On limits of triply periodic minimal surfaces

Norio Ejiri,* Shoichi Fujimori,† and Toshihiro Shoda‡

April 11, 2018

Abstract

In this paper, we introduce generic limits of triply periodic minimal surfaces and consider the genus three case. We will prove that generic limits of such minimal surfaces consist of a one-parameter family of Karcher's saddle towers and Rodríguez' standard examples.

2010 Mathematics Subject Classification. Primary 53A10; Secondary 49Q05, 53C42.

Key words and phrases. minimal surface, triply periodic, limit.

1 Introduction

It is known that properly immersed triply periodic minimal surfaces in $\mathbb{R}^3$ have been used for the description of lipids or synthetic surfactants in physics, chemistry, and so on. Moreover, there are transitions from these membranes to lamellar phases, which are periodic parallel planes by environment conditions. In the previous paper [1], we showed that Rodríguez' standard examples [16] appear as limits of the real five-dimensional family of triply periodic minimal surfaces given by Meeks [9]. Limits of the standard examples include the Riemann minimal examples look like singly periodic parallel planes similar to lamellar phases. This suggests that limits of triply periodic minimal

*Partly supported by JSPS Grant-in-Aid for Scientific Research (C) 15K04859.
†Partly supported by JSPS Grant-in-Aid for Scientific Research (C) 17K05219.
‡Partly supported by JSPS Grant-in-Aid for Scientific Research (C) 16K05134.

surfaces might be mathematical description of the transition. This paper continues this work.

Let $f : M \to \mathbb{R}^3/\Lambda$ be a compact oriented minimal surface in a flat three-torus. By the isothermal coordinates, M can be reconsidered as a Riemann surface, and we call f a *conformal minimal immersion.* The following theorem gives an explicit description for a conformal minimal immersion. (See for instance [9].)

Theorem 1.1 (Weierstrass representation formula)**.** *Let $f : M \to \mathbb{R}^3/\Lambda$ be a conformal minimal immersion. Then, up to translations, f can be represented by the following path-integrals*:

$$f(p) = \Re \int_{p_0}^{p} {}^t(\omega_1,\, \omega_2,\, \omega_3) \mod \Lambda, \tag{1.1}$$

where p_0 is a fixed point on M and the ω_i's are holomorphic differentials on M satisfying the following three conditions.

$$\omega_1^2 + \omega_2^2 + \omega_3^2 = 0, \tag{1.2}$$

$$\omega_1,\, \omega_2,\, \omega_3 \text{ have no common zeros,} \tag{1.3}$$

$$\left\{ \Re \int_C {}^t(\omega_1,\, \omega_2,\, \omega_3) \,\middle|\, C \in H_1(M,\, \mathbb{Z}) \right\} \text{ is a sublattice of } \Lambda. \tag{1.4}$$

Conversely, the real part of path-integrals of holomorphic differentials satisfying the above three conditions defines a conformal minimal immersion.

Remark 1.1. We can write out $\omega_1,\, \omega_2,\, \omega_3$ in Theorem 1.1 as $(\omega_1,\, \omega_2,\, \omega_3) = (1 - g^2,\, i(1 + g^2),\, 2g)\omega$ for some holomorphic one-form ω and meromorphic function g which is the Gauss map composed with stereographic projection onto $\mathbb{C} \cup \{\infty\}$.

A minimal surface in $\mathbb{R}^3$ is said to be *periodic* if it is connected and invariant under a group Γ of isometries of $\mathbb{R}^3$ that acts properly discontinuously and freely (see [11]). Γ can be chosen to be a rank three lattice Λ in $\mathbb{R}^3$ (the triply periodic case), a rank two lattice $\Lambda \subset \mathbb{R}^2 \times \{0\}$ generated by two linearly independent translations (the doubly periodic case), or a cyclic group Λ generated by a screw motion symmetry, that is, a rotation around the x_3-axis composed with a non-trivial translation by a vector on the x_3-axis (the singly periodic case). The geometry of a periodic minimal surface in $\mathbb{R}^3$ can usually be described in terms of the geometry of its quotient surface M

in the flat three-manifold $\mathbb{R}^3/\Lambda$. Hence a triply periodic minimal surface is a minimal surface in a flat three-torus $\mathbb{T}^3$, a doubly periodic minimal surface is a minimal surface in $\mathbb{T}^2 \times \mathbb{R}$ where $\mathbb{T}^2$ is a flat two-torus, and a singly periodic minimal surface is a minimal surface in $S^1 \times \mathbb{R}^2$.

