r-1}+1$ for some i then we have $(\alpha_{1,r-1}, \ldots, \alpha_{d,r-1}) = (\alpha_{1,r}, \ldots, \alpha_{i,r} - 1, \ldots, \alpha_{d,r})$ for such i. Suppose $n_{i,r} = n_{i,r-1}$ for all i. $\sum_{i=1}^{d} n_{i,j} \leq r - 2$ by assumption (1). Hence we have weaker condition $\alpha_{i,r-1} \geq n_{i,r-1}$ for some i. So we can take $(\alpha_{1,r-1}, \ldots, \alpha_{d,r-1}) = (\alpha_{1,r}, \ldots, \alpha_{i,r} - 1, \ldots, \alpha_{d,r})$ for such i. □

Now we come back to the proof of the theorem. Let us construct a permissible sequence $\underline{m}_\sigma$ with Step$(\underline{m}_\sigma) = r$. Let $\underline{m}_{\sigma t} = \underline{m}_\emptyset + u_{i_1} + \ldots + u_{i_t}$ with Step$(\underline{m}_{\sigma t}) = t+1$ be subsequnce of $\underline{m}_\sigma$. It suffices to determine $\underline{m}_{\sigma t}$ for $t = 1, 2, \ldots, r-1$ from $\{\alpha_{i,j}\}$ as in the following Lemma. If $\alpha_{i,t} = \sharp\{j \mid i_j = i, 1 \le j \le r-1\}$, then $\underline{m}_\sigma$ is determined from $\alpha_{i,t}$ $(t = 1, \ldots, r-1)$ and we have constructed a permissible sequence with Step$(\underline{m}_\sigma) = r$. □

In the case $d = 2$, we have $\mu(\mathfrak{a}(I, J)) = \mathrm{Step}(S)$, where $\mu(I)$ denotes the minimal number of generators of I. Hence we have the following.

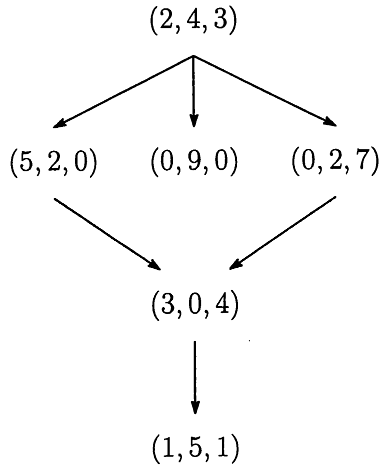
Corollary 5.5. *If $d = 2$, then $\mu(\mathfrak{a}(I, J)) = r_J(I) + 1$.*

6. Examples

In this section, we give some examples. Let $R = k[X_1, \ldots, X_d]$ be a polynomial ring over a field k with maximal ideal $\mathfrak{m} = (X_1, \ldots, X_d)$.

Example 6.1. Let $R = k[X, Y]$ and $J = (X^n, Y^n) \subset I = (J, X^s Y^{n-s})$ with $(n, s) = 1$. Then by 3.2, we have $\mathfrak{a}(I, J) \supset \mathfrak{m}^{n-1}$. On the other hand, $\left[\frac{ts}{n}\right] + \left\lceil\frac{t(n-s)}{n}\right\rceil \ge t$ if and only if t is a multiple of n. Hence the $r_J(I) = n - 1$ and by 5.5, $\mu(\mathfrak{a}(I, J)) = n$. Hence we have $\mathfrak{a}(I, J) = \mathfrak{m}^{n-1}$.

Example 6.2. Let $R = k[X, Y, Z]$ and $J = (X^5, Y^7, Z^7) \subset I = (J, X^2Y^2Z^3)$. Then $\underline{m}_\emptyset = (2, 4, 3), \underline{u}_1 = (3, -2, -3), \underline{u}_2 = (-2, 5, -3), \underline{u}_3 = (-2, -2, 4)$. So the socle sequence of I with respect to J is as follows;

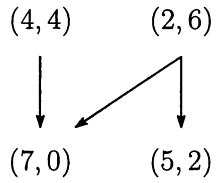


Thus, $\mathfrak{a}(I, J) = (X^3, Y^5, Z^4) \cap (X^6, Y^3, Z) \cap (X, Y^{10}, Z) \cap (X, Y^3, Z^8) \cap (X^4, Y, Z^5) \cap (X^2, Y^6, Z^2) = (X^6, Y^{10}, Z^8, XY^6, XZ^5, X^2Y^5, X^3Y^3, X^4Z, Y^3Z^4, Y^5Z^2, Y^6Z, XYZ^4, X^3YZ)$. The reduction number $r_J(I) = 4$.

Example 6.3. If $I = (J, F_1, F_2)$ and if we put $I_i = (J.F_i)$ $(i = 1, 2)$, then by 2,2, we have $\mathfrak{a}(I, J) \subset \mathfrak{a}(I_1, J) \cap \mathfrak{a}(I_2, J)$. The following example shows that the inclusion above is strict in general.

Let $R = k[X, Y], J = (X^{11}, Y^{11}), I_1 = (J, X^6Y^6), I_2 = (J, X^8Y^4)$ and $I = (J, X^6Y^6, X^8Y^4)$. Then $\mathfrak{a}(I_1, J) = (X^5, Y^5)$ and $\mathfrak{a}(I_2, J) = (X^6, X^3Y^3, Y^7)$. Hence

$\mathfrak{a}(I_1, J) \cap \mathfrak{a}(I_2, J) = (X^6, X^5Y^3, X^3Y^5, Y^7)$. On the other hand, the socle for $\mathfrak{a}(I, J)$ is computed as follows;



Therefore

$\mathfrak{a}(I, J) = (X^8, Y) \cap (X^6, Y^3) \cap (X^5, Y^5) \cap (X^2, Y^6) = (X^8, X^6Y, X^5Y^3, X^3Y^5, Y^7)$.

Hence $X^6 \in \mathfrak{a}(I_1, J) \cap \mathfrak{a}(I_2, J)$ but $X^6 \notin \mathfrak{a}(I, J)$.

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ON A CATEGORY OF COFINITE MODULES

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We assume that all rings are commutative and noetherian with identity throughout this report.

1. Introduction

Our aims in this report are to prove the following theorems:

Theorem 1. *Let $(A, \mathfrak{m})$ be a local ring, and I an ideal of A. We denote by $\mathcal{M}(A, I)_{cof}$ the collection of A-modules N satisfying the condition*

$(*)$ $\quad \mathrm{Supp}_A(N) \subseteq V(I)$ and $\mathrm{Ext}^j_A(A/I, N)$ is of finite type, for all j.

If I is an ideal of A of dimension one, then the collection $\mathcal{M}(A, J)_{cof}$ is a category, which is closed under kernels and cokernels, and so forms an abelian subcategory of all A-modules $\mathcal{M}(A)$.

Theorem 2. *Let $(R, \mathfrak{n})$ be a regular local ring, J an ideal of R of dimension one. Let $N^\bullet$ be in the derived category $\mathcal{D}^+(R)$. Then $N^\bullet$ is cofinite if and only if $H^i(N^\bullet)$ is in $\mathcal{M}(R, J)_{cof}$ for all i.*

These results are concerned with some questions given in [7].

Remark 1. Some results were known under the condition that $\dim A \leq 2$ (cf. [16, Theorem 7.4, p. 664]).

Let R be a regular ring, and J an ideal of R. The J-cofiniteness is defined as follows:

Definition 1. Let $N^\bullet$ be in the derived category $\mathcal{D}(R)$. We say $N^\bullet$ is **J-cofinite**, or for short **cofinite**, if there exists $M^\bullet \in \mathcal{D}_{ft}(R)$, such that $N^\bullet \simeq D_J(M^\bullet)$ in $\mathcal{D}(R)$. Here $D_J(\ -\ )$ is the J-dualizing functor which we shall introduce below.

2. Preliminaries

Let A be a ring. In this section, we recall terminologies on derived categories and derived functors. In this report, we follow the notations by that of [8]:

$\mathcal{M}(A)$	:	Category of A-modules and A-homomorphisms,
$C^*(A)$	:	Category of complexes of A-modules and homomorphisms between complexes,
$K^*(A)$	:	Category of complexes of A-modules and homomorphisms up to homotopies,
$\mathcal{D}^*(A)$	:	Derived category from $K^*(A)$ localized by quasi-isomorphisms,

where we write * in spite of $+$, $-$, b or $\emptyset$. Let A' be a thick abelian subcategory of $\mathcal{M}(A)$ (i.e. any extension in $\mathcal{M}(A)$ of two objects of A' is in A'). We define $K^*_{A'}(A)$ (and $\mathcal{D}^*_{A'}(A)$) to be the full subcategory of $K^*(A)$ (respectively $\mathcal{D}^*(A)$) consisting of these complexes $X^\bullet$ whose cohomology objects $H^i(X^\bullet)$ are all in A'. In this report, we denote $K^*_{ft}(A)$

(respectively $\mathcal{D}^*_{ft}(A)$) for $K^*_{A'}(A)$ (respectively $\mathcal{D}^*_{A'}(A)$) in the case that A' is the category of all A-modules of finite type.

Let R be a regular ring of finite Krull dimension d (complete with respect to the J-adic topology). Let $E^\bullet_J = \Gamma_J(E^\bullet)$, where $E^\bullet$ is an injective resolution of R and $\Gamma_J(\ -\ )$ is the J-power torsion subfunctor of the identity functor on $\mathcal{M}(R)$ (cf. [14, §1, p. 41]). Let $D_J(\ -\ )$ be the functor $\mathrm{Hom}^\bullet(\ -\ , E^\bullet_J)$ (or $\mathbb{R}\,\mathrm{Hom}^\bullet(\ -\ , \mathbb{R}\Gamma_J(R))$) on the derived category $\mathcal{D}(R)$. In this report, we call this functor $D_J(\ -\ )$ the **J-dualizing functor** (or the **dualizing functor** on J) according to [14, § 4.3, p. 70]. Further we often denote its cohomologies $H^i(D_J(\ -\ ))$ by $D^i_J(\ -\ )$ for some i, according to the notations in [7].

3. The Mayer-Vietoris spectral sequence

We shall propose that the Mayer-Vietoris spectral sequence for the dualizing functor.

Proposition 3. *Let R be a regular ring, let $I_1, I_2, \dots, I_n \subset R$ be ideals and let M be an R-module. There is a spectral sequence in the second quadrant:*

$$E_1^{-p,q} = \oplus_{1\le i_0<i_1<\cdots<i_p\le n} D^q_{I_{i_0}+I_{i_1}+\cdots+I_{i_p}}(M) \Longrightarrow D^{(-p)+q}_{I_1\cap I_2\cap\cdots\cap I_n}(M).$$

Proof. The proof is based on that given by G. Lyubeznik [15]. □

Remark 2. Applying the case of $n = 2$ to Proposition 3, we have so-called the 'Mayer-Vietoris exact sequence'.

Lemma 4. *Let A be a ring, I an ideal of A, $N^\bullet \in \mathcal{D}^+(A)$ a complex. And let $W^\bullet$ be in $\mathcal{D}^-(A/I)$ (we also see this complex over A via the natural map $A \longrightarrow A/I$). Then there is a spectral sequence between the hyperexts:*

$$E_2^{p,q} = \mathrm{Ext}^p(W^\bullet, \mathrm{Ext}^q(A/I, N^\bullet)) \Longrightarrow H^{p+q} = \mathrm{Ext}^{p+q}(W^\bullet, N^\bullet).$$

Remark 3. As a special case, we have well-known spectral sequences for ordinal extension modules.

We refine the lemma of [12, lemma 1] in terms of the derived category:

Lemma 5. *Let $N^\bullet \in \mathcal{D}^+(R)$ a complex, J an ideal of R. Then the following conditions are equivalent:*

(i) $\mathrm{Ext}^j(R/J, N^\bullet)$ *is of finite type over R for all $j \geq 0$;*
(ii) $\mathrm{Ext}^j(R/\sqrt{J}, N^\bullet)$ *is of finite type over R for all $j \geq 0$;*
(iii) $\mathrm{Ext}^j(R/P, N^\bullet)$ *is of finite type over R for all $j \geq 0$ and for all $P \in \mathrm{Min}(R/J)$;*
(iv) $\mathrm{Ext}^j(W, N^\bullet)$ *is of finite type over R for all $j \geq 0$ and for all finitely generated R-modules W such that* $\mathrm{Supp}\, W \subseteq V(J)$.
(v) $\mathrm{Ext}^j(W^\bullet, N^\bullet)$ *is of finite type over R for all $j \geq 0$ and for each $W^\bullet \in \mathcal{D}^b_{ft}(R)$ such that* $\mathrm{Supp}\, H^l(W^\bullet) \subseteq V(J)$ *for all $l \geq 0$.*
(vi) $\mathrm{Ext}^j(W^\bullet, N^\bullet)$ *is of finite type over R for all $j \geq 0$ and for each $W^\bullet \in \mathcal{D}^-_{ft}(R)$ such that* $\mathrm{Supp}\, H^l(W^\bullet) \subseteq V(J)$ *for all $l \geq 0$.*

Proof. The following implications are clear: (iv) ⇒ (i), (iv) ⇒ (ii), (iv) ⇒ (iii), (v) ⇒ (iv) and (vi) ⇒ (v).

The program for the proof is as follows: First we prove the implication (i) ⇒ (iv). Secondly we prove the implication (ii) ⇒ (i). Next we prove the implication (iii) ⇒ (ii). Further we shall prove the implication (iv) ⇒ (v). Finally we prove the implication (v) ⇒ (vi). □

Finally we introduce the following theorem (cf. [7, Theorem 5.1]):

Theorem 6. *Let $N^\bullet \in \mathcal{D}^+(R)$ be a complex, J an ideal of a regular ring R. And we suppose that R is J-adic complete. Then the following conditions are equivalent:*

(i) *$N^\bullet$ is J-cofinite;*

(ii) (a) $\mathrm{Supp}(H^i(N^\bullet)) \subseteq V(J)$ *for all $i \geq 0$, and*

(b) *The equivalent conditions in Lemma 5.*

Proof. See [7, Theorem 5.1] for the proof. □

Remark 4. Let J be an ideal of a regular ring R. We suppose that R is J-adic complete. If M is an R-module of finite type, then $\mathrm{Ext}^i(R/J, D_J(M))$ is of finite type over R.

Proof. Since M is an R-module of finite type, $D_J(M)$ is a cofinite complex. It follows from Theorem 6 that $\mathrm{Ext}^i(R/J, D_J(M))$ is of finite type for all i. □

4. Dualizing functors and associated prime ideals

In this section, we prove several lemmas which we need to prove Theorem 1. Throughout this section, we assume that $(R, \mathfrak{m})$ is a regular local ring and J is an ideal of R. Let $(E^\bullet, \{d^\bullet\})$ be a minimal injective resolution of R. If J is an ideal of dimension one, then $\Gamma_J(E^\bullet)$ is a complex consisting of injective modules in degree $d-1$ and d. We note that $\Gamma_J(d^{d-1})$ is surjective by the local Lichtenbaum-Hartshorne vanishing theorem. So the complex $\Gamma_J(E^\bullet)$ is an injective resolution of $\ker \Gamma_J(d^{d-1})$ (We note that $\ker \Gamma_J(d^{d-1})$ is just the local cohomology module $H_J^{d-1}(R)$).

Now we start to prove the following lemma.

Lemma 7. *let A be a ring, let $\mathfrak{p}$ be a prime ideal of R. Let M be an A-module of finite type. Then the following conditions are equivalent:*

(i) $M_\mathfrak{p} = 0$;

(ii) *no associated prime of M is contained in $\mathfrak{p}$;*

(iii) $\mathrm{Hom}_A(M, E(A/\mathfrak{p})) = 0$.

Proof. The proof is elementary and not so difficult. □

Lemma 8. *Let R be a regular local ring, J an ideal of R. Let M be an R-module of finite type. If the complex $D_J(M)$ is isomorphic to the complex consisting of a single module, then $H^i(D_J(M)) \in \mathcal{M}(R, J)_{cof}$ for all i (it is non-zero for only one i).*

The following lemmas are proved by using the affine duality theorem.

Lemma 9 (Compare Lemma 7.3 in [7]). *Let $(R, \mathfrak{m})$ be a regular local ring, J an ideal of R of dimension one. Let M be an R-module of finite type. Then $D_J^{d-1}(M) = 0$ if and only if no associated prime of M is contained in $\mathfrak{p}_1 \cup \mathfrak{p}_2 \cup \cdots \cup \mathfrak{p}_r$, where $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r$ is the minimal prime ideals containing J.*

Remark 5. The conditions in Lemma 9 are equivalent to $\mathrm{Hom}_R(M, \Gamma_J(I^{d-1})) = 0$.

Lemma 10 (Compare Lemma 7.4 in [7]). *Let $(R, \mathfrak{m})$ be a regular local ring, J an ideal of R of dimension one. Let M be a non-zero R-module of finite type. If every associated prime of M is contained in $\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_r$ then $D_J^d(M) = 0$. Further if $D_J^d(M) = 0$, then some associated prime of M is contained in $\mathfrak{p}_1 \cup \mathfrak{p}_2 \cup \cdots \cup \mathfrak{p}_r$, where $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r$ is the minimal prime ideals containing J (that is, some associated prime $\mathfrak{q}$ of M is contained in $\mathfrak{p}_i$ for some $i = i(\mathfrak{q})$).*

Lemma 11. *Let $(R,\mathfrak{m})$ be a regular local ring, J_1 an ideal of R and M an R-module of finite type. Suppose that P_1 a prime ideal of R of dimension one which does not contain J_1. If P_1 is not in* $\mathrm{Supp}(M)$, *then cohomology modules $H^i(\mathbb{R}\operatorname{Hom}^\bullet(R/P_1, D_{J_1}(M)))$ are of finite type over R for all i.*

5. Proof of the main theorems

Now we shall prove the main theorems.

Proof of Theorem 2.

We may assume that J is a radical ideal. We set $J_i = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \check{\mathfrak{p}}_i \cap \cdots \cap \mathfrak{p}_r$ for $1 \leq i \leq r$, where the check $\check{}$ means the omission. We note that $J = J_i \cap \mathfrak{p}_i$. Then we have the natural map between complexes: $D_{J_i}(M) \longrightarrow D_J(M)$. We shall be able to prove the theorem by the induction on r. To prove the theorem, we use several lemmas in [10], which we must refine. □

Corollary 12. *Let $(R,\mathfrak{n})$ be a regular local ring, and J an ideal of R. If J is an ideal of R of dimension one, then the set $\mathcal{M}(R,J)_{cof}$ is closed under kernels and cokernels, and so forms an abelian subcategory of the category $\mathcal{M}(R)$ of all R-modules.*

Proof. Let $f : N_1 \to N_2$ be in $\mathcal{M}(R,J)_{cof}$, that is, N_i $(i=1,2)$ are R-modules satisfying the condition $(*)$: $\mathrm{Supp}(N_i) \subseteq V(J)$ and $\mathrm{Ext}^j_R(R/I, N_i)$ is of finite type, for all j. Consider the R-homomorphism $f : N_1 \to N_2$ as the map between complexes, each of which consists of a single module: We denote this complex by $f^\bullet : N_1^\bullet \longrightarrow N_2^\bullet$, that is, $f^i : N_1^i \longrightarrow N_2^i$ is $f : N_1 \to N_2$ if $i=0$, zero otherwise. Then each of them is a cofinite complex by Theorem 2. Hence the third side of a triangle constructed by f, which is also a cofinite complex. The third side of a triangle constructed by f is the mapping cone $C_f^\bullet$. It follows from Theorem 2 again that the cohomology modules of the mapping cone by f are in $\mathcal{M}(R,J)_{cof}$. The differential $d_{C_f}^{i-1}$ is just f if $i=0$ and the zero map otherwise. From Theorem 2, the cohomology modules of $C_f^\bullet$ are in $\mathcal{M}(R,J)_{cof}$. These cohomology modules are just the kernel and cokernel of f. Therefore $\mathcal{M}(R,J)_{cof}$ is closed under kernels and cokernels, and so forms an abelian subcategory of the category $\mathcal{M}(R)$. □

Proof of Theorem 1

Now we prove Theorem 1. We may assume that A is a complete local ring $\hat{A}$, since the natural map $A \to \hat{A}$ is a faithfully flat. So we have a surjection from a complete regular local ring R to A. We suppose that I is an ideal of A of dimension one. Let J be the pre-image of I in R, so J is an ideal of R and also of dimension one. Further we assume that R is J-adic complete. We proceed by Delfino's argument (cf. [4]). □

Our theorems and corollaries shall produce the following corollaries.

Corollary 13. *Let $(A,\mathfrak{m})$ be a local ring, and I an ideal of A. Let M,N be A-modules of finite type. If I is an ideal of A of dimension one, then the generalized local cohomology module $H^j_I(M,N)$ is I-cofinite for all j.*

Proof. One can give the proof of the corollary according to the method in [2] (cf. [17]). □

Corollary 14. *Let $(R, \mathfrak{n})$ be a regular local ring, and J an ideal of R. Let M be an R-module of finite type. If J is of dimension one, then the cohomology modules $D^j_J(M)$ applying the J-dualizing functor to M is J-cofinite for all j.*

Proof. Apply $N = R$ to Corollary 13. The assertion follows from [7, Proposition 4.2]. □

The following theorem is investigated by several authors (cf. [10], [4], [5], and [18]) concerning the first question in [7] and the question in [6].

Theorem 15. *Let $(A, \mathfrak{m})$ be a local ring, and I an ideal of A. Let M be an A-module of finite type. If I is an ideal of A of dimension one, then the local cohomology module $H^j_I(M)$ is I-cofinite for all j.*

Proof. It is proved if we apply $M = A$ to Corollary 13. □

The following results on Bass numbers are quickly obtained from our Theorems and Corollaries by [10, Lemma 4·2].

Corollary 16. *Let $(A, \mathfrak{m})$ be a local ring, and I an ideal of A. Let M, N be A-modules of finite type. If I is an ideal of A of dimension one, then all the Bass numbers of the generalized local cohomology module $H^j_I(M, N)$ are finite for all j.*

Corollary 17. *Let $(R, \mathfrak{n})$ be a regular local ring, and J an ideal of R. Let M be an R-module of finite type. If J is of dimension one, then all the Bass numbers of the cohomology modules $D^j_J(M)$ applying the J-dualizing functor to M are finite for all j.*

Theorem 18. *Let $(A, \mathfrak{m})$ be a local ring, and I an ideal of A. Let M be an A-module of finite type. If I is an ideal of A of dimension one, then all the Bass numbers of the local cohomology module $H^j_I(M)$ are finite for all j.*

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The 30th Conference on Commutative Algebra, Japan, during November 18–21, 2008, at Niji-no-Matsubara Hotel (Address: 4 Higashi-Karatsu, Karatsu, Saga 847-0017, Japan).

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LYUBEZNIK RESOLUTION AND THE ARITHMETICAL RANK OF MONOMIAL IDEALS

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1. Introduction

Let S be a polynomial ring over a field K, and let I be a monomial ideal of S. We write $G(I) = \{m_1, m_2, \dots, m_\mu\}$ as the minimal set of monomial generators of I and $\mu(I) = \mu$ as the cardinality of $G(I)$. It is a difficult problem to construct a minimal graded free resolution of S/I. But Taylor [13] constructed an explicit graded free resolution $T_\bullet$ of S/I, which is called the *Taylor resolution* of I:

$$T_\bullet : 0 \longrightarrow T_\mu \xrightarrow{d_\mu} T_{\mu-1} \xrightarrow{d_{\mu-1}} \cdots \xrightarrow{d_1} T_0 \longrightarrow S/I \longrightarrow 0,$$

where free basis of T_s are

$$e_{i_1 i_2 \cdots i_s}, \quad 1 \leq i_1 < i_2 < \cdots < i_s \leq \mu$$

with the degree

$$\deg e_{i_1 i_2 \cdots i_s} = \deg \operatorname{lcm}(m_{i_1}, m_{i_2}, \dots, m_{i_s}),$$

and the differential d_s is given by

$$d_s(e_{i_1 i_2 \cdots i_s}) = \sum_{j=1}^{s} (-1)^{j-1} \frac{\operatorname{lcm}(m_{i_1}, \dots, m_{i_s})}{\operatorname{lcm}(m_{i_1}, \dots, \widehat{m_{i_j}}, \dots, m_{i_s})} e_{i_1 \cdots \widehat{i_j} \cdots i_s}.$$

In 1988, Lyubeznik [10] also found an explicit graded free resolution of S/I, which is called a *Lyubeznik resolution* of I. Actually, a Lyubeznik resolution of I is a subcomplex of the Taylor resolution of I generated by all L-admissible symbols. Here, a symbol $e_{i_1 i_2 \cdots i_s}$ is said to be L-admissible if and only if m_q does not divide $\operatorname{lcm}(m_{i_t}, m_{i_{t+1}}, \dots, m_{i_s})$ for all $t < s$ and for all $q < i_t$. In general, the length of a Lyubeznik resolution of I is rather shorter than that of the Taylor resolution of I, and often coincides with the projective dimension of S/I. Note that the Taylor resolution of I is determined uniquely by the monomial ideal I, but a Lyubeznik resolution of I, even the length of it, depends on a order of elements of $G(I)$.

On the other hand, the *arithmetical rank* of I is defined by

$$\operatorname{ara} I := \min \left\{ r \ : \ \text{there exist } a_1, \dots, a_r \in I \text{ such that } \sqrt{(a_1, \dots, a_r)} = \sqrt{I} \right\}.$$

A trivial upper bound for the arithmetical rank of I is $\mu(I)$, which is equal to the length of the Taylor resolution of I. The main theorem in this report is the following one:

Theorem 1. *Let I be a monomial ideal of S. If the length of a Lyubeznik resolution of I is λ, then*

$$\operatorname{ara} I \leq \lambda.$$

Moreover, we assume that I is squarefree. In this case, Lyubeznik [9] proved that $\operatorname{ara} I \geq \operatorname{pd}_S S/I$, where $\operatorname{pd}_S S/I$ denotes the projective dimension of S/I. The author expects that $\operatorname{ara} I = \operatorname{pd}_S S/I$ holds for almost all squarefree monomial ideals, and proved that equality for some classes of squarefree monomial ideals with Terai and Yoshida ([7, 8]). For example, we proved $\operatorname{ara} I = \operatorname{pd}_S S/I$ when $\mu(I) - \operatorname{height} I = 2$ ([8]). Barile [1, 3, 4], Barile–Terai [5], Morales [11] also proved that equality for some classes of squarefree monomial ideals. From our theorem, we have the following corollary:

Corollary 2. *Let I be a squarefree monomial ideal of S. If the length of the Lyubeznik resolution of I with respect to some order of monomial generators is equal to the projective dimension of S/I, then*

$$\operatorname{ara} I = \operatorname{pd}_S S/I.$$

In particular, if a Lyubeznik resolution of I is minimal, then the same assertion is true.

Barile [1, 2, 3] provided some classes of squarefree monomial ideals whose Lyubeznik resolution can be minimal. In particular, in [2, Remark 1], she pointed out a necessary and sufficient condition for a Lyubeznik resolution to be minimal. Novik [12] proved that a Lyubeznik resolution is minimal for the matroid ideal of a finite projective space.

Theorem 1 is proved by taking λ elements which generate I up to radical. In this report, we provide those λ elements and explain key points of this taking. But we omit the detailed proof of Theorem 1, which can be obtained in [6]. In Section 3, we give some examples of squarefree monomial ideals which satisfy the assumption of the Corollary 2.

2. How to find λ elements

Let λ be the length of the Lyubeznik resolution of I with respect to some order of monomial generators $m_1, m_2, \ldots, m_\mu$ of I. For simplicity, we set

$$L_s := \left\{ [i_1, i_2, \ldots, i_s] \in \mathbb{N}^s \; : \; \begin{array}{l} 1 \leq i_1 < i_2 < \cdots < i_s \leq \mu(I) \\ e_{i_1 i_2 \cdots i_s} \text{ is } L\text{-admissible} \end{array} \right\}.$$

To prove Theorem 1, we show that $\sqrt{(a_1, \ldots, a_\lambda)} = \sqrt{I}$ holds for

$$\begin{cases} a_1 = m_1, \\ a_2 = m_2 + \displaystyle\sum_{\substack{[i_1,i_2,\ldots,i_{\lambda-1}]\in L_{\lambda-1} \\ i_1 \geq 3}} m_{i_1} m_{i_2} \cdots m_{i_{\lambda-1}}, \\ \vdots \\ a_\ell = m_\ell + \displaystyle\sum_{\substack{[i_1,i_2,\ldots,i_{\lambda-\ell+1}]\in L_{\lambda-\ell+1} \\ i_1 \geq \ell+1}} m_{i_1} m_{i_2} \cdots m_{i_{\lambda-\ell+1}}, \\ \vdots \\ a_\lambda = m_\lambda + \displaystyle\sum_{\substack{[i_1]\in L_1 \\ i_1 \geq \lambda+1}} m_{i_1} = m_\lambda + m_{\lambda+1} + \cdots + m_\mu. \end{cases}$$

To explain the idea of this taking, we consider the following ideal:

$$I = (x_1x_2x_3, x_1x_4x_5, x_2x_4x_6x_7, x_3x_8, x_1x_2x_5x_6x_9).$$

We label generators of I with this order as $m_1, m_2, \ldots, m_5$. First, we see L-admissible symbols. For example, let us consider e_{45} and e_{134}. Since m_4m_5 is divisible by m_1, we have that e_{45} is not L-admissible. On the other hand, e_{134} is L-admissible. To see this, we must check the following 3 conditions:

(a) m_q does not divide m_4 for all $q < 4$;
(b) m_q does not divide $\operatorname{lcm}(m_3, m_4)$ for all $q < 3$;
(c) m_q does not divide $\operatorname{lcm}(m_1, m_3, m_4)$ for all $q < 1$.

The condition (a) is trivial because $m_1, m_2, \ldots, m_5$ is a minimal system of monomial generators of I. Also, $\operatorname{lcm}(m_3, m_4) = x_2x_3x_4x_6x_7x_8$ is not divisible by $m_1 = x_1x_2x_3$ and $m_2 = x_1x_4x_5$. Thus the condition (b) is satisfied. Note that the condition (c) says nothing because there are no integers q with $(1\leq)q < 1$. These observations yield the following lemma:

Lemma 3. *Suppose* $[i_1, i_2, \ldots, i_s] \in L_s$.

(1) $[i_{j_1}, \ldots, i_{j_t}] \in L_t$ *for all* $1 \leq j_1 < \cdots < j_t \leq s$.
(2) *If* $i_1 > 1$, *then* $[1, i_1, i_2, \ldots, i_s] \in L_{s+1}$. *In particular, if* $[i_1, i_2, \ldots, i_\lambda] \in L_\lambda$, *then* $i_1 = 1$.
(3) *Suppose* $\ell < i_1$. *If* $[\ell, i_1, i_2, \ldots, i_s] \notin L_{s+1}$, *then* $m_\ell m_{i_1} m_{i_2} \cdots m_{i_s}$ *is divisible by at least one of* $m_1, m_2, \ldots, m_{\ell-1}$.

This lemma is easy to see from the definition of the L-admissibleness, but it plays a key role in the proof of Theorem 1.

Now, we return to the above ideal I. Sets L_s are given as follow:

$$L_1 = \{[1], [2], [3], [4], [5]\},$$
$$L_2 = \{[1,2], [1,3], [1,4], [1,5], [2,3], [2,4], [2,5], [3,4]\},$$
$$L_3 = \{[1,2,3], [1,2,4], [1,2,5], [1,3,4]\}.$$

Note that all elements in L_3 contain 1 by Lemma 3 (2). Thus we take $a_1 = m_1$. Next in L_2, we ignore elements which contain 1, and the rest elements are ones which contain 2, and [3, 4]. Thus we take $a_2 = m_2 + m_3 m_4$. Finally in L_1, we ignore [1] and [2], and take $a_3 = m_3 + m_4 + m_5$.

Next, we verify that a_1, a_2, a_3 generate I up to radical. Set $J = (a_1, a_2, a_3)$. We only need to show that $m_1, m_2, \ldots, m_5 \in \sqrt{J}$. Since $a_1 = m_1$, we have $m_1 \in J$. To see $m_2 \in \sqrt{J}$, we consider $m_2 a_2$. Then $m_2 a_2 \in \sqrt{J}$ yields that

$$m_2^2 + m_2 m_3 m_4 \in \sqrt{J}.$$

Since $[3,4] \in L_2$ and $[2,3,4] \notin L_3$, we have that $m_2 m_3 m_4$ is divisible by m_1 by Lemma 3 (2), (3). Thus we have $m_2^2 \in J$ and $m_2 \in \sqrt{J}$. Then we also have $m_3 m_4 = a_2 - m_2 \in \sqrt{J}$. Similarly, $m_3 a_3 \in J$ yields that

$$m_3^2 + m_3 m_4 + m_3 m_5 \in \sqrt{J}.$$

Because $[3,5] \notin L_2$, we have that $m_3 m_5$ is divisible by m_1 or m_2 by Lemma 3 (3). Thus $m_3 m_5 \in \sqrt{J}$. Therefore $m_3 \in \sqrt{J}$. Then we have $a_3' := m_4 + m_5 = a_3 - m_3 \in \sqrt{J}$, and $m_4 a_3' \in \sqrt{J}$ yields

$$m_4^2 + m_4 m_5 \in \sqrt{J}.$$

By a similar argument as above, we have $m_4 \in \sqrt{J}$, and $m_5 = a_3' - m_4 \in \sqrt{J}$. Hence, $\sqrt{J} = \sqrt{I}$ holds.

Actually, the Lyubeznik resolution of I with respect to the above order is minimal and we have $\operatorname{pd}_S S/I = 3 = \operatorname{ara} I$. But if we change the order of generators, it is not necessarily minimal. For example, we change the order of generators of I as follow:

$$m_1 = x_1x_4x_5,\ m_2 = x_1x_2x_3,\ m_3 = x_2x_4x_6x_7,\ m_4 = x_3x_8,\ m_5 = x_1x_2x_5x_6x_9,$$

Then $[1,2,3,4] \in L_4$, and the Lyubeznik resolution of I with respect to this order is not minimal. More precisely, its length is bigger than the projective dimension of S/I.

3. Examples

First, we give squarefree monomial ideals I which satisfy the assumption of Corollary 2. That is, the length of a Lyubeznik resolution of I is equal to the projective dimension of S/I.

Example 4. Let I be a squarefree monomial ideal with $\mu(I) - \operatorname{pd}_S S/I \leq 1$. Then the length of the Lyubeznik resolution of I with respect to some order of monomial generators is equal to the projective dimension of S/I. Let us check it.

When $\mu(I) - \operatorname{pd}_S S/I = 0$, the assertion is clear because the Taylor resolution of I is minimal.

Next, we consider the case $\mu(I) - \operatorname{pd}_S S/I = 1$. Set $\mu = \mu(I)$. In this case, we can assume that m_1 divides $m_2 \cdots m_\mu$ because the Taylor resolution of I is not minimal (see also [7, Lemma 2.2]). Then $[1, 2, \ldots, \mu] \notin L_\mu$ and $L_\mu = \emptyset$.

Therefore, the length of the Lyubeznik resolution of I with respect to this order is at most $\mu(I) - 1 = \mathrm{pd}_S S/I$.

Moreover if $\mu(I) - \mathrm{height}\, I \leq 1$, then a Lyubeznik resolution of I can be minimal with respect to the order of monomial generators mentioned in above. It can be checked by the classification of these ideals; see [7, Theorem 4.4].

But when $\mu(I) - \mathrm{height}\, I = 2$, there exists an ideal I such that all of Lyubeznik resolutions of I are not minimal.