We will focus on the genus three case because of the following motivation in terms of the Morse index of a minimal surface. The Morse index of a compact oriented minimal surface in a flat three-torus is defined as the sum of the dimensions of the eigenspaces corresponding to negative eigenvalues of the Jacobi operator of the area. A minimal surface is said to be *stable* if it has Morse index zero. It is well-known that a compact oriented stable minimal surface in a flat three-torus must be a totally geodesic subtorus in the torus. Thus a compact oriented minimal surface in a flat three-torus which is not totally geodesic must have Morse index at least one, that is, the Morse index one case is the least one. In 2006, Ros [17] proved that a compact oriented minimal surface in a flat three-torus with Morse index one has genus three. So the genus three case might be important for natural phenomena. In fact, many one-parameter families of compact oriented minimal surfaces of genus three in flat three-tori have been studied in physics and chemistry. (See for example [8], [18].)

Recall that a compact oriented minimal surface of genus three in a flat three-torus is hyperelliptic, that is, it can be represented as a two-sheeted branched covering of the sphere. (See p. 49 in [13] or Corollary 3.2 in [9].) In this case, the Riemann surface M in Theorem 1.1 can be given by $w^2 = \prod_{i=1}^{8}(z-a_i)$ for distinct eight complex numbers $a_1, \ldots, a_8$ or $w^2 = \prod_{i=1}^{7}(z-b_i)$ for distinct seven complex numbers $b_1, \ldots, b_7$. (See p. 102 in [2] or p. 254 in [3].) We consider the former case since we can apply the similar arguments to the latter case. We define *generic limits* of triply periodic minimal surfaces as limits of f in Theorem 1.1 for the following three cases: (i) the case $a_2 \to a_1$, (ii) the case $(a_2, a_4) \to (a_1, a_3)$, (iii) the case $(a_2, a_4, a_6) \to (a_1, a_3, a_5)$. (See § 2 for the details.) Our main result is as follows.

Main Theorem. For a compact oriented embedded minimal surface of genus three in a flat three-torus, generic limits of the minimal surface consists of a one-parameter family of Karcher's saddle towers and Rodríguez' standard examples.

The paper is organized as follows. In § 2, we discuss generic limits of triply periodic minimal surfaces and give their properties. In § 3, we prove our main result, and finally in § 4, we introduce some singly periodic examples derived from generic limits of triply periodic minimal surfaces.

2 Generic limits

In this section, we will refer to the details of generic limits of triply periodic minimal surfaces for the genus three case. The arguments essentially appear in Section three of [1].

For eight distinct complex numbers $a_1, \ldots, a_8$, let M be the hyperelliptic Riemann surface of genus three defined by $w^2 = \prod_{i=1}^{8}(z - a_i)$. We can write out a basis of holomorphic differentials on M by $\{dz/w,\, zdz/w,\, z^2dz/w\}$. (See p. 255 in [3].) Let $f : M \to \mathbb{R}^3/\Lambda$ be a conformal minimal immersion of M into a flat three-torus, and we can choose $g = z$ and $\omega = dz/w$ in Theorem 1.1. (See Theorem 3.1 in [9]. See also Corollary of Theorem 2 in [13].)

We now consider each behavior of f for the following three cases: (i) the case $a_2 \to a_1$, (ii) the case $(a_2, a_4) \to (a_1, a_3)$, (iii) the case $(a_2, a_4, a_6) \to (a_1, a_3, a_5)$. We only treat the case (i) since the similar arguments work for the other two cases.

We first construct M as a two-sheeted branched covering of the sphere by the Gauss map $M \ni (z, w) \mapsto z \in \mathbb{C} \cup \{\infty\} \cong S^2$. The branch locus of the Gauss map consists of the following eight points on S^2:

$$a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6,\ a_7,\ a_8.$$

We prepare two copies of $\mathbb{C} \cup \{\infty\} \cong S^2$ and take two closed curves passing through the eight points, respectively. So we can divide S^2 into two domains and label "+″ and "−″. (See Figure 2.1.) Slit thick curves as in the upper pictures in Figure 2.1 and glue (i) and (ii) as in the lower pictures in Figure 2.1. The thin curves in the upper pictures in Figure 2.1 correspond to the thin curves in the lower pictures in Figure 2.1. By this procedure, we obtain the hyperelliptic Riemann surface M of genus three.