Example 5. Let us consider the following 2 ideals I_1, I_2:

$$I_1 = (x_1x_2x_3, x_1x_2x_4x_5x_6x_8, x_1x_3x_4x_5x_7x_9, x_2x_3x_4x_6x_7x_{10}),$$
$$I_2 = (x_1x_2x_3x_5, x_1x_2x_4x_6, x_1x_3x_4x_6x_7x_8, x_2x_3x_4x_5x_7x_9).$$

Both of these ideals I satisfy $\mu(I) = 4$, $\mathrm{pd}_S S/I = \mathrm{height}\, I = 2$, and $\mu(I) - \mathrm{height}\, I = 2$. The Lyubeznik resolution of I_1 with respect to this order of generators is minimal. In fact,

$$L_2 = \{[1,2], [1,3], [1,4]\}$$

and $L_3 = \emptyset$. On the other hand, the Lyubeznik resolution of I_2 with respect to this order of generators is not minimal. Since $[1,2,3] \in L_3$, even the length of it is bigger than the projective dimension of S/I. In fact, it is also true for arbitrary orders of generators of I_2.

Lastly, we give a squarefree monomial ideal whose Lyubeznik resolution has the quite short length.

Example 6. Let I be the squarefree monomial ideal generated by the following 10 elements:

$$x_1x_2x_3, x_1x_2x_5, x_1x_3x_6, x_1x_4x_5, x_1x_4x_6, x_2x_3x_4, x_2x_4x_6, x_2x_5x_6, x_3x_4x_5, x_3x_5x_6.$$

Then I is the Stanley–Reisner ideal of the Reisner's triangulation of the projective plane. The projective dimension of S/I is given by

$$\mathrm{pd}_S S/I = \begin{cases} 3 & \text{when } \mathrm{char}\, K \neq 2, \\ 4 & \text{when } \mathrm{char}\, K = 2. \end{cases}$$

The length of the Taylor resolution of I is equal to $\mu(I) = 10$, and this is quite bigger than the projective dimension of S/I. On the other hand, the length of the Lyubeznik resolution of I with respect to this order of generators is equal to 4. This number is very close to the projective dimension of S/I. In fact, it is equal to the arithmetical rank of I (see Yan [14]).

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Ideals generated by some 2-minors

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For a natural number n, let S be a polynomial ring $k[X_1, ..., X_n, Y_1, ..., Y_n]$ over a field k with $2n$ variables, $[n]$ the set of natural numbers $\{1, 2, ..., n\}$. For $i, j \in [n]$, $[i, j]$ denotes the 2-minor

$$\det \begin{pmatrix} X_i & X_j \\ Y_i & Y_j \end{pmatrix} = X_i Y_j - Y_i X_j.$$

In this article, assume that all graphs are simple, i.e., they do not have multiple edges or loops. For a graph D whose vertex set is $[n]$, we define an ideal I_D of S as follows :

$$I_D := \big([i, j] \mid D \text{ has an edge which connects } i \text{ and } j\big).$$

Problem 1. What are the ideal I_D and the ring S/I_D?

First, we compute a Gröbner basis of I_D as the set of "minimal irreducible paths" of D.

1 Gröbner basis

Recall that the definition of a path of a graph.

Definition 2. A *walk* P of a graph D is $v_0 e_1 v_1 e_2 ... e_m v_m$ such that (1) each v_i is a vertex of D, that (2) e_i is an edge of D which connects v_{i-1} and v_i and that (3) $v_0 \leq v_m$.

In addition, P is called a *path* if $v_i \neq v_j$ for each $i \neq j$.

Remark 3. This definition is unusual. We usually $P = v_0 e_1 v_1 e_2 ... e_m v_m$ a walk if conditions (1) and (2) hold. So 1-2-3 is a path but 3-2-1 is not.

For a path $P = v_0 e_1 v_1 e_2 ... e_m v_m$, we call an element of the set $J(P) := \{v_1, v_2, ..., v_{m-1}\}$ a *joint* of P, and m is called the *length* of P.

Now we assume that D is simple, so an edge is determined by vertices which are connected by it. So we express a path $P = v_0 e_1 v_1 e_2 ... e_m v_m$ as $P = v_0 v_1 ... v_m$ simply.

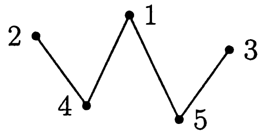
We define an order on the set of walks of D.

Definition 4. For two walks $P = v_0v_1...v_m$ and $Q = u_0u_1...u_\ell$ of D, we say that $P \leq Q$ if $v_0 = u_0$, $v_m = u_\ell$ and $J(P) \subseteq J(Q)$ hold.

A walk $P = v_0v_1...v_m$ is minimal if and only if P is a path and there never exists an edge which connects v_i and v_j where i and j do not adjoin.

Definition 5. We say that a minimal path $P = v_0v_1...v_m$ is *irreducible* if there never exists a joint v_i such that $v_0 < v_i < v_m$.

Example 6. Let D be the graph



This graph D has 10 paths and all paths are minimal. A path 2-4-1-5 is not irreducible because there is the joint 4 which is more than 2 and is less than 5.

For a minimal irreducible path $P = v_0v_1...v_m$, we define a binomial g_P by $g_P = M_P \cdot [v_0, v_m]$, where M_P is a monomial

$$M_P := \prod_{i=1}^{m-1} Z_{p_i}, \qquad Z_{p_i} := \begin{cases} Y_{p_i}, & \text{if } p_i < p_0 \\ X_{p_i}, & \text{if } p_i > p_m \end{cases}.$$

Theorem 7. *$\mathcal{G} := \{g_P \mid P$ is a minimal irreducible path of $D\}$ is the reduced Gröbner basis of I_D with respect to the reverse-lexicographic order which is defined by $Y_1 > Y_2 > \cdots > Y_n > X_1 > X_2 > \cdots > X_n$.*

Example 8. Let D be the graph in Example 6. The ideal I_D is generated by 4 binomials $[1,4]$, $[1,5]$, $[2,4]$ and $[3,5]$. By the theorem, the following $\mathcal{G}$ is the reduced Gröbner basis :

$$\mathcal{G} = \left\{ \begin{array}{l} [1,4],\ [1,5],\ [2,4],\ [3,5] \\ X_4[1,2],\ X_5[1,3],\ Y_1[4,5],\ Y_1X_5[3,4],\ Y_1X_4X_5[2,3] \end{array} \right\}.$$

$P =$ 2-4-1-5 is not irreducible and the corresponding binomial is unnecessary. In fact, the equation $Y_1X_4[2,5] = X_4 \cdot Y_1[4,5] + Y_1X_5 \cdot [2,4]$ holds.

By the theorem, we can prove that I_D is a radical ideal because of the following easy lemma.

Lemma 9. *Let I be an ideal of a polynomial ring S over a field. Assume that there is a monomial order $<$ such that the initial ideal $\mathrm{in}_< I_D$ of I_D with respect to it is generated by square-free monomials. Then I is a radical ideal.*

Corollary 10. *The ideal I_D is a radical ideal.*

So I_D is expressed as the intersection of some prime ideals of S. In the next section, we construct the prime decomposition of I_D.

2 Prime decomposition

First, we determine whether I_D is prime. For a vertex v of D, $N(v)$ denotes the neighborhood of v

$$\{x \mid x \text{ and } v \text{ are connected by an edge of } D\}.$$

Definition 11. For a vertex v of D, D is *complete around* v if there is an edge of D which connects x and y for any elements $x \neq y$ of $N(v)$.

Proposition 12. *For a graph D, the following conditions are equivalent :*
(1) D is complete around all vertices of D,
(2) D is the direct sum of complete graphs,
(3) I_D is a prime ideal.

From now on, assume that I_D is not prime. By the proposition, D has a vertex v around which D is not complete. For this v, we make two ideals as follows :

$$(1)\ \ I_D + (X_v, Y_v), \qquad\qquad (2)\ \ I_D + ([x,y] \mid x, y \in N(v)).$$

These ideals respectively correspond to graphs which made by following operations :

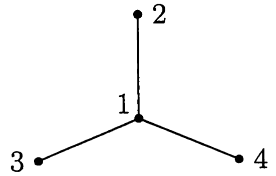
(1) Clearing the vertex v and all edges such that v is its end,
(2) Adding all edges which connect two vertices in $N(v)$. (♣)

In both operation, the number of vertices around which the graph is not complete decrease.

Proposition 13. $I_D = \big(I_D + (X_v, Y_v)\big) \cap \big(I_D + ([x,y] \mid x, y \in N(v))\big)$

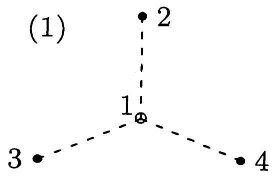
Applying operations (♣) until there are vertices around which the graph is not complete, we can express I_D as the intersection of the ideals which correspond to direct sums of complete graphs. So we can get the prime decomposition of I_D.

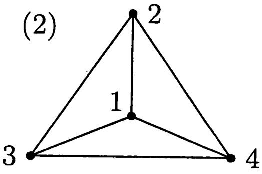
Example 14. Let D be the following graph :



D is not complete around the vertex 1 since $N(1) = \{2, 3, 4\}$ holds.

Apllying the operations (♣) to 1, we get the following graphs :





$$I_D + (X_1, Y_1) = (X_1, Y_1), \qquad I_D + ([2,3],\ [3,4],\ [2,4]) = I_2(X)$$

where X is the matrix $\begin{pmatrix} X_1 & X_2 & X_3 & X_4 \\ Y_1 & Y_2 & Y_3 & Y_4 \end{pmatrix}$ and $I_2(X)$ denotes the ideal generated by all 2-minors of X. These ideals are prime, so we get the prime decomposition $I_D = (X_1, Y_1) \cap I_2(X)$.

STRUCTURE THEOREMS OF PROJECTED VARIETIES ACCORDING TO MOVING THE CENTER

SIJONG KWAK

1. Introduction

Let V be an $(n+1)$-dimensional vector space over an algebraically closed field k and let X be a non-degenerate reduced closed subscheme in a projective space $\mathbb{P}^n = \mathbb{P}(V)$ and $R = k[x_0, \ldots, x_n]$ be the coordinate ring of $\mathbb{P}(V)$. We are mainly interested in the geometric properties of X and its projections carried by the minimal free resolution of the saturated ideal I_X. In general, there is a basic exact sequence of graded R-modules associated to the embedding $X \hookrightarrow \mathbb{P}^n$ as follows:

$$(1.1) \qquad 0 \longrightarrow R/I_X \longrightarrow E \longrightarrow H^1_*(\mathcal{I}_{X/\mathbb{P}^n}) \longrightarrow 0$$

where $E = \bigoplus_{\ell \geq 0} H^0(X, \mathcal{O}_X(\ell))$ the graded module of twisted sections of $\mathcal{O}_X$ and $H^1_*(\mathcal{I}_{X/\mathbb{P}^n}) = \bigoplus_{\ell \geq 0} H^1(\mathbb{P}^n, \mathcal{I}_{X/\mathbb{P}^n}(\ell))$ is the Hartshorne-Rao module. For a projectively normal embedding of X, $R/I_X = E$ and $H^1_*(\mathcal{I}_{X/\mathbb{P}^n}) = 0$. There are many classical conjectures and known results about simplest linear syzygies (i.e. property N_p) for highly positive embeddings of projective varieties ([5], [12]).

First of all, consider the minimal free resolution of R/I_X as follows;

$$(1.2) \qquad \cdots \to L_i \to L_{i-1} \to \cdots \to L_1 \to R \to R/I_X \to 0 \text{ (as R-modules)}$$

where $L_i = \bigoplus_j R(-i-j)^{\oplus \beta_{i,j}}$. Then, one can define that X(or R/I_X) satisfies property $N_{d,p}$ (cf. [5]) if one of the following conditions holds:

(a) $\beta_{i,j} = 0$ for $1 \leq i \leq p$ and all $j \geq d$ in the minimal free resolution (1.2);
(b) the truncation $(I_X)_{\geq d}$ of I_X in degrees $\geq d$ is generated in degree d and the minimal free resolution of $(I_X)_{\geq d}$ is linear until p-th step, namely,

$$\cdots \to R(-d-p+1)^{\oplus \beta_{p,d-1}} \to \cdots \to R(-d)^{\oplus \beta_{1,d-1}} \to R \to R/(I_X)_{\geq d} \to 0.$$

The case of $d = 2$ has been of particular interest. For $d = 2, p = 1$, I_X is generated by quadrics and $N_{2,2}$ means that I_X is generated by quadrics and there are only linear relations on quadrics. Note that property $N_{2,p}$ is the same as property N_p (defined by Green-Lazarsfeld) if the given variety is projectively normal.

One more important projective invariant of a projective scheme X is the Castelnuovo-Mumford regularity which measures the maximal degree of all syzygy modules in the whole minimal free resolution of R/I_X. A closed subscheme $X \subset \mathbb{P}^n$ is said to be m-regular if one of the following conditions holds (see [4]):

(a) $\beta_{i,j} = 0$ for all $j \geq m$, $i \geq 0$ in the minimal free resolution (1.2), that is, the i-th syzygy module L_i is generated by elements of degree $\leq i + m - 1$;

(b) $H^i(\mathbb{P}^n, \mathcal{I}_{X/\mathbb{P}^n}(m-i)) = 0$ for every $i \geq 1$. More precisely,
- $i = 1$, $H^i(\mathbb{P}^n, \mathcal{I}_{X/\mathbb{P}^n}(m-1)) = 0$ (Castelnuovo-normality)
- $i \geq 1$, $H^i(\mathcal{O}_X(m-1-i)) = 0$ (vanishing not depending on the embedding).

We define $\operatorname{reg}(X) := \min\{m \mid X \subset \mathbb{P}^n \text{ is } m\text{-regular}\}$. For an integral projective scheme X in $\mathbb{P}^n$, it has been a long open problem to show that

$$\operatorname{reg}(X) \leq \deg(X) - \operatorname{codim}(X) + 1 \quad (\text{Eisenbud-Goto conjecture}).$$

Remark that the condition $N_{2,p}$, $N_{d,p}$ and $\operatorname{reg}(X)$ have their own algebraic and geometric properties respectively. For history and summary, also see [5],[11].

On the other hand, the simplest type of the minimal free resolution of $E = \bigoplus_{\ell \geq 0} H^0(X, \mathcal{O}_X(\ell))$ until p-th step should be of the form:

$$\to R(-p-1)^{\oplus \beta_{p,1}} \to \cdots \to R(-3)^{\oplus \beta_{2,1}} \to R(-2)^{\oplus \beta_{1,1}} \to R \oplus R(-1)^{\oplus c} \to E \to 0$$

where $c = \operatorname{codim}(V, H^0(\mathcal{O}_X(1)))$. In this case, we can say that E satisfies property N_p^S (cf. [12]). The property N_p^S of E gives us interesting geometric information on X by using cohomological methods as follows:

(a) Let $X \subset \mathbb{P}(V)$ be a reduced, non-degenerate projective variety and $c = \operatorname{codim}(V, H^0(\mathcal{O}_X(1)))$. If E satisfies property N_1^S as a R-module, i.e.

$$\cdots \to R(-2)^{\oplus \beta_{1,1}} \to R \oplus R(-1)^{\oplus c} \to E \to 0,$$

then X is k-normal for all $k \geq c+1$ and cut out by hypersurfaces of degree $\leq c+2$. (Theorem 1.1 in [12]).

(b) Furthermore, if E satisfies property N_p^S for some $p \geq 1$ and X is k-normal for all $k \geq k_0$ for some $k_0 \leq c+1$. Then I_X is generated by forms of degrees $\leq k_0 + 1$ and satisfies property $N_{k_0+1,p}$ (Theorem 3 in [2]).

In this note, we are more naturally interested in the simplest syzygies of R/I_X and geometric properties of both X and its projections by using the mapping cone construction and the Koszul cohomology method.

Acknowledgements I would like to express my special thanks to Professors C. Miyazaki and N. Terai for their invitation for me to be able to attend "The 30-th conference on Commutative Algebra, Japan" held in Fukuoka (2008.11.18-21) and for their hospitalities during my stay there.

2. Graded mapping cone construction under projections

The mapping cone under projection and its related long exact sequence is our starting point to understand algebraic and geometric structures of projections.

- $W = \bigoplus_{i=1}^n k \cdot x_i \subset V = \bigoplus_{i=0}^n k \cdot x_i$: vector spaces over k.
- $S_1 = k[x_1, \ldots, x_n] \subset R = k[x_0, \ldots, x_n]$: polynomial rings.
- M : a graded R-module (which is also a graded S_1-module).
- $K_\bullet^{S_1}(M)$: the graded Koszul complex of M as follows:

$$0 \to \wedge^n W \otimes M \to \cdots \to \wedge^2 W \otimes M \to W \otimes M \to M \to 0$$

whose graded components are $K_i^{S_1}(M)_{i+j} = \wedge^i W \otimes M_j$.

Consider the multiplicative map $\varphi : M(-1) \xrightarrow{\times x_0} M$ as a graded S_1-module homomorphism such that $\varphi(m) = x_0 \cdot m$. Then we have the induced map

$$\overline{\varphi} : \mathbb{F}_\bullet = K_\bullet^{S_1}(M(-1)) \xrightarrow{\times x_0} \mathbb{G}_\bullet = K_\bullet^{S_1}(M)$$

between graded complexes. In each degree, we have

$$K_i^{S_1}(M(-1))_{i+j} = \wedge^i W \otimes M_{j-1} \xrightarrow{\overline{\varphi}} \wedge^i W \otimes M_j = K_i^{S_1}(M)_{i+j}$$

given by $\overline{\varphi}(e_I \otimes m) = e_I \otimes x_0 m$ where $e_I = x_{s_1} \wedge \cdots \wedge x_{s_i}$ for $1 \leq s_1, \ldots, s_i \leq n$, and $m \in M_{j-1}$. Then $\overline{\varphi} = \times x_0$ also induces the map on homology as follows:

$$\begin{array}{ccc} \operatorname{Tor}_i^{S_1}(M,k)_{i+j-1} & \xrightarrow{\overline{\varphi}} & \operatorname{Tor}_i^{S_1}(M,k)_{i+j} \\ \| & & \| \\ H_i[K_\bullet^{S_1}(M(-1))]_{i+j} & \longrightarrow & H_i[K_\bullet^{S_1}(M)]_{i+j}. \end{array}$$

Now, we construct the mapping cone $(C_\bullet(\overline{\varphi}), \partial_{\overline{\varphi}})$ induced by $\overline{\varphi}$ such that $C_\bullet(\overline{\varphi}) = \mathbb{G}_\bullet \bigoplus \mathbb{F}_\bullet[-1]$ and

- $C_i(\overline{\varphi})_{i+j} = [\mathbb{G}_i]_{i+j} \bigoplus [\mathbb{F}_{i-1}]_{i+j} = \wedge^i W \otimes M_j \oplus \wedge^{i-1} W \otimes M_j$.
- the differential $\partial_{\overline{\varphi}} : C_i(\overline{\varphi}) \to C_{i-1}(\overline{\varphi})$ is given by

$$\partial_{\overline{\varphi}} = \begin{pmatrix} \partial & \overline{\varphi} \\ 0 & -\partial \end{pmatrix},$$

where ∂ is the differential of Koszul complex $K_\bullet^{S_1}(M)$ and $\partial_{\overline{\varphi}} \circ \partial_{\overline{\varphi}} = 0$.

Finally, the mapping cone $(C_\bullet(\overline{\varphi}), \partial_{\overline{\varphi}})$ becomes a complex over S_1 and we have the exact sequence of complexes

$$0 \longrightarrow \mathbb{G}_\bullet \longrightarrow C_\bullet(\overline{\varphi}) \longrightarrow \mathbb{F}_\bullet[-1] \longrightarrow 0. \tag{2.1}$$

From the exact sequence (2.1), we have a long exact sequence in homology:

$$\begin{array}{llll} \to \operatorname{Tor}_i^{S_1}(M,k)_{i+j} & \to H_i(C_\bullet(\overline{\varphi}))_{i+j} & \to & \\ & \operatorname{Tor}_{i-1}^{S_1}(M,k)_{i+j-1} & \xrightarrow{\delta} & \operatorname{Tor}_{i-1}^{S_1}(M,k)_{i+j} \to \end{array} \tag{2.2}$$

and the connecting homomorphism δ is the multiplicative map induced by $\overline{\varphi}$.

In the following Lemma 2.1, we claim that $\operatorname{Tor}^R(M,k)$ can be obtained by the homology of the mapping cone.

Lemma 2.1. *Let M be a graded R-module. Then we have the following natural isomorphism:*

$$\operatorname{Tor}_i^R(M,k)_{i+j} \simeq H_i(C_\bullet(\overline{\varphi}))_{i+j}.$$

From the long exact sequence (2.2) and Lemma 2.1, we obtain the following useful Theorem.

Theorem 2.2. *Let $S_1 = k[x_1, \ldots, x_n] \subset R = k[x_0, x_1 \ldots, x_n]$ be polynomial rings. For a graded R-module M, we have the following long exact sequence:*

$$\longrightarrow \operatorname{Tor}_i^{S_1}(M,k)_{i+j} \longrightarrow \operatorname{Tor}_i^R(M,k)_{i+j} \longrightarrow \operatorname{Tor}_{i-1}^{S_1}(M,k)_{i+j-1} \longrightarrow$$

$$\xrightarrow{\delta} \operatorname{Tor}_{i-1}^{S_1}(M,k)_{i+j} \longrightarrow \operatorname{Tor}_{i-1}^R(M,k)_{i+j} \longrightarrow \operatorname{Tor}_{i-2}^{S_1}(M,k)_{i+j-1} \xrightarrow{\delta} \cdots$$

whose connecting homomorphism δ is the multiplicative map $\times x_0$.

Proof. It is clear from (2.2) and Lemma 2.1. □

Note that Theorem 2.2 gives us an useful information about syzygies of outer projections (i.e. isomorphic or birational projections) of projective varieties.
As a first step, we obtain the following interesting Corollary.

Corollary 2.3. *Let $I \subset R$ be a homogeneous ideal such that R/I is a finitely generated S_1-module. Assume that I admits d-linear resolution up to p-th step for $p \geq 2$. Then, for $1 \leq i \leq p-1$,*

(a) *the minimal free resolution of R/I as a graded S_1-module is given as follows:*
$\to L_{p-1} \to \cdots \to S_1(-d)^{\oplus \beta^{S_1}_{1,d-1}} \to \bigoplus_{i=1}^{d} S_1(-d+i) \to R/I \to 0,$
where $L_i = S_1(-d+1-i)^{\oplus \beta^{S_1}_{i,d-1}}, 1 \leq i \leq p-1$;

(b) *in particular, $\beta^{S_1}_{i,d-1} = (-1)^i + \sum_{1\leq j\leq i}(-1)^{j+i}\beta^{R}_{j,d-1}(R/I)$.*

Proof. (a) First, consider the exact sequence

$$\to \mathrm{Tor}_1^R(R/I,k)_j \to \mathrm{Tor}_0^{S_1}(R/I,k)_{j-1} \xrightarrow{\delta} \mathrm{Tor}_0^{S_1}(R/I,k)_j \to \mathrm{Tor}_0^R(R/I,k)_j \to 0.$$

Since $\mathrm{Tor}_1^R(R/I)_j = 0$ for all $j \neq d$ and $\mathrm{Tor}_0^R(R/I)_j = 0$ for all $j \neq 0$, we obtain that $\beta^R_{0,0} = \beta^{S_1}_{0,j} = 1$ for all $0 \leq j \leq d-1$ and $\beta^{S_1}_{0,j} = 0$ for all $j \notin \{0,1,\ldots,d-1\}$.

Note that $\mathrm{Tor}_i^R(R/I)_{i+j} = 0$ for $1 \leq i \leq p$ and $j \neq d-1$ by assumption that I is d-linear up to p-th step. Applying Theorem 2.2 for $M = R/I$, we have an isomorphism induced by $\delta = \times x_0$

$$\mathrm{Tor}^{S_1}_{i-1}(R/I,k)_{(i-1)+j} \xrightarrow{\delta} \mathrm{Tor}^{S_1}_{i-1}(R/I,k)_{(i-1)+(j+1)},$$

for $1 \leq i \leq p$ and for all $j \notin \{d-2, d-1\}$. Hence we conclude that

$$\mathrm{Tor}^{S_1}_{i-1}(R/I,k)_{(i-1)+j} = 0 \ \text{ for } 1 \leq i \leq p \ \text{ and } \ j \neq d-1$$

since R/I is finitely generated as an S_1-module, which means that

$$L_i = S_1(-d-i+1)^{\oplus \beta^{S_1}_{i,d-1}} \ \text{ for } \ 1 \leq i \leq p-1.$$

(b) Note that we have

(2.3) $0 \to \mathrm{Tor}_i^{S_1}(R/I,k)_{i+d-1} \to \mathrm{Tor}_i^R(R/I,k)_{i+d-1} \to \mathrm{Tor}^{S_1}_{i-1}(R/I,k)_{i+d-2} \to 0$

for $1 \leq i \leq p-1$ such that

$$\beta^{S_1}_{i,d-1}(R/I) = \beta^R_{i,d-1}(R/I) - \beta^{S_1}_{i-1,d-1}(R/I).$$

Then, by induction on p, we get the desired result. □

Notation 2.4. In this paper, we use the following notations:

- $R = k[x_0,\ldots,x_n] = \mathrm{Sym}(V)$ and $S_t = k[x_t, x_{t+1}\ldots,x_n] = \mathrm{Sym}(W)$: two polynomial rings where $W \subset V$, $\mathrm{codim}(W,V) = t$.
- $\Lambda = \mathbb{P}(U) = Z(x_t, x_{t+1},\cdots,x_n)$ is a linear space in $\mathbb{P}^n$ where U is a t-dimensional vector space with a basis $\{x_0,x_1,\cdots,x_{t-1}\}$.
- $\pi_\Lambda : X \to Y_t = \pi_\Lambda(X) \subset \mathbb{P}^{n-t} = \mathbb{P}(W)$ is the projection from the center Λ and $\Lambda \cap X = \phi$.
- $\beta^R_{i,j}(M) := \dim_k \mathrm{Tor}_i^R(M,k)_{i+j}$ for a finitely generated R-module M.

- $H^i_*(\mathcal{F}) := \bigoplus_{\ell\in\mathbb{Z}} H^i(\mathcal{F}(\ell))$ and $h^i(\mathcal{F}) = \dim H^i(\mathcal{F})$ for a coherent sheaf $\mathcal{F}$.

From now on, we consider a projection $\pi_\Lambda : X \to Y_t = \pi_\Lambda(X) \subset \mathbb{P}(W)$ where $\dim\Lambda = t-1 \geq 0, \Lambda\cap X = \phi$. Then, the following basic sequence

$$0 \longrightarrow R/I_X \longrightarrow E \longrightarrow H^1_*(\mathcal{I}_X) \longrightarrow 0 \quad (\text{as } S_t\text{-modules})$$

is also exact as finitely generated S_t-modules as Lemma ?? shows. Furthermore, it would be very useful to compare their graded Betti tables by the mapping cone as we see in the subsequent sections.

Proposition 2.5. *Let X be a reduced non-degenerate projective variety in $\mathbb{P}^n = \mathbb{P}(V)$. Consider the linear projection $\pi_\Lambda{:}X \to \mathbb{P}(W), \mathrm{Sym}(W) = S_t$ from the center Λ such that $\Lambda\cap X = \phi$, $\Lambda = \mathbb{P}(U) = \mathbb{P}^{t-1}$. Then, we have the following results:*

(a) *If X satisfies property $N_{2,p}$, then R/I_X satisfies property N^S_{p-t} as S_t-module for $1\leq t\leq p$, i.e. it has the simplest syzygies up to $(p-t)$-th step as follows:*

$$\cdots \to S_t(-p+t-1)^{\oplus\beta^{S_t}_{p-t,1}} \to \cdots \to S_t(-2)^{\oplus\beta^{S_t}_{1,1}} \to S_t\oplus S_t(-1)^{\oplus\, t} \to R/I_X \to 0.$$

(b) *More generally, if X satisfies property $N_{d,p}$, then $R/(I_X)_{\geq d}$ has the simplest syzygies up to $(p-t)$-th step as S_t-module for $1\leq t\leq p$,*

$$\to L_{p-t} \to \cdots \to L_1 = S_t(-d)^{\oplus\beta^{S_t}_{1,d-1}} \to \bigoplus_{i=0}^{d-1}\mathrm{Sym}^i(U)\otimes S_t(-i) \to R/(I_X)_{\geq d} \to 0.$$

where $L_i = S_t(-i-d+1)^{\oplus\beta^{S_t}_{i,d-1}}$ for $1\leq i\leq p-t$ and $\mathrm{Sym}^i(U) = H^0(\mathcal{O}_\Lambda(i))$ is a vector space of homogeneous forms of degree i generated by U.

On the other hand, we have the similar result for $E = \bigoplus_{\ell\in\mathbb{Z}} H^0(X, \mathcal{O}_X(\ell))$ as the following proposition shows.

The following theorem gives us a geometric meaning of property $N_{d,p}$ and note that part (b) was also proved in Theorem 1.1 in [5] with a different method.

Theorem 2.6. *Let X be a reduced non-degenerate projective variety satisfying property $N_{d,p}$ in $\mathbb{P}^n$. Consider the linear projection $\pi_\Lambda{:}X \to Y_t \subset \mathbb{P}^{n-t} = \mathbb{P}(W)$ from the center Λ such that $\Lambda\cap X = \phi$, $\Lambda = \mathbb{P}(U) = \mathbb{P}^{t-1}, t\leq p$. Then, we have the following results:*

(a) *every fiber of π_Λ is $(d-1)$-normal, i.e. $\mathrm{reg}(\pi_\Lambda^{-1}(y)) \leq d$ for all $y\in Y_t$;*
(b) *$\mathrm{reg}(X\cap L) \leq d$ for any linear section $X\cap L$ as a finite scheme where $L = \mathbb{P}^{k_0}$, $1\leq k_0\leq p$. In particular, for a projective variety satisfying property $N_{2,p}$, there is no $(p+2)$-secant p-plane.*

Proof. For a proof of (a), consider the minimal free resolution of $R/(I_X)_{\geq d}$ in Proposition 2.5.(b), namely,

$$\cdots \to S_t(-d)^{\oplus\beta^{S_t}_{1,d-1}} \to \bigoplus_{i=0}^{d-1}\mathrm{Sym}^i(U)\otimes S_t(-i) \to R/(I_X)_{\geq d} \to 0$$

where $\mathrm{Sym}^i(U) = H^0(\mathcal{O}_\Lambda(i))$ is a vector space of homogeneous forms of degree i generated by U. By sheafifying this exact sequence and tensoring $\bigotimes \mathcal{O}_{\mathbb{P}^{n-t}}(d-1)$, we have the surjective morphism of sheaves

$$\cdots \longrightarrow \bigoplus_{i=0}^{d-1}\mathrm{Sym}^i(U)\otimes\mathcal{O}_{\mathbb{P}^{n-t}}(d-1-i) \longrightarrow \pi_{\Lambda *}\mathcal{O}_X(d-1) \longrightarrow 0.$$

For all $y \in Y_t$, we have the following surjective commutative diagram $(*)$ by Nakayama's lemma:

$$\begin{array}{ccccc}
\bigoplus_{i=0}^{d-1} \mathrm{Sym}^i(U) \otimes \mathcal{O}_{\mathbb{P}^{n-t}}(d-1-i) \otimes k(y) & \longrightarrow & \pi_{\Lambda *}\mathcal{O}_X(d-1) \otimes k(y) & \longrightarrow 0 \\
\| & (*) & \| & \\
H^0(\langle \Lambda, y\rangle, \mathcal{O}_{\langle \Lambda, y\rangle}(d-1)) & \longrightarrow & H^0(\mathcal{O}_{\pi_\Lambda{}^{-1}(y)}(d-1)) & \longrightarrow 0.
\end{array}$$

Therefore, as a finite scheme, ${\pi_\Lambda}^{-1}(y)$ is $(d-1)$-normal for all $y \in Y_t$.

For a proof of (b), suppose that $\mathrm{reg}(X \cap L) > d$ for some linear section $X \cap L$ as a finite scheme where $L = \mathbb{P}^{k_0}$ for some $1 \leq k_0 \leq p$. Then we can take a linear subspace $\Lambda_1 \subset L$ of dimension $k_0 - 1$ disjoint from $X \cap L$. Then $X \cap L$ is a fiber of projection $\pi_{\Lambda_1} : X \to \mathbb{P}^{n-k_0-1}$ at $\pi_{\Lambda_1}(L)$. However, this is a contradiction by (a). □

3. Structure theorems of projected varieties according to moving the center

For a projective variety $X \subset \mathbb{P}^n$, property $N_{2,p}$ is a natural generalization of property N_p. The following theorems show that property $N_{2,p}$ plays an important role to control the normality and defining equations of projected varieties under isomorphic and birational projections up to $(p-1)$-th step.

Theorem 3.1. (Isomorphic projections of varieties satisfying $N_{2,p}$) *Let $X \subset \mathbb{P}^n$ be a reduced non-degenerate projective variety satisfying property $N_{2,p}$ for some $p \geq 2$. Consider any isomorphic projection $\pi_\Lambda : X \to Y_t \subset \mathbb{P}^{n-t}, t \leq p-1$. Suppose X is m-normal for all $m \geq n_0(X)$. Then we have the following:*

(a) *$H^1(\mathcal{I}_X(m)) = H^1(\mathcal{I}_{Y_t}(m)) \; \forall\, m \geq t+1$. Consequently Y_t is m-normal if and only if X is m-normal, and Y_t is m-normal for $\forall\, m \geq \max\{n_0(X), t+1\}$;*

(b) *Y_t is cut out by equations of degree at most $t+2$ and satisfies property $N_{t+2,p-t}$;*

(c) *$\mathrm{reg}(Y_t) \leq \max\{\mathrm{reg}(X), t+2\}$.*

In the complete embedding of $X \subset \mathbb{P}(H^0(\mathcal{O}_X(1)))$, property $N_{2,p}$ is the same as property N_p. In this case, we have the following Corollary which is already given in Theorem 1.2 in [12] and Corollary 3 in [2].

Corollary 3.2. *Let $X \subset \mathbb{P}(H^0(\mathcal{O}_X(1))) = \mathbb{P}^n$ be a reduced non-degenerate projective variety with property N_p for some $p \geq 2$. Consider an isomorphic projection $\pi_\Lambda : X \to Y_t \subset \mathbb{P}(W) = \mathbb{P}^{n-t}, t = \mathrm{codim}(W, H^0(\mathcal{O}_X(1))), 1 \leq t \leq p-1$. The projected variety $Y_t \subset \mathbb{P}(W)$ satisfies the following:*

(a) *Y_t is m-normal for all $m \geq t+1$,*

(b) *Y_t is cut out by equations of degree at most $(t+2)$ and moreover Y_t satisfies property $N_{2+t,p-t}$,*

(c) *$\mathrm{reg}(Y_t) \leq \max\{\mathrm{reg}(X), t+2\}$.*

Theorem 3.3. (Birational projections of varieties satisfying $N_{2,p}$) *Let $X \subset \mathbb{P}^n$ be a reduced non-degenerate projective variety satisfying property $N_{2,p}$ for some $p \geq 2$. Suppose X is m-normal for all $m \geq n_0(X)$ and consider a birational projection $\pi_q : X \to Y_1 \subset \mathbb{P}^{n-1}$ where $q \in \mathrm{Sec}(X) \cup \mathrm{Tan}(X) \setminus X$. Then we have the following:*

(a) $H^1_*(\mathcal{I}_X) = H^1_*(\mathcal{I}_{Y_1})$. *Consequently,* Y_1 *is* m*-normal if and only if* X *is* m*-normal for all* $m \geq 1$*, and* Y_1 *is also* m*-normal for all* $m \geq n_0(X)$*;*
(b) Y_1 *is cut out by at most cubic hypersurfaces and satisfies property* $N_{3,p-1}$*;*
(c) $\mathrm{reg}(Y_1) \leq \max\{n_0(X)+1, \mathrm{reg}(\mathcal{O}_{Y_1})+1\}$

- **Moving the center of projection**

So far, we have shown the uniform properties for any projection morphism of a projective variety with the condition $N_{2,p}, p \geq 2$. On the other hand, the following propositions show that the number of quadric equations and the depth of projected varieties depend on moving of the center of projections. For a complete embedding $X \subset \mathbb{P}(H^0(\mathcal{L}))$, the same result is given in [16].

Proposition 3.4. *Let* $X \subset \mathbb{P}^n$ *be a reduced non-degenerate projective variety satisfying property* $N_{2,p}$*,* $p \geq 2$*. Consider the projection* $\pi_q : X \to Y_1 \subset \mathbb{P}^{n-1}$ *where* $q \notin X$*. Let* $\Sigma_q(X)$ *is the secant locus of the projection* π_q*. Then the following holds:*
(a) $h^0(\mathbb{P}^{n-1}, \mathcal{I}_{Y_1}(2)) = h^0(\mathbb{P}^n, \mathcal{I}_X(2)) - n + s$ *where* $s = \dim \Sigma_q(X)$*,*
(b) $\mathrm{depth}(Y_1) = \min\{\mathrm{depth}(X), s+2\}$ *under the condition that* $H^i(\mathcal{O}_X(j)) = 0, \forall j \leq -i, 1 \leq i \leq \dim(X)$*.*

We give some examples related to our proposition.

Example 3.5. (A non-normal variety with non-vanishing cohomology) For a projective normal variety X, let $\delta(X) := \min\{\mathrm{depth}\, \mathcal{O}_{X,x} | x \text{ is a closed point}\}$. Then $H^i(\mathcal{O}_X(\ell)) = 0$ for all $\ell << 0$ and $i < \delta(X)$ by vanishing theorem of Enriques-Severi-Zariski-Serre. In the proof of proposition 3.4, for $s = 0$ we have an interesting example Y_1 such that Y_1 has only one isolated non-normal singular point and $H^1(\mathcal{O}_{Y_1}(\ell)) \neq 0$ for all $\ell \leq 0$. In this case, $\pi_q : X \to Y_1$ is a smooth normalization of Y_1.

The Proposition 3.4 can be extended to the inner projections by letting $s = \dim(X)$ as follows.

Proposition 3.6. *Let* X *be a smooth projective variety,* $\mathcal{L}$ *a very ample line bundle and* $(X, \mathcal{L})$ *satisfies property* $N_p, p \geq 1$*. For the complete embedding* $X \hookrightarrow \mathbb{P}^n = \mathbb{P}(H^0(\mathcal{L}))$*, consider the inner projection* $Y_1 = \overline{\pi_q(X \setminus \{q\})}$ *in* $\mathbb{P}^{n-1}$ *for a point* $q \in X \setminus \mathrm{Trisec}(X)$*. Then we have the following:*
(a) $h^0(\mathbb{P}^{n-1}, \mathcal{I}_{Y_1}(2)) = h^0(\mathbb{P}^n, \mathcal{I}_X(2)) - n + \dim(X)$*;*
(b) $\mathrm{depth}(Y_1) = \mathrm{depth}(X)$*.*

On the other hand, for a projective variety $X \subset \mathbb{P}^n$ with the condition $N_{2,p}, p \geq 2$ and $q \notin X$, we obtained that $\pi_q(X)$ satisfies at least property $N_{3,p-1}$ by Theorem 3.7 and Theorem 3.3. Thus, it is quite natural to ask property $N_{2,p-1}$ for the projected varieties under some assumptions. Property $N_{2,p}$ is *rigid* : if X is a reduced subscheme in $\mathbb{P}^n$ with the condition $N_{2,p}, p = \mathrm{codim}(X, \mathbb{P}^n)$, then X is 2-regular([5]), but Property $N_{2,p}$ is very subtle under outer projections in general. However, for general inner projections, the projected varieties satisfy property $N_{2,p-1}$. As an example, consider the rational normal curve $C = \nu_d(\mathbb{P}^1)$ in $\mathbb{P}^d$ we have the following equivalent condition as follows (see [1], [15]): let $S^\ell(C)$ be the ℓ-th higher secant variety of C with $\dim S^\ell(C) = \min\{2\ell - 1, d\}$. Then we have the following filtration on higher secant varieties of C:

$$C \subsetneq \mathrm{Sec}(C) = S^2(C) \subsetneq S^3(C) \subsetneq \cdots \subsetneq S^{\lfloor \frac{d}{2} \rfloor}(C) \subsetneq S^{\lfloor \frac{d}{2} \rfloor + 1}(C) = \mathbb{P}^d \text{ and}$$

(a) $\overline{\pi_q(C)} \subset \mathbb{P}^{d-1}$ is a rational normal curve with property $N_{2,d-2}$ for $q \in C$,
(b) $\pi_q(C) \subset \mathbb{P}^{d-1}$ is a rational curve with one node satisfying property $N_{2,d-3}$ for $q \in \mathrm{Sec}(C) \setminus C$,
(c) $\pi_q(C) \subset \mathbb{P}^{d-1}$ is a smooth rational curve with property $N_{2,\ell-3}$ for $q \in S^{\ell}(C) \setminus S^{\ell-1}(C)$.

Note that all projected curves are m-normal for all $m \geq 2$ and thus 3-regular.

On the other hand, the inner projection of X from q is a rational map defined on $X \setminus \{q\}$, so we take Zariski closure of the image Y. Geometrically, this is just adding points, which is the image of tangential projection from q, to $\pi_q(X \setminus \{q\})$. Algebraically, this process corresponds to just the elimination of 1st variable x_0 of ideal I_X (so, Y is defined by $I_Y = I_X \bigcap S$ ideal-theoretically).

$$\begin{array}{ccc} X & \overset{\pi_q}{\dashrightarrow} & \pi_q(X \setminus \{q\}) \\ \uparrow \sigma & \searrow^{\pi_q} & \downarrow \textit{Zariski closure} \\ \mathrm{Bl}_q(X) & \xrightarrow{\widetilde{\pi_q}} & Y \subset \mathbb{P}(W) \end{array}$$

From [1], we know that this morphism $\widetilde{\pi_q}$ is an embedding if $q \in X \setminus \mathrm{Trisec}(X)$. We also have the following interesting behavior for syzygies of inner projections.

Theorem 3.7. (inner projection of varieties satisfying $N_{2,p}$)
Let $X \subset \mathbb{P}^n$ be a non-degenerate projective variety satisfying property $N_{2,p}$ for some $p \geq 2$ and q be a smooth point of X. Consider the inner projection $\pi_q : X \to Y \subset \mathbb{P}^{n-1}$. Then we have

(a) *There is a surjection of syzygies up to $p-1$-th step, i.e.*

$$\exists \ \mathrm{Tor}_i^R(I_X)_{i+j} \twoheadrightarrow \mathrm{Tor}_{i-1}^S(I_Y)_{i-1+j} \ \ \textit{for } 0 \leq i \leq p-1$$

(b) *The projected variety Y is cut out by quadrics and satisfies property $N_{2,p-1}$.*

Proof. See [10] for details. □

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DUALIZING COMPLEX OF A TORIC FACE RING —NORMAL AND NON-NORMAL CASES—

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1. Introduction

Stanley-Reisner rings and affine semigroup rings are important subjects of combinatorial commutative algebra. The notion of *toric face rings*, which originated in an earlier work of Stanley [8], generalizes both of them, and has been studied by Bruns, Römer and their coauthors recently (e.g. [1, 3, 5]). Contrary to these classical examples, a toric face ring does not admit a nice multi-grading in its most general setting. It makes the study of this ring complicated. It is also noteworthy that a toric face ring is an example of Yuzvinsky's "ring of sections" ([10]).

A toric face ring $\Bbbk[\mathcal{M}]$ supported by a finite regular cell complex $\mathcal{X}$ is built of affine semigroup rings $\Bbbk[\mathbf{M}_\sigma]$ with $\dim \Bbbk[\mathbf{M}_\sigma] = \dim \sigma + 1$ for each cell $\sigma \in \mathcal{X}$. To get $\Bbbk[\mathcal{M}]$, we "glue" $\Bbbk[\mathbf{M}_\sigma]$ along with $\mathcal{X}$.

In this article, we describe a dualizing complex of $\Bbbk[\mathcal{M}]$. When $\Bbbk[\mathbf{M}_\sigma]$ is normal for all $\sigma \in \mathcal{X}$, then the description is pretty concise. The results in this part are joint work with R. Okazaki ([7]). In the general case, our description is a generalization of "Ishida complex" ([6]) for an affine semigroup ring.

For the proof of the former case, we introduce the notion of *squarefree modules* over $R := \Bbbk[\mathcal{M}]$. As in the case of affine semigroup rings ([9]), from a squarefree R-module M, we assign a constructible sheaf M^+ on (the underlying topological space of) $\mathcal{X}$. Our dualizing complex (more precisely, a complex which is quasi-isomorphic to the dualizing complex) $I_R^\bullet$ is composed of squarefree modules. In our context, $(I_R^\bullet)^+$ is Verdier's dualizing complex of $\mathcal{X}$ with coefficients in $\Bbbk$, and $\mathrm{RHom}_R(-, I_R^\bullet)$ corresponds to Poincaré-Verdier duality on $\mathcal{X}$. From this observation, we see that the Cohen-Macaulay (resp. Gorenstein*, Buchsbaum) property of R is a topological property of the underlying space of $\mathcal{X}$. Unfortunately, there is not enough space to introduce this direction here. Consult [7] for detail.

2. Notation and Construction

Let $\mathcal{X}$ be a finite regular cell complex with the intersection property, and X its underlying topological space. More precisely, the following conditions are satisfied.

(1) $\emptyset \in \mathcal{X}$, $X = \bigcup_{\sigma \in \mathcal{X}} \sigma$, and $\sigma \in \mathcal{X}$ are pairwise disjoint;
(2) If $\emptyset \neq \sigma \in \mathcal{X}$, then, for some $i \in \mathbb{N}$, there exists a homeomorphism from an i-dimensional ball $\{x \in \mathbb{R}^i \mid ||x|| \leq 1\}$ to the closure $\overline{\sigma}$ of σ which maps $\{x \in \mathbb{R}^i \mid ||x|| < 1\}$ onto σ (In this case, set $\dim \sigma = i$ and call σ an i-cell);

The second author is partially supported by Grant-in-Aid for Scientific Research (c) (no.19540028).

(3) For $\sigma \in \mathcal{X}$, the closure $\overline{\sigma}$ can be written as the union of some cells in $\mathcal{X}$;
(4) For $\sigma, \tau \in \mathcal{X}$, there is a cell $v \in \mathcal{X}$ such that $\overline{v} = \overline{\sigma} \cap \overline{\tau}$ (here v can be $\emptyset$).

We regard $\mathcal{X}$ as a partially ordered set by $\sigma \geq \tau \overset{\text{def}}{\Longleftrightarrow} \overline{\sigma} \supset \tau$.

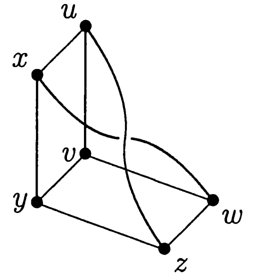
Example 2.1. We shall give two typical examples of such a cell complex. One is associated with an (abstract) simplicial complex $\Delta \subset 2^{[n]}$. Take its geometric realization $|\Delta|$, and let ρ be the map giving the realization. Then $X := |\Delta|$ together with $\{$ the relative interior of $\rho(F) \mid F \in \Delta \}$ is a cell complex with the above conditions. The other example is a polytope P. In this case, P itself is the underlying topological space; the cells are the relative interiors of its faces.

Definition 2.2. A *conical complex* $(\Sigma, \mathcal{X})$ on $\mathcal{X}$ consists of the following data.

(1) $\Sigma = \{ C_\sigma \mid \sigma \in \mathcal{X} \}$ is a set such that $C_\sigma \subset \mathbb{R}^{\dim \sigma + 1}$ is a finitely generated pointed cone with $\dim C_\sigma = \dim \sigma + 1$. (The word "pointed" means that $\{0\}$ is a face of C_σ.)
(2) An injection $\iota_{\sigma,\tau} : C_\tau \to C_\sigma$ for $\sigma, \tau \in \mathcal{X}$ with $\sigma \geq \tau$ satisfying the following.
   (a) $\iota_{\sigma,\tau}$ can be lifted up to a linear map $\mathbb{R}^{\dim \tau + 1} \to \mathbb{R}^{\dim \sigma + 1}$.
   (b) The image $\iota_{\sigma,\tau}(C_\tau)$ is a face of C_σ. Conversely, for a face C' of C_σ, there is a sole cell τ with $\tau \leq \sigma$ such that $\iota_{\sigma,\tau}(C_\tau) = C'$. Thus we have a one-to-one correspondence between $\{$ faces of $C_\sigma \}$ and $\{ \tau \in \mathcal{X} \mid \tau \leq \sigma \}$.
   (c) $\iota_{\sigma,\sigma} = \mathrm{id}_{C_\sigma}$ and $\iota_{\sigma,\tau} \circ \iota_{\tau,v} = \iota_{\sigma,v}$ for $\sigma, \tau, v \in \mathcal{X}$ with $\sigma \geq \tau \geq v$.

A typical example of a conical complex is a pointed fan, i.e., a finite collection Σ of pointed cones in $\mathbb{R}^n$ satisfying the following properties: (1) for $C' \subset C \in \Sigma$, C' is a face of C if and only if $C' \in \Sigma$; (2) for $C, C' \in \Sigma$, $C \cap C'$ is a face of both C and C'. In this case, as an underlying cell complex, we can take $\{ \text{rel-int}(C \cap \mathbb{S}^{n-1}) \mid C \in \Sigma \}$, where $\mathbb{S}^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$, and the injections ι are inclusion maps.

Example 2.3. There exists a conical complex which is not a fan. In fact, consider the Möbius strip as follows. Regarding each rectangles as the cross-sections of



3-dimensional cones, we have a conical complex that is not a fan (see [2]).

Definition 2.4. A *monoidal complex* $\mathcal{M}$ supported by a conical complex $(\Sigma, \mathcal{X})$ is a set of monoids $\{\mathbf{M}_\sigma\}_{\sigma \in \mathcal{X}}$ with the following conditions:

(1) $\mathbf{M}_\sigma \subset \mathbb{Z}^{\dim \sigma + 1}$ for each $\sigma \in \mathcal{X}$, and it is a finitely generated additive submonoid (so $\mathbf{M}_\sigma$ is an affine semigroup) with $\mathbb{Z}\mathbf{M}_\sigma = \mathbb{Z}^{\dim \sigma + 1}$;

(2) $\mathbf{M}_\sigma \subset C_\sigma$ and $\mathbb{R}_{\geq 0}\mathbf{M}_\sigma = C_\sigma$ for each $\sigma \in \mathcal{X}$ (hence the cone C_σ is automatically rational);
(3) for $\sigma, \tau \in \mathcal{X}$ with $\sigma \geq \tau$, the map $\iota_{\sigma,\tau} : C_\tau \to C_\sigma$ induces an isomorphism $\mathbf{M}_\tau \cong \mathbf{M}_\sigma \cap \iota_{\sigma,\tau}(C_\tau)$ of monoids.

For example, let Σ be a rational pointed fan in $\mathbb{R}^n$. Then $\{ C \cap \mathbb{Z}^n \mid C \in \Sigma \}$ gives a monoidal complex. More generally, a family of affine semigroups $\{ \mathbf{M}_C \subset \mathbb{Z}^n \mid C \in \Sigma \}$ satisfying the following conditions, forms a monoidal complex; (1) $\mathbb{R}_{\geq 0}\mathbf{M}_C = C$ for each $C \in \Sigma$; (2) $\mathbf{M}_C \cap C' = \mathbf{M}_{C'}$ for $C, C' \in \Sigma$ with $C' \subset C$.

For a conical complex $(\Sigma, \mathcal{X})$ and a monoidal complex $\mathcal{M}$ supported by Σ, we set

$$|\mathcal{M}| := \varinjlim_{\sigma \in \mathcal{X}} \mathbf{M}_\sigma, \quad |\mathbb{Z}\mathcal{M}| := \varinjlim_{\sigma \in \mathcal{X}} \mathbb{Z}\mathbf{M}_\sigma,$$

where the direct limits are taken with respect to the inclusions $\iota_{\sigma,\tau} : \mathbf{M}_\tau \to \mathbf{M}_\sigma$ and induced map $\mathbb{Z}\mathbf{M}_\tau \to \mathbb{Z}\mathbf{M}_\sigma$ respectively, for $\sigma, \tau \in \mathcal{X}$ with $\sigma \geq \tau$. Let $a, b \in |\mathbb{Z}\mathcal{M}|$. If there is some $\sigma \in \mathcal{X}$ with $a, b \in \mathbb{Z}\mathbf{M}_\sigma$, there is a unique minimal cell among these σ's by our assumption on $\mathcal{X}$. Hence we can define $a \pm b \in |\mathbb{Z}\mathcal{M}|$.

Definition 2.5. Let $(\Sigma, \mathcal{X})$ be a conical complex, $\mathcal{M}$ a monoidal complex supported by Σ, and $\Bbbk$ a field. Then the $\Bbbk$-vector space

$$\Bbbk[\mathcal{M}] := \bigoplus_{a \in |\mathcal{M}|} \Bbbk\, t^a$$

equipped with the following multiplication

$$t^a \cdot t^b = \begin{cases} t^{a+b} & \text{if } a, b \in \mathbf{M}_\sigma \text{ for some } \sigma \in \mathcal{X}; \\ 0 & \text{otherwise,} \end{cases}$$

has a $\Bbbk$-algebra structure. We call $\Bbbk[\mathcal{M}]$ the *toric face ring* of $\mathcal{M}$ over $\Bbbk$.

It is easy to see that $\dim R = \dim \mathcal{X} + 1$. When Σ is a rational pointed fan in $\mathbb{R}^n$ and $\mathbf{M}_\sigma = C_\sigma \cap \mathbb{Z}^n$ for each σ, $\Bbbk[\mathcal{M}]$ is just an earlier version due to Stanley ([8]). Henceforth we refer a toric face ring of $\mathcal{M}$ supported by a fan as an *embedded* toric face ring. Every Stanley-Reisner ring and every affine semigroup ring (associated with a positive affine semigroup) can be established as embedded toric face rings.

An embedded toric face ring always has the natural $\mathbb{Z}^n$-grading such that the dimension, as a $\Bbbk$-vector space, of each homogeneous component is less than or equal to 1. However a non-embedded one does not have such a grading.

Example 2.6. Consider the conical complex given in Example 2.3, and choose each rectangles to be a unit square. In this case, we can construct a monoidal complex $\mathcal{M}$ such that u, v, w, x, y, z are generators of $\mathcal{M}$. We set $S := \Bbbk[X_u, X_v, X_w, X_x, X_y, X_z]$, where $X_u, \dots, X_z$ are variables. Clearly, $\Bbbk[\mathbf{M}_\sigma]$ is a polynomial ring if $\dim \sigma \leq 1$, and a 3 dimensional normal semigroup ring of the form $\Bbbk[a,b,c,d]/(ac - bd)$ if $\dim \sigma = 2$. Therefore we conclude that

$$\Bbbk[\mathcal{M}] \cong S/(X_xX_v - X_uX_y, X_vX_z - X_yX_w, X_xX_z - X_uX_w, X_uX_vX_w, X_uX_vX_z).$$

Let $R := \Bbbk[\mathcal{M}]$ be a toric face ring, $\operatorname{Mod} R$ the category of R-modules, and $\operatorname{mod} R$ its full subcategory consisting of finitely generated modules.

Definition 2.7. $M \in \operatorname{Mod} R$ is said to be *$\mathbb{Z}\mathcal{M}$-graded* if the following conditions are satisfied;

(1) $M = \bigoplus_{a \in |\mathbb{Z}\mathcal{M}|} M_a$ as $\Bbbk$-vector spaces;
(2) $t^a \cdot M_b \subset M_{a+b}$ if $a \in \mathbf{M}_\sigma$ and $b \in \mathbb{Z}\mathbf{M}_\sigma$ for some $\sigma \in \mathcal{X}$, and $t^a \cdot M_b = 0$ otherwise.

Clearly, R itself is $\mathbb{Z}\mathcal{M}$-graded. An ideal of R is $\mathbb{Z}\mathcal{M}$-graded if and only if it is generated by monomials (i.e., elements of the form t^a). Of course, since R is not a graded ring in the usual sense, the word "$\mathbb{Z}\mathcal{M}$-graded" is abuse of terminology.

Let $\operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R$ (resp. $\operatorname{mod}_{\mathbb{Z}\mathcal{M}} R$) denote the subcategory of $\operatorname{Mod} R$ (resp. $\operatorname{mod} R$) whose objects are $\mathbb{Z}\mathcal{M}$-graded and morphisms are degree preserving (i.e., $f : M \to N$ with $f(M_a) \subset N_a$ for all $a \in |\mathbb{Z}\mathcal{M}|$). It is clear that $\operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R$ and $\operatorname{mod}_{\mathbb{Z}\mathcal{M}} R$ are abelian.

3. Čech complex and Dualizing complex (Normal Case)

In the rest of this article, $\mathcal{M}$ is a monoidal complex supported by a cell complex $\mathcal{X}$, and $R := \Bbbk[\mathcal{M}]$ is its toric face ring. Set $d := \dim R$ (hence $\dim \mathcal{X} = d - 1$).

Unless otherwise specified, the results in this section are taken from the joint work [7] with Okazaki.

Lemma 3.1. *Let $M \in \operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R$, and let T be a multiplicatively closed subset of R consisting of monomials. Then $T^{-1}M \in \operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R$.*

Proof. Take any $x/t^a \in T^{-1}M$ with $a \in |\mathcal{M}|$, $b \in |\mathbb{Z}\mathcal{M}|$, and $x \in M_b$. If there is no $\sigma \in \mathcal{X}$ with $a, b \in \mathbb{Z}\mathbf{M}_\sigma$, then $x/t^a = (xt^a)/t^{2a} = 0$; otherwise, $b - a$ is well-defined. So set $\deg(x/t^a) = b - a$. □

For $\sigma \in \mathcal{X}$, set $T_\sigma := \{\, t^a \mid a \in \mathbf{M}_\sigma \,\} \subset R$. Then T_σ forms a multiplicatively closed subset consisting of monomials. Well, set

$$L_R^i := \bigoplus_{\substack{\sigma \in \mathcal{X} \\ \dim \sigma = i-1}} T_\sigma^{-1} R$$

and define $\partial : L_R^i \to L_R^{i+1}$ by

$$\partial(x) = \sum_{\substack{\tau \geq \sigma \\ \dim \tau = i}} \varepsilon(\tau, \sigma) \cdot f_{\tau,\sigma}(x)$$

for $x \in T_\sigma^{-1} R \subset L_R^i$, where $\varepsilon(\sigma, \tau) : \mathcal{X} \times \mathcal{X} \to \{0, \pm 1\}$ is an incidence function on $\mathcal{X}$ and $f_{\tau,\sigma}$ is a natural map $T_\sigma^{-1} R \to T_\tau^{-1} R$ for $\sigma \leq \tau$. Then $(L_R^\bullet, \partial)$ forms a complex in $\operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R$:

$$L_R^\bullet : 0 \longrightarrow L_R^0 \xrightarrow{\partial} L_R^1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} L_R^d \longrightarrow 0.$$

We set $\mathfrak{m} := (t^a \mid 0 \neq a \in |\mathcal{M}|)$. This is a maximal ideal of R.

Proposition 3.2 (cf. [5, Theorem 4.2]). *For any R-module M,*

$$H^i_{\mathfrak{m}}(M) \cong H^i(L^\bullet_R \otimes_R M) \quad (\forall i \in \mathbb{Z}).$$

The proof for the $\mathbb{Z}^n$-graded case given in [5] also works here. (Moreover, the proof is essentially same to that of the corresponding statement for an affine semigroup ring.)

For $\sigma \in \mathcal{X}$, a monomial ideal $\mathfrak{p}_\sigma := (t^a \mid a \in |\mathcal{M}| \setminus \mathbf{M}_\sigma)$ of R is prime. In fact, the quotient ring $\Bbbk[\sigma] := R/\mathfrak{p}_\sigma$ is isomorphic to the affine semigroup ring $\Bbbk[\mathbf{M}_\sigma]$.

We say R is *cone-wise normal*, if $\Bbbk[\sigma]$ is normal for all $\sigma \in \mathcal{X}$. We now describe the dualizing complex of R in the cone-wise normal case. Set

$$I^i_R := \bigoplus_{\substack{\sigma \in \mathcal{X} \\ \dim \Bbbk[\sigma] = -i}} \Bbbk[\sigma]$$

for $i = 0, \ldots, d$, and define $I^{-i}_R \to I^{-i+1}_R$ by

$$x \longmapsto \sum_{\substack{\dim \Bbbk[\tau] = i-1 \\ \tau \leq \sigma}} \varepsilon(\sigma, \tau) \cdot p_{\tau,\sigma}(x)$$

for $x \in \Bbbk[\sigma] \subset I^{-i}_R$, where $p_{\tau,\sigma}$ is the natural surjection $\Bbbk[\sigma] \to \Bbbk[\tau]$. Then

$$I^\bullet_R : 0 \longrightarrow I^{-d}_R \longrightarrow I^{-d+1}_R \longrightarrow \cdots \longrightarrow I^0_R \longrightarrow 0$$

is a complex.

Theorem 3.3. *If R is cone-wise normal, then $I^\bullet_R$ is quasi-isomorphic to the normalized dualizing complex $D^\bullet_R$ of R.*

If $\mathcal{M}$ is embedded (i.e., R is $\mathbb{Z}^n$-graded), the assertion can be easily proved by usual "graded" argument. However, in the general case, we need much more technical argument. The rest of this section is devoted to a sketch of the proof.

The dualizing complex $D^\bullet_R$ is of the form

$$0 \to \bigoplus_{\substack{\mathfrak{p} \in \operatorname{Spec} R, \\ \dim R/\mathfrak{p} = d}} E_R(R/\mathfrak{p}) \to \bigoplus_{\substack{\mathfrak{p} \in \operatorname{Spec} R, \\ \dim R/\mathfrak{p} = d-1}} E_R(R/\mathfrak{p}) \to \cdots \to \bigoplus_{\substack{\mathfrak{p} \in \operatorname{Spec} R, \\ \dim R/\mathfrak{p} = 0}} E_R(R/\mathfrak{p}) \to 0,$$

where $E_R(R/\mathfrak{p})$ is the injective envelope of $R/\mathfrak{p}$ and cohomological degrees are given by a similar way to $I^\bullet_R$.

Lemma 3.4. *For $\sigma \in \mathcal{X}$ with $d_\sigma :=$ $\dim \Bbbk[\sigma]$, there is a* <u>canonical</u> *embedding $\Bbbk[\sigma] \hookrightarrow D^{-d_\sigma}_R$. Via this embedding, $I^\bullet_R$ is a subcomplex of $D^\bullet_R$.*

Proof. Since $\Bbbk[\sigma]$ is normal, its canonical module is just the ideal $J_\sigma := \langle t^a \mid a \in \text{rel-int}(C_\sigma) \cap \mathbf{M}_\sigma \rangle$ of $\Bbbk[\sigma]$. By the exact sequence $0 \to J_\sigma \to \Bbbk[\sigma] \to \Bbbk[\sigma]/J_\sigma \to 0$ with $\dim(\Bbbk[\sigma]/J_\sigma) < d_\sigma$, we have

$$\begin{aligned} \Bbbk[\sigma] \cong \operatorname{Ext}^{-d_\sigma}_R(J_\sigma, D^\bullet_R) &= \operatorname{Ker}(\operatorname{Hom}_R(J_\sigma, D^{-d_\sigma}_R) \to \operatorname{Hom}_R(J_\sigma, D^{-d_\sigma+1}_R)) \\ &= \{ x \in D^{-d_\sigma}_R \mid \mathfrak{p}_\sigma x = 0 \text{ and } \partial_{D^\bullet_R}(J_\sigma x) = 0 \} \subset D^{-d_\sigma}_R. \end{aligned}$$

So we get the first assertion.

To prove the second assertion, note that $\Bbbk[\sigma]$ is a $\mathbb{Z}^{d_\sigma}$-graded ring. Let $D^\bullet_{\Bbbk[\sigma]}$ (resp. ${}^*D^\bullet_{\Bbbk[\sigma]}$) be the usual (resp. $\mathbb{Z}^{d_\sigma}$-graded) dualizing complex of $\Bbbk[\sigma]$. It is well-known that $D^\bullet_{\Bbbk[\sigma]}$ is an injective resolution of ${}^*D^\bullet_{\Bbbk[\sigma]}$, and the latter is a subcomplex of the former. Since $D^\bullet_{\Bbbk[\sigma]} \cong \operatorname{Hom}_R(\Bbbk[\sigma], D^\bullet_R)$, $D^\bullet_{\Bbbk[\sigma]}$ can be regarded as a subcomplex of $D^\bullet_R$. Hence ${}^*D^\bullet_{\Bbbk[\sigma]}$ is a subcomplex of $D^\bullet_R$, and we have a (non-canonical) inclusion ${}^*D^{-d_\sigma}_{\Bbbk[\sigma]} = {}^*E_{\Bbbk[\sigma]}(\Bbbk[\sigma]) \hookrightarrow D_R^{-d_\sigma}$, where ${}^*E_{\Bbbk[\sigma]}(\Bbbk[\sigma])$ is the injective envelope of $\Bbbk[\sigma]$ in the category of $\mathbb{Z}^{d_\sigma}$-graded $\Bbbk[\sigma]$-modules. Note that the "positive part" $\bigoplus_{a\in \mathbf{M}_\sigma}[{}^*E_{\Bbbk[\sigma]}(\Bbbk[\sigma])]_a$ of ${}^*E_{\Bbbk[\sigma]}(\Bbbk[\sigma])$ is isomorphic to $\Bbbk[\sigma]$. Since the inclusion ${}^*D^\bullet_{\Bbbk[\sigma]} \to D^\bullet_R$ induces quasi-isomorphism $\operatorname{Hom}_{\Bbbk[\sigma]}(J_\sigma, {}^*D^\bullet_{\Bbbk[\sigma]}) \cong \operatorname{Hom}_R(J_\sigma, D^\bullet_R) \cong \Bbbk[\sigma]$, the canonical inclusion $\Bbbk[\sigma] \hookrightarrow D_R^{-d_\sigma}$ is nothing other than the positive part $\Bbbk[\sigma] \cong \bigoplus_{a\in \mathbf{M}_\sigma}[{}^*E_{\Bbbk[\sigma]}(\Bbbk[\sigma])]_a$ through the inclusion ${}^*E_{\Bbbk[\sigma]}(\Bbbk[\sigma]) \hookrightarrow D_R^{-d_\sigma}$. For $\tau \in \mathcal{X}$ with $\tau \le \sigma$, $\Bbbk[\tau]$ is a quotient ring of $\Bbbk[\sigma]$, and ${}^*D^\bullet_{\Bbbk[\tau]} \cong \operatorname{Hom}_{\Bbbk[\sigma]}(\Bbbk[\tau], {}^*D^\bullet_{\Bbbk[\sigma]})$ is a *graded* subcomplex of ${}^*D^\bullet_{\Bbbk[\sigma]}$. The canonical inclusion $\Bbbk[\tau] \hookrightarrow D_R^{-d_\tau}$ also given by the positive part of ${}^*D^{-d_\tau}_{\Bbbk[\tau]} = {}^*E_{\Bbbk[\tau]}(\Bbbk[\tau]) \hookrightarrow D_R^{-d_\tau}$. Summing up these facts, we see that $I^\bullet_R$ is a subcomplex of $D^\bullet_R$. See [7] for detail. □

Definition 3.5. We say $M \in \operatorname{mod}_{\mathbb{Z}\mathcal{M}} R$ is *squarefree*, if there is an exact sequence of the form

$$0 \to M \to \bigoplus_{\sigma\in\mathcal{X}} \Bbbk[\sigma]^{n_\sigma} \to \bigoplus_{\sigma\in\mathcal{X}} \Bbbk[\sigma]^{m_\sigma} \qquad (\exists n_\sigma, m_\sigma \in \mathbb{N})$$

in $\operatorname{mod}_{\mathbb{Z}\mathcal{M}} R$. Let $\operatorname{Sq} R$ be the full subcategory of $\operatorname{mod}_{\mathbb{Z}\mathcal{M}} R$ consisting of squarefree modules.

Lemma 3.6. *$\operatorname{Sq} R$ is an abelian subcategory of $\operatorname{mod}_{\mathbb{Z}\mathcal{M}} R$ with enough injectives, and indecomposable injectives in this category are $\Bbbk[\sigma]$'s.*

Let Inj-Sq be the full subcategory of $\operatorname{Sq} R$ consisting of all injective objects, that is, finite direct sums of $\Bbbk[\sigma]$ for various $\sigma \in \mathcal{X}$. As is well-known, the bounded homotopy category $K^b(\text{Inj-Sq})$ is equivalent to $D^b(\operatorname{Sq} R)$. We see that $K^b(\text{Inj-Sq}) \ni J^\bullet \mapsto \operatorname{Hom}^\bullet_R(J^\bullet, I^\bullet_R) \in K^b(\text{Inj-Sq})$ defines an exact functor $\mathbb{D} : K^b(\text{Inj-Sq}) \to K^b(\text{Inj-Sq})^{\mathrm{op}}$.

Lemma 3.7. *There is the following commutative diagram;*

$$\begin{array}{ccc} D^b(\operatorname{Sq} R) \cong K^b(\text{Inj-Sq}) & \xrightarrow{\ \mathbb{U}\ } & D^b(\operatorname{Mod} R) \\ \mathbb{D}\big\downarrow & & \big\downarrow \operatorname{RHom}(-,D^\bullet_R) \\ K^b(\text{Inj-Sq})^{\mathrm{op}} & \xrightarrow[\mathbb{U}]{} & D^b(\operatorname{Mod} R)^{\mathrm{op}}, \end{array}$$

where $\mathbb{U}$ is the functor induced by the forgetful functor $\operatorname{Sq} R \to \operatorname{Mod} R$.

Proof. Since $I^\bullet_R$ is a subcomplex of $D^\bullet_R$ by Lemma 3.4, we have a chain map

$$\mathbb{D}(J^\bullet) = \operatorname{Hom}^\bullet_R(J^\bullet, I^\bullet_R) \longrightarrow \operatorname{Hom}^\bullet_R(J^\bullet, D^\bullet_R) = \operatorname{RHom}_R(J^\bullet, D^\bullet_R),$$

which gives a natural transformation $\Psi : \mathbb{U}\circ\mathbb{D} \to \operatorname{RHom}_R(-, D^\bullet_R)\circ\mathbb{U}$. It is easy to see that $\Psi(\Bbbk[\sigma])$ is isomorphism for all $\sigma \in \mathcal{X}$ ($\Bbbk[\sigma]$ is a normal semigroup ring, and we can use "graded" argument). Hence Ψ is a natural isomorphism by a well-known result [4, Proposition 7.1] on derived categories. $\square$

Since $R \in \operatorname{Sq} R$, we have $I_R^\bullet = \mathbb{D}(R) \cong \operatorname{RHom}(R, D_R^\bullet) \cong D_R^\bullet$ in $D^b(\operatorname{Mod} R)$ by Lemma 3.7. This completes the proof of Theorem 3.3.