We now consider the case $a_2 \to a_1$. Let α be a closed curve enclosing a_1 and a_2 in the z-plane. Lift α to closed curves in M, name them $\hat{\alpha}$ and $\hat{\alpha}'$. Choosing suitable α, we can divide M into two disjoint sets in M with the following properties: One set contains $(a_1, 0)$ and $(a_2, 0)$ whose boundary consists of $\hat{\alpha}$ and $\hat{\alpha}'$, and the other set is the remaining one. (See Figure 2.2.) Let M_α denote the latter.

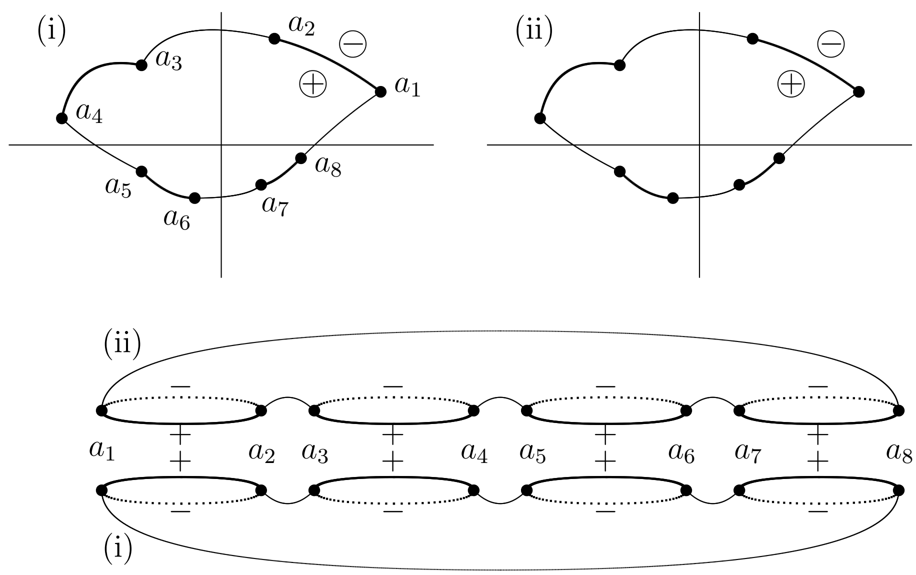

Figure 2.1: The hyperelliptic Riemann surface M of genus three

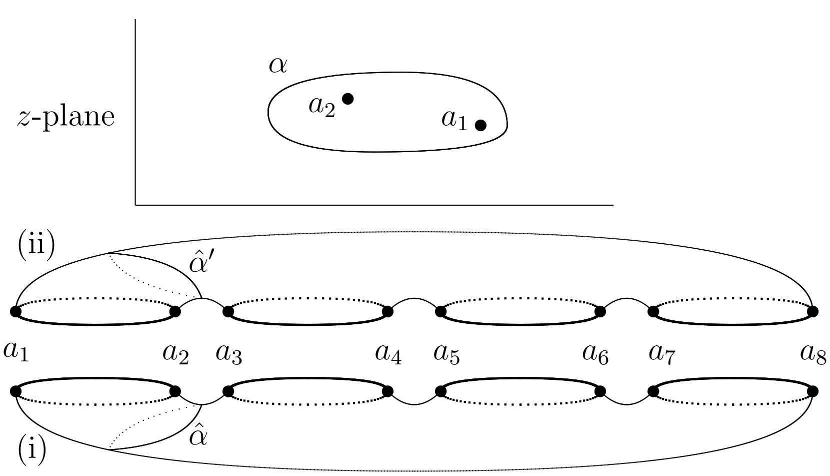

Figure 2.2: Closed curves α, $\hat{\alpha}$, $\hat{\alpha}'$.