4. Dualizing Complex (Non-Normal Case)

We can define the $\mathbb{Z}\mathcal{M}$-graded Matlis duality $(-)^\vee : \operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R \to (\operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R)^{\mathrm{op}}$ in a natural way. By an argument similar to the last step of the proof of Theorem 3.3, we have the following. Since R is not (graded) local, this is not very trivial.

Proposition 4.1. *If R is cone-wise normal, the Matlis dual $(L_R^\bullet)^\vee$ of the Čech complex $L_R^\bullet$ is quasi-isomorphic to the dualizing complex $D_R^\bullet$.*

In this section, we show that the above fact holds in general (i.e., even if R is not cone-wise normal).

Set

$$\mathbf{M}_\sigma - \mathcal{M} := \{\, a - b \mid a \in \mathbf{M}_\sigma,\ b \in \mathbf{M}_\tau \text{ for some } \tau \geq \sigma \,\} \subset |\mathbb{Z}\mathcal{M}|$$

For $c \in \mathbf{M}_\sigma - \mathcal{M}$, let t_σ^c be a basis element with degree c, and

$$E_\sigma(\mathcal{M}) := \langle\, t_\sigma^c \mid c \in \mathbf{M}_\sigma - \mathcal{M} \,\rangle$$

the $\Bbbk$-vector space spanned by these elements.

We can regard $E_\sigma(\mathcal{M})$ as a $\mathbb{Z}\mathcal{M}$-graded $\Bbbk[\mathcal{M}]$-module by

$$t^a \cdot t_\sigma^c = \begin{cases} t_\sigma^{a+c} & \text{if } a + c \text{ exists and } a + c \in \mathbf{M}_\sigma - \mathcal{M} \\ 0 & \text{otherwise.} \end{cases}$$

Then $E_\sigma(\mathcal{M})$ is the $\mathbb{Z}\mathcal{M}$-graded Matlis dual of $R_{T\sigma}$.

Remark 4.2. As shown in the proof of [5, Theorem 5.1], if $\mathcal{M}$ is embedded, $E_\sigma(\mathcal{M})$ is the injective envelope of $\Bbbk[\sigma]$ in the category of $\mathbb{Z}^n$-graded R-modules. For $\tau \in \mathcal{X}$ with $\tau \geq \sigma$, the submodule $\operatorname{Hom}_R(\Bbbk[\tau], E_\sigma(\mathcal{M}))$ of $E_\sigma(\mathcal{M})$ is

$$\langle\, t_\sigma^{a-b} \mid a \in \mathbf{M}_\sigma,\ b \in \mathbf{M}_\tau \,\rangle,$$

which is isomorphic to the injective envelope of $\Bbbk[\sigma]$ in the category of $\mathbb{Z}^{d_\tau}$-graded $\Bbbk[\tau]$-modules (note that $\Bbbk[\sigma]$ is a quotient ring of $\Bbbk[\tau]$ in this case).

For $\sigma, \tau \in \mathcal{X}$ with $\sigma \geq \tau$,

$$t_\sigma^a \longmapsto \begin{cases} t_\tau^a & \text{if } a \in \mathbf{M}_\tau - \mathcal{M}, \\ 0 & \text{otherwise} \end{cases}$$

gives an R-homomorphism $g_{\tau,\sigma} : E_\sigma(\mathcal{M}) \to E_\tau(\mathcal{M})$. In fact, $g_{\tau,\sigma}$ is the $\mathbb{Z}\mathcal{M}$-graded Matlis dual of the natural map $R_{T_\tau} \to R_{T_\sigma}$ (note that $T_\sigma \supset T_\tau$ in this case).

We have a complex $J_R^\bullet$ of the form

$$0 \to \bigoplus_{\substack{\sigma \in \mathcal{X} \\ \dim \sigma = d-1}} E_\sigma(\mathcal{M}) \to \bigoplus_{\substack{\sigma \in \mathcal{X} \\ \dim \sigma = d-1}} E_\sigma(\mathcal{M}) \to \cdots \to \bigoplus_{\substack{\sigma \in \mathcal{X} \\ \dim \sigma = 0}} E_\sigma(\mathcal{M}) \to E_\emptyset(\mathcal{M}) \to 0,$$

$$\partial_{J^\bullet_R} : E_\sigma(\mathcal{M}) \ni x \longmapsto \sum_{\substack{\dim \tau = i-1 \\ \tau \leq \sigma}} \varepsilon(\sigma, \tau) \cdot g_{\tau,\sigma}(x) \in \bigoplus_{\dim \tau = i-1} E_\tau(\mathcal{M}).$$

We put the cohomological degree of $\bigoplus_{\dim \sigma = i-1} E_\sigma(\mathcal{M})$ to $-i$. Clearly, $J^\bullet_R$ is the $\mathbb{Z}\mathcal{M}$-graded Matlis dual of the Cěch complex $L^\bullet_R$.

Theorem 4.3. *The complex $J^\bullet_R$ is quasi-isomorphic to the normalized dualizing complex $D^\bullet_R$ of R.*

Outline of the proof. For each $\sigma \in \mathcal{X}$, set $\overline{\mathbf{M}}_\sigma := \mathbb{Z}^{d_\sigma} \cap C_\sigma$. Then $\overline{\mathcal{M}} := \{ \overline{\mathbf{M}}_\sigma \}_{\sigma \in \mathcal{X}}$ is a monoidal complex supported by $\mathcal{X}$ again. Let $\widetilde{R} := \Bbbk[\overline{\mathcal{M}}]$ be the toric face ring of $\overline{\mathcal{M}}$. Naturally, R is a subring of $\widetilde{R}$ and $J^\bullet_R$ is a subcomplex of $J^\bullet_{\widetilde{R}}$ (here we regard $J^\bullet_{\widetilde{R}}$ as a complex of R-modules). Since $\widetilde{R}$ is cone-wise normal, $J^\bullet_{\widetilde{R}}$ is quasi-isomorphic to the dualizing complex $D^\bullet_{\widetilde{R}}$ by Proposition 4.1. Note that $D^\bullet_{\widetilde{R}} = \operatorname{Hom}^\bullet_R(\widetilde{R}, D^\bullet_R)$. Via the injection $R \hookrightarrow \widetilde{R}$, we have a chain map $\phi : D^\bullet_{\widetilde{R}} = \operatorname{Hom}^\bullet_R(\widetilde{R}, D^\bullet_R) \longrightarrow \operatorname{Hom}_R(R, D^\bullet_R) = D^\bullet_R$. Repeating the argument in the previous section (after suitable modification), we can prove that the composition

$$J^\bullet_R \to J^\bullet_{\widetilde{R}} \xrightarrow{\cong} D^\bullet_{\widetilde{R}} \xrightarrow{\phi} D^\bullet_R$$

is a quasi-isomorphism. □

Corollary 4.4. *R is Cohen-Macaulay (resp. Gorenstein) if and only if so is the local ring $R_\mathfrak{m}$.*

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Integral closure algorithms

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Let's first look at three examples of integral closure:

1. (Nagata) Let k be a field of characteristic 2, let X, Y, Z be indeterminates over k, and let $b_1, b_2, \ldots$ be countably many elements of k such that for all n, $[k^2(b_1, \ldots, b_n) : k^2] = 2^n$. Then the integral closure of the three-dimensional Noetherian local domain
$$k^2[[X,Y,Z]][k][Y\sum_i b_{2i}X^i + Z\sum_i b_{2i+1}Z^i]$$
is not Noetherian.

2. Let $R = \mathbb{Z}[\sqrt{D}]$, where D is a non-zero integer. Write $D = qn^2$ for some square-free integer q and some integer n. Then
$$\overline{R} = \begin{cases} \mathbb{Z}\left[\frac{1+\sqrt{q}}{2}\right] & \text{if } q \equiv 1 \mod 4; \\ \mathbb{Z}\left[\sqrt{q}\right] & \text{if } q \not\equiv 1 \mod 4. \end{cases}$$

3. Let R be a monomial algebra, i.e., $R = k[\underline{x}^{\underline{a}} : \underline{a} \in E] \subseteq k[x_1, \ldots, x_d]$, where E is a subsemigroup of $\mathbb{N}^d$. Then $\overline{R}$ is also a monomial algebra, and
$$\overline{R} = k[\underline{x}^{\underline{a}} : \underline{a} \in \mathbb{Z}E \cap \mathbb{Q}_{\geq 0}E].$$

If we are interested in algorithms for computing the integral closures of rings, we probably want to find a finite set of generators of $\overline{R}$ over R. Thus, the input rings had better not be as in Example 1 above; not only is the integral closure of that ring not module-finite over R, it is not even Noetherian.

The integral closure of the ring in Example 2 above was determined directly, not via a step-by-step algorithm. Many other examples of integral closures by theorems are known, starting with Dedekind, then more examples in the 1930s by Albert [1], [2], and many more by others. A more recent paper is due to Tan and Zhang [15]. In the rest of this paper, we concentrate not on such theorems, but on step-by-step algorithmic procedures.

Bruns and Koch in Normaliz [3] implemented a program that computes efficiently the integral closures of finitely generated monomial algebras, as in Example 3. The theory behind Normaliz is explained in [4]. However, that algorithm does not extend to the computation of the integral closures of general affine domains.

Our interest lies in describing algorithms for the integral closure of general computable integral domains, such as affine domains. The first general method was described by Stolzenberg in 1968 [14], and was improved by Seidenberg in [11, 12]. Their method was not very algorithmically constructive or effective. The first effective procedure, from the mid-1990s, is due to de Jong [5], based on the 1971, 1984 work of Grauert and Remmert [7, 8]. However, computation of integral closure is very time and memory-consuming, and many times the established symbolic computer algebra computer systems do not return an answer, so a search for different algorithms continues.

In this article we start by reviewing the history of the computation of integral closure in greater detail, and at the end we present our more recent algorithm [13], based on the work of Leonard and Pellikaan [9]. All the methods up to the Leonard–Pellikaan one successively approximated the integral closure "from below," namely, by building successively strictly larger rings contained in the integral closure. The new algorithm instead starts with a finitely generated module over R that contains the integral closure, and the successive steps produce strictly smaller submodules, eventually terminating in the integral closure "from above." This algorithm only works in prime characteristic. We implemented our algorithm in Macaulay2, and sometimes our algorithm computes the integral closure faster than de Jong's. We show computation comparisons at the end.

Here is a summary of general procedures for computing the integral closure:

Stolzenberg-Seidenberg:

1. Find a module-finite extension R' of R that satisfies Serre's condition (R_1). **Details.** We assume that there exist units $u_1, u_2, \ldots$ in R whose differences are also units. Let P be a prime ideal of height 1 such that R_P is not regular. Let a, b be part of a minimal generating set of PR_P. Then there exists i such that $\frac{a}{u_i b + a}$ is integral over R_P and is not in R_P.
2. Compute a primary decomposition of cR', where c is an arbitrary non-zero element in the conductor of R'.
3. Under the assumption that R' satisfies (R_1), the integral closure of R equals

$$\overline{R} = \frac{\text{the intersection of the minimal components of } cR'}{c}.$$

Note that the existence of infinitely many units as in 1. is not constructive. However, the finding of integral elements that are not in the ring, as in 1., as given in the Stolzenberg and Seidenberg articles, is constructive, after establishing a Noether normalization A of R and a conductor element of c of R that is a non-zero element of A. It relies on the structure theorem of finitely generated modules over principal ideal domains (which A is after localizing at any prime ideal minimal over (c). The existence of c is guaranteed if R is separably generated over a subfield, and in general, Seidenberg's [12] gives a method for handling the non-separably generated case. Neither Stolzenberg nor Seidenberg discuss how one computes primary decompositions as in 2.

Grauert-Remmert-de Jong:

The crucial theorem is the following. Let J be an integrally closed ideal such that $V(J)$ contains the non-normal locus. Then $\mathrm{Hom}_R(J,J) = R$ if and only if R is integrally closed.

In case of affine domains over perfect fields, by the Jacobian criterion, one can take in the theorem $J = \sqrt{J_{R/k}}$ or $J = \overline{J_{R/k}}$. Then $\mathrm{Hom}_R(J,J)$ is a ring between R and $\overline{R}$, and if $R \neq \overline{R}$, then $\mathrm{Hom}_R(J,J)$ is strictly larger than R. Repeat with R' in place of R. This is the algorithm.

The computation of the integral closure of an ideal in general requires the computation of the integral closure of the Rees algebra of an ideal; thus if in the procedure above we used $J = \overline{J_{R/k}}$, we would have to compute the integral closure of a ring for which the integral closure is probably harder to compute. Thus instead one always takes $J = \sqrt{J_{R/k}}$. This is still a difficult computation, but more doable, due to the work of Eisenbud, Huneke, and Vasconcelos [6]. The more variables there are, the more time-consuming it is to compute $J_{R/k}$. After one passes to the ring $\mathrm{Hom}_R(J,J)$, the number of variables typically increases, making it even more difficult to compute the integral closure.

A simplification in characteristic zero is due to Lipman [10]:

Lipman:

In characteristic 0, the Grauert-Remmert-de Jong conditions are equivalent to $\mathrm{Hom}_R(J^{-1}, J^{-1}) = R$, where $J = J_{R/k}$ and $J^{-1} = \mathrm{Hom}_R(J,R)$.

Vasconcelos's algorithms in characteristic 0:

Vasconcelos [16, 17] proved two different methods for computing the integral closure. He replaced some steps in the de Jong's algorithm, and for his second algorithm, Vasconcelos established an upper bound on the number of steps.

1. Compute a Noether normalization A of R.
2. Set $R^{**} = \mathrm{Hom}_A(\mathrm{Hom}_A(R,A),A)$. This is a subring of $\overline{R}$ that satisfies (S_2).
3. If R satisfies (R_1), so does R^{**}, hence R^{**} is the integral closure of R. If R does not satisfy (R_1), then apply the Grauert–Remmert–de Jong or Lipman's step to get a strictly larger ring, then proceed with double dual as above.

Here is another version, with effective bound $\sum_{\mathrm{ht}\, P=1} \lambda(A_P/J_{A/k}A_P)$ on the number of steps:

1. Get a Noether normalization A' of R, and $r \in R$ such that $Q(A')(r) = Q(R)$. Set $A = A'[r]$. Then A is Gorenstein, $A \subseteq R$ is module-finite, $Q(A) = Q(R)$.
2. Set $R' = \mathrm{Hom}_A(\mathrm{Hom}_A(R,A),A)$. As before, $R \subseteq R' \subseteq \overline{R}$, R' satisfies (S_2).

A different algorithm in prime characteristic:

(Leonard–Pellikaan 03, Singh–Swanson 08). This is a new method that first computes a module-finite extension V_0 of R that contains $\overline{R}$, and then successively computes submodules. For simplicity we state the result for affine domains R in characteristic p.

1. Let D be a non-zero conductor element of R.
2. Set $V_0 = \frac{1}{D}R$. For $e \geq 0$, set $V_{e+1} = \{f \in V_e : f^p \in V_e\}$. Then

$$V_0 \supseteq V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots$$

   are algorithmically constructible R-modules.
3. The descending chain stabilizes, and if $V_e = V_{e+1}$, then $\overline{R} = V_e$.

In general, there is no descending chain condition for modules between $\overline{R}$ and V_0, but the given construction does terminate. There is a theoretical upper bound on the number of steps needed for termination, namely, if D is a non-zero conductor element such that $V_0 = \frac{1}{D}R$, and if for all Rees valuations v of the ideal DR, $v(D) \leq p^e$, then V_e in the construction equals the integral closure of R.

We end this paper with a table of performance comparisons between the de Jong's algorithm and ours. As our algorithm uses the Frobenius homomorphism (raising ring elements to pth powers, where p is the characteristic of the ring), it is expected that as p increases, our algorithm performs worse. But it is not always too bad, see below. All computations were done in Macaulay 2 on a Mac Book Pro, and the times are measured in seconds.

Computing the integral closure of $\mathbb{Z}_p[x,y,u,v]/(x^2v - y^2u)$:

characteristic p	2	3	5	7	11	13	17	37	97
LPSS	0.04	0.03	0.04	0.04	0.04	0.05	0.05	0.13	0.59
de Jong	0.08	0.09	0.09	0.09	0.14	0.15	0.15	0.15	0.15

There are many cases where our algorithm performs much better. In the subsequent tables we give examples of this. Note that $*$ denotes that the computation did not finish in 6 hours.

Computing the integral closure of $\mathbb{Z}_p[u,v,w,x,y,z]/(u^2x^4 + uvy^4 + v^2z^4)$

characteristic p	2	3	5	7	11
LPSS	0.07	0.22	9.67	143	12543
de Jong	1.16	*	*	*	*

Computing the integral closure of $\mathbb{Z}_p[u,v,w,x,y,z]/(u^2x^p + uvy^p + v^2z^p)$

characteristic p	2	3	5	7	11	13	17	19	23
LPSS	0.06	0.07	0.09	0.27	1.81	4.89	26	56	225
de Jong	0.16	1.49	75.00	4009	*	*	*	*	*

Computing the integral closure of $\mathbb{Z}_p[x_1,\ldots,x_6]$ modulo the kernel of the map to $\mathbb{Z}_p[x,y,z,t]$ mapping $x_1 \mapsto x^3$, $x_2 \mapsto y-z$, $x_3 \mapsto z^2$, $x_4 \mapsto x^2t$, $x_5 \mapsto yt$, $x_6 \mapsto xzt$:

characteristic p	2	3	5
LPSS	0.4	2.08	198.17
de Jong	*	*	*

Computing the integral closure of $\mathbb{Z}_p[x_1,\ldots,x_5]$ modulo the kernel of the map to $\mathbb{Z}_p[x,y,z]$ mapping $x_1 \mapsto x^3-y^3$, $x_2 \mapsto y-z^2$, $x_3 \mapsto z^3$, $x_4 \mapsto x^2$, $x_5 \mapsto y^2$:

characteristic p	2	3	5
LPSS	0.11	0.21	1744.27
de Jong	*	0.32	*

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DIOPHANTINE INEQUALITY FOR EQUICHARACTERISTIC EXCELLENT HENSELIAN LOCAL DOMAINS

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2000 MSCN: 13F30, 13F40, 13H99, 11D75, 11J25

KEY WORDS: Diophantine inequality, m-valuation, product inequality, linear Artin approximation

The famous result in Diophantine approximation is Roth's theorem:

If $z \in \mathbb{R} \setminus \mathbb{Q}$ *is an algebraic number,*

$$\forall \epsilon > 0\ \exists c(z,\ \epsilon) > 0\ \forall x \in \mathbb{Z}\ \forall y \in \mathbb{Z}^* :\ \left|z - \frac{x}{y}\right| > c(z,\ \epsilon)|y|^{-2-\epsilon}.$$

This means that a non-rational real algebraic number can not be efficiently approximated by rationals in comparison to the denominator of the latter. There is a quite analogous result for the field of quotients (the Laurent series field) in a single variable. Recently Rond [Ro2] obtained a Diophantine inequality for the field of quotients of the convergent or formal power series ring in multivariables in connection with the *linear Artin approximation property* (Spivakovsky, cf. [Ro1]).

In this talk we assert that Diophantine inequality holds for the field of quotients of a good local domain, a generalization of Rond's theorem:

Theorem . *Let* $(A,\ \mathfrak{m})$ *be an equicharacteristic excellent Henselian local domain and* v *an* $\mathfrak{m}$*-valuation (a good valuation defined by Rees* [Re4]*) on the field* $K := Q(A)$ *of quotients of* A*. If* $z \in \hat{K} \setminus K$ *is algebraic over* K*, then we have the following:*

$$\exists a > 0\ \exists c > 0\ \forall x \in A\ \forall y \in A^* :\ \left|z - \frac{x}{y}\right|_{\hat{v}} > c\,|y|_v^a.$$

Note that $\hat{K}$ is not generally the field quotients of $\hat{A}$ (cf. [Ro1], **2.4**). Our proof is quite similar to Rond's. He used the *product inequality* [Iz1] for the order function v on an analytic integral domain. We need *Rees's inequality* [Re4] *for* $\mathfrak{m}$*-valuations* on complete local rings, a generalised and stronger variant of the product inequality. To be precise, we use its

Date: December 26, 2008.

further generalization to analytically irreducible excellent domains by Hübl-Swanson [HS]. We also need Rees's strong valuation theorem [Re2], [Re3] to connect valuations to the maximal-ideal-adic order. Combining these results we have the following:

Fact. *Let $(A, \mathfrak{m})$ be an equicharacteristic analytically irreducible excellent local domain and let v be an $\mathfrak{m}$-valuation on A. Then we have:*

$$\exists C > 0\ \exists s > 0\ \exists t > 0\ \forall x \in A :\ s\,v(x) \leq \bar{\nu}_{\mathfrak{m}}(x) \leq \nu_{\mathfrak{m}}(x) + C \leq t\,v(x) + C.$$

This can be also used to show that Theorem implies the following in the same way as [Ro2], **3.1**.

Corollary . *Let $(A, \mathfrak{m})$ be an equicharacteristic analytically irreducible excellent Henselian domain and let $P(X, Y) \in A[X, Y]$ be a homogeneous polynomial. Then the Artin function of $P(X, Y)$ is majorised by an affine function, i.e.*

$$\exists \alpha\ \exists \beta\ \forall x \in A\ \forall y \in A :\ \nu_{\mathfrak{m}}(P(x, y)) \geq \alpha i + \beta$$

$$\Longrightarrow \exists \bar{x} \in A\ \exists \bar{y} \in A :\ \nu_{\mathfrak{m}}(\bar{x} - x) \geq i,\ \nu_{\mathfrak{m}}(\bar{y} - y) \geq i,\ P(\bar{x}, \bar{y}) = 0.$$

The case $P(X, Y) = XY$ is nothing but the product inequality [Iz1].

Note. After proof of our [ItIz], we have found that Hickel [Hi] proves a result more general than our Corollary. He treats linear Artin approximation for simultaneous equations, whereas we treat only a single equation.

We assume the ring to be analytically irreducible in Theorem and Corollary above. This is, however, unnecessry as in [Hi].

We wish to express our deep thanks to Professor I. Swanson for important suggestion to improve our result on the occasion of the symposium.

We shall explain these things elsewhere.

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STABLE CATEGORIES AND DERIVED CATEGORIES

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1. Introduction

Let $R = \bigoplus_{n\geq 0} R_n$ be a d-dimensional graded Gorenstein ring with $R_0 = k$ be an algebraically closed field of characteristic 0. We denote by $\mathrm{mod}^{\mathbb{Z}} R$ the category of finitely generated graded R-modules with degree preserving morphisms, by $\mathrm{CM}^{\mathbb{Z}} R$ the full subcategory of $\mathrm{mod}^{\mathbb{Z}} R$ consisting of all graded maximal Cohen-Macaulay modules and by $\underline{\mathrm{CM}}^{\mathbb{Z}} R$ the stable category of $\mathrm{CM}^{\mathbb{Z}} R$.

For a path algebra kQ given by a finite quiver Q, we denote by $\mathrm{mod} kQ$ the category of finitely generated left kQ-modules and by $\mathcal{D}^b(\mathrm{mod}\, kQ)$ the bounded derived category.

In this lecture, we shall show the following theorem.

Theorem 1. *Suppose R has a simple singularity, then there exists a Dynkin quiver Q such that $\underline{\mathrm{CM}}^{\mathbb{Z}} R$ is equivalent to $\mathcal{D}^b(\mathrm{mod}\, kQ)$ as triangulated categories.*

Since R is a d-dimensional graded Gorenstein ring with a simple singularity, R is isomorphic to $k[x, y, z_2, z_3, \cdots, z_d]/(f)$ where f is a one of the following polynomials (cf. [6, Theorem 8.8]);

$$f = \begin{cases} y^2 - x^{n+1} + z_2^2 + z_3^2 + \cdots + z_d^2 & (A_n) \\ xy^2 + x^{n-1} + z_2^2 + z_3^2 + \cdots + z_d^2 & (D_n) \\ x^3 + y^4 + z_2^2 + z_3^2 + \cdots + z_d^2 & (E_6) \\ x^3 + xy^3 + z_2^2 + z_3^2 + \cdots + z_d^2 & (E_7) \\ x^3 + y^5 + z_2^2 + z_3^2 + \cdots + z_d^2 & (E_8) \end{cases}$$

The following theorem which is called *Knörrer's periodicity* is a one of key theorem to show Theorem 1 (cf [5], [6, Theorem 12.10]).

Theorem 2 (Knörrer's periodicity). *If R is a graded Gorenstein ring with a simple singularity, then $\underline{\mathrm{CM}}^{\mathbb{Z}} R$ is equivalent to $\underline{\mathrm{CM}}^{\mathbb{Z}} R^{\sharp\sharp}$ as triangulated categories. Here, $R^{\sharp} = k[x, y, z_2, z_3, \cdots, z_d, z_{d+1}]/(f + z_{d+1}^2)$.*

The detailed version of this paper will be submitted for publication elsewhere.

By using Knörrer's periodicity, we may check only the cases of $\dim R = 1, 2$ and Kajiura, Saito and Takahashi proved the case of $\dim R = 2$ case.

Theorem 3. [3, Theorem 3.1] *Let Q be a Dynkin quiver of corresponding type of f. Then, $\underline{\mathrm{CM}}^{\mathbb{Z}} R$ is equivalent to $\mathcal{D}^b(\mathrm{mod}\, kQ)$ as triangulated categories.*

Thus, it is enough to show the case of $\dim R = 1$.

2. Proof of Theorem 1

Through in this section, we assume that R is a 1-dimensional graded Gorenstein ring with a simple singularity and isomorphic to $k[x, y]/(f)$ where f is a one of the following polynomials;

$$f = \begin{cases} y^2 - x^{n+1} & (A_n) \\ xy^2 + x^{n-1} & (D_n) \\ x^3 + y^4 & (E_6) \\ x^3 + xy^3 & (E_7) \\ x^3 + y^5 & (E_8) \end{cases}$$

We prepare the following theorem to show Theorem 1.

Theorem 4 ([5], [2]). *Let $\mathcal{T}$ be an algebraic triangulated category with a tilting object $T \in \mathcal{T}$. Then there exists a triangle equivalence*

$$\mathbf{K}^b(\mathrm{pr}\,\mathrm{End}_{\mathcal{T}}(T)) \to \mathcal{T}$$

where $\mathbf{K}^b(\mathrm{pr}\,\mathrm{End}_{\mathcal{T}}(T))$ is the homotopy category of bounded complexes of finitely generated projective $\mathrm{End}_{\mathcal{T}}(T)$-modules.

If we find a tilting object $T \in \underline{\mathrm{CM}}^{\mathbb{Z}} R$ with $\underline{\mathrm{End}}(T) \cong kQ$ for some Dynkin quiver Q, then we get the Theorem 1 by Theorem 4. From now on, we shall find a tilting object for each case.

We denote by $(A, (i_1, i_2, \cdots, i_r))$ the cokernel of

$$\bigoplus_{t=1}^{s} R(j_t) \xrightarrow{A} \bigoplus_{t=1}^{r} R(i_t)$$

for some integers $i_1, i_2, \cdots, i_r, j_1, j_2, \cdots, j_s$ and $s \times r$-matrix A. Note that the integers $j_1, j_2, \cdots, j_s$ are determined by $i_1, i_2, \cdots, i_r$ and A.

1. $R = k[x, y]/(y^2 - x^{2m+1}) \qquad (\deg x = 2,\ \deg y = 2m + 1)$.

We put X_i as follows;

$$X_i = \begin{cases} \left(\begin{pmatrix} y & x^i \\ -x^{2m-i+1} & -y \end{pmatrix}, (-2i, -2m-1) \right) & (1 \leq i \leq m). \\ X_{2m-i+1}(2m-2i+1) & (m+1 \leq i \leq 2m). \end{cases}$$

In this case, we can see that $T := \bigoplus_{i=1}^{2m} X_i$ is a tilting object and $\underline{\mathrm{End}}(T)$ is isomorphic to the path algebra kQ of Dynkin quiver Q of type (A_{2m}).

$$X_{2m} \longrightarrow X_{2m-1} \longrightarrow \cdots \longrightarrow X_1$$

2. $R = k[x,y]/(y^2 - x^{2m})$ $\qquad$ $(\deg x = 1,\ \deg y = m)$.

We put X_i as follows;

$$X_i = \begin{cases} \left(\begin{pmatrix} y & x^i \\ -x^{2m-i} & -y \end{pmatrix}, (-i, -m) \right) & (1 \leq i \leq m-1). \\ ((\, y \pm x^m \,), (-m)) & (i = \pm). \end{cases}$$

In this case, we can see that $T := \left(\bigoplus_{i=1}^{m-1} X_i\right) \oplus X_+ \oplus X_-$ is a tilting object and $\underline{\mathrm{End}}(T)$ is isomorphic to the path algebra kQ of Dynkin quiver Q of type (D_{m+1}).

$$\begin{array}{ccccccc} X_+ & & & & & & \\ & \searrow & & & & & \\ & & X_{m-2} \longrightarrow & X_{m-1} & \longrightarrow \cdots \longrightarrow & X_1 \\ & \nearrow & & & & & \\ X_- & & & & & & \end{array}$$

3. $R = k[x,y]/(xy^2 + x^{2m})$ $\qquad$ $(\deg x = 2,\ \deg y = 2m-1)$.

We put X_i and Y_i as follows;

$$X_1 = ((\, x \,), (-2m+1)). \qquad Y_1 = ((\, y^2 + x^{2m-1} \,), (0)).$$

$$\begin{cases} X_{2j} = \left(\begin{pmatrix} y & x^j \\ x^{2m-j} & -xy \end{pmatrix}, (-2j, -2m+1) \right). \\ Y_{2j} = \left(\begin{pmatrix} xy & x^j \\ x^{2m-j} & -y \end{pmatrix}, (-2j+2, -2m+1) \right). \\ X_{2j+1} = \left(\begin{pmatrix} xy & x^{j+1} \\ x^{2m-j} & -xy \end{pmatrix}, (-2j, -2m+1) \right). \\ Y_{2j+1} = \left(\begin{pmatrix} y & x^j \\ x^{2m-j-1} & -y \end{pmatrix}, (-2j, -2m+1) \right). \end{cases} \qquad (1 \leq j \leq m)$$

$$
\begin{cases}
X_{2m+2i-1} = X_{2m-2i+1}(-2i+1). \\
X_{2m+2i} = X_{2m-2i}(-2i-1). \\
Y_{2m+2i-1} = Y_{2m-2i+1}(-2i+1). \\
Y_{2m+2i} = Y_{2m-2i}(-2i+1).
\end{cases}
\qquad (1 \leq i \leq m-1)
$$

$$
X_{4m-1} = X_1(-2m+1). \qquad\qquad Y_{4m-1} = Y_1(-2m+1).
$$

In this case, we can see that $T := \bigoplus_{i=1}^{4m-1} X_i$ is a tilting object and $\underline{\mathrm{End}}\,(T)$ is isomorphic to the path algebra kQ of Dynkin quiver Q of type (A_{4m-1}).

$$
X_{4m-1} \longrightarrow X_{4m-2} \longrightarrow \cdots \longrightarrow X_1
$$

4. $R = k[x,y]/(xy^2 + x^{2m+1}) \qquad (\deg x = 1,\ \deg y = m).$

We put X_i and Y_i as follows;

$$
X_1 = ((\,x\,), (-m)). \qquad\qquad Y_1 = ((\,y^2+x^{2m}\,), (0)).
$$

$$
\begin{cases}
X_{2j} = \left(\begin{pmatrix} y & x^j \\ x^{2m-j+1} & -xy \end{pmatrix}, (-j, -m)\right). \\
Y_{2j} = \left(\begin{pmatrix} xy & x^j \\ x^{2m-j+1} & -y \end{pmatrix}, (-j+1, -m)\right).
\end{cases}
\qquad (1 \leq j \leq m)
$$

$$
\begin{cases}
X_{2j+1} = \left(\begin{pmatrix} xy & x^{j+1} \\ x^{2m-j+1} & -xy \end{pmatrix}, (-j, -m)\right). \\
Y_{2j+1} = \left(\begin{pmatrix} y & x^j \\ x^{2m-j} & -y \end{pmatrix}, (-j, -m)\right).
\end{cases}
\qquad (1 \leq j \leq m-1)
$$

$$
X_\pm = ((\,xy \pm \sqrt{-1}x^{m+1}\,), (-m)). \qquad\qquad Y_\pm = ((\,y \pm \sqrt{-1}x^m\,), (-m)).
$$

In this case, we can see that $T := (\bigoplus_{i=1}^{2m} X_i) \oplus X_+ \oplus X_-$ is a tilting object and $\underline{\mathrm{End}}\,(T)$ is isomorphic to the path algebra kQ of Dynkin quiver Q of type (D_{2m+2}).

$$
\begin{array}{lllllll}
X_+ & & & & & & \\
 & \searrow & & & & & \\
 & & X_{2m} & \longrightarrow X_{2m-1} & \longrightarrow & \cdots \longrightarrow & X_1 \\
 & \nearrow & & & & & \\
X_- & & & & & &
\end{array}
$$

5. $R = k[x,y]/(x^3 + y^4) \qquad (\deg x = 4,\ \deg y = 3).$

We put X_i and Y_i as follows;

$$
\begin{aligned}
X_1 &= \left(\begin{pmatrix} x & y \\ y^3 & -x^2 \end{pmatrix}, (-3,-4)\right). \\
Y_1 &= \left(\begin{pmatrix} x^2 & y \\ y^3 & -x \end{pmatrix}, (0,-5)\right). \\
X_2 &= \left(\begin{pmatrix} y^3 & x^2 & xy^2 \\ xy & -y^2 & x^2 \\ x^2 & -xy & -y^3 \end{pmatrix}, (-3,-4,-2)\right). \\
Y_2 &= \left(\begin{pmatrix} y & 0 & x \\ x & -y^2 & 0 \\ 0 & x & y \end{pmatrix}, (-5,-3,-4)\right). \\
X_3 &= \left(\begin{pmatrix} x & y^2 & 0 & y \\ y^2 & -x^2 & -xy & 0 \\ 0 & 0 & x^2 & y^2 \\ 0 & 0 & y^2 & -x \end{pmatrix}, (-5,-3,-4,-6)\right). \quad & Y_3 &= X_3(1). \\
X_4 &= \left(\begin{pmatrix} x & y^2 \\ y^2 & -x^2 \end{pmatrix}, (-6,-4)\right). & Y_4 &= X_4(1). \\
X_5 &= X_2(-1). & Y_5 &= Y_2(-1). \\
X_6 &= Y_1(-3). & Y_6 &= X_1(-1).
\end{aligned}
$$

In this case, we can see that $T := \bigoplus_{i=1}^{6} X_i$ is a tilting object and $\underline{\mathrm{End}}\,(T)$ is isomorphic to the path algebra kQ of Dynkin quiver Q of type (E_6).

$$
\begin{array}{ccccccccc}
 & & & & X_4 & & & & \\
 & & & & \downarrow & & & & \\
X_6 & \longrightarrow & X_5 & \longrightarrow & X_3 & \longrightarrow & X_2 & \longrightarrow & X_1
\end{array}
$$

6. $R = k[x,y]/(x^3 + xy^3)$ $\qquad$ $(\deg x = 3,\ \deg y = 2)$.

We put X_i and Y_i as follows;

$$
\begin{aligned}
X_1 &= \left(\begin{pmatrix} x & y \\ xy^2 & -x^2 \end{pmatrix}, (-2,-3)\right). \\
Y_1 &= \left(\begin{pmatrix} x^2 & y \\ xy^2 & -x \end{pmatrix}, (0,-4)\right). \\
X_2 &= \left(\begin{pmatrix} xy^2 & -x^2 & -x^2y \\ xy & y^2 & -x^2 \\ x^2 & xy & xy^2 \end{pmatrix}, (-2,-3,-1)\right). \\
Y_2 &= \left(\begin{pmatrix} y & 0 & x \\ -x & xy & 0 \\ 0 & -x & y \end{pmatrix}, (-4,-2,-3)\right). \\
X_3 &= \left(\begin{pmatrix} x & y & -y^2 & 0 \\ y^2 & -x & 0 & -y^2 \\ 0 & 0 & x^2 & xy \\ 0 & 0 & xy^2 & -x^2 \end{pmatrix}, (-3,-4,-2,-3)\right). \\
Y_3 &= \left(\begin{pmatrix} x^2 & xy & y^2 & 0 \\ xy^2 & -x^2 & 0 & y^2 \\ 0 & 0 & x & y \\ 0 & 0 & y^2 & -x \end{pmatrix}, (-1,-2,-3,-4)\right). \\
X_4 &= \left(\begin{pmatrix} x^2 & xy \\ xy^2 & -x^2 \end{pmatrix}, (-2,-3)\right). \\
Y_4 &= \left(\begin{pmatrix} x & y \\ y^2 & -x \end{pmatrix}, (-3,-4)\right).
\end{aligned}
$$

$$X_5 = \left(\begin{pmatrix} x^2 & -y^2 & -xy \\ xy & x & -y^2 \\ xy^2 & xy & x^2 \end{pmatrix}, (-2,-4,-3) \right).$$
$$Y_5 = \left(\begin{pmatrix} x & 0 & y \\ -xy & x^2 & 0 \\ 0 & -xy & x \end{pmatrix}, (-3,-2,-4) \right).$$
$$X_6 = \left(\begin{pmatrix} x & y^2 \\ xy & -x^2 \end{pmatrix}, (-4,-3) \right).$$
$$Y_6 = \left(\begin{pmatrix} x^2 & y^2 \\ xy & -x \end{pmatrix}, (-2,-4) \right).$$
$$X_7 = ((x^2+y^3), (-3)).$$
$$Y_7 = ((x), (-4)).$$

In this case, we can see that $T := \bigoplus_{i=1}^{7} X_i$ is a tilting object and $\underline{\mathrm{End}}(T)$ is isomorphic to the path algebra kQ of Dynkin quiver Q of type (E_7).

$$\begin{array}{cccccccccccc} & & & & & & X_4 & & & & & \\ & & & & & & \downarrow & & & & & \\ X_7 & \longrightarrow & X_6 & \longrightarrow & X_5 & \longrightarrow & X_3 & \longrightarrow & X_2 & \longrightarrow & X_1 \end{array}$$

7. $R = k[x,y]/(x^3+y^5)$ $\qquad (\deg x = 5,\ \deg y = 3)$.

We put X_i and Y_i as follows;

$$X_1 = \left(\begin{pmatrix} x^2 & y \\ y^4 & -x \end{pmatrix}, (0,-7) \right).$$
$$Y_1 = \left(\begin{pmatrix} x & y \\ y^4 & -x^2 \end{pmatrix}, (-2,-4) \right).$$
$$X_2 = \left(\begin{pmatrix} y^4 & xy^3 & x^2 \\ -x^2 & y^4 & xy \\ -xy & -x^2 & y^2 \end{pmatrix}, (-4,-2,-6) \right).$$
$$Y_2 = \left(\begin{pmatrix} y & -x & 0 \\ 0 & y & -x \\ x & 0 & y^3 \end{pmatrix}, (-8,-6,-4) \right).$$
$$X_3 = \left(\begin{pmatrix} y & -x & 0 & y^3 \\ x & 0 & -y^3 & 0 \\ -y^2 & 0 & -x^2 & 0 \\ 0 & -y^2 & -xy & -x^2 \end{pmatrix}, (-7,-5,-3,-1) \right).$$
$$Y_3 = \left(\begin{pmatrix} 0 & x^2 & -y^3 & 0 \\ -x^2 & xy & 0 & -y^3 \\ 0 & -y^2 & -x & 0 \\ y^2 & 0 & y & -x \end{pmatrix}, (-2,-4,-5,-3) \right).$$
$$X_4 = \left(\begin{pmatrix} y^4 & x^2 & 0 & -xy^2 & 0 \\ -x^2 & xy & 0 & -y^3 & 0 \\ 0 & -y^2 & -x & 0 & y^3 \\ -xy^2 & y^3 & 0 & x^2 & 0 \\ -y^3 & 0 & -y^2 & xy & -x^2 \end{pmatrix}, (-3,-5,-6,-4,-2) \right).$$
$$Y_4 = \left(\begin{pmatrix} y & -x & 0 & 0 & 0 \\ x & 0 & 0 & y^2 & 0 \\ -y^2 & 0 & -x^2 & 0 & -y^3 \\ 0 & -y^2 & 0 & x & 0 \\ 0 & 0 & y^2 & y & -x \end{pmatrix}, (-7,-5,-3,-6,-4) \right).$$

$$X_5 = \left(\begin{pmatrix} y^4 & xy^2 & x^2 & 0 & 0 & xy \\ -x^2 & y^3 & xy & -x & 0 & 0 \\ -xy^2 & -x^2 & y^3 & 0 & -xy & 0 \\ 0 & 0 & 0 & y & -x & 0 \\ 0 & 0 & 0 & 0 & y^2 & -x \\ 0 & 0 & 0 & x & 0 & y^2 \end{pmatrix}, (-2,-3,-4,-7,-5,-6) \right).$$

$$Y_5 = \left(\begin{pmatrix} y & -x & 0 & 0 & 0 & -x \\ 0 & y^2 & -x & xy & 0 & 0 \\ x & 0 & y^2 & 0 & xy & 0 \\ 0 & 0 & 0 & y^4 & xy^2 & x^2 \\ 0 & 0 & 0 & -x^2 & y^3 & xy \\ 0 & 0 & 0 & -xy^2 & -x^2 & y^3 \end{pmatrix}, (-6,-4,-5,-2,-3,-4) \right).$$

$$X_6 = \left(\begin{pmatrix} y & -x & 0 \\ 0 & y^2 & -x \\ x & 0 & y^2 \end{pmatrix}, (-7,-5,-6) \right).$$

$$Y_6 = \left(\begin{pmatrix} y^4 & xy^2 & x^2 \\ -x^2 & y^3 & xy \\ -xy^2 & -x^2 & y^3 \end{pmatrix}, (-2,-3,-4) \right).$$

$$X_7 = \left(\begin{pmatrix} x^2 & y^2 & 0 & xy \\ y^3 & -x & -y^2 & 0 \\ 0 & 0 & x & y^2 \\ 0 & 0 & y^3 & -x^2 \end{pmatrix}, (-3,-7,-6,-5) \right).$$

$$Y_7 = \left(\begin{pmatrix} x & y^2 & 0 & y \\ y^3 & -x^2 & -xy^2 & 0 \\ 0 & 0 & x^2 & y^2 \\ 0 & 0 & y^3 & -x \end{pmatrix}, (-5,-4,-3,-7) \right).$$

$$X_8 = \left(\begin{pmatrix} x & y^2 \\ y^3 & -x^2 \end{pmatrix}, (-6,-5) \right).$$

$$Y_8 = \left(\begin{pmatrix} x^2 & y^2 \\ y^3 & -x \end{pmatrix}, (-3,-7) \right).$$

In this case, we can see that $T := \bigoplus_{i=1}^{8} X_i$ is a tilting object and $\underline{\mathrm{End}}(T)$ is isomorphic to the path algebra kQ of Dynkin quiver Q of type (E_8).

$$\begin{array}{ccccccccccccc} & & & & X_6 & & & & & & & & \\ & & & & \downarrow & & & & & & & & \\ X_8 & \longrightarrow & X_7 & \longrightarrow & X_5 & \longrightarrow & X_4 & \longrightarrow & X_3 & \longrightarrow & X_2 & \longrightarrow & X_1 \end{array}$$

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PICARD GROUPS AND AUTOMORPHISM GROUPS OF CATEGORIES

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1. Introduction

Let k be a commutative ring and let A be a commutative k-algebra. We denote by A-Mod the category of all A-modules and all A-homo morphisms. Let $\mathfrak{C}$ be an additive full subcategory of A-Mod. When we say that $\mathfrak{C}$ is a full subcategory of A-Mod, we always assume that $\mathfrak{C}$ is closed under isomorphisms, and we simply write $X \in \mathfrak{C}$ to indicate that X is an object of $\mathfrak{C}$. Since A is a k-algebra, every additive full subcategory $\mathfrak{C}$ is a k-category. Recall that $F : \mathfrak{C} \to \mathfrak{C}$ is a k-linear functor if it induces k-linear mappings $\mathrm{Hom}_A(X, Y) \to \mathrm{Hom}_A(F(X), F(Y))$ for all $X, Y \in \mathfrak{C}$. A covariant functor $F : \mathfrak{C} \to \mathfrak{C}$ is called a k-linear automorphism of $\mathfrak{C}$ if it is a k-linear functor giving an auto-equivalence of the category $\mathfrak{C}$.

In this note, we study the automorphism groups of additive full subcategories. We denote the set of all the isomorphism classes of k-linear automorphisms of $\mathfrak{C}$ by $\mathrm{Aut}_k(\mathfrak{C})$, which forms a group by defining the multiplication to be the composition of functors. We will show a certain structure theorem for $\mathrm{Aut}_k(\mathfrak{C})$ (Theorem 12). To this end we shall define the Picard groups for arbitrary additive full subcategories $\mathfrak{C}$ and we will give a certain presentation theorem for k-linear automorphisms of $\mathfrak{C}$.

2. Picard group of additive categories

Throughout the note, we always assume that $\mathfrak{C}$ is an additive full subcategory of A-Mod that contains A as an object. We note that we will be able to prove the following result which enables us to describe the form of any <u>A-linear</u> automorphisms of $\mathfrak{C}$.

Theorem 1 ([2, Corollary 3.1]). *For any element $[F] \in \mathrm{Aut}_A(\mathfrak{C})$, there is an isomorphism of functors $F \cong \mathrm{Hom}_A(N, -)|_{\mathfrak{C}}$ for some $N \in \mathfrak{C}$.*

Later we shall give a proof for this theorem in more general form. Taking the theorem into consideration, we make the following definition.

The detailed version of this paper will be submitted for publication elsewhere.

Definition 2. We define $\mathrm{Pic}(\mathfrak{C})$ to be the set of all the isomorphism classes of A-modules $M \in \mathfrak{C}$ such that $\mathrm{Hom}_A(M,-)|_{\mathfrak{C}}$ gives an auto-equivalence of the category $\mathfrak{C}$. That is,

$$\mathrm{Pic}(\mathfrak{C}) = \{M \in \mathfrak{C} \mid \begin{array}{l} \mathrm{Hom}_A(M,-)|_{\mathfrak{C}} : \mathfrak{C} \to \mathfrak{C} \\ \text{gives an } (A\text{-linear}) \text{ equivalence} \end{array} \}/\cong .$$

We define the group structure on $\mathrm{Pic}(\mathfrak{C})$ as in the following manner: Let $[M]$ and $[N]$ be in $\mathrm{Pic}(\mathfrak{C})$. Since the composition $\mathrm{Hom}_A(M,-)|_{\mathfrak{C}} \circ \mathrm{Hom}_A(N,-)|_{\mathfrak{C}}$ is also an A-linear automorphism, it follows from Theorem 1 that there exists an $L \in \mathfrak{C}$ such that

$$\mathrm{Hom}_A(L,-)|_{\mathfrak{C}} \cong \mathrm{Hom}_A(M,-)|_{\mathfrak{C}} \circ \mathrm{Hom}_A(N,-)|_{\mathfrak{C}}.$$

We define the multiplication $[M]\cdot[N]$ by $[L]$ in $\mathrm{Pic}(\mathfrak{C})$. We remark that

$$\begin{aligned} \mathrm{Hom}_A(M,-)|_{\mathfrak{C}} \circ \mathrm{Hom}_A(N,-)|_{\mathfrak{C}} &\cong \mathrm{Hom}_A(M \otimes_A N,-)|_{\mathfrak{C}} \\ &\cong \mathrm{Hom}_A(N,-)|_{\mathfrak{C}} \circ \mathrm{Hom}_A(M,-)|_{\mathfrak{C}}, \end{aligned}$$

and hence we have $[M]\cdot[N] = [N]\cdot[M]$. In such a way $\mathrm{Pic}(\mathfrak{C})$ is an abelian group with the identity element $[A]$. We call $\mathrm{Pic}(\mathfrak{C})$ the Picard group of $\mathfrak{C}$.

Note from Yoneda's lemma that the multiplication in $\mathrm{Pic}(\mathfrak{C})$ is well-defined. Furthermore, the mapping $\mathrm{Pic}(\mathfrak{C}) \to \mathrm{Aut}_A(\mathfrak{C})$ which sends $[M]$ to $\mathrm{Hom}_A(M,-)|_{\mathfrak{C}}$ is an isomorphism of groups by Theorem 1. Since $\mathrm{Aut}_A(\mathfrak{C})$ is naturally a subgroup of $\mathrm{Aut}_k(\mathfrak{C})$, we can regard $\mathrm{Pic}(\mathfrak{C})$ as a subgroup of $\mathrm{Aut}_k(\mathfrak{C})$ through the isomorphism $\mathrm{Pic}(\mathfrak{C}) \cong \mathrm{Aut}_A(\mathfrak{C})$.

In the rest of this section, we shall give several examples of $\mathrm{Pic}(\mathfrak{C})$. Recall that the classical Picard group of the ring A, which is denoted by $\mathrm{Pic}\ A$, is the set of isomorphism classes of invertible A-modules i.e. $\mathrm{Pic}\ A = \{\text{invertible } A\text{-modules}\}/\cong$. The multiplication in $\mathrm{Pic}\ A$ is defined by tensor product.

Example 3 ([2, Example 3.8, 3.11])**.** We denote by A-mod the full subcategory consisting of all finitely generated A-modules. We also denote by $\mathrm{Proj}(A)$ (resp. $\mathrm{proj}(A)$) the full subcategory consisting of all projective A-modules (resp. all finitely generated projective A-modules). If A is an integral domain, we denote by $\mathrm{Tf}(A)$ (resp. $\mathrm{tf}(A)$) the full subcategory consisting of all torsion free A-modules (resp. all finitely generated torsion free A-modules). Let $\mathfrak{C}$ be one of the full subcategories A-Mod, A-mod, $\mathrm{Proj}(A)$, $\mathrm{proj}(A)$, $\mathrm{Tf}(A)$ and $\mathrm{tf}(A)$. Then we have an isomorphism $\mathrm{Pic}(\mathfrak{C}) \cong \mathrm{Pic}\ A$. See also [2, Proposition 3.7].

Example 4 ([2, Example 3.9, 3.10])**.** Let A be a Krull domain and let $\mathrm{Ref}(A)$ be the full subcategory consisting of all reflexive A-lattices. (Respectively, let A be a Noetherian normal domain and let $\mathrm{ref}(A)$ be the full subcategory consisting of all finitely generated reflexive A-modules.) Then there is an isomorphism $\mathrm{Pic}(\mathrm{Ref}(A)) \cong C\ell(A)$ (resp.

$\mathrm{Pic}(\mathrm{ref}(A)) \cong C\ell(A))$, where $C\ell(A)$ denotes the divisor class group of A.

Example 5 ([2, Example 3.12]). Let $(A, \mathfrak{m})$ be a Noetherian local ring. We consider the full subcategory $d^{\geq 1}(A)$ of A-Mod which consists of all the finitely generated A-modules M satisfying depth $M \geq 1$. If depth $A \geq 1$, then $\mathrm{Pic}(d^{\geq 1}(A))$ is a trivial group.

Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local k-algebra, that is, A is a Noetherian local k-algebra with maximal ideal $\mathfrak{m}$ and satisfies the equality depth $A = \dim A$. We denoted by $\mathrm{CM}(A)$ the category consisting of all the maximal Cohen-Macaulay modules over A. See [3] for the details of $\mathrm{CM}(A)$.

Theorem 6 ([2, Theorem 5.2]). *Let A be a Cohen-Macaulay local k-algebra of* $\dim A = d \geq 3$. *Suppose that A is regular in codimension two, i.e. $A_\mathfrak{p}$ is a regular local ring for any prime ideal $\mathfrak{p}$ with* $\mathrm{ht}(\mathfrak{p}) = 2$. *Then* $\mathrm{Pic}(\mathrm{CM}(A))$ *is a trivial group.*

Proof. Let $[M]$ be an arbitrary element in $\mathrm{Pic}(\mathrm{CM}(A))$. Assuming that M is not free, we shall show a contradiction. Take a free cover F of M and we obtain an exact sequence $0 \longrightarrow \Omega(M) \longrightarrow F \longrightarrow M \longrightarrow 0$. We remark that the first syzygy module $\Omega(M)$ belongs to $\mathrm{CM}(A)$. Apply $\mathrm{Hom}_A(M, -)$ to the sequence, and we get an exact sequence:

$$0 \to \mathrm{Hom}_A(M, \Omega(M)) \to \mathrm{Hom}_A(M, F) \\ \to \mathrm{Hom}_A(M, M) \xrightarrow{f} \mathrm{Ext}^1_A(M, \Omega(M)).$$

Notice that $f \neq 0$, since we have assumed that M is not free. Because of the assumption, we see that $\mathrm{Ext}^1_A(M, \Omega(M))_\mathfrak{p} = 0$ for all prime ideals $\mathfrak{p}$ with $\mathrm{ht}(\mathfrak{p}) = 2$. This implies that $\dim \mathrm{Ext}^1_A(M, \Omega(M)) \leq d-3$, hence the image $\mathrm{Im}(f)$ is a nontrivial A-module of dimension at most $d-3$. In particular, we have depth $\mathrm{Im}(f) \leq d-3$.

On the other hand, since $\mathrm{Hom}_A(M, -)|_{\mathrm{CM}(A)}$ is an automorphism of $\mathrm{CM}(A)$, the modules $\mathrm{Hom}_A(M, \Omega(M))$, $\mathrm{Hom}_A(M, F)$ and $\mathrm{Hom}_A(M, M)$ have depth d. Hence we conclude from the depth argument [1, Proposition 1.2.9] that depth $\mathrm{Im}(f) \geq d-2$. This is a contradiction, and the proof is completed. □

3. Structure of $\mathrm{Aut}_k(\mathfrak{C})$

Let $\mathfrak{C}$ be an additive full subcategory of A-Mod. In this section, we study the group of all the <u>k-linear</u> automorphisms of $\mathfrak{C}$.

Definition 7. $\mathrm{Aut}_k(\mathfrak{C})$ is the group of all the isomorphism classes of k-linear automorphisms of $\mathfrak{C}$, i.e.

$$\mathrm{Aut}_k(\mathfrak{C}) = \{\, F : \mathfrak{C} \to \mathfrak{C} \mid \begin{array}{l} F \text{ is a } k\text{-linear covariant functor that} \\ \text{gives an equivalence of the category } \mathfrak{C} \end{array} \} / \cong .$$

Note that the multiplication in $\mathrm{Aut}_k(\mathfrak{C})$ is defined to be the composition of functors, hence the identity element of $\mathrm{Aut}_k(\mathfrak{C})$ is represented by the class of the identity functor on $\mathfrak{C}$.

We denote by $\mathrm{Aut}_{k\text{-alg}}(A)$ the group of all the k-algebra automorphisms of A. For $\sigma \in \mathrm{Aut}_{k\text{-alg}}(A)$, we can define a covariant k-linear functor $\sigma_* : A\text{-Mod} \to A\text{-Mod}$ as in the following manner. For each A-module M, we define $\sigma_* M$ to be M as an abelian group on which the A-module structure is defined by $a \circ m = \sigma^{-1}(a)m$ for $a \in A$, $m \in M$. For an A-homomorphism $f : M \to N$, we define $\sigma_* f : \sigma_* M \to \sigma_* N$ to be the same mapping as f. Note that $\sigma_* f$ is an A-homomorphism, since $(\sigma_* f)(a \circ m) = f(\sigma^{-1}(a)m) = \sigma^{-1}(a)f(m) = a \circ \sigma_* f(m)$ for all $a \in A$ and $m \in M$. Notice that σ_* is a k-automorphism of the category A-Mod.

Definition 8. Let $\mathfrak{C}$ be an additive full subcategory of A-Mod. Then $\mathfrak{C}$ is said to be stable under $\mathrm{Aut}_{k\text{-alg}}(A)$ if $\sigma_*(\mathfrak{C}) \subseteq \mathfrak{C}$ for all $\sigma \in \mathrm{Aut}_{k\text{-alg}}(A)$.

All the full subcategories we have shown in the previous section (e.g. A-Mod, $\mathrm{Ref}(A)$, $d^{\geq 1}(A)$, $\mathrm{CM}(A)$, etc.) are stable under $\mathrm{Aut}_{k\text{-alg}}(A)$. Note that if $\mathfrak{C}$ is stable under $\mathrm{Aut}_{k\text{-alg}}(A)$ then $\sigma_*|_{\mathfrak{C}}$ gives a k-linear auto-equivalence of $\mathfrak{C}$ for all $\sigma \in \mathrm{Aut}_{k\text{-alg}}(A)$. Therefore we have a natural group homomorphism $\Psi : \mathrm{Aut}_{k\text{-alg}}(A) \to \mathrm{Aut}_k(\mathfrak{C})$ which maps σ to the class of $\sigma_*|_{\mathfrak{C}}$. It is easy to verify the following lemma holds.

Lemma 9. *Assume that $\mathfrak{C}$ is stable under $\mathrm{Aut}_{k\text{-alg}}(A)$ and that $A \in \mathfrak{C}$. Then the natural group homomorphism $\Psi : \mathrm{Aut}_{k\text{-alg}}(A) \to \mathrm{Aut}_k(\mathfrak{C})$ is an injection.*

By this lemma, we can regard $\mathrm{Aut}_{k\text{-alg}}(A)$ as a subgroup of $\mathrm{Aut}_k(\mathfrak{C})$.

Definition 10. Let N be an A-module. Given a k-algebra homomorphism $\sigma : A \to A$, we define an $(A \otimes_k A)$-module N_σ by $N_\sigma = N$ as an abelian group on which the ring action is defined by $(a \otimes b) \cdot n = a\sigma(b)n$ for $a \otimes b \in A \otimes_k A$ and $n \in N$. In such a case, we can define a k-linear functor $\mathrm{Hom}_A(N_\sigma, -) : A\text{-Mod} \to A\text{-Mod}$, for which the A-module structure on $\mathrm{Hom}_A(N_\sigma, X)$ ($X \in A$-Mod) is defined by $(b \cdot f)(n) = f((1 \otimes b) \cdot n)$ for $f \in \mathrm{Hom}_A(N_\sigma, X)$, $b \in A$ and $n \in N$.

If σ is a k-algebra automorphism of A, then it is easy to see the following equality of functors holds:

$$(\sigma^{-1})_* \circ \mathrm{Hom}_A(N,\ ) = \mathrm{Hom}_A(N_\sigma,\ ).$$

The following theorem enables us to describe the forms of k-linear automorphisms of $\mathfrak{C}$.

Theorem 11 ([2, Theorem 2.5]). *Let A be a commutative k-algebra and let $\mathfrak{C}$ be an additive full subcategory of* A-Mod *such that $A \in \mathfrak{C}$.*

For a given k-linear automorphism $F \in \mathrm{Aut}_k(\mathfrak{C})$, there is a k-algebra automorphism $\sigma \in \mathrm{Aut}_{k\text{-alg}}(A)$ such that F is isomorphic to the composition of functors $\sigma_* \circ \mathrm{Hom}_A(N, -)|_{\mathfrak{C}}$, where N is any object in $\mathfrak{C}$ satisfying $F(N) \cong A$ in $\mathfrak{C}$.

Proof. We give below an outline of the proof. See [2, Theorem 2.5] for the detail.

Since A is commutative, the multiplication map $a_X : X \to X$ by an element $a \in A$ is an A-homomorphism for all objects $X \in \mathfrak{C}$. Thus we can define a natural transformation $\alpha(a) : F \to F$ by $\alpha(a)(X) = F(a_X) : F(X) \to F(X)$. Denote by $\mathrm{End}(F)$ the set of all the natural transformations $F \to F$, and this induces the mapping

$$\alpha : A \to \mathrm{End}(F)\ ; \quad a \mapsto F(a_{(\ )}).$$

Note that $\mathrm{End}(F)$ is a ring by defining the composition of natural transformations as the multiplication and it is also a k-algebra, since F is a k-linear functor. By using the fact that F is an auto-equivalence, it is straightforward to see that α is a k-algebra isomorphism.

Since F is a dense functor and $A \in \mathfrak{C}$, there is an object $N \in \mathfrak{C}$ such that $F(N) \cong A$. For such an object N, we can identify $\mathrm{End}_A(F(N))$ with A as k-algebra through the mapping $A \to \mathrm{End}_A(F(N))$ which sends $a \in A$ to the multiplication mapping $a_{F(N)}$ by a on $F(N)$. Thus we have a k-algebra homomorphism

$$\beta : \mathrm{End}(F) \to \mathrm{End}_A(F(N)) \cong A\ ; \quad \varphi \mapsto \varphi(N).$$

We easily see that β is a k-algebra isomorphism.

Now define a k-algebra automorphism $\sigma : A \to A$ as the composition of α and β;

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha} & \mathrm{End}(F) & \xrightarrow{\beta} & \mathrm{End}_A(F(N)) & \xrightarrow{\cong} & A \\ a & \longrightarrow & F(a_{(\ )}) & \longrightarrow & F(a_{(N)}) & \longrightarrow & \sigma(a). \end{array}$$

Then, for each object $X \in \mathfrak{C}$, we have isomorphisms of k-modules;

$$\begin{array}{ccccc} F(X) & \xrightarrow{\cong} & \mathrm{Hom}_A(F(N), F(X)) & \xrightarrow{\cong} & \mathrm{Hom}_A(N_{\sigma^{-1}}, X) \\ x & \longrightarrow & (x_{F(N)} : 1 \mapsto x) & \longrightarrow & F^{-1}(x_{F(N)}), \end{array}$$

whose composition we denote by φ_X. Since $F^{-1}(\sigma(a)_{F(N)}) = a_{(N)}$ holds for $a \in A$, we can show that φ_X is an A-module isomorphism for all $X \in \mathfrak{C}$. Since it is easily verified that φ_X is functorial in X, we have the isomorphism of functors $F \cong \mathrm{Hom}_A(N_{\sigma^{-1}},\ \ )$, and the proof is completed. □

Remark that Theorem 1 is just a special case of this theorem where $k = A$.

Assume furthermore that an additive full subcategory $\mathfrak{C}$ is stable under $\mathrm{Aut}_{k\text{-alg}}(A)$. Then we have shown by the above argument together with Lemma 9 that $\mathrm{Aut}_k(\mathfrak{C})$ contains two subgroups, $\mathrm{Pic}(\mathfrak{C})$

and $\mathrm{Aut}_{k\text{-alg}}(A)$. Moreover, Theorem 11 implies that these two subgroups generate the group $\mathrm{Aut}_k(\mathfrak{C})$. Thus it is straightforward to see that the following theorem holds.

Theorem 12 ([2, Theorem 4.9]). *Assume that an additive full subcategory $\mathfrak{C}$ is stable under $\mathrm{Aut}_{k\text{-}alg}(A)$ and assume that $A \in \mathfrak{C}$. Then there is an isomorphism of groups*

$$\mathrm{Aut}_k(\mathfrak{C}) \cong \mathrm{Aut}_{k\text{-}alg}(A) \ltimes \mathrm{Pic}(\mathfrak{C}).$$

As an application, we can prove the following structure theorem of $\mathrm{Aut}_k(\mathrm{CM}(A))$. Recall that a local ring $(A, \mathfrak{m})$ is said to have only an isolated singularity if $A_\mathfrak{p}$ is a regular local ring for all prime ideals $\mathfrak{p}$ except $\mathfrak{m}$.

Theorem 13. *Let A be a Cohen-Macaulay local k-algebra with dimension d. Suppose that A has only an isolated singularity. Then there are isomorphisms of groups*

$$\mathrm{Aut}_k(\mathrm{CM}(A)) \cong \begin{cases} \mathrm{Aut}_{k\text{-}alg}(A) & \text{if } \dim A \neq 2, \\ \mathrm{Aut}_{k\text{-}alg}(A) \ltimes C\ell(A) & \text{if } \dim A = 2. \end{cases}$$

Proof. If $\dim A = 0$, then $\mathrm{CM}(A) = A\text{-mod}$ and hence $\mathrm{Pic}(\mathrm{CM}(A)) = \mathrm{Pic}\, A$ is a trivial group by Example 3. If $\dim A = 1$, then $\mathrm{CM}(A) = d^{\geq 1}(A)$ and we have shown in Example 5 that $\mathrm{Pic}(\mathrm{CM}(A))$ is again a trivial group. If $d = 2$ then A is a normal domain and we have $\mathrm{CM}(A) = \mathrm{ref}(A)$, hence $\mathrm{Pic}(\mathrm{CM}(A)) \cong C\ell(A)$ by Example 4. If $d \geq 3$, then we see from Theorem 6 that $\mathrm{Pic}(\mathrm{CM}(A))$ is a trivial group. Therefore the assertion holds by Theorem 12. □

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1. INTRODUCTION

This is a work still in progress, based on the joint research of Rodney Y. Sharp, Yuji Yoshino and Takeshi Yoshizawa.

In this note R always denotes a commutative Noetherian complete local ring of prime characteristic $p > 0$ and f is the Frobenius map over R, i.e. $f(r) = r^p$ for all $r \in R$.

The skew polynomial ring $R[x, f]$ is the set of all polynomials $\sum_{i=0}^{n} r_i x^i$ ($r_i \in R$) with commuting relations $xs = s^p x$ for all $s \in R$. Note that a left $R[x, f]$-module $\mathbf{H}$ is a left R-module H with action of x from the left so that $x \cdot rh = r^p x \cdot h$ for $r \in R$ and $h \in H$. Similarly, a right $R[x, f]$-module $\mathbf{M}$ is a right R-module M with action of x from the right so that $mr^p \cdot x = m \cdot xr$ for $r \in R$ and $m \in M$.

The main result of this note is concerning the relationship between the category of left $R[x, f]$-modules and the category of right $R[x, f]$-modules, where R is an F-finite complete local ring.

We define an (R, R)-bimodule R^1 to be $R^1 = R$ as an abelian group with the (R, R)-bimodule structure defined by

$$r \cdot t \cdot s = rs^p t \quad \text{for all } r, s \in R \text{ and } t \in R^1.$$

We denote the set of all left R-homomorphisms by $\mathrm{Hom}_{lR}(M, N)$ for left R-modules M and N. Similarly, for right R-modules M and N, we denote the set of all right R-homomorphisms by $\mathrm{Hom}_{rR}(M, N)$.

If a left R-module $\mathbf{H}$ has a left action of x, then a left R-homomorphism $\alpha : R^1 \otimes_R H \to H$ is specified, i.e. $\alpha(r \otimes h) = rx \cdot h$ for $r \in R$ and $h \in H$. Conversely, if an R-module H is given together with a left R-homomorphism $\alpha : R^1 \otimes_R H \to H$, then H has a left action of x by $x \cdot h = \alpha(1 \otimes h)$ for $h \in H$. Therefore we denote by (H, α) to describe a left $R[x, f]$-module $\mathbf{H}$, where H is an underlying R-module and $\alpha \in \mathrm{Hom}_{lR}(R^1 \otimes_R H, H)$. Note that we have an isomorphism of (R, R)-bimodules $\mathrm{Hom}_{lR}(R^1 \otimes_R H, H) \cong \mathrm{Hom}_{lR}(H, \mathrm{Hom}_{lR}(R^1, H))$. Thus, specifying $\alpha \in \mathrm{Hom}_{lR}(R^1 \otimes_R H, H)$ is equivalent to specifying

The detailed version of this paper will be submitted for publication elsewhere.

the corresponding element $\tilde{\alpha}$ in $\mathrm{Hom}_{lR}(H, \mathrm{Hom}_{lR}(R^1, H))$. We denote $\mathbf{H} = (H, \alpha) = [H, \tilde{\alpha}]$.

Similarly, a right $R[x, f]$-module $\mathbf{M}$ is described as (M, β) (resp. $[M, \tilde{\beta}]$) where M is an underlying R-module and $\beta \in \mathrm{Hom}_{rR}(M \otimes_R R^1, M)$ (resp. $\tilde{\beta} \in \mathrm{Hom}_{rR}(M, \mathrm{Hom}_{rR}(R^1, M))$).

Example 1.1. The Frobenius mapping on R defines a natural structure of a left $R[x, f]$-module on R, i.e.

$$x \cdot r = f(r) = r^p \quad (r \in R).$$

Example 1.2. Let K be a perfect field with a ring homomorphism $R \to K$. Then K is a right $R[x, f]$-module by the following right action of x.

$$r \cdot x = r^{1/p} \quad (r \in K).$$

A left $R[x, f]$-homomorphism $\mathbf{H} = (H, \alpha) \to \mathbf{H}' = (H', \alpha')$ of left $R[x, f]$-modules is an R-homomorphism $\varphi : H \to H'$ which commutes with the left action of x. In other words, φ makes the following diagram commutative:

$$\begin{array}{ccc} R^1 \otimes_R H & \xrightarrow{\alpha} & H \\ {\scriptstyle 1\otimes\varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ R^1 \otimes_R H' & \xrightarrow{\alpha'} & H'. \end{array}$$

A right $R[x, f]$-homomorphism is defined similarly.

2. CATEGORY EQUIVALENCE

For a local ring $(R, \mathfrak{m})$, we denote the injective hull of the R-module $R/\mathfrak{m}$ by $E_R(R/\mathfrak{m})$. First we note the following result.

Lemma 2.1. [3, Lemma 3.6] *Let $g : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism of local rings and assume that g is finite. Then there is an isomorphism of S-modules* $\mathrm{Hom}_R(S, E_R(R/\mathfrak{m})) \cong E_S(S/\mathfrak{n})$.

We note that $\mathrm{Hom}_{lR}(R^1, E_R(R/\mathfrak{m}))$ is an (R, R)-bimodule on which the action of R is given by

$$(s\varphi s')(r) = \varphi(r \cdot s)s' = rs^p s' \varphi(1)$$

for $s, s' \in R$, $r \in R^1$ and $\varphi \in \mathrm{Hom}_{lR}(R^1, E_R(R/\mathfrak{m}))$.

Similarly, $\mathrm{Hom}_{rR}(R^1, E_R(R/\mathfrak{m}))$ is an (R, R)-bimodule on which the action of R is given by

$$(s\psi s')(r) = s\psi(s'r)$$

for $s, s' \in R$, $r \in R^1$ and $\psi \in \mathrm{Hom}_{rR}(R^1, E_R(R/\mathfrak{m}))$.

Using the lemma above, we can show the following lemma which enables us to convert left R-homomorphisms to right R-homomorphisms.