The Weierstrass integral (1.1) along any closed curve in M_α is contained in the lattice of the target torus. So the Weierstrass integral depends only on the endpoint of a path in M_α. We now assume $p_0 \in M_\alpha$. Taking t as a local complex coordinate on M_α, we can write

$$\frac{1-z^2}{w}dz = \varphi_1(t)dt,\ \frac{i(1+z^2)}{w}dz = \varphi_2(t)dt,\ \frac{2z}{w}dz = \varphi_3(t)dt$$

for some holomorphic functions $\varphi_1(t)$, $\varphi_2(t)$, $\varphi_3(t)$. Then, the three functions $\varphi_1(t)$, $\varphi_2(t)$, $\varphi_3(t)$ converge uniformly on M_α as $a_2 \to a_1$. Hence the limits can be moved inside the integrals.

After that, letting α shrink to a point, we define generic limits of f for the case $a_2 \to a_1$ as a limit of f.

For the case $a_2 \to a_1$, M converges to the following Riemann surface with node, denoted by M':

$$w^2 = (z-a_1)^2 \prod_{j=3}^{8}(z-a_j).$$

Let M_1 be the Riemann surface of genus two defined by

$$v^2 = (z-a_3)(z-a_4)(z-a_5)(z-a_6)(z-a_7)(z-a_8).$$

Then there exists the following reparametrization of M':

$$M_1 \ni (z,\, v) \mapsto (z,\, (z-a_1)v) \in M'.$$

By using them, the limit, as $a_2 \to a_1$, of f is

$$\Re \int_{p_0}^{p} {}^t(1-z^2,\, i(1+z^2),\, 2z)\frac{dz}{(z-a_1)v} \tag{2.1}$$

on $M_1 \setminus \{(a_1,\, \pm v(a_1))\}$.

For the case $(a_2,\, a_4) \to (a_1,\, a_3)$, M converges to the following Riemann surface with nodes, denoted by M'':

$$w^2 = (z-a_1)^2(z-a_3)^2 \prod_{j=5}^{8}(z-a_j).$$

Let M_2 be the torus defined by

$$v^2 = (z - a_5)(z - a_6)(z - a_7)(z - a_8).$$

Then there exists the following reparametrization of M'':

$$M_2 \ni (z,\, v) \mapsto (z,\, (z - a_1)(z - a_3)v) \in M''.$$

By using them, the limit, as $(a_2,\, a_4) \to (a_1,\, a_3)$, of f is

$$\Re \int_{p_0}^{p} {}^t(1 - z^2,\, i(1 + z^2),\, 2z) \frac{dz}{(z - a_1)(z - a_3)v} \tag{2.2}$$

on $M_2 \setminus \{(a_1,\, \pm v(a_1)),\, (a_3,\, \pm v(a_3))\}$.

For the case $(a_2,\, a_4,\, a_6) \to (a_1,\, a_3,\, a_5)$, M converges to the following Riemann surface with nodes, denoted by M''':

$$w^2 = (z - a_1)^2(z - a_3)^2(z - a_5)^2(z - a_7)(z - a_8).$$

Let M_3 be the sphere defined by

$$v^2 = (z - a_7)(z - a_8).$$

Then there exists the following reparametrization of M''':

$$M_3 \ni (z,\, v) \mapsto (z,\, (z - a_1)(z - a_3)(z - a_5)v) \in M'''.$$

By using them, the limit, as $(a_2,\, a_4,\, a_6) \to (a_1,\, a_3,\, a_5)$, of f is

$$\Re \int_{p_0}^{p} {}^t(1 - z^2,\, i(1 + z^2),\, 2z) \frac{dz}{(z - a_1)(z - a_3)(z - a_5)v} \tag{2.3}$$

on $M_3 \setminus \{(a_1,\, \pm v(a_1)),\, (a_3,\, \pm v(a_3)),\, (a_5,\, \pm v(a_5))\}$.

Next we consider the asymptotic behavior of ends for (2.1), (2.2), (2.3). We set

$$\Phi = \begin{cases} {}^t(1 - z^2,\, i(1 + z^2),\, 2z) \dfrac{dz}{(z - a_1)v} & \text{(for (2.1))} \\ {}^t(1 - z^2,\, i(1 + z^2),\, 2z) \dfrac{dz}{(z - a_1)(z - a_3)v} & \text{(for (2.2))} \\ {}^t(1 - z^2,\, i(1 + z^2),\, 2z) \dfrac{dz}{(z - a_1)(z - a_3)(z - a_5)v} & \text{(for (2.3))} \end{cases}$$

It is straightforward to check that we have the Laurent expansion of Φ at $(a_1,\, v(a_1))$

$$\frac{1}{z}\,\alpha + \beta + \gamma\, z + \cdots\,,$$

where

$$\alpha = \begin{cases} {}^t(1-a_1^2,\, i(1+a_1^2),\, 2a_1)\frac{1}{v(a_1)} & \text{(for (2.1))} \\ {}^t(1-a_1^2,\, i(1+a_1^2),\, 2a_1)\frac{1}{(a_1-a_3)v(a_1)} & \text{(for (2.2))} \\ {}^t(1-a_1^2,\, i(1+a_1^2),\, 2a_1)\frac{1}{(a_1-a_3)(a_1-a_5)v(a_1)} & \text{(for (2.3))} \end{cases}$$

Let $\langle\cdot,\,\cdot\rangle$ be the complex bilinear inner product. Then we have $\langle\alpha,\,\alpha\rangle = 0$. Thus α is not real nor pure imaginary, and so $\Re(2\pi i\alpha) \neq \mathbf{0}$. We call the end *Scherk type end* with period $\Re(2\pi i\alpha)$. (See, for instance, Theorem 5 in [11].)

Note that there exists the hyperelliptic involution $(z,\, v) \mapsto (z,\, -v)$ on each M_j $(1 \leq j \leq 3)$. Hence we obtain a pair of parallel Scherk type ends given by $\alpha/z+\beta+\gamma\, z+\cdots$ and $-(\alpha/z+\beta+\gamma\, z+\cdots)$ with non-zero periods $\pm\Re(2\pi i\alpha)$.

Lemma 2.1. *For* (2.1)*,* (2.2)*, and* (2.3)*, ends consist of pairs of parallel Scherk type ends. For each pair of parallel Scherk type ends, up to translation in $\mathbb{R}^3$, one end can be transformed to the other end by a suitable inversion. In particular, the Gaussian curvature at each end converges to zero.*

We conclude this section with the following proposition.

Proposition 2.1. *Generic limits of triply periodic embedded minimal surfaces of genus three consist of singly periodic or doubly periodic properly embedded minimal surfaces in $\mathbb{R}^3$ with pairs of Scherk type ends. For each pair of Scherk type ends, the Gauss image at one end coincides with the Gauss image at the other end.*

Proof. We use the notation as above. Let $f_j : M_j \setminus E_j \to \mathbb{R}^3$ $(1 \leq j \leq 3)$ be an element of generic limits of triply periodic embedded minimal surfaces of genus three defined by (2.1), (2.2), (2.3), where E_j is a set of ends for each f_j. Note that f_j might be multivalued in general. Thus, by taking a suitable covering space $\tilde{M}_j \to M_j \setminus E_j$, there exists a single-valued minimal immersion $\tilde{f}_j : \tilde{M}_j \to \mathbb{R}^3$. By similar arguments to the proof of Theorem 7.1 in [9], $\hat{M}_j = \tilde{f}_j(\tilde{M}_j) = f_j(M_j \setminus E_j)$ is embedded.

It is straightforward to check that each $\hat{M}_j$ is complete from (2.1), (2.2), (2.3). By Lemma 2.1, $\hat{M}_j$ has only Scherk type ends, and thus it has bounded Gaussian curvature. Combining the two facts yields $\hat{M}_j$ is proper (see [12]), and in particular, $\hat{M}_j$ is orientable.

Recall that each $\tilde{f}_j : \tilde{M}_j \to \mathbb{R}^3$ is derived from a triply periodic minimal surface $f : M \to \mathbb{R}^3/\Lambda$. To obtain $\tilde{f}_j$, we collapse some one-cycles on M. By (1.4), $\tilde{f}_j$ must be triply periodic or doubly periodic or singly periodic or non-periodic minimal embedding. Since $\tilde{f}_j$ has Scherk type ends with non-zero periods and is proper, it cannot be triply periodic nor be non-periodic minimal embedding. Therefore $\tilde{f}_j$ is singly periodic or doubly periodic.