Lemma 2.2. *Suppose that R is an F-finite local ring with maximal ideal $\mathfrak{m}$. Then there is an (R,R)-bimodule isomorphism*

$$\mathrm{Hom}_{rR}(R^1, E_R(R/\mathfrak{m})) \cong \mathrm{Hom}_{lR}(R^1, E_R(R/\mathfrak{m})).$$

Remark 2.3. (1) Let R be F-finite as in Lemma 2.2. Then, by Lemma 2.1, there exists a right R-module isomorphism $\tilde{\beta} : E_R(R/\mathfrak{m}) \to \mathrm{Hom}_{rR}(R^1, E_R(R/\mathfrak{m}))$. Therefore the mapping

$$\tilde{\beta} \in \mathrm{Hom}_{rR}(E_R(R/\mathfrak{m}), \mathrm{Hom}_{rR}(R^1, E_R(R/\mathfrak{m}))$$

defines a right $R[x,f]$-module structure on $E_R(R/\mathfrak{m})$.

(2) Let k be a perfect field, $S = k[[t_1, t_2, \dots, t_n]]$ and $\mathfrak{n}$ be maximal ideal of S. Then it is well known that $E_S(S/\mathfrak{n})$ is the inverse polynomial system $k[t_1^{-1}, t_2^{-1}, \dots, t_n^{-1}]$. We set $E_S^{(p)}(S/\mathfrak{n}) = k[t_1^{-p}, t_2^{-p}, \dots, t_n^{-p}]$ as the p-th Veronese subring of $k[t_1^{-1}, t_2^{-1}, \dots, t_n^{-1}]$. We define the mapping $\pi : E_S(S/\mathfrak{n}) \to E_S^{(p)}(S/\mathfrak{n})$ which sends a polynomial $\sum c_{i_1,\dots,i_n} t_1^{-i_1} \cdots t_n^{-i_n}$ to its p-multiple part $\sum_{i_1 \equiv \dots \equiv i_n \equiv 0 \pmod p} c_{i_1,\dots,i_n} t_1^{-i_1} \cdots t_n^{-i_n}$. Under these circumstances, we should note that $E_S(S/\mathfrak{n})$ has a right x-action by defining

$$e \cdot x = \pi(e)^{1/p} \quad (e \in E_S(S/\mathfrak{n})).$$

This action is called the standard right x-action on $E_S(S/\mathfrak{n})$. Moreover, given a residue ring $R = S/I$ where I is an ideal of S, the standard right x-action on $E_S(S/\mathfrak{n})$ induces a right x-action on $E_R(S/\mathfrak{n}) = (0 :_{E_S(S/\mathfrak{n})} I)$. Hence $E_R(S/\mathfrak{n})$ naturally has a structure of right $R[x,f]$-module.

In the rest of this section, let R be an F-finite complete local ring with maximal ideal $\mathfrak{m}$ and let E be the injective hull of the R-module $R/\mathfrak{m}$. We denote by $(-)^\vee$ the Matlis-duality functor $\mathrm{Hom}_R(-,E)$. Recall that an R-module M is called a Matlis reflexive R-module if the natural homomorphism $M \to M^{\vee\vee}$ is an isomorphism. All Artinian (resp. Noetherian) R-modules are Matlis reflexive. Furthermore it is known by Enochs [1] that an R-module M is Matlis reflexive if and only if it can be embedded into a short exact sequence

$$0 \longrightarrow A \longrightarrow M \longrightarrow N \longrightarrow 0,$$

where A is an Artinian R-module and N is a Noetherian R-module.

Let $\mathcal{L}$ be the category of all left $R[x,f]$-modules which are Matlis reflexive as R-modules and left $R[x,f]$-homomorphisms. Similarly let $\mathcal{R}$ be the category of right $R[x,f]$-modules which are Matlis reflexive as R-modules and right $R[x,f]$-homomorphisms.

Now suppose we are given a left $R[x, f]$-module $\mathbf{H} = (H, \alpha)$ with $\alpha \in \mathrm{Hom}_{lR}(R^1 \otimes_R H, H)$. We shall construct a right $R[x, f]$-module structure on $H^\vee$. Taking the Matlis dual of the left R-homomorphism $\alpha : R^1 \otimes_R H \to H$, we have a right R-homomorphism

$$\alpha^\vee : H^\vee \to \mathrm{Hom}_{lR}(R^1 \otimes_R H, E).$$

Combine this with the isomorphisms of (R, R)-bimodules which obtained from the adjoint formula and Lemma 2.2, we can get a right R-homomorphism

$$D(\alpha) : H^\vee \otimes_R R^1 \to H^\vee,$$

which makes $H^\vee$ a right $R[x, f]$-module. We define $\mathbf{D}(\mathbf{H}) = \mathbf{D}(H, \alpha) = (H^\vee, D(\alpha))$.

In such a way we have defined the contravariant functor $\mathbf{D}$ from the category of left $R[x, f]$-modules to the category of right $R[x, f]$-modules.

As in the same manner as above, we can define a contravariant functor $\mathbf{D}'$ from the category of right $R[x, f]$-modules to the category of left $R[x, f]$-modules.

We can show that these contravariant functors give a duality between the categories $\mathcal{L}$ and $\mathcal{R}$. The following theorem is a main result of this note.

Theorem 2.4. *Let R be an F-finite complete local ring. Then there exist contravariant functors $\mathbf{D} : \mathcal{L} \to \mathcal{R}^{op}$ and $\mathbf{D}' : \mathcal{R} \to \mathcal{L}^{op}$ which give the equivalences of categories.*

3. AN APPLICATION

Definition 3.1. (1) A left $R[x, f]$-module $\mathbf{H}$ is called x-torsion free if it satisfies the condition that $x \cdot h = 0$ implies $h = 0$ for $h \in \mathbf{H}$.

(2) A right $R[x, f]$-module $\mathbf{M}$ is called x-divisible if it satisfies the condition that for any $m \in \mathbf{M}$ there is $m' \in \mathbf{M}$ such that $m = m' \cdot x$.

Using Theorem 2.4 we can show the following theorem, in which the implication (1) $\Rightarrow$ (2) is known to hold by R. Y. Sharp [2, Theorem 3.5] without the assumption that R is F-finite.

Theorem 3.2. *Let R be an F-finite complete local ring with maximal ideal $\mathfrak{m}$, and let E be the injective hull of $R/\mathfrak{m}$. Then the following two conditions are equivalent.*

(1) *E has a left $R[x, f]$-module structure by which E is x-torsion free.*

(2) *R is a reduced F-pure ring.*

The following lemma is necessary to prove Theorem 3.2.

Lemma 3.3. *The R-module E has a left $R[x,f]$-module structure by which E is x-torsion free if and only if R has a right $R[x,f]$-module structure by which R is x-divisible.*

Proof. Assume that E is a x-torsion free left $R[x,f]$-module. As noted in Section 1, it is described as $(E,\alpha)=[E,\tilde{\alpha}]$, where $\alpha\in \mathrm{Hom}_{lR}(R^1\otimes E,E)$ and $\tilde{\alpha}\in \mathrm{Hom}_{lR}(E,\mathrm{Hom}_{lR}(R^1,E))$ such that

$$\tilde{\alpha}(e)(r)=rx\cdot e \quad (r\in R,\ e\in E).$$

Since E is a x-torsion free, $\tilde{\alpha}$ is injective. It follows from the definition of $\mathbf{D}'$ that $\tilde{\alpha}$ is a composition of $\mathbf{D}(\alpha)^{\vee}$ with various isomorphisms. Therefore $\tilde{\alpha}$ is injective if and only if so is $\mathbf{D}(\alpha)^{\vee}$, which is equivalent to that $\mathbf{D}(\alpha)$ is surjective. $\mathbf{D}(\alpha)$ induces a right $R[x,f]$-module structure on R such that

$$r\cdot x=\mathbf{D}(\alpha)(r\otimes 1) \quad (r\in R).$$

Since $\mathbf{D}(\alpha)$ is surjective, R is x-divisible. The converse is proved in a similar manner. □

Now we are able to prove Theorem 3.2.

Proof. (1) $\Rightarrow$ (2): By the above lemma R is a x-divisible right $R[x,f]$-module. To prove that R is reduced, we have only to show that $r^p=0$ implies that $r=0$ for $r\in R$. Suppose $r^p=0$. Since R is x-divisible, we can find an element $s\in R$ with $s\cdot x=1$. Then we have $r=(s\cdot x)r=sr^p\cdot x=0\cdot x=0$.

To show that R is F-pure, define a mapping $\psi: R\to R^p$ by $\psi(r)=(r\cdot x)^p$ for any $r\in R$. Since $\psi(rs^p)=(rs^p\cdot x)^p=(r\cdot xs)^p=(r\cdot x)^p s^p=\psi(r)s^p$ holds for $s\in R$, ψ is an R^p-linear mapping. Furthermore ψ is surjective. In fact, for any $s^p\in R^p$, we can take $r\in R$ with $r\cdot x=s$, then $\psi(r)=s^p$. Thus the R^p-linear mapping ψ gives a splitting of the natural embedding $R^p\to R$ as R^p-modules. Hence R is F-pure.

(2) $\Rightarrow$ (1): Now assume that R is a reduced F-pure ring. It is enough to show that R has a structure of x-divisible right $R[x,f]$-module. Since R is F-pure, there exists a surjective R^p-linear mapping $\varphi: R\to R^p$. We define the right action of x on R by $r\cdot x=\varphi(r)^{1/p}$ for all $r\in R$. This is well-defined, since R is reduced. Choose $r\in R$ so that $\varphi(r)=1$. Then, for any $s\in R$, we have $rs^p\cdot x=r\cdot xs=1s=s$. Therefore R is x-divisible. □

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MODULES IN RESOLVING SUBCATEGORIES

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1. Introduction

In the 1960s, Auslander and Bridger [2] introduced the notion of a resolving subcategory of an abelian category with enough projectives. They proved that in the category of finitely generated modules over a left and right noetherian ring, the full subcategory consisting of all modules of Gorenstein dimension zero, which are now also called totally reflexive modules, is resolving.

In this note, we study the distribution of modules in a given resolving subcategory of $\operatorname{mod} R$. To be more precise, let $\mathcal{X}$ be a resolving subcategory of $\operatorname{mod} R$. There are three main purposes. The first purpose is to find a better module $X' \in \mathcal{X}$ when a module $X \in \mathcal{X}$ is given. The second purpose is to count the (minimum) number of steps required to construct such X' from X. The third purpose is to consider how many nonisomorphic indecomposable modules are in $\mathcal{X}$.

Throughout this note, let R be a commutative noetherian ring. All R-modules considered in this note are assumed to be finitely generated. We denote by $\operatorname{mod} R$ the category of finitely generated R-modules. By a *subcategory* of $\operatorname{mod} R$, we always mean a full subcategory of $\operatorname{mod} R$ which is closed under isomorphisms.

In this note, many things are omitted for lack of space. For details, see [7].

2. Foundations

In this section, we define the resolving closures and the nonfree loci of an R-module and a subcategory of $\operatorname{mod} R$, and study their basic properties.

Definition 2.1. A subcategory $\mathcal{X}$ of $\operatorname{mod} R$ is called *resolving* if $\mathcal{X}$ satisfies the following conditions.

(1) $\mathcal{X}$ contains all projective R-modules.
(2) $\mathcal{X}$ is closed under direct summands: if M is in $\mathcal{X}$ and N is a direct summand of M, then N is also in $\mathcal{X}$.
(3) $\mathcal{X}$ is closed under extensions: for any exact sequence $0 \to L \to M \to N \to 0$ in $\operatorname{mod} R$, if L and N are in $\mathcal{X}$, then so is M.

(4) $\mathcal{X}$ is closed under kernels of epimorphisms: for any exact sequence $0 \to L \to M \to N \to 0$ in $\operatorname{mod} R$, if M and N are in $\mathcal{X}$, then so is L.

A lot of important subcategories of $\operatorname{mod} R$ are known to be resolving. Here, let us make a list of examples.

Example 2.2. (1) It is trivial that the subcategory $\operatorname{mod} R$ of $\operatorname{mod} R$ is resolving.

(2) It is obvious that the subcategory $\operatorname{proj} R$ of $\operatorname{mod} R$ consisting of all projective R-modules is resolving.

(3) Let I be an ideal of R. Then the subcategory of $\operatorname{mod} R$ consisting of all R-modules M with $\operatorname{grade}(I, M) \geq \operatorname{grade}(I, R)$ is resolving. This can be shown by using the equality $\operatorname{grade}(I, M) = \inf\{i \in \mathbb{Z} \mid \operatorname{Ext}_R^i(R/I, M) \neq 0\}$.

(4) Let R be a Cohen-Macaulay local ring. Then, letting I be the maximal ideal of R in (3), we see that the subcategory $\operatorname{CM}(R)$ of $\operatorname{mod} R$ consisting of all maximal Cohen-Macaulay R-modules is resolving.

(5) An R-module C is called *semidualizing* if the natural homomorphism $R \to \operatorname{Hom}_R(C, C)$ is an isomorphism and $\operatorname{Ext}_R^i(C, C) = 0$ for every $i > 0$. An R-module M is called *totally C-reflexive*, where C is a semidualizing R-module, if the natural homomorphism $M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, C), C)$ is an isomorphism and $\operatorname{Ext}_R^i(M, C) = \operatorname{Ext}_R^i(\operatorname{Hom}_R(M, C), C) = 0$ for every $i > 0$. The subcategory $\mathcal{G}_C(R)$ of $\operatorname{mod} R$ consisting of all totally C-reflexive R-modules is resolving by [1, Theorem 2.1].

(6) A totally R-reflexive R-module is simply called *totally reflexive*. The subcategory $\mathcal{G}(R)$ of $\operatorname{mod} R$ consisting of all totally reflexive R-modules is resolving by (5); see also [2, (3.11)].

(7) Let n be a nonnegative integer, and let K be an R-module (which is not necessarily finitely generated). Then the subcategory of $\operatorname{mod} R$ consisting of all R-modules M with $\operatorname{Tor}_i^R(M, K) = 0$ for $i > n$ (respectively, $i \gg 0$) and the subcategory of $\operatorname{mod} R$ consisting of all R-modules M with $\operatorname{Ext}_R^i(M, K) = 0$ for $i > n$ (respectively, $i \gg 0$) are both resolving.

(8) Let R be a local ring. We say that an R-module M is *bounded* if there is an integer s such that $\beta_i^R(M) \leq s$ for all $i \geq 0$, where $\beta_i^R(M)$ denotes the ith Betti number of M. The subcategory of $\operatorname{mod} R$ consisting of all bounded R-modules is resolving. This can be shown by using the equality $\beta_i^R(M) = \dim_k \operatorname{Tor}_i^R(M, k)$, where k is the residue field of R.

(9) Let R be local. We say that an R-module M has *complexity* c if c is the least nonnegative integer d such that there exists a real number r satisfying the inequality $\beta_i^R(M) \le r i^{d-1}$ for $i \gg 0$. The subcategory of $\operatorname{mod} R$ consisting of all R-modules having finite complexity is resolving.

(10) Let R be local. We say that an R-module M has *lower complete intersection zero* if M is totally reflexive and has finite complexity. The subcategory of $\operatorname{mod} R$ consisting of all R-modules of lower complete intersection dimension zero is resolving by (6) and (9).

Now we define the resolving closures of a subcategory of $\operatorname{mod} R$ and an R-module.

Definition 2.3. For a subcategory $\mathcal{X}$ of $\operatorname{mod} R$, we denote by $\operatorname{res} \mathcal{X}$ (or $\operatorname{res}_R \mathcal{X}$ when there is some fear of confusion) the resolving sucategory of $\operatorname{mod} R$ generated by $\mathcal{X}$, namely, the smallest resolving subcategory of $\operatorname{mod} R$ containing $\mathcal{X}$. If $\mathcal{X}$ consists of a single module X, then we simply write $\operatorname{res} X$ (or $\operatorname{res}_R X$).

Next we recall the definition of the nonfree locus of an R-module and define the nonfree locus of a subcategory of $\operatorname{mod} R$.

Definition 2.4. (1) We denote by $\mathrm{NF}(X)$ (or $\mathrm{NF}_R(X)$) the *nonfree locus* of an R-module X, namely, the set of prime ideals $\mathfrak{p}$ of R such that the $R_\mathfrak{p}$-module $X_\mathfrak{p}$ is nonfree.

(2) We define the *nonfree locus* of a subcategory $\mathcal{X}$ of $\operatorname{mod} R$ as the union of $\mathrm{NF}(X)$ where X runs through all (nonisomorphic) R-modules in $\mathcal{X}$, and denote it by $\mathrm{NF}(\mathcal{X})$ (or $\mathrm{NF}_R(\mathcal{X})$).

Example 2.5. Let R be a Cohen-Macaulay local ring. Then the nonfree locus $\mathrm{NF}(\mathrm{CM}(R))$ coincides with the singular locus $\operatorname{Sing} R$ of R.

The nonfree locus of a module can be described as the support of an Ext module.

Proposition 2.6. *Let* $\sigma : 0 \to Y \to P \to X \to 0$ *be an exact sequence of R-modules such that P is projective. Then one has* $\mathrm{NF}(X) = \operatorname{Supp} \operatorname{Ext}_R^1(X, Y)$. *Hence* $\mathrm{NF}(X) = \operatorname{Supp} \operatorname{Ext}_R^1(X, \Omega X)$.

Recall that a subset Z of $\operatorname{Spec} R$ is called *specialization-closed* provided that if $\mathfrak{p} \in Z$ and $\mathfrak{q} \in \operatorname{Spec} R$ with $\mathfrak{p} \subseteq \mathfrak{q}$ then $\mathfrak{q} \in Z$. Note that every closed subset of $\operatorname{Spec} R$ is specialization-closed.

Corollary 2.7. (1) *The nonfree locus of an R-module is a closed subset of* $\operatorname{Spec} R$ *in the Zariski topology.*

(2) *The nonfree locus of a subcategory of* $\operatorname{mod} R$ *is specialization-closed.*

3. THE STRUCTURE OF RESOLVING CLOSURES

In this section, we investigate the structure of the resolving closure of an R-module. For this, we first build a filtration of subcategories in the resolving closure of a subcategory of $\operatorname{mod} R$, and inductively construct the resolving closure. This is an imitation of the notion of thickenings in the thick closure of a subcategory of a triangulated category, which were introduced by Avramov, Buchweitz, Iyengar and Miller [3]. Recall that the *additive closure* $\operatorname{add}\mathcal{X}$ (or $\operatorname{add}_R \mathcal{X}$) of a subcategory $\mathcal{X}$ of $\operatorname{mod} R$ is defined to be the subcategory of $\operatorname{mod} R$ consisting of all direct summands of finite direct sums of modules in $\mathcal{X}$.

Definition 3.1. Let $\mathcal{X}$ be a subcategory of $\operatorname{mod} R$. For a nonnegative integer n, we inductively define a subcategory $\operatorname{res}^n \mathcal{X}$ (or $\operatorname{res}^n_R \mathcal{X}$) of $\operatorname{mod} R$ as follows:

(1) Set $\operatorname{res}^0 \mathcal{X} = \operatorname{add}(\mathcal{X} \cup \{R\})$.
(2) For $n \geq 1$, let $\operatorname{res}^n \mathcal{X}$ be the additive closure of the subcategory of $\operatorname{mod} R$ consisting of all R-modules Y having an exact sequence of either of the following two forms:

$$0 \to A \to Y \to B \to 0,$$
$$0 \to Y \to A \to B \to 0$$

where $A, B \in \operatorname{res}^{n-1} \mathcal{X}$.

If $\mathcal{X}$ consists of a single module X, then we simply write $\operatorname{res}^n X$ instead of $\operatorname{res}^n \mathcal{X}$.

Remark 3.2. Let $\mathcal{X}, \mathcal{Y}$ be subcategories of $\operatorname{mod} R$, and let n be a nonnegative integer. Then the following hold.

(1) There is an ascending chain $\{0\} \subseteq \operatorname{res}^0 \mathcal{X} \subseteq \operatorname{res}^1 \mathcal{X} \subseteq \cdots \subseteq \operatorname{res}^n \mathcal{X} \subseteq \cdots \subseteq \operatorname{res} \mathcal{X}$ of subcategories of $\operatorname{mod} R$.
(2) The equality $\operatorname{res} \mathcal{X} = \bigcup_{n \geq 0} \operatorname{res}^n \mathcal{X}$ holds.

For a subcategory $\mathcal{X}$ of $\operatorname{mod} R$, we denote by $\operatorname{ind}\mathcal{X}$ (or $\operatorname{ind}_R \mathcal{X}$) the set of nonisomorphic indecomposable R-modules in $\mathcal{X}$.

Example 3.3. (1) Let us consider the 1-dimensional complete local hypersurface $R = \mathbb{C}[[x, y]]/(x^2)$ over the complex number field. Then the subcategory $\operatorname{res}^1(xR)$ coincides with $\operatorname{CM}(R)$, and there exist infinitely many nonisomorphic indecomposable R-modules in $\operatorname{res}^1(xR)$.
(2) Let k be a field. We consider the 2-dimensional hypersurface $R = k[[x, y, z]]/(x^2)$. Put $\mathfrak{p}(f) = (x, y-zf)R$ for an element $f \in k[[z]] \subseteq R$. Then $\mathfrak{p}(f)$ is an indecomposable R-module in $\operatorname{res}^1(xR)$, and

there exist uncountably many nonisomorphic indecomposable R-modules in $\operatorname{res}^1(xR)$.

Next, we study the structure of the nonfree locus of an R-module. We establish the following lemma, which is proved by taking advantage of an idea used in the proof of [4, Theorem 1].

Lemma 3.4. *Let R be a local ring with maximal ideal $\mathfrak{m}$. Let*

$$\sigma : 0 \to L \xrightarrow{f} M \to N \to 0$$

be an exact sequence of R-modules. Let x be an element in $\mathfrak{m}$. Then there is an exact sequence

$$0 \to L \xrightarrow{\binom{x}{f}} L \oplus M \to K \to 0.$$

If this splits, then so does σ.

Using the above lemma, one can prove the following proposition, which will play an essential role in the proofs of the main results.

Proposition 3.5. *Let X be an R-module. Let $\mathfrak{p}$ be a prime ideal in $\mathrm{NF}(X)$ and x an element in $\mathfrak{p}$. Then there is a commutative diagram*

$$\text{(3.5.1)} \qquad \begin{array}{ccccccccc} \sigma : 0 & \longrightarrow & \Omega X & \xrightarrow{f} & R^n & \longrightarrow & X & \longrightarrow & 0 \\ & & {\scriptstyle x}\downarrow & & \downarrow & & \| & & \\ x\sigma : 0 & \longrightarrow & \Omega X & \longrightarrow & X_1 & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

of R-modules with exact rows, and the following statements hold:

(1) $X_1 \in \operatorname{res}^2 X$,
(2) $V(\mathfrak{p}) \subseteq \mathrm{NF}(X_1) \subseteq \mathrm{NF}(X)$,
(3) $D(x) \cap \mathrm{NF}(X_1) = \emptyset$.

Using this proposition, one can prove the following theorem.

Theorem 3.6. *For any R-module X and any subset W of $\mathrm{NF}(X)$ which is closed in $\operatorname{Spec} R$, there exists an R-module $Y \in \operatorname{res} X$ such that $W = \mathrm{NF}(Y)$.*

Let X be an R-module. For an R-module $Y \in \operatorname{res} X$, we now consider how many resolving operations are needed to take to construct Y from X. Here, resolving operations mean extensions and kernels of epimorphisms. For this, we introduce the following invariant which measures the minimum number of required resolving operations. This is an imitation of a level in a triangulated category defined in [3].

Definition 3.7. For two R-modules X and Y, we define

$$\mathrm{step}(X,Y) = \inf\{n \geq 0 \mid Y \in \mathrm{res}_R^n X\}.$$

Remark 3.8. Let X be an R-module.

(1) One has $\mathrm{step}(X,Y) = 0$ for every R-module $Y \in \mathrm{res}^1 X = \mathrm{add}(X \oplus R)$. In particular, $\mathrm{step}(X,X) = \mathrm{step}(X,R) = \mathrm{step}(X,0) = 0$.

(2) One has $\mathrm{step}(X,Y) < \infty$ for an R-module Y if and only if Y belongs to $\mathrm{res}\, X$.

In general, the invariant $\mathrm{step}(-,-)$ does not induce a distance function. However, it satisfies the triangle inequality.

Proposition 3.9. *Let X, Y, Z be R-modules.*

(1) *Let m, n be nonnegative integers. If $Y \in \mathrm{res}^m X$ and $Z \in \mathrm{res}^n Y$, then $Z \in \mathrm{res}^{m+n} X$.*

(2) *The inequality $\mathrm{step}(X,Z) \leq \mathrm{step}(X,Y) + \mathrm{step}(Y,Z)$ holds.*

Let Z be a subset of $\mathrm{Spec}\, R$. For a prime ideal $\mathfrak{p}$ in Z, we define the *height* of $\mathfrak{p}$ with respect to Z as the supremum of $\mathrm{ht}(\mathfrak{p}/\mathfrak{q})$ where $\mathfrak{q}$ runs through all prime ideals in Z that are contained in $\mathfrak{p}$. We denote it by $\mathrm{ht}_Z(\mathfrak{p})$.

The following theorem is one of the main results.

Theorem 3.10. *Let X be an R-module and let $\mathfrak{p}$ be a prime ideal in $\mathrm{NF}(X)$. Then there exists an R-module $Y \in \mathrm{res}\, X$ satisfying the following three conditions:*

(1) $\mathrm{step}(X,Y) \leq 2\, \mathrm{ht}_{\mathrm{NF}(X)}(\mathfrak{p})$,

(2) $\mathfrak{p} \in \mathrm{NF}(Y)$,

(3) $\mathrm{ht}_{\mathrm{NF}(Y)}(\mathfrak{p}) = 0$.

Applying this theorem to a local ring R, we get the following result.

Corollary 3.11. *Let R be a local ring. Then for every nonfree R-module X, there exists a nonfree R-module Y in $\mathrm{res}\, X$ satisfying the following conditions:*

(1) $\mathrm{step}(X,Y) \leq 2 \dim \mathrm{NF}(X)$,

(2) *Y is free on the punctured spectrum of R.*

Restricting the above corollary to the Cohen-Macaulay case, we obtain the following result on maximal Cohen-Macaulay modules.

Corollary 3.12. *Let R be a Cohen-Macaulay local ring. Then for any nonfree maximal Cohen-Macaulay R-module X, there exists a nonfree maximal Cohen-Macaulay R-module Y satisfying the following two conditions:*

(1) $\mathrm{step}(X,Y) \leq 2\dim \mathrm{Sing}\, R$,
(2) Y *is free on the punctured spectrum of* R.

Forgetting the first condition on the module Y in Corollary 3.11, we obtain the following result.

Corollary 3.13. *Let* R *be a local ring and* $\mathcal{X}$ *a resolving subcategory of* $\mathrm{mod}\, R$. *If there exists a nonfree* R*-module in* $\mathcal{X}$, *then there exists a nonfree* R*-module in* $\mathcal{X}$ *which is free on the punctured spectrum of* R.

4. Resolving subcategories of countable type

In this section, we investigate resolving subcategories in which there exist only countably many nonisomorphic indecomposable modules. The following proposition plays a key role for this goal, which is proved by using Theorem 3.6.

Proposition 4.1. *For a subcategory* $\mathcal{X}$ *of* $\mathrm{mod}\, R$ *one has an inclusion of sets:*

$$\mathrm{NF}(\mathcal{X}) \subseteq \left\{ \sqrt{\mathrm{Ann}\,\mathrm{Ext}_R^1(Y,Z)} \;\middle|\; Y,Z \in \mathrm{ind}(\mathrm{res}\,\mathcal{X}) \right\}.$$

Definition 4.2. We say that a subcategory $\mathcal{X}$ of $\mathrm{mod}\, R$ has *coubtable type* if the set $\mathrm{ind}\,\mathcal{X}$ is countable.

We say that a Cohen-Macaulay local ring R has *countable Cohen-Macaulay representation type* if $\mathrm{CM}(R)$ has countable type.

The result below is a direct consequence of Proposition 4.1.

Corollary 4.3. *Let* $\mathcal{X}$ *be a subcategory of* $\mathrm{mod}\, R$. *If* $\mathrm{res}\,\mathcal{X}$ *has countable type, then* $\mathrm{NF}(\mathcal{X})$ *is at most a countable set.*

The converse of this corollary does not necessarily hold.

The lemma below is proved by using so-called countable prime avoidance; see [6, Lemma 2.2] for the proof.

Lemma 4.4. *Let* R *be a local ring with residue field* k, *and assume either that* R *is complete or that* k *is uncountable. Let* Z *be a specialization-closed subset of* $\mathrm{Spec}\, R$. *If* Z *is at most countable, then* $\dim Z \leq 1$.

Corollaries 4.3, 2.7(2) and Lemma 4.4 yield the following theorem, which is one of the main results of this note.

Theorem 4.5. *Let* R *be a local ring with residue field* k, *and assume either that* R *is complete or that* k *is uncountable. Let* $\mathcal{X}$ *be a subcategory of* $\mathrm{mod}\, R$ *such that* $\mathrm{res}\,\mathcal{X}$ *has countable type. Then* $\dim \mathrm{NF}(\mathcal{X}) \leq 1$.

Combining this theorem with Corollary 3.11 gives the following result.

Corollary 4.6. *Let R be a local ring with residue field k, and assume either that R is complete or that k is uncountable. Let X be a nonfree R-module such that* $\operatorname{res} X$ *has countable type. Then there exists a nonfree R-module $Y \in \operatorname{res} X$ which is free on the punctured spectrum of R and satisfies* $\operatorname{step}(X,Y) \leq 2$.

We immediately get the following corollary from Theorem 4.5.

Corollary 4.7. *Let R be a local ring with residue field k, and assume either that R is complete or that k is uncountable. Let $\mathcal{X}$ be a resolving subcategory of* $\operatorname{mod} R$ *of countable type. Then* $\dim \operatorname{NF}(\mathcal{X}) \leq 1$.

Applying this corollary to the subcategory of maximal Cohen-Macaulay modules over a Cohen-Macaulay local ring (cf. Example 2.2(4)), we can recover a theorem of Huneke and Leuschke.

Corollary 4.8. [4, Theorem 1.3][6, Theorem 2.4] *Let R be a Cohen-Macaulay local ring of countable Cohen-Macaulay representation type. Assume either that R is complete or that the residue field is uncountable. Then* $\dim \operatorname{Sing} R \leq 1$.

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On the third symbolic powers of prime ideals defining space monomial curves

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Let $A = k[X,Y,Z]$ and $k[t]$ be the polynomial rings over a field k. We take positive integers n_1, n_2, n_3 such that $\mathrm{GCD}\{n_1, n_2, n_3\} = 1$. Let $\varphi : A \longrightarrow k[t^{n_1}, t^{n_2}, t^{n_3}]$ be the epimorphism of k-algebras such that

$$\varphi(X) = t^{n_1}, \varphi(Y) = t^{n_2} \text{ and } \varphi(Z) = t^{n_3}.$$

We put $P = \mathrm{Ker}\,\varphi$, which is a prime ideal in A of $\mathrm{ht}_A P = 2$. The n-th symbolic power of P is $P^{(n)} = P^n A_P \cap A$. Ein - Lazarsfeld - Smith [1] and Hochster - Huneke [3] proved that the following inclusions hold true.

Theorem 1 *$P^{(2n)} \subseteq P^n$ for any $n > 0$.*

In this talk, we consider the following question.

Question $P^{(3)} \subseteq P^2$? $\quad P^{(4)} \subseteq P^3$?

These inclusions obviously hold if P is a complete intersection, and so we assume that P is not a complete intersection in the rest of this report. Then we have the following result due to Herzog [4].

Theorem 2 *P is generated by the maximal minors of the matrix of the form*

$$\begin{pmatrix} X^{\alpha} & Y^{\beta'} & Z^{\gamma'} \\ Y^{\beta} & Z^{\gamma} & X^{\alpha'} \end{pmatrix},$$

where $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ are positive integers.

We put $a = Z^{\gamma+\gamma'} - X^{\alpha'}Y^{\beta'}$, $b = X^{\alpha+\alpha'} - Y^{\beta}Z^{\gamma'}$ and $c = Y^{\beta+\beta'} - X^{\alpha}Z^{\gamma}$. Then we have the following two relations:

$$X^{\alpha}a + Y^{\beta'}b + Z^{\gamma'}c = 0 \quad \text{and} \quad Y^{\beta}a + Z^{\gamma}b + X^{\alpha'}c = 0.$$

We will find certain elements of the symbolic powers from relations derived from the relations above. For example, if $\gamma \le \gamma'$,

$$(X^{\alpha}a + Y^{\beta'}b + Z^{\gamma'}c)\,b - (Y^{\beta}a + Z^{\gamma}b + X^{\alpha'}c)\,Z^{\gamma'-\gamma}c = 0\,.$$

$$\therefore \quad X^{\alpha}ab + Y^{\beta'}b^2 - Y^{\beta}Z^{\gamma'-\gamma}ac - X^{\alpha'}Z^{\gamma'-\gamma}c^2 = 0\,.$$

$$\therefore \quad X^{\alpha}ab - X^{\alpha'}Z^{\gamma'-\gamma}c^2 = Y^{\beta}Z^{\gamma'-\gamma}ac - Y^{\beta'}b^2\,.$$

We put $\alpha'' = \min\{\alpha, \alpha'\}$ and assume $\beta \le \beta'$. Then

$$X^{\alpha''}\,(X^{\alpha-\alpha''}ab - X^{\alpha'-\alpha''}Z^{\gamma'-\gamma}c^2) = Y^{\beta}\,(Z^{\gamma'-\gamma}ac - Y^{\beta'-\beta}b^2)\,.$$

Because $X^{\alpha''}, Y^{\beta}$ is an A-regular sequence, $\exists d_2 \in P^{(2)}$ such that

$$X^{\alpha''}d_2 = Z^{\gamma'-\gamma}ac - Y^{\beta'-\beta}b^2 \quad \text{and} \quad Y^{\beta}d_2 = X^{\alpha-\alpha''}ab - X^{\alpha'-\alpha''}Z^{\gamma'-\gamma}c^2\,.$$

Lemma 3 *We may assume that one of the following conditions is satisfied;*

Type 1: $\alpha \le \alpha'\,,\ \beta \le \beta'$ *and* $\gamma \le \gamma'\,,$

Type 2: $\alpha > \alpha'\,,\ \beta < \beta'$ *and* $\gamma < \gamma'\,.$

Proof. For example, if $\alpha \le \alpha'$, $\beta > \beta'$ and $\gamma > \gamma'$, then

$$\begin{pmatrix} X^{\alpha} & Y^{\beta'} & Z^{\gamma'} \\ Y^{\beta} & Z^{\gamma} & X^{\alpha'} \end{pmatrix} \longrightarrow \begin{pmatrix} Y^{\beta} & Z^{\gamma} & X^{\alpha'} \\ X^{\alpha} & Y^{\beta'} & Z^{\gamma'} \end{pmatrix} \longrightarrow \begin{pmatrix} X^{\alpha'} & Z^{\gamma} & Y^{\beta} \\ Z^{\gamma'} & Y^{\beta'} & X^{\alpha} \end{pmatrix}.$$

Change the variables Y and Z. Then the matrix is of Type 1 if $\alpha = \alpha'$, and Type 2 if $\alpha < \alpha'$. ∎

In the rest of this talk we always assume that the matrix is of Type 1 or Type 2.