It follows that the minimal embedding $\tilde{f}_j : \tilde{M}_j \to \mathbb{R}^3$ gives rise to a conformal minimal embedding $\overline{f_j}$ of a Riemann surface $\overline{M_j}$ into an $S^1 \times \mathbb{R}^2$ or a $\mathbb{T}^2 \times \mathbb{R}$. Also, f_j is a conformal minimal immersion of $M_j \setminus E_j$ into the $S^1 \times \mathbb{R}^2$ or the $\mathbb{T}^2 \times \mathbb{R}$, and there exists a covering map $\pi : M_j \setminus E_j \to \overline{M_j}$ such that $f_j = \overline{f_j} \circ \pi$. Hence $\overline{f_j}$ has finite total curvature, and so $\overline{M_j}$ is biholomorphic to a compact Riemann surface $\overline{N_j}$ with a finite number of points removed. Moreover, π extends to a covering map $\pi : M_j \to \overline{N_j}$.

Let G and $\overline{G}$ be the Gauss map of f_j and $\overline{f_j}$, respectively. Note that G and $\overline{G}$ can be extended to holomorphic maps $M_j \to S^2$ and $\overline{N_j} \to S^2$, respectively. Then we have $G = \overline{G} \circ \pi$. Note that G can be obtained by $(z, v) \mapsto z$, and so $\deg(G) = 2$. To show $\deg(\pi) = 1$, we now assume $\deg(\overline{G}) = 1$, and thus $\overline{G}$ is biholomorphic.

For each $k = 1, 3, 5$, the equation $G(a_k, v(a_k)) = G(a_k, -v(a_k))$ implies that $\overline{f_j}$ is a singly periodic or doubly periodic properly minimal embedding of genus zero with at most three ends. However, for each case, $\overline{f_j}$ has at least four ends ([10], [15]), and it leads to a contradiction.

Therefore we have $\deg(\pi) = 1$, and so f_j coincides with $\overline{f_j}$. Again, by the equation $G(a_k, v(a_k)) = G(a_k, -v(a_k))$, the Gauss image at one end coincides with the Gauss image at the other end for each pair of ends. $\square$

3 Proof of Main Theorem

In this section, we will prove Main Theorem in the introduction. We use the notation as in the previous section.

We first consider f_1 defined by (2.1), and then f_1 is a properly embedded singly or doubly periodic minimal surface with a pair of Scherk type ends with periods $\pm\Re(2\pi i\alpha)$. Suppose that f_1 is singly periodic. Let $\{E_1, E_1'\}$ be the pair of Scherk type ends and $G(E_1)$, $G(E_1')$ their Gauss images. In this case, there exists a plane Π such that its normal vector is parallel to $G(E_1)$ and, by the strong halfspace theorem [4], it intersects the minimal surface at an interior point. (See Figure 3.1.) However, it contradicts the maximum principle for minimal surfaces. Thus f_1 is doubly periodic, but it contradicts Meeks-Rosenberg's result [10]. Hence this case is excluded.

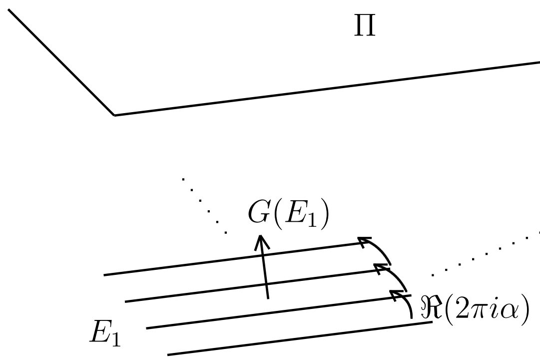

Figure 3.1: A singly periodic minimal surface with a pair of Scherk type ends

We next consider f_2 defined by (2.2), and in this case, f_2 gives a properly embedded singly periodic or doubly periodic minimal surface of genus one with two pairs of Scherk type ends. If f_2 is singly periodic, then by Proposition 2.1, there are two essential cases as in Figure 3.2. For both cases, there

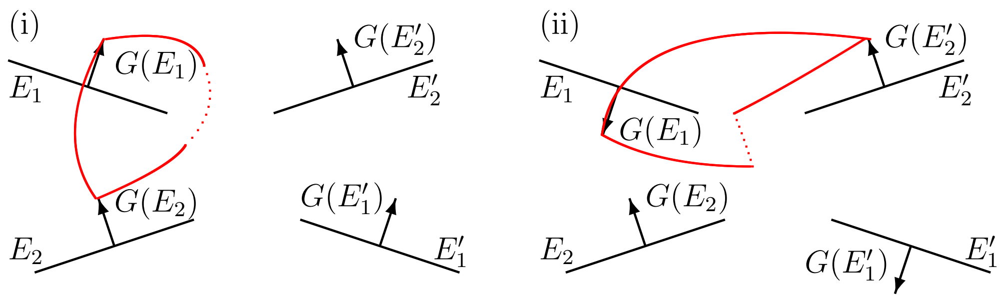

Figure 3.2: Two pairs of Scherk type ends $\{E_1, E_1'\}$ and $\{E_2, E_2'\}$

exists a closed curve which intersects the minimal surface once (see the red curves in Figure 3.2), and it contradicts the classical result in surface theory. Suppose that f_2 is doubly periodic. If the ends are not parallel, then we can apply the same arguments as in Figure 3.2, and it contradicts. Hence the ends are parallel, and by the result of Pérez-Rodríguez-Traizet [14], it must be Rodríguez' standard example.

We finally consider f_3 defined by (2.3), and then f_3 is a singly or doubly periodic minimal embedding of genus zero with six Scherk type ends. Combining Lazard-Holly and Meeks' result [7] and Pérez-Traizet's result [15] yields that f_3 must be a singly periodic Scherk surface or Karcher saddle tower or doubly periodic Scherk surface. It has six ends, and thus it must be the Karcher saddle tower.

Remark 3.1. There exists a doubly periodic Scherk surface with a handle [5]. It has two pairs of Scherk type ends and its ends are not parallel. However, for each pair of ends, the Gauss image at one end is distinct from the Gauss image at the other end. By Proposition 2.1, this surface cannot be contained in generic limits of triply periodic minimal surfaces of genus three.

4 Appendix (related examples)

We now consider singly periodic minimal surfaces given by (2.3). By the period condition for singly periodic minimal surfaces, it can be reduced to two cases essentially. One is given by

$$\begin{gathered} a_1 = -e^{i\pi/4}, \qquad a_3 = e^{i(\theta+\pi/4)}, \qquad a_5 = e^{i(-\theta+\pi/4)}, \\ a_7 = e^{i(\varphi+\pi/4)}, \qquad a_8 = e^{i(-\varphi+\pi/4)} \end{gathered} \tag{4.1}$$

in (2.3), where $\theta \in (\pi/2, \pi)$ and

$$\varphi = \arccos\left(\frac{2\sin^2\theta}{1-2\cos\theta} - 1\right)$$

are constants (see the left hand side of Figure 4.1). One can see that none of this family is embedded, and thus it cannot be included in generic limits of embedded triply periodic minimal surfaces of genus three. The other is embedded (see the right hand side of Figure 4.1), more precisely, these singly periodic minimal surfaces were found by Karcher. See § 2.5.1 in [6].

It is known that the symmetric Karcher saddle tower (§ 2.5.1 in [6] with $r = 0$) is a limit of a family of embedded triply periodic minimal surfaces of genus three, called Schwarz H family. However, a family of embedded triply periodic minimal surfaces of genus three whose limit is non-symmetric Karcher saddle tower is yet to be found, leading us to the following open problem:

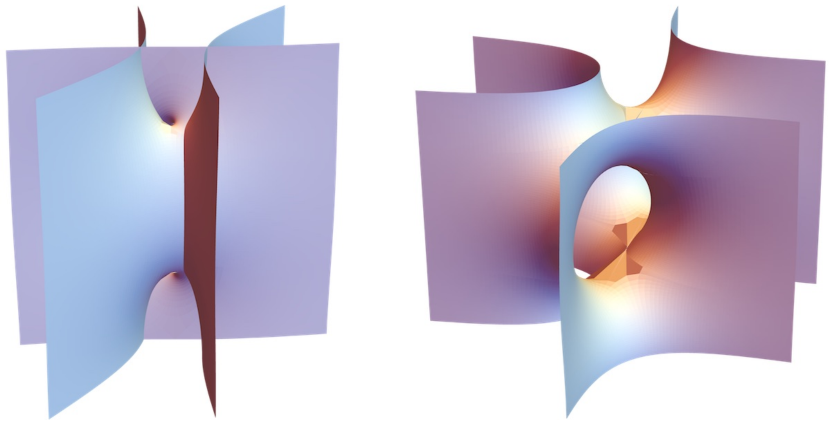

Figure 4.1: Left: Singly periodic minimal surface given by (4.1) with $\theta = 3\pi/4$. Right: Singly periodic minimal surface given by § 2.5.1 in [6] with $r = 1/2$.