Lemma 4 *Let $\mathfrak{a}$ be an ideal in $k[X,Y]$ generated by monomials*

$$X^{p_0}\,,\ X^{p_1}Y^{q_1}\,,\ \ldots\,,\ X^{p_{r-1}}Y^{q_{r-1}}\,,\ Y^{q_r}\,,$$

where $p_0 \ge p_1 \ge \cdots \ge p_{r-1} > 0$ and $0 = q_0 < q_1 \le \cdots \le q_{r-1} \le q_r$. Then

$$\ell_{k[X,Y]}(\,k[X,Y]/\mathfrak{a}\,) = \sum_{i=0}^{r-1} p_i\,(\,q_{i+1} - q_i\,)\,.$$

Theorem 5 (cf. [5]) *We put $\alpha'' = \min\{\alpha, \alpha'\}$. Then there exists $d_2 \in P^{(2)}$ such that $P^{(2)} = P^2 + (d_2)$ and*

$$\begin{aligned} X^{\alpha''} d_2 &= Z^{\gamma'-\gamma} ac - Y^{\beta'-\beta} b^2 \\ Y^{\beta} d_2 &= X^{\alpha-\alpha''} ab - X^{\alpha'-\alpha''} Z^{\gamma'-\gamma} c^2 \\ Z^{\gamma} d_2 &= -X^{\alpha-\alpha''} a^2 + X^{\alpha'-\alpha''} Y^{\beta'-\beta} bc . \end{aligned}$$

Proof. Recall $X^{\alpha} a + Y^{\beta'} b + Z^{\gamma'} c = 0$ and $Y^{\beta} a + Z^{\gamma} b + X^{\alpha'} c = 0$.

$$\begin{aligned} Z^{\gamma} \cdot X^{\alpha''} d_2 &= a \cdot Z^{\gamma'} c - Y^{\beta'-\beta} b \cdot Z^{\gamma} b \\ &= a(-X^{\alpha} a - Y^{\beta'} b) - Y^{\beta'-\beta} b(-Y^{\beta} a - X^{\alpha'} c) \\ &= -X^{\alpha} a^2 - Y^{\beta'} ab + Y^{\beta'} ab + X^{\alpha'} Y^{\beta'-\beta} bc \\ &= -X^{\alpha} a^2 + X^{\alpha'} Y^{\beta'-\beta} bc . \end{aligned}$$

We put $\mathfrak{a} = P^2 + (d_2)$. It is easy to show $\alpha \neq \alpha'$ or $\beta \neq \beta'$ or $\gamma \neq \gamma'$.

First we consider the case where $\gamma \neq \gamma'$. Because $X^{\alpha''} d_2 = Z^{\gamma'-\gamma} ac - Y^{\beta'-\beta} b^2$ and $\gamma' - \gamma > 0$, we have $X^{\alpha''} d_2 \underset{(Z)}{\equiv} -Y^{\beta'-\beta} b^2 \underset{(Z)}{\equiv} -Y^{\beta'-\beta} X^{2\alpha+2\alpha'}$.

$$\therefore \quad d_2 \underset{(Z)}{\equiv} -X^{2\alpha+2\alpha'-\alpha''} Y^{\beta'-\beta} .$$

Then, as $(Z) + P = (Z) + (X^{\alpha+\alpha'}, X^{\alpha'} Y^{\beta'}, Y^{\beta+\beta'})$, we see

$$(Z) + \mathfrak{a} = (Z) + \begin{pmatrix} X^{2\alpha+2\alpha'}, X^{2\alpha+2\alpha'-\alpha''} Y^{\beta'-\beta}, X^{\alpha+2\alpha'} Y^{\beta'}, X^{\alpha+\alpha'} Y^{\beta+\beta'}, \\ X^{\alpha'+\alpha''} Y^{2\beta'}, X^{\alpha'} Y^{\beta+2\beta'}, Y^{2\beta+2\beta'} \end{pmatrix} .$$

Therefore by Lemma 4, we get

$$\ell_A(A/(Z) + \mathfrak{a}) = 3(\alpha\beta' + \alpha'\beta + \alpha'\beta') .$$

On the other hand, as $\mathfrak{a} \subseteq P^{(2)}$,

$$\ell_A(A/(Z) + \mathfrak{a}) \geq \ell_A(A/(Z) + P^{(2)}) = \mathrm{e}_{(Z)}(A/P^{(2)}) =$$

$$\ell_{A_P}(A_P/P^2 A_P) \cdot \mathrm{e}_{(Z)}(A/P) = 3 \cdot \ell_A(A/(Z) + P) = 3(\alpha\beta' + \alpha'\beta + \alpha'\beta') .$$

$$\therefore \quad (Z) + \mathfrak{a} = (Z) + P^{(2)} .$$

$$\therefore \quad P^{(2)} = (Z) \cap P^{(2)} + \mathfrak{a} = Z P^{(2)} + \mathfrak{a} .$$

$$\therefore \quad P^{(2)} = \mathfrak{a} .$$

Similarly, if $\beta \neq \beta'$ (resp. $\alpha \neq \alpha'$), we see

$$\ell_A(A/(Y) + \mathfrak{a}) = \ell_A(A/(Y) + P^{(2)}) = 3(\alpha\gamma + \alpha'\gamma + \alpha\gamma') ,$$

$$(\text{ resp. } \ell_A(A/(X) + \mathfrak{a}) = \ell_A(A/(X) + P^{(2)}) = 3(\beta\gamma + \beta\gamma' + \beta'\gamma)) ,$$

and we get $\mathfrak{a} = P^{(2)}$. ∎

Theorem 6 (cf. [2], [6]) *Suppose that the matrix*

$$\begin{pmatrix} X^{\alpha} & Y^{\beta'} & Z^{\gamma'} \\ Y^{\beta} & Z^{\gamma} & X^{\alpha'} \end{pmatrix}$$

is of Type 1. Put $\alpha'' = \min\{\alpha, \alpha'-\alpha\}$, $\beta'' = \min\{\beta, \beta'-\beta\}$ *and* $\gamma'' = \min\{\gamma, \gamma'-\gamma\}$. *Then there exist* d_3 *and* d_3' *in* $P^{(3)}$ *such that* $P^{(3)} = PP^{(2)} + (d_3, d_3')$ *and*

$$\begin{aligned} X^{\alpha} d_3 &= -Y^{\beta-\beta''} Z^{\gamma'-\gamma-\gamma''} a^2 c - Y^{\beta'-\beta-\beta''} Z^{\gamma-\gamma''} b^3 \\ Y^{\beta''} d_3 &= Z^{\gamma-\gamma''} b d_2 + X^{\alpha'-\alpha} Z^{\gamma'-\gamma-\gamma''} a c^2 \\ Z^{\gamma''} d_3 &= -Y^{\beta-\beta''} a d_2 + X^{\alpha'-\alpha} Y^{\beta'-\beta-\beta''} b^2 c \end{aligned}$$

$$\begin{aligned} X^{\alpha''} d_3' &= Y^{\beta-\beta''} Z^{\gamma'-\gamma} c d_2 - Y^{\beta'-\beta-\beta''} b^3 \\ Y^{\beta''} d_3' &= X^{\alpha-\alpha''} b d_2 - X^{\alpha'-\alpha-\alpha''} Z^{2\gamma'-2\gamma} c^3 \\ Z^{\gamma} d_3' &= -X^{\alpha-\alpha''} Y^{\beta-\beta''} a d_2 - X^{\alpha'-\alpha-\alpha''} Y^{\beta'-\beta} Z^{\gamma'-\gamma} b c^2 + X^{\alpha'-\alpha''} Y^{\beta'-\beta-\beta''} b^2 c. \end{aligned}$$

Proof. Because the matrix is of Type 1, we have

$$\begin{aligned} X^{\alpha} d_2 &= Z^{\gamma'-\gamma} a c - Y^{\beta'-\beta} b^2 & \cdots \quad (1) \\ Y^{\beta} d_2 &= ab - X^{\alpha'-\alpha} Z^{\gamma'-\gamma} c^2 & \cdots \quad (2) \\ Z^{\gamma} d_2 &= -a^2 + X^{\alpha'-\alpha} Y^{\beta'-\beta} bc & \cdots \quad (3). \end{aligned}$$

In order to find d_3, we notice $a^2 \cdot b = a \cdot ab$. Then

$$(X^{\alpha'-\alpha} Y^{\beta'-\beta} bc - Z^{\gamma} d_2)\, b = a\,(Y^{\beta} d_2 + X^{\alpha'-\alpha} Z^{\gamma'-\gamma} c^2)$$

by (2) and (3). $\therefore \quad -Y^{\beta} a d_2 + X^{\alpha'-\alpha} Y^{\beta'-\beta} b^2 c = Z^{\gamma} b d_2 + X^{\alpha'-\alpha} Z^{\gamma'-\gamma} a c^2$.

$$\therefore \quad Y^{\beta''}(-Y^{\beta-\beta''} a d_2 + X^{\alpha'-\alpha} Y^{\beta'-\beta-\beta''} b^2 c) = Z^{\gamma''}(b d_2 + X^{\alpha'-\alpha} Z^{\gamma'-\gamma-\gamma''} a c^2).$$

Thus we get d_3 as $Y^{\beta''}, Z^{\gamma''}$ is an A-regular sequence.

Similarly, we get d_3' from (1) and (2).

We put $\mathfrak{a} = PP^{(2)} + (d_3, d_3') \subseteq P^{(3)}$. In order to show $\mathfrak{a} = P^{(3)}$, we notice $2\alpha \neq \alpha'$ or $2\beta \neq \beta'$ or $2\gamma \neq \gamma'$. If $2\gamma \neq \gamma'$, then we have

$$\ell_A(A/(Z) + \mathfrak{a}) = 6(\alpha\beta' + \alpha'\beta + \alpha'\beta') = \ell_A(A/(Z) + P^{(3)}),$$

and so $(Z) + \mathfrak{a} = (Z) + P^{(3)}$, which means $\mathfrak{a} = P^{(3)}$. Similarly we get $\mathfrak{a} = P^{(3)}$ in the case where $2\beta \neq \beta'$ or $2\gamma \neq \gamma'$. ∎

Theorem 7 (cf. [2], [6]) *Suppose that the matrix*

$$\begin{pmatrix} X^{\alpha} & Y^{\beta'} & Z^{\gamma'} \\ Y^{\beta} & Z^{\gamma} & X^{\alpha'} \end{pmatrix}$$

is of Type 2. Put $\alpha'' = \min\{\alpha', \alpha-\alpha'\}$, $\beta'' = \min\{\beta, \beta'-\beta\}$ *and* $\gamma'' = \min\{\gamma, \gamma'-\gamma\}$. *Then there exist* d_3, d_3' *and* d_3'' *in* $P^{(3)}$ *such that* $P^{(3)} = PP^{(2)} + (d_3, d_3', d_3'')$ *and*

$$\begin{aligned} X^{\alpha'} d_3 &= -Y^{\beta-\beta''} Z^{\gamma'-\gamma-\gamma''} a^2 c - Y^{\beta'-\beta-\beta''} b^3 \\ Y^{\beta''} d_3 &= Z^{\gamma-\gamma''} b d_2 + Z^{\gamma'-\gamma-\gamma''} ac^2 \\ Z^{\gamma''} d_3 &= -Y^{\beta-\beta''} a d_2 + Y^{\beta'-\beta-\beta''} b^2 c \end{aligned}$$

$$\begin{aligned} X^{\alpha''} d_3' &= -Z^{\gamma-\gamma''} b d_2 + Z^{\gamma'-\gamma-\gamma''} ac^2 \\ Y^{\beta} d_3' &= -X^{\alpha-\alpha'-\alpha''} Z^{\gamma-\gamma''} ab^2 - X^{\alpha'-\alpha''} Z^{\gamma'-\gamma-\gamma''} c^3 \\ Z^{\gamma''} d_3' &= X^{\alpha'-\alpha''} c d_2 + X^{\alpha-\alpha'-\alpha''} a^2 b \end{aligned}$$

$$\begin{aligned} X^{\alpha''} d_3'' &= Y^{\beta-\beta''} a d_2 + Y^{\beta'-\beta-\beta''} b^2 c \\ Y^{\beta''} d_3'' &= -X^{\alpha'-\alpha''} c d_2 + X^{\alpha-\alpha'-\alpha''} a^2 b \\ Z^{\gamma} d_3'' &= -X^{\alpha-\alpha'-\alpha''} Y^{\beta-\beta''} a^3 - X^{\alpha'-\alpha''} Y^{\beta'-\beta-\beta''} bc^2 . \end{aligned}$$

Theorem 8 $P^{(3)} \subseteq P^2$.

Proof. We prove in the case where the matrix is of Type 1. Because $P^{(3)} = PP^{(2)} + (d_3, d_3')$, it is enough to show $d_3, d_3' \in P^2$. Let us prove $d_3 \in P^2$. We use $Y^{\beta''} d_3 = Z^{\gamma-\gamma''} b d_2 + X^{\alpha'-\alpha} Z^{\gamma'-\gamma-\gamma''} ac^2$.

$$Z^{\gamma-\gamma''} b d_2 = Z^{\gamma-\gamma''} (X^{\alpha+\alpha'} - Y^{\beta} Z^{\gamma'}) d_2 = X^{\alpha+\alpha'} Z^{\gamma-\gamma''} d_2 - Y^{\beta} Z^{\gamma+\gamma'-\gamma''} d_2 .$$

$$\begin{aligned} X^{\alpha+\alpha'} Z^{\gamma-\gamma''} d_2 &= X^{\alpha'} Z^{\gamma-\gamma''} \cdot X^{\alpha} d_2 \\ &= X^{\alpha'} Z^{\gamma-\gamma''} (Z^{\gamma'-\gamma} ac - Y^{\beta'-\beta} b^2) \\ &\equiv X^{\alpha'} Z^{\gamma'-\gamma''} ac \mod Y^{\beta''} P^2 . \end{aligned}$$

$$Y^{\beta} Z^{\gamma+\gamma'-\gamma''} d_2 = Y^{\beta} Z^{\gamma'-\gamma''} \cdot Z^{\gamma} d_2 \in Y^{\beta''} P^2 .$$

$$\therefore \quad Z^{\gamma-\gamma''} b d_2 \equiv X^{\alpha'} Z^{\gamma'-\gamma''} ac \mod Y^{\beta''} P^2 .$$

On the other hand

$$\begin{aligned} X^{\alpha'-\alpha} Z^{\gamma'-\gamma-\gamma''} ac^2 &= X^{\alpha'-\alpha'} Z^{\gamma'-\gamma-\gamma''} ac (Y^{\beta+\beta'} - X^{\alpha} Z^{\gamma}) \\ &\equiv -X^{\alpha'} Z^{\gamma'-\gamma''} ac \mod Y^{\beta''} P^2 . \end{aligned}$$

Thus we get $Y^{\beta''} d_3 \in Y^{\beta''} P^2$, and so $d_3 \in P^2$.

Similarly, we can prove $d_3' \in P^2$. ∎

Theorem 9 $(P^{(2)})^2 \subseteq P^3$.

Proof. It is enough to show ${d_2}^2 \in P^3$. ∎

Example 10 *Suppose* $\operatorname{ch} k = 2$. *If* $\alpha \le \alpha'$, $3\beta < \beta'$, $\gamma \le \gamma' < 2\gamma$ *and* $3\gamma \le 2\gamma'$, *then* $P^{(4)} \not\subseteq P^3$. *For example, setting* $n_1 = 28$, $n_2 = 11$, $n_3 = 9$, *we have*

$$P = I_2 \begin{pmatrix} X & Y^4 & Z^5 \\ Y & Z^3 & X \end{pmatrix}.$$

Proof. In this case $d_2 \in P^{(2)}$ satisfies

$$Y^{\beta} d_2 = ab - Z^{\gamma'-\gamma} c^2 \quad \text{and} \quad Z^{\gamma} d_2 = -a^2 + Y^{\beta'-\beta} bc.$$

Because $\operatorname{ch} k = 2$, the first equality implies $Y^{2\beta} {d_2}^2 = a^2 b^2 - Z^{2\gamma'-2\gamma} c^4$. On the other hand, from the second equality we get

$$a^2 b^2 = (Y^{\beta'-\beta} bc - Z^{\gamma} d_2)\, b^2 = Y^{\beta'-\beta} b^3 c - Z^{\gamma} b^2 d_2.$$

$$\therefore \quad Y^{2\beta} {d_2}^2 + Z^{2\gamma'-2\gamma} c^4 = Y^{\beta'-\beta} b^3 c - Z^{\gamma} b^2 d_2.$$

$$\therefore \quad Y^{2\beta} (Y^{\beta'-3\beta} b^3 c - {d_2}^2) = Z^{\gamma} (b^2 d_2 + Z^{2\gamma'-3\gamma} c^4).$$

$$\therefore \quad \exists e_4 \in P^{(4)} \text{ such that } Z^{\gamma} e_4 = Y^{\beta'-3\beta} b^3 c - {d_2}^2.$$

Then $Z^{\gamma} e_4 \underset{(Y)}{\equiv} -{d_2}^2 \underset{(Y)}{\equiv} -Z^{2\gamma+4\gamma'}$, so $e_4 \underset{(Y)}{\equiv} -Z^{\gamma+4\gamma'}$. However, as $(Y) + P = (Y) + (X^{\alpha+\alpha'}, X^{\alpha} Z^{\gamma}, Z^{\gamma+\gamma'})$, we have

$$(Y) + P^3 = (Y) + (\quad \cdots \quad, Z^{3\gamma+3\gamma'}).$$

This means $e_4 \notin P^3$, since $(3\gamma + 3\gamma') - (\gamma + 4\gamma') = 2\gamma - \gamma' > 0$. ∎

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STRONGLY EDGE DECOMPOSABLE SIMPLICIAL COMPLEXES ARE LEFSCHETZ

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ABSTRACT. We study a relation between the strong Lefschetz property of Stanley–Reisner rings and contractions of simplicial complexes. We define edge decomposable simplicial complexes by generalizing Nevo's definition of edge decomposable spheres, and prove that every edge decomposable simplicial complex has the strong Lefschetz property.

1. INTRODUCTION

In this paper, we study a relation between the strong Lefschetz property of Stanley–Reisner rings and a combinatorial operation, called *contractions.*

We first recall the basis on simplicial complexes. Let Δ be a simplicial complex on $[n] = \{1, 2, \dots, n\}$. Thus Δ is a collection of subsets of $[n]$ satisfying that (i) $\{i\} \in \Delta$ for all $i \in [n]$ and (ii) if $F \in \Delta$ and $G \subset F$ then $G \in \Delta$. An element F of Δ is called a *face* of Δ and maximal faces of Δ under inclusion are called *facets* of Δ. A simplicial complex is said to be *pure* if all its facets have the same cardinality. Let $f_k(\Delta)$ be the number of faces $F \in \Delta$ with $|F| = k+1$. The *dimension* of Δ is the maximal integer k such that $f_k(\Delta) \neq 0$. The vector $f(\Delta) = (f_0(\Delta), f_1(\Delta), \dots, f_{d-1}(\Delta))$ is called the *f-vector* of Δ, where $d = \dim \Delta + 1$. The *h-vector* $h(\Delta) = (h_0(\Delta), h_1(\Delta), \dots, h_d(\Delta))$ of Δ is defined by the relations

$$h_i(\Delta) = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_{j-1}(\Delta) \quad \text{and} \quad f_{i-1}(\Delta) = \sum_{j=0}^{i} \binom{d-j}{d-i} h_j(\Delta),$$

where we set $f_{-1}(\Delta) = 1$. Note that knowing $f(\Delta)$ is equivalent to knowing $h(\Delta)$. The *link* of Δ w.r.t. a face $F \in \Delta$ is the simplicial complex $\mathrm{lk}_\Delta(F) = \{G \subset [n] \setminus F : G \cup F \in \Delta\}$. To simplify, we write $\mathrm{lk}_\Delta(\{i\}) = \mathrm{lk}_\Delta(i)$ and $\mathrm{lk}_\Delta(\{i,j\}) = \mathrm{lk}_\Delta(i,j)$.

Next, we define contractions of simplicial complexes. The *contraction* of Δ w.r.t. an edge $\{i,j\} \in \Delta$, where $i < j$, is the simplicial complex

$$\mathcal{C}_\Delta(i,j) = \{F \in \Delta : i \notin F\} \cup \{(F \setminus \{i\}) \cup \{j\} : i \in F \in \Delta\}.$$

Note that the contraction $\mathcal{C}_\Delta(i,j)$ is obtained from Δ by identifying two vertices i and j. We say that Δ satisfies the *Link condition* w.r.t. $\{i,j\} \in \Delta$ if

$$\mathrm{lk}_\Delta(i) \cap \mathrm{lk}_\Delta(j) = \mathrm{lk}_\Delta(i,j).$$

The next result of Nevo [N] shows an importance of the Link condition when we consider contractions.

The author is supported by JSPS Research Fellowships for Young Scientists.

Theorem 1.1 (Nevo). *Let Δ be a PL-manifold without boundary. Then Δ is PL-homeomorphic to $C_\Delta(i,j)$ if and only if* $\mathrm{lk}_\Delta(i) \cap \mathrm{lk}_\Delta(j) = \mathrm{lk}_\Delta(i,j)$.

One of the main open problem in f-vector theory is the g-conjecture for simplicial spheres, that is, the characterization of h-vectors of triangulations of spheres (or more generally Gorenstein* complexes). Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring over a field K. The *Stanly–Reisner ideal* $I_\Delta \subset S$ of Δ is the ideal generated by all squarefree monomials $x_F = \prod_{i \in F} x_i$ with $F \notin \Delta$. The ring $K[\Delta] = S/I_\Delta$ is called the *Stanley–Reisner ring* of Δ over a field K. A $(d-1)$-dimensional simplicial complex Δ is said to be *Gorenstein** (or homology sphere) if $K[\Delta]$ is Gorenstein and $h_d(\Delta) = 1$. On h-vectors of Gorenstein* complexes, there is the following conjecture.

Conjecture 1.2 (g-conjecture for homology spheres). If Δ is a $(d-1)$-dimensional Gorenstein* complex, then $(h_0, h_1 - h_0, \ldots, h_{\lfloor \frac{d}{2} \rfloor} - h_{\lfloor \frac{d}{2} \rfloor - 1})$ is an M-vector (that is, the Hilbert function of a standard graded K-algebra).

The above conjecture is important since if the conjecture is true then it yields the complete characterization of h-vectors of Gorenstein* complexes (see [S]).

An algebraic approach for Conjecture 1.2 is to study the Lefschetz property of Stanley–Reisner rings. Let $I \subset S$ be a homogeneous ideal and $A = S/I$. Let d be the Krull dimension of A. We say that A has the *strong Lefschetz property* if A is Cohen–Macaulay and there exist a linear system of parameters (l.s.o.p. for short) $\theta_1, \ldots, \theta_d \in S_1$ of A and a linear form $\omega \in S_1$ such that the multiplication map

$$\omega^{s-2i} : \big(A/(\theta_1, \ldots, \theta_d)A\big)_i \to \big(A/(\theta_1, \ldots, \theta_d)A\big)_{s-i}$$

is bijective for $i = 0, 1, \ldots, \lfloor \frac{s}{2} \rfloor$, where $s = \max\{k : \dim_K (A/(\theta_1, \ldots, \theta_d)A)_k \neq 0\}$. The element ω is called a *strong Lefschetz element* of $A/(\theta_1, \ldots, \theta_d)A$. A $(d-1)$-dimensional simplicial complex Δ is said to have the *strong Lefschetz property* (over K) if Δ is Cohen–Macaulay and there exist an l.s.o.p. $\Theta = \theta_1, \ldots, \theta_d$ of $K[\Delta]$ and a linear form ω such that the multiplication map $\omega^{d-2i} : (K[\Delta]/\Theta)_i \to (K[\Delta]/\Theta)_{d-i}$ is bijective for $i = 0, 1, \ldots, \lfloor \frac{d}{2} \rfloor$ (equivalently, $K[\Delta]$ has the strong Lefschetz property and $h_d(\Delta) \neq 0$).

Stanley [S] proved the g-conjecture for boundary complexes of simplicial polytopes by proving that they have the strong Lefschetz property in characteristic 0. Stanley's result leads the following conjecture.

Conjecture 1.3 (algebraic g-conjecture). Every Gorenstein* complex has the strong Lefschetz property.

It is known that Conjecture 1.3 implies Conjecture 1.2. On the Lefschetz property and contractions, we prove the following result.

Theorem 1.4. *Let Δ be a pure simplicial complex on $[n]$ satisfying the Link condition with respect to $\{i, j\} \in \Delta$. If $C_\Delta(i,j)$ and $\mathrm{lk}_\Delta(i,j)$ have the strong Lefschetz property then Δ has the strong Lefschetz property.*

Note that the above theorem was also proved by Babson-Nevo [BN] in the special case when $\mathrm{char}(K) = 0$ in a different way.

Since the boundary of a simplex has the strong Lefschetz property, Theorem 1.4 proves the strong Lefschetz property for the following class of simplicial complexes.

Definition 1.5. The boundary of a simplex is *edge decomposable* and, recursively, a pure simplicial complex Δ is said to be edge decomposable if there exists $\{i,j\} \in \Delta$ such that Δ satisfies the Link condition with respect to $\{i,j\}$ and both $\mathcal{C}(i,j)$ and $\mathrm{lk}_\Delta(i,j)$ are edge decomposable.

Corollary 1.6. *If Δ is edge decomposable then Δ has the strong Lefschetz property.*

Example 1.7. Let Γ be the simplicial complex generated by $\{1,2\},\{2,3\},\{3,4\}$ and $\{1,4\}$ (that is, Γ is a cycle of length 4). Then Γ satisfies the Link condition with respect to $\{1,2\}$. Also, $\mathcal{C}_\Delta(12)$ is the boundary of the simplex $\{2,3,4\}$ and $\mathrm{lk}_\Delta(1,2) = \{\emptyset\}$. Hence Γ is strongly edge decomposable.

Similarly, consider the simplicial complex Γ' generated by $\{1,2\},\{2,3\},\{3,4\}$ and $\{2,4\}$. Then Γ' also satisfies the Link condition with respect to $\{1,2\}$ and we have $\mathcal{C}_{\Gamma'}(1,2) = \mathcal{C}_\Delta(1,2)$ and $\mathrm{lk}_{\Gamma'}(1,2) = \mathrm{lk}_\Delta(1,2)$. Thus Γ' is also strongly edge decomposable. Note that Γ' is not Gorenstein*.

The simplicial complex $\Sigma = \Gamma \cup \{\{1,3\}\}$ is not strongly edge decomposable since Σ does not satisfy the Link condition with respect to any $\{i,j\} \in \Sigma$.

On edge decomposable complexes, the following facts are known:

- 1-dimensional and 2-dimensional spheres are edge decomposable (see [N]).
- the join of edge decomposable complexes is edge decomposable. In particular, a squarefree complete intersection is edge decomposable (if it is not a cone).
- there exists a 3-sphere which is not edge decomposable ([DEGN, §7]).
- there are many edge decomposable spheres which are not the boundary of a simplicial polytope ([M]).
- A shifted complex has the strong Lefschetz property if and only if it is edge decomposable.

Theorem 1.4 and Corollary 1.6 do not depend on the characteristic of the base field. In particular, these results show that 2-dimensional spheres and squarefree complete intersections have the strong Lefschetz property in any characteristic, while Stanley's theorem only guarantees the Lefschetz property in characteristic 0.

2. How to use the Link condition

In this section, we study an algebraic meaning of the Link condition. For a monomial ideal I, let $G(I)$ be the unique minimal set of monomial generators of I.

Lemma 2.1. *Let Δ be a simplicial complex on $[n]$ and let $1 \le i < j \le n$ be integers. Then Δ satisfies the Link condition with respect to $\{i,j\}$ if and only if I_Δ has no generators which are divisible by $x_i x_j$.*

Proof. We first show the 'only if' part. Let $x_i x_j x_F \in I_\Delta$ with $F \subset [n] \setminus \{i,j\}$. Since $F \notin \mathrm{lk}_\Delta(i,j) = \mathrm{lk}_\Delta(i) \cap \mathrm{lk}_\Delta(j)$, $x_i x_F \in I_\Delta$ or $x_j x_F \in I_\Delta$. Thus $x_i x_j x_F \notin G(I_\Delta)$.

Next, we prove the 'if' part. The inclusion $\mathrm{lk}_\Delta(i) \cap \mathrm{lk}_\Delta(j) \supset \mathrm{lk}_\Delta(i,j)$ is obvious. What we must prove is $\mathrm{lk}_\Delta(i) \cap \mathrm{lk}_\Delta(j) \subset \mathrm{lk}_\Delta(i,j)$. Let $F \in \mathrm{lk}_\Delta(i) \cap \mathrm{lk}_\Delta(j)$. Suppose $F \notin \mathrm{lk}_\Delta(i,j)$. Then $x_i x_j x_F \in I_\Delta$ and there exists $x_G \in G(I_\Delta)$ such that $G \subset \{i,j\} \cup F$. Since $\{i\} \cup F \in \Delta$ and $\{j\} \cup F \in \Delta$, we have $G \not\subset \{i\} \cup F$ and $G \not\subset \{j\} \cup F$. Thus we have $\{i,j\} \subset G$, however, this contradicts the assumption that $x_G \in G(I_\Delta)$ is not divisible by $x_i x_j$. Hence $F \in \mathrm{lk}_\Delta(i,j)$. □

For a simplicial complex Δ and a face $F \in \Delta$, the simplicial complex $\mathrm{st}_\Delta(F) = \{F' \cup G : F' \subset F,\ G \in \mathrm{lk}_\Delta(F)\}$ is called the *star* of Δ w.r.t. a face $F \in \Delta$. For integers $1 \le i < j \le n$, let φ_{ij} be the graded K-algebra automorphism of S induced by $\varphi_{ij}(x_k) = x_k$ for $k \ne j$ and $\varphi_{ij}(x_j) = x_i + x_j$. We write $\mathrm{in}(I)$ for the initial ideal of a homogeneous ideal I of S w.r.t. the degree reverse lexicographic order induced by $x_1 > \cdots > x_n$ (see [E, §15] for the definition). Algebraically, the benefit of Lemma 2.1 can be explained by the following fact.

Lemma 2.2. *Let Δ be a simplicial complex on $[n]$ and let $1 \le i < j \le n$ be integers. Let $\Gamma = \mathcal{C}_\Delta(i,j) \cup \mathrm{st}_\Delta(i,j)$. If I_Δ has no generators which are divisible by $x_i x_j$ then*

$$\mathrm{in}(\varphi_{ij}(I_\Delta)) = I_\Gamma.$$

Proof. We claim that Δ and Γ have the same f-vector. Let $\Delta_k = \{F \in \Delta : |F| = k+1\}$. A routine computation shows

$$\begin{aligned} f_k(\Delta) = |\Delta_k| &= |\{F \in \Delta_k : i \notin F \text{ or } i \in F,\ j \notin F,\ (F \setminus \{i\}) \cup \{j\} \notin \Delta\}| \\ &\quad + |\{\{i\} \cup F \in \Delta_k : \{j\} \in F \text{ or } F \in \mathrm{lk}_\Delta(i) \cap \mathrm{lk}_\Delta(j)\}|. \end{aligned}$$

By the definition of contractions

$$f_{k-1}(\mathcal{C}_\Delta(i,j)) = |\{F \in \Delta_k : i \notin F \text{ or } i \in F,\ j \notin F,\ (F \setminus \{i\}) \cup \{j\} \notin \Delta\}|.$$

Also, since Δ satisfies the Link condition w.r.t. $\{i,j\}$, it follows that

$$\{\{i\} \cup F \in \Delta_k : \{j\} \in F \text{ or } F \in \mathrm{lk}_\Delta(i) \cap \mathrm{lk}_\Delta(j)\} = \{F \in \mathrm{st}_\Delta(i,j)_k : i \in F\}.$$

Since $f_k(\Gamma) = f_k(\mathcal{C}_\Delta(i,j)) + |\{F \in \mathrm{st}_\Delta(i,j)_k : i \in F\}|$, the above equations show that Δ and Γ have the same f-vector.

Since Δ and Γ have the same f-vector, I_Δ and I_Γ have the same Hilbert function. Since I_Δ and $\mathrm{in}(\varphi_{ij}(I_\Delta))$ also have the same Hilbert function, what we must prove is $G(I_\Gamma) \subset \mathrm{in}(\varphi_{ij}(I_\Delta))$. Let $x_F \in G(I_\Gamma)$.

Case 1: Suppose $i \notin F$. If $j \notin F$ then $x_F \in I_\Delta$. Thus $\mathrm{in}(\varphi_{ij}(x_F)) = x_F \in \mathrm{in}(\varphi_{ij}(I_\Delta))$ as desired. If $j \in F$ then $F \notin \Delta$ and $(F \setminus \{j\}) \cup \{i\} \notin \Delta$ by the definition of contractions. Thus $\mathrm{in}(\varphi_{ij}(x_F - x_{(F\setminus\{j\})\cup\{i\}})) = x_F \in \mathrm{in}(\varphi_{ij}(I_\Delta))$.

Case 2: Suppose $i \in F$. If $j \notin F$, then since $F \notin \mathrm{st}_\Delta(i,j)$, $F \setminus \{i\} \notin \mathrm{lk}_\Delta(i)$ or $F \setminus \{i\} \notin \mathrm{lk}_\Delta(j)$ by the Link condition. Thus $x_F \in I_\Delta$ or $x_{(F\setminus\{i\})\cup\{j\}} \in I_\Delta$. In both cases we have $x_F \in \mathrm{in}(\varphi_{ij}(I_\Delta))$ since $\mathrm{in}(\varphi_{ij}(x_F)) = \mathrm{in}(\varphi_{ij}(x_{(F\setminus\{i\})\cup\{j\}})) = x_F$. If $j \in F$ then $x_F \in I_\Delta$. Since $\{i,j\} \subset F$, by the assumption, there exists $x_G \in G(I_\Delta)$ such that $G \subset F$ and $\{i,j\} \not\subset G$. Then $\mathrm{in}(\varphi_{ij}(x_G))$ is either x_G or $x_{(G\setminus\{j\})\cup\{i\}}$. In both cases $\mathrm{in}(\varphi_{ij}(x_G))$ divides x_F. Hence $x_F \in \mathrm{in}(\varphi_{ij}(I_\Delta))$ as desired. □

Lemma 2.2 is false if I_Δ has a generator which is divisible by $x_i x_j$. Indeed, it is easy to see that if $x_i x_j x_F \in G(I_\Delta)$ then $x_i^2 x_F$ is a generator of $\mathrm{in}(\varphi_{ij}(I_\Delta))$.

The next Lemma and Lemma 2.2 give a nice relation between $\mathcal{C}_\Delta(i,j)$ and Δ.

Lemma 2.3. *Let Δ be a simplicial complex satisfying the link condition w.r.t. $\{i,j\} \in \Delta$ and $\Gamma = \mathcal{C}_\Delta(i,j) \cup \mathrm{st}_\Delta(i,j)$. Then*

$$(1) \qquad 0 \longrightarrow K[\mathrm{st}_\Delta(i,j)] \xrightarrow{\times x_i} K[\Gamma] \to K[\mathcal{C}_\Delta(i,j)] \longrightarrow 0$$

is exact, where the first map is the multiplication by x_i and where the second map is a natural surjection induced by the inclusion $\mathcal{C}_\Delta(i,j) \subset \Gamma$.

Proof. For any simplicial complex Σ, it is easy to see that

$$0 \longrightarrow K[\mathrm{st}_\Sigma(i)] \xrightarrow{\times x_i} K[\Sigma] \to K[\Sigma - \{i\}] \longrightarrow 0$$

is exact, where $\Sigma - \{i\} = \{F \in \Sigma : i \notin F\}$. On the other hand, by the definition of Γ, $\mathrm{st}_\Gamma(i) = \mathrm{st}_\Delta(i,j)$ and $\Gamma - \{i\} = \mathcal{C}_\Delta(i,j)$. □

Finally, we remark the following simple fact.