Problem. Whether there exists a family of embedded triply periodic minimal surfaces of genus three whose limit is non-symmetric Karcher saddle tower (§ 2.5.1 in [6] with $r \neq 0$) or not?

References

[1] N. Ejiri, S. Fujimori, and T. Shoda, *A remark on limits of triply periodic minimal surfaces of genus* 3, Topology Appl. **196** (2015), part B, 880–903.

[2] H. M. Farkas and I. Kra, *Riemann surfaces*, Second edition, Graduate Texts in Mathematics, **71**. Springer-Verlag, New York, 1992.

[3] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Reprint of the 1978 original. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994.

[4] D. Hoffman and W. H. Meeks III, *The strong halfspace theorem for minimal surfaces*, Invent. Math. **101**, (1990), 373–377.

[5] D. Hoffman, F. Wei, and H. Karcher, *The genus one Helicoid and the minimal surfaces that led to its discovery*, Global analysis in modern

mathematics (Orono, ME, 1991; Waltham, MA, 1992), 119–170, Publish or Perish, Houston, TX, 1993.

[6] H. Karcher, *Embedded minimal surfaces derived from Scherk's examples*, Manuscripta Math. **62** (1988), no. 2, 83–114.

[7] H. Lazard-Holly and W. H. Meeks III, *Classification of doubly-periodic minimal surfaces of genus zero*, Invent. Math. **143**, (2001), no. 1, 1–27.

[8] S. Lidin and S. Larsson, *Bonnet Transformation of Infinite Periodic Minimal Surfaces with Hexagonal Symmetry*, J. Chem. Soc. Farday trans **86**(5) (1990), 769–775.

[9] W. H. Meeks III, *The theory of triply periodic minimal surfaces*, Indiana Univ. Math. J. **39** (1990), no. 3, 877–936.

[10] W. H. Meeks III and H. Rosenberg, *The global theory of doubly periodic minimal surfaces*, Invent. Math. **97** (1989), no. 2, 351–379.

[11] W. H. Meeks III and H. Rosenberg, *The geometry of periodic minimal surfaces*, Comment. Math. Helv. **68** (1993), no. 4, 538–578.

[12] W. H. Meeks III and H. Rosenberg, *The uniqueness of the helicoid*, Ann. of Math. (2) **161** (2005), no. 2, 727–758.

[13] T. Nagano and B. Smyth, *Periodic minimal surfaces*, Comment. Math. Helv. **53** (1978), no. 1, 29–55.

[14] J. Pérez, M. Magdalena Rodríguez, and M. Traizet, *The classification of doubly periodic minimal tori with parallel ends*, J. Differential Geom. **69**, (2005), no. 3, 523–577.

[15] J. Pérez and M. Traizet, *The classification of singly periodic minimal surfaces with genus zero and Scherk-type ends*, Trans. Amer. Math. Soc. **359** (2007), no. 3, 965–990.

[16] M. Magdalena Rodríguez, *The space of doubly periodic minimal tori with parallel ends: standard examples*, Michigan Math. J. **55** (2007), no. 1, 103–122.

[17] A. Ros, *One-sided complete stable minimal surfaces*, J. Differential Geom. **74** (2006), no. 1, 69-92.

[18] G. E. Schröder-Turk, A. Fogden, and S. T. Hyde, *Bicontinuous geometries and molecular self-assembly: comparison of local curvature and global packing variations in genus-three cubic, tetragonal and rhombohedral surfaces*, Eur. Phys. J. B. **54** (2006), 509–524.

Norio Ejiri
Department of Mathematics, Meijo University
Tempaku, Nagoya 468-8502, Japan.
ejiri@meijo-u.ac.jp

Shoichi Fujimori
Department of Mathematics, Okayama University
Tsushimanaka, Okayama 700-8530, Japan.
fujimori@math.okayama-u.ac.jp

Toshihiro Shoda
Faculty of Education, Saga University
1 Honjo-machi, Saga-city, Saga, 840-8502, Japan.
tshoda@cc.saga-u.ac.jp