Lemma 2.4. *Let Δ be a $(d-1)$-dimensional simplicial complex on $[n]$ and $\{i,j\} \in \Delta$. If $\dim \mathcal{C}_\Delta(i,j) \neq d-1$ then $\mathcal{C}_\Delta(i,j)$ is a cone.*

Proof. Suppose $\dim \mathcal{C}_\Delta(i,j) < d-1$. Then all facets of Δ contain $\{i,j\}$. Thus $\mathcal{C}_\Delta(i,j)$ is a cone. □

If a $(d-1)$-dimensional simplicial complex Δ is a cone, then $h_d(\Delta) = 0$. In particular Δ cannot have the strong Lefschetz property.

3. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. The original proof given in [M] is rather technical. Here, we give a simpler proof.

We identify a sequence of linear forms $\theta_1, \dots, \theta_d \in S_1$ with an element of $K^{n\times d}$. We require the following known facts.

Lemma 3.1. *Let $I \subset S$ be a homogeneous ideal and d the Krull dimension of S/I. If $A = S/I$ has the strong Lefschetz property, then there exits a nonempty Zariski open subset $U \subset K^{n\times(d+1)}$ such that, for any sequence of linear forms $\theta_1, \dots, \theta_d, \theta_{d+1} \in U$, $\theta_1, \dots, \theta_d$ is an l.s.o.p. of A and θ_{d+1} is a strong Lefschetz element of $A/(\theta_1, \dots, \theta_d)A$.*

Lemma 3.2 (Wiebe [W]). *Let $I \subset S$ be a homogeneous ideal. If $S/\mathrm{in}(I)$ has the strong Lefschetz property then S/I has the strong Lefschetz property.*

Proof of Theorem 1.4. Let $\Gamma = \mathcal{C}_\Delta(i,j) \cup \mathrm{st}_\Delta(i,j)$. By Lemmas 2.2 and 3.2, it is enough to prove that Γ has the strong Lefschetz property.

We first prove that Γ is Cohen–Macaulay. By the assumption, $\mathcal{C}_\Delta(i,j)$ and $\mathrm{st}_\Delta(i,j)$ are $(d-1)$-dimensional Cohen–Macaulay complexes. Indeed, $\dim \mathrm{st}_\Delta(i,j) = d-1$ since Δ is pure, and $\dim \mathcal{C}_\Delta(i,j) = d-1$ follows from Lemma 2.4 since $\mathcal{C}_\Delta(i,j)$ has the strong Lefschetz property. Also, $\mathcal{C}_\Delta(i,j)$ and $\mathrm{lk}_\Delta(i,j)$ are Cohen–Macaulay by the assumption, and $\mathrm{st}_\Delta(i,j)$ is Cohen–Macaulay since $K[\mathrm{st}_\Delta(i,j)] = K[\mathrm{lk}_\Delta(i,j)][x_i, x_j]$.

Consider the exact sequence (1) given in Lemma 2.3. Then since every ring appearing in (1) is a $(d-1)$-dimensional and since $K[\mathrm{st}_\Delta(i,j)]$ and $K[\mathcal{C}_\Delta(i,j)]$ are Cohen–Macaulay, it follows that $K[\Gamma]$ is Cohen–Macaulay.

Let $\theta_1, \dots, \theta_d, \omega$ be generic linear forms. Then $\Theta = \theta_1, \dots, \theta_d$ is a common l.s.o.p. of $K[\mathrm{st}_\Delta(i,j)]$, $K[\mathcal{C}_\Delta(i,j)]$ and $K[\Gamma]$. Since these rings are Cohen–Macaulay,

$$0 \longrightarrow K[\mathrm{st}_\Delta(i,j)]/\Theta \xrightarrow{\times x_i} K[\Gamma]/\Theta \to K[\mathcal{C}_\Delta(i,j)]/\Theta \longrightarrow 0$$

is exact. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & (K[\mathrm{st}_\Delta(i,j)]/\Theta)_{i-1} & \xrightarrow{\times x_i} & (K[\Gamma]/\Theta)_i & \to & (K[\mathcal{C}_\Delta(i,j)]/\Theta)_i & \to & 0 \\ & & \downarrow \times\omega^{d-2i} & & \downarrow \times\omega^{d-2i} & & \downarrow \times\omega^{d-2i} & & \\ 0 & \to & (K[\mathrm{st}_\Delta(i,j)]/\Theta)_{(d-2)-(i-1)} & \xrightarrow{\times x_i} & (K[\Gamma]/\Theta)_{d-i} & \to & (K[\mathcal{C}_\Delta(i,j)]/\Theta)_{d-i} & \to & 0 \end{array}$$

Then, since $\mathcal{C}_\Delta(i,j)$ has the strong Lefschetz property, by Lemma 3.1, the third vertical map in the commutative diagram is bijective. Also, since $\mathrm{lk}_\Delta(i,j)$ has the strong Lefschetz property, the multiplication map $\omega^{d-2i} : (K[\mathrm{lk}_\Delta(i,j)]/\Theta)_{i-1} \to (K[\mathrm{lk}_\Delta(i,j)]/\Theta)_{(d-2)-(i-1)}$ is bijective. Since $K[\mathrm{lk}_\Delta(i,j)][x_i,x_j] = K[\mathrm{st}_\Delta(i,j)]$, by Lemma 3.1, $K[\mathrm{st}_\Delta(i,j)]$ has the strong Lefschetz property and the first vertical map in the commutative diagram is bijective. Then the second vertical map must be bijective. Hence Γ has the strong Lefschetz property. □

4. Open problems

Here, we give a few open problems.

Problem 4.1. Let Δ be a Gorenstein* complex which satisfies the Link condition w.r.t. $\{i,j\} \in \Delta$. Prove that if Δ has the strong Lefschetz property then $\mathcal{C}_\Delta(i,j)$ also has the strong Lefschetz property.

The above problem is important since if it is true then the algebraic g-conjecture holds for all PL-spheres (see [BN] for details).

We say that a Gorenstein* complex Δ is *irreducible* if it does not satisfies the Link condition for any edge in Δ. It is not difficult to prove that if Δ is Gorenstein* and satisfies the Link condition w.r.t. $\{i,j\}$ then $\mathcal{C}_\Delta(i,j)$ is also Gorenstein*. Then Theorem 1.4 shows that, to prove the algebraic g-conjecture, it is enough to prove

Problem 4.2. Prove that every irreducible Gorenstein* complex has the strong Lefschetz property.

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A NOTE ON THE BUCHSBAUM-RIM MULTIPLICITY

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This note is a summary of the paper [7] with Eero Hyry (University of Tampere). In this note we prove that the Buchsbaum-Rim multiplicity $e(F/N)$ of a parameter module N in a free module $F = A^r$ is bounded above by the colength $\ell_A(F/N)$. Moreover, we prove that once the equality $\ell_A(F/N) = e(F/N)$ holds true for some parameter module N in F, then the base ring A is Cohen-Macaulay.

1. Introduction

Let $(A, \mathfrak{m})$ be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Let $F = A^r$ be a free module of rank $r > 0$, and let M be a submodule of F such that F/M has finite length and $M \subseteq \mathfrak{m}F$.

In their article [4] from 1964 Buchsbaum and Rim introduced and studied a multiplicity associated to a submodule of finite colength in a free module. This multiplicity, which generalizes the notion of Hilbert–Samuel multiplicity for ideals, is nowadays called the Buchsbaum–Rim multiplicity. In more detail, it first turns out that the function

$$\lambda(n) := \ell_A(\mathcal{S}_n(F)/\mathcal{R}_n(M))$$

is eventually a polynomial of degree $d+r-1$, where $\mathcal{S}_A(F) = \bigoplus_{n \geq 0} \mathcal{S}_n(F)$ is the symmetric algebra of F and $\mathcal{R}(M) = \bigoplus_{n \geq 0} \mathcal{R}_n(M)$ is the image of the natural homomorphism from $\mathcal{S}_A(M)$ to $\mathcal{S}_A(F)$. The polynomial $P(n)$ corresponding to $\lambda(n)$ can then be written in the form

$$P(n) = \sum_{i=0}^{d+r-1} (-1)^i e_i \binom{n+d+r-1-i}{d+r-1-i}$$

with integer coefficients e_i. The *Buchsbaum-Rim multiplicity of M in F*, denoted by $e(F/M)$, is now defined to be the coefficient e_0.

Buchsbaum and Rim also introduced in their article the notion of a parameter module (matrix), which generalizes the notion of a parameter ideal (system of parameters). The module N in F is said to be *a parameter module in F*, if the following three conditions are satisfied: (i) F/N

has finite length, (ii) $N \subseteq \mathfrak{m}F$, and (iii) $\mu_A(N) = d+r-1$, where $\mu_A(N)$ is the minimal number of generators of N.

Buchsbaum and Rim utilized in their study the relationship between the Buchsbaum-Rim multiplicity and the Euler-Poincaré characteristic of a certain complex, and proved the following:

Theorem 1.1 (Buchsbaum-Rim [4, Corollary 4.5]). *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$. Then the following statements are equivalent:*

(1) *A is a Cohen-Macaulay local ring;*
(2) *For any rank $r > 0$, the equality $\ell_A(F/N) = e(F/N)$ holds true for every parameter module N in $F = A^r$.*

Then it is natural to ask the following:

Question 1.2.

(1) *Does the inequality $\ell_A(F/N) \geq e(F/N)$ hold true for any parameter module N in F?*
(2) *Does the equality $\ell_A(F/N) = e(F/N)$ for some parameter module N in F imply that the ring A is Cohen-Macaulay?*

The purpose of this note is to give a complete answer to Question 1.2. Our results can be summarized as follows:

Theorem 1.3. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$.*

(1) *For any rank $r > 0$, the following two inequalities*

$$\ell_A(F/N) \geq e(F/N) \text{ and } \ell_A(A/I(N)) \geq e(F/N)$$

always hold true for every parameter module N in $F = A^r$, where $I(N)$ is the 0-th Fitting ideal of F/N.

(2) *The following statements are equivalent:*
   (i) *A is a Cohen-Macaulay local ring;*
   (ii) *For some rank $r > 0$, there exists a parameter module N in $F = A^r$ such that the equality $\ell_A(F/N) = e(F/N)$ holds true;*
   (iii) *For some rank $r > 0$, there exists a parameter module N in $F = A^r$ such that the equality $\ell_A(A/I(N)) = e(F/N)$ holds true.*

   When this is the case, the equality $\ell_A(F/N) = \ell_A(A/I(N)) = e(F/N)$ holds true for all parameter modules N in $F = A^r$ of any rank $r > 0$.

Note that the equality $\ell_A(F/N) = \ell_A(A/I(N))$ is known by [1, 2.10].

2. Higher Euler-Poincaré characteristics

In order to prove Theorem 1.3, we will investigate higher Euler-Poincaré characteristics of a generalized Koszul complex.

Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$. Let $F = A^r$ be a free module of rank $r > 0$ with a basis $\{t_1, \dots, t_r\}$. Let M be a submodule of F generated by $c_1, c_2, \dots, c_n$, where $n = \mu_A(M)$ is the minimal number of generators of M. Writing $c_j = c_{1j}t_1 + \cdots + c_{rj}t_r$ for some $c_{ij} \in A$, we have an $r \times n$ matrix (c_{ij}) associated to M. We call this matrix the matrix of M, and denote it by $\widetilde{M}$. Let $I(M) = \mathrm{Fitt}_0(F/M)$ be the 0-th Fitting ideal of F/M. Let $K_\bullet(\widetilde{M}; t)$ be the generalized Koszul complex associated to a matrix $\widetilde{M}$ and an integer $t \in \mathbb{Z}$. For the definition and basic facts about the generalized Koszul complex, we refer the reader to [2, 8] and [6, Appendix A2.6]. We assume that F/M has finite length and $M \subseteq \mathfrak{m}F$. Then $I(M)$ is an $\mathfrak{m}$-primary ideal, because $\sqrt{I(M)} = \sqrt{\mathrm{Ann}_A(F/M)}$. Hence each homology module $H_p(K_\bullet(\widetilde{M}; t))$ has finite length, because the module $H_p(K_\bullet(\widetilde{M}; t))$ is annihilated by the ideal $I(M)$. So the Euler-Poincaré characteristics of $K_\bullet(\widetilde{M}; t)$ can be defined as follows:

Definition 2.1. *For any integer $q \geq 0$, we set*

$$\chi_q(K_\bullet(\widetilde{M}; t)) := \sum_{p \geq q} (-1)^{p-q} \ell_A(H_p(K_\bullet(\widetilde{M}; t)))$$

and call it the q-th partial Euler-Poincaré characteristic of $K_\bullet(\widetilde{M}; t)$. When $q = 0$, we simply write $\chi(K_\bullet(\widetilde{M}; t))$ for $\chi_0(K_\bullet(\widetilde{M}; t))$, and call it the Euler-Poincaré characteristic of $K_\bullet(\widetilde{M}; t)$.

Buchsbaum and Rim studied in [4] the Euler-Poincaré characteristic of the Buchsbaum-Rim complex $K_\bullet(\widetilde{M}; 1)$ in analogy with the Euler-Poincaré characteristic of the ordinary Koszul complex in the case of usual multiplicities. In 1985 Kirby investigated in [9] Euler-Poincaré characteristics of the complex $K_\bullet(\widetilde{M}; t)$ for all t and proved the following:

Theorem 2.2 (Buchsbaum-Rim, Kirby). *For any integer $t \in \mathbb{Z}$, we have*

$$\chi(K_\bullet(\widetilde{M}; t)) = \begin{cases} e(F/M) & (n = d + r - 1), \\ 0 & (n > d + r - 1). \end{cases}$$

In particular, $\chi(K_\bullet(\widetilde{M}; t)) \geq 0$ for all $t \in \mathbb{Z}$.

The last statement holds for the higher Euler-Poincaré characteristics, too:

Theorem 2.3. *For any $q \geq 0$ and any $t \geq -1$, we have*

$$\chi_q(K_\bullet(\widetilde{M};t)) \geq 0.$$

Proof. Let $\widetilde{M} = (c_{ij}) \in \mathrm{Mat}_{r\times n}(A)$ be the matrix of M and $X = (X_{ij})$ be the generic matrix of the same size $r \times n$. Let $A[X] = A[X_{ij} \mid 1 \leq i \leq r,\ 1 \leq j \leq n]$ be a polynomial ring over A and let $B = A[X]_{(\mathfrak{m},X)}$. We will consider the ring A as a B-algebra via the substitution homomorphism $\phi : B \longrightarrow A\ ; X_{ij} \mapsto c_{ij}$. Let $\mathfrak{b} := \mathrm{Ker}\,\phi = (X_{ij} - c_{ij} \mid 1 \leq i \leq r,\ 1 \leq j \leq n)B$ and let $C_t(X) := H_0(K_\bullet(X;t))$. Then, using ideas from [5], we can prove the isomorphism

$$H_p(K_\bullet(\widetilde{M};t)) \cong H_p(K_\bullet(\mathfrak{b}) \otimes_B C_t(X))$$

for any $p \geq 0$ and any $t \geq -1$, where $K_\bullet(\mathfrak{b})$ is the ordinary Koszul complex associated to a sequence $\widetilde{\mathfrak{b}}$. Hence we have

$$\chi_q(K_\bullet(\widetilde{M};t)) = \chi_q(K_\bullet(\mathfrak{b}) \otimes_B C_t(X))$$

for any $q \geq 0$ and any $t \geq -1$. Here the right hand side is non–negative by Serre's Theorem ([11, Ch. IV Appendix II]). Thus $\chi_q(K_\bullet(\widetilde{M};t)) \geq 0$. □

3. Proof of Theorem 1.3

Theorem 1.3 will be a consequence of the following more general result:

Theorem 3.1. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$.*

(1) *For any rank $r > 0$, the inequality $\ell_A(H_0(K_\bullet(\widetilde{N};t))) \geq e(F/N)$ holds true for any integer $t \geq -1$ and any parameter module N in $F = A^r$.*

(2) *The following statements are equivalent:*
   (i) *A is a Cohen-Macaulay local ring;*
   (ii) *For some rank $r > 0$, there exist an integer $-1 \leq t \leq d$ and a parameter module N in $F = A^r$ such that the equality $\ell_A(H_0(K_\bullet(\widetilde{N};t))) = e(F/N)$ holds true.*

   When this is the case, the equality $\ell_A(H_0(K_\bullet(\widetilde{N};t))) = e(F/N)$ holds true for any integer $-1 \leq t \leq d$ and any parameter module N in $F = A^r$ of any rank $r > 0$.

Proof. (1): Let N be a parameter module in $F = A^r$, and let $t \geq -1$. By Theorem 2.2 we obtain that

$$e(F/N) = \chi(K_\bullet(\widetilde{N};t)) = \ell_A(H_0(K_\bullet(\widetilde{N};t))) - \chi_1(K_\bullet(\widetilde{N};t)).$$

Since $\chi_1(K_\bullet(\widetilde{N};t)) \geq 0$ by Theorem 2.3, the desired inequality follows.

(2): Assume that the ring A is Cohen-Macaulay. Let N be any parameter module in $F = A^r$ of any rank $r > 0$. Let $n = \mu_A(N) = d + r - 1$. Then, since $\operatorname{grade} I(N) = \operatorname{ht} I(N) = d = n - r + 1$, $K_\bullet(\widetilde{N};t)$ is acyclic for all $-1 \leq t \leq n - r + 1 = d$. Therefore, by Theorem 2.2, we have $e(F/N) = \chi(K_\bullet(\widetilde{N};t)) = \ell_A(H_0(K_\bullet(\widetilde{N};t)))$. This proves the implication (i) $\Rightarrow$ (ii), and also the last assertion.

It remains to show the implication (ii) $\Rightarrow$ (i). Assume that there exist integers $r > 0$, $-1 \leq t \leq d$, and a parameter module N in $F = A^r$ such that $\ell_A(H_0(K_\bullet(\widetilde{N};t))) = e(F/N)$. Arguing as in the proof of Theorem 2.3 and using the same notation, we get

$$\begin{aligned}\chi_1(K_\bullet(\mathfrak{b}) \otimes_B C_t(X)) &= \chi_1(K_\bullet(\widetilde{N};t)) \\ &= \ell_A(H_0(K_\bullet(\widetilde{N};t))) - e(F/N) \;=\; 0.\end{aligned}$$

We note here that $\dim_B C_t(X) = rn$ and hence $\mathfrak{b}$ is a parameter ideal of $C_t(X)$. Therefore the equality $\chi_1(K_\bullet(\mathfrak{b}) \otimes_B C_t(X)) = 0$ implies that $C_t(X)$ is a Cohen-Macaulay B-module. On the other hand, $\operatorname{pd}_B C_t(X) = d$, because the complex $K_\bullet(X;t)$ is a minimal B-free resolution of $C_t(X)$ of length $n - r + 1 = d$. Hence, by the Auslander-Buchsbaum formula, we have

$$\begin{aligned}d + rn &= \operatorname{pd}_B C_t(X) + \operatorname{depth}_B C_t(X) \\ &= \operatorname{depth} B \\ &\leq \dim B \;=\; d + rn.\end{aligned}$$

Thus $\operatorname{depth} B = \dim B$ so that B is Cohen-Macaulay. Therefore A is also a Cohen-Macaulay local ring. □

Taking $t = 0, 1$ in Theorem 3.1, now readily gives Theorem 1.3. We close this note with the following question.

Question 3.2. *Let F be a fixed free module of rank $r > 0$. Assume that A is a Buchsbaum local ring. Then the difference $\ell_A(F/N) - e(F/N)$ of length and multiplicity of a parameter module N is independent of the choice of a parameter module N in F?*

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A criterion of Gorenstein property of a Doset Hibi ring

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1 Introduction

Grassmannians and their Schubert subvarieties are fascinating objects and attract many mathematicians. Let $G_{m,n}$ be the Grassmannian consisting of all m-dimensional vector subspaces of an n-dimensional vector space V over a field K. Then the homogeneous coordinate ring of $G_{m,n}$ is $K[\Gamma(X)]$, where $X = (X_{ij})$ is the $m \times n$ matrix of indeterminates. See (2.2) for the notation.

For a Schubert subvariety of $G_{m,n}$, there correspond integers $b_1, \ldots, b_m$ with $1 \leq b_1 < \cdots < b_m \leq n$ and the universal $m \times n$ matrix Z with the condition

$$I_i(Z_{\leq b_i - 1}) = (0) \quad \text{for } i = 1, \ldots, m$$

and the homogeneous coordinate ring of the Schubert subvariety of $G_{m,n}$ is $K[\Gamma(Z)]$.

On the other hand, because of the universal property of Z, any subgroup of $\mathrm{GL}(m, K)$ acts on $K[Z]$. The present author defined the notion of doset Hibi ring in order to study the rings of $\mathrm{O}(m)$ and $\mathrm{SO}(m)$ invariants of $K[Z]$.

A (generalized) doset Hibi ring is a normal affine semigroupring and therefore is Cohen-Macaulay. In this note, we state a criterion of a (generalized) doset Hibi ring to be Gorenstein.

2 Preliminaries

(2.1) In this note, all rings and algebras are commutative with identity element.

We denote by $\boldsymbol{N}$ the set of non-negative integers, by $\boldsymbol{Z}$ the set of integers, by $\boldsymbol{R}$ the set of real numbers and by $\boldsymbol{R}_{\geq 0}$ the set of non-negative real numbers.

Let K be a field and $X_1, \ldots, X_r$ indeterminates, S a finitely generated additive submonoid of $\boldsymbol{N}^r$. We set $K[S] := K[X^{\boldsymbol{s}} \mid \boldsymbol{s} \in S]$ where $X^{\boldsymbol{s}} = X_1^{s_1} \cdots X_r^{s_r}$ for $\boldsymbol{s} = (s_1, \ldots, s_r)$. Then

Theorem 2.1 ([Hoc]) *(1) $K[S]$ is normal if and only if $S = \boldsymbol{Z}S \cap \boldsymbol{R}_{\geq 0}S$.*

(2) If $K[S]$ is normal, then it is Cohen-Macaulay.

(2.2) Let m, n be integers with $0 < m \leq n$. For an $m \times n$ matrix M with entries in a K algebra S, we denote by $I_t(M)$ the ideal of S generated by the t-minors of M, by $M^{\leq i}$ the $i \times n$ matrix consisting of first i-rows of M, by $M_{\leq j}$ the $m \times j$ matrix consisting of first j-columns of M, by $\Gamma(M)$ the set of maximal minors of M and by $K[M]$ the K-subalgebra of S generated by the entries of M.

(2.3) Let P be a finite partially ordered set (poset for short).

The length of a chain (totally ordered subset) X of P is $\#X - 1$, where $\#X$ is the cardinality of X.

The rank of P, denoted by $\mathrm{rank}P$, is the maximum of the lengths of chains in P.

A poset is said to be pure if its all maximal chains have the same length.

A poset ideal of P is a subset I of P such that $x \in I$, $y \in P$ and $y \leq x$ imply $y \in I$.

For x, $y \in P$, y covers x, denoted by $x \lessdot y$, means $x < y$ and there is no $z \in P$ such that $x < z < y$.

For x, $y \in P$ with $x \leq y$, we set $[x, y]_P := \{z \in P \mid x \leq z \leq y\}$.

(2.4) Let H be a finite distributive lattice. A join-irreducible element in H is an element $\alpha \in H$ such that α can not be expressed as a join of two elements different from α. That is, $\alpha = \beta \vee \gamma \Rightarrow \alpha = \beta$ or $\alpha = \gamma$. Note that we treat the unique minimal element of H as a join-irreducible element.

Let P be the set of all join-irreducible elements in H. Then it is known that H is isomorphic to $J(P) \setminus \{\emptyset\}$ ordered by inclusion, where $J(P)$ is the set of all poset ideals of P. The isomorphisms $\Phi\colon H \to J(P) \setminus \{\emptyset\}$ and $\Psi\colon J(P) \setminus \{\emptyset\} \to H$ are given by

$$\begin{aligned} \Phi(\alpha) &:= \{x \in P \mid x \leq \alpha \text{ in } H\} && \text{for } \alpha \in H \text{ and} \\ \Psi(I) &:= \bigvee_{x \in I} x && \text{for } I \in J(P) \setminus \{\emptyset\}. \end{aligned}$$

(2.5) Let L be another distributive lattice, Q the set of join-irreducible elements in L and $\varphi\colon H \to L$ a surjective lattice homomorphism. We set

$$\varphi^*(\beta) := \bigwedge_{\alpha\in\varphi^{-1}(\beta)} \alpha.$$

Then

Lemma 2.2 *(1)* $\varphi^*(\varphi(\alpha)) \le \alpha$ *for any* $\alpha \in H$*.*

(2) $\varphi(\varphi^*(\beta)) = \beta$ *for any* $\beta \in L$*.*

(3) $\beta_1 \le \beta_2 \Rightarrow \varphi^*(\beta_1) \le \varphi^*(\beta_2)$*.*

(4) For $\alpha \in H$ *and* $\beta \in L$*,* $\varphi(\alpha) \ge \beta \iff \alpha \ge \varphi^*(\beta)$*.*

(5) $\varphi^*(\beta_1 \vee \beta_2) = \varphi^*(\beta_1) \vee \varphi^*(\beta_2)$*.*

(6) $y \in Q \Rightarrow \varphi^*(y) \in P$*.*

Remark 2.3 $\varphi^*(\beta_1) \wedge \varphi^*(\beta_2) \ne \varphi^*(\beta_1) \wedge \varphi^*(\beta_2)$ in general.

By (2) and (3) of Lemma 2.2, we may regard L as a subposet (not a sublattice) of H by φ^*. Then, by (6) of Lemma 2.2, Q is identified as a subset of P.

Lemma 2.4 *By the identification above, the composition map* $J(P)\setminus\{\emptyset\} \simeq H \xrightarrow{\varphi} L \simeq J(Q)\setminus\{\emptyset\}$ *is identical to the map* $I \mapsto I \cap Q$*.*

3 Hibi rings, doset Hibi rings and generalized doset Hibi rings

Let K be a field, H a finite distributive lattice, P the set of join-irreducible elements of H, $J(P)$ the set of poset ideals of P and $\{T_x\}_{x\in P}$ a family of indeterminates indexed by P. Hibi [Hib] defined the ring $\mathcal{R}_K(H)$, which is called the Hibi ring nowadays, as follows.

Definition 3.1 ([Hib]) $\mathcal{R}_K(H) := K[\prod_{x\le\alpha} T_x \mid \alpha \in H]$.

Then

Theorem 3.2 **([Hib])** *$\mathcal{R}_K(H)$ is an ASL over K generated by H with structure map $\alpha \mapsto \prod_{x \le \alpha} T_x$. The straightening law is $\alpha\beta = (\alpha \wedge \beta)(\alpha \vee \beta)$ for α, $\beta \in H$ with $\alpha \not\sim \beta$.*

For a map $\nu \colon P \to \boldsymbol{N}$, we set $T^\nu := \prod_{x \in P} T_x^{\nu(x)}$. We also set $\overline{\mathcal{T}}(P) := \{\nu \colon P \to \boldsymbol{N} \mid a \le b \Rightarrow \nu(a) \ge \nu(b)\}$ and $\mathcal{T}(P) := \{\nu \colon P \to \boldsymbol{N} \setminus \{0\} \mid a < b \Rightarrow \nu(a) > \nu(b)\}$. Then

Theorem 3.3 **([Hib])** *$\mathcal{R}_K(H)$ is a free K-module with basis $\{T^\nu \mid \nu \in \overline{\mathcal{T}}(P)\}$.*

Remark 3.4 By Theorem 2.1, $\mathcal{R}_K(H)$ is a normal affine semigroup ring and hence is Cohen-Macaulay.

Now we recall the definition of a doset by DeConcini-Eisenbud-Procesi.

Definition 3.5 **([DEP, Section 18])** A subset D of $H \times H$ is called a doset if

(1) $\{(\alpha, \alpha) \mid \alpha \in H\} \subset D \subset \{(\alpha, \beta) \mid \alpha, \beta \in H,\ \alpha \le \beta\}$ and

(2) if $\alpha_1 \le \alpha_2 \le \alpha_3$, then

$$(\alpha_1, \alpha_3) \in D \Leftrightarrow (\alpha_1, \alpha_2) \in D \text{ and } (\alpha_2, \alpha_3) \in D.$$

Let L be another distributive lattice and $\varphi \colon H \to L$ a surjective lattice homomorphism. We set $D := \{(\alpha, \beta) \mid \alpha \le \beta,\ \varphi(\alpha) = \varphi(\beta)\}$. Then it is easily verified that D is a doset.

Definition 3.6 Doset Hibi ring over K defined by φ, denoted by $\mathcal{D}_K(\varphi)$, is the subalgebra of $\mathcal{R}_K(H)$ generated by $\{\alpha\beta \mid (\alpha, \beta) \in D\}$

Remark 3.7 If (α, β), $(\alpha', \beta') \in D$, then

$$\begin{aligned}
& (\alpha\beta)(\alpha'\beta') \\
= \; & (\alpha \wedge \alpha')(\alpha \vee \alpha')(\beta \wedge \beta')(\beta \vee \beta') \\
= \; & (\alpha \wedge \alpha')((\alpha \vee \alpha') \wedge (\beta \wedge \beta'))((\alpha \vee \alpha') \vee (\beta \wedge \beta'))(\beta \vee \beta')
\end{aligned}$$

and

$$\begin{aligned}
\varphi((\alpha \vee \alpha') \wedge (\beta \wedge \beta')) &= \varphi(\alpha \wedge \alpha') \\
\varphi((\alpha \vee \alpha') \vee (\beta \wedge \beta')) &= \varphi(\beta \vee \beta').
\end{aligned}$$

So $(\alpha \wedge \alpha', (\alpha \vee \alpha') \wedge (\beta \wedge \beta')), ((\alpha \vee \alpha') \vee (\beta \wedge \beta'), \beta \vee \beta') \in D$. Therefore, we see, by repeated application of straightening law, that $\{\alpha_1\alpha_2 \cdots \alpha_{2r-1}\alpha_{2r} \mid \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{2r}, (\alpha_{2i-1}, \alpha_{2i}) \in D \text{ for } i = 1, \ldots, r\}$ is a K-free basis of the doset Hibi ring.

Now let Q be the set of join-irreducible elements of L. As mentioned in (2.5), we regard Q as a subset of P.

Regarding $\mathcal{R}_K(H)$ as a subring of $K[T_x \mid x \in P]$, a standard monomial $\alpha_1\alpha_2 \cdots \alpha_s$ with $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_s$ corresponds to T^ν, where $\nu(x) = \#\{i \mid x \leq \alpha_i\}$. On the other hand, by the identification $L \simeq J(Q) \setminus \{\emptyset\}$ and $Q \subset P$, $\varphi(\alpha)$ corresponds to $\{y \in Q \mid y \leq \alpha \text{ in } H\}$ by Lemma 2.4. So a standard monomial $\alpha_1\alpha_2 \cdots \alpha_{2r}$ on H with $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{2r}$ satisfies $(\alpha_{2i-1}, \alpha_{2i}) \in D$ for $i = 1, \ldots, r$ if and only if $\nu(y) \equiv 0 \pmod 2$ for any $y \in Q$, where ν is an element of $\overline{\mathcal{T}}(P)$ corresponding to $\alpha_1 \cdots \alpha_{2r}$. Therefore we have the following

Theorem 3.8 *$\mathcal{D}_K(\varphi)$ is a free K-module with basis $\{T^\nu \mid \nu \in \overline{\mathcal{T}}(P), \nu(y) \equiv 0 \pmod 2$ for any $y \in Q\}$. In particular, by Theorem 2.1, $\mathcal{D}_K(\varphi)$ is a normal affine semigroup ring, and therefore, is Cohen-Macaulay.*

Note the description of $\mathcal{D}_K(\varphi)$ in the theorem above depends only on H and Q. Therefore we can generalize the notion of doset Hibi ring as follows.

Definition 3.9 Let H be a finite distributive lattice, P the set of join-irreducible elements of H, Q a subset of P. We define the generalized doset Hibi ring defined by H and Q, denoted by $\mathcal{D}_K(H, Q)$, as $\mathcal{D}_K(H, Q) := K[T^\nu \mid \nu \in \overline{\mathcal{T}}(P), \nu(y) \equiv 0 \pmod 2 \text{ for any } y \in Q]$.

The following is the direct consequence of the definition.

Theorem 3.10 *The generalized doset Hibi ring $\mathcal{D}_K(H, Q)$ is a free K-module with basis $\{T^\nu \mid \nu \in \overline{\mathcal{T}}(P), \nu(y) \equiv 0 \pmod 2$ for any $y \in Q\}$. In particular, by Theorem 2.1, $\mathcal{D}_K(H, Q)$ is a normal affine semigroup ring and hence is Cohen-Macaulay.*

Note that if Q contains the minimal element x_0 of H, then by setting $L = J(Q) \setminus \{\emptyset\}$ and φ the map corresponding to $J(P) \setminus \{\emptyset\} \to J(Q) \setminus \{\emptyset\}$, $(I \mapsto I \cap Q)$, $\mathcal{D}_K(H, Q)$ is equal to $\mathcal{D}_K(\varphi)$. In particular, $\mathcal{D}_K(H, Q)$ is the ordinal doset Hibi ring.

4 Gorenstein property

In this section, we state a criterion for a (generalized) doset Hibi ring to be Gorenstein.

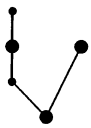
Set $P^+ := P \cup \{\infty\}$, where ∞ is a new element such that $x < \infty$ for any $x \in P$, $Q^+ := Q \cup \{\infty\}$ and $\tilde{P} := P^+ \cup \{(y_1, y_2) \in Q^+ \times Q^+ \mid y_1 \lessdot y_2 \text{ in } P^+\}$. We define the order on $\tilde{P}$ by by extending $y_1 < (y_1, y_2) < y_2$ for $y_1,\ y_2 \in Q^+$ with $y_1 \lessdot y_2$ in P^+. Then

Theorem 4.1 *$\mathcal{D}_K(H, Q)$ is Gorenstein if and only if*

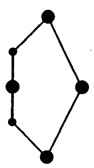
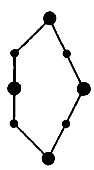
(1) $\tilde{P}$ is pure and

(2) for any $y_1,\ y_2 \in Q^+$ with $y_1 < y_2$, $\operatorname{rank}[y_1, y_2]_{\tilde{P}} \equiv 0 \pmod 2$.

Example 4.2 (1) If

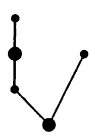
$P =$  ,

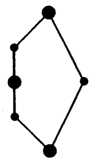
where big dots express elements of Q, then

$P^+ =$  and $\tilde{P} =$  .

Therefore, $\mathcal{D}_K(H, Q)$ is Gorenstein.

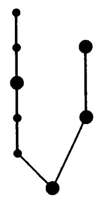
(2) If

$P =$  , then

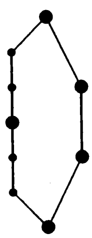
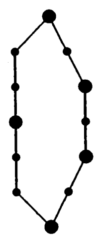
$P^+ =$  $= \tilde{P}.$

Therefore, $\tilde{P}$ is not pure and $\mathcal{D}_K(H, Q)$ is not Gorenstein.

(3) If

$P =$  ,

then

$P^+ =$  and $\tilde{P} =$  .

So there are y, $y' \in Q^+$ such that $y < y'$ and $\mathrm{rank}[y, y']_{\tilde{P}} = 3$. Therefore, $\mathcal{D}_K(H, Q)$ is not Gorenstein.

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