MAXIMAL SURFACES IN MINKOWSKI 3-SPACE WITH NON-TRIVIAL TOPOLOGY AND CORRESPONDING CMC 1 SURFACES IN DE SITTER 3-SPACE

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1. Introduction

With the recent interest in finding Weierstrass-type representations for surfaces other than minimal surfaces in Euclidean 3-space, the case of (spacelike) constant mean curvature (CMC) 1 surfaces in de-Sitter 3-space $\mathbb{S}^{2,1}$ has been undergoing investigation. (Throughout this paper, we treat only spacelike surfaces with singularities.) Having more methods available for producing surfaces of this type is useful, and is the goal of this paper.

In the recent work by Fujimori, Rossman, Umehara, Yamada and Yang [FRUYY], the method by Rossman, Umehara and Yamada in [RUY] was adapted to the case of maximal surfaces in Minkowski 3-space $\mathbb{R}^{2,1}$ and their cousin CMC 1 surfaces in de Sitter 3-space, for the purpose of producing some specific examples of surfaces with particular geometric properties of interest. Here we reformulate that result in [FRUYY] to apply to other surfaces as well, using a non-degeneracy condition like that used in [RUY], see Theorem 3.2.

Although in Euclidean 3-space $\mathbb{R}^3$ every direction is geometrically the same, this is not the case in $\mathbb{R}^{2,1}$. For this reason, when we formulate the non-degeneracy condition in Section 3, we use only two timelike planes in general position, rather than the three planes in general position that were used in [RUY].

In section 4 we give new examples of genus 1 maxfaces in $\mathbb{R}^{2,1}$ with two or three ends and apply Theorem 3.2 to produce corresponding genus 1 CMC 1 surfaces in $\mathbb{S}^{2,1}$. With the final two examples, we are able to provide an answer to Problem 2 raised in [FRUYY]. In fact, one of those two examples has all ends embedded.

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2. CMC surfaces in de Sitter spaces

Let $\mathbb{R}^{3,1}$ be Minkowski 4-space with the metric of signature $(-,+,+,+)$. We define de Sitter space of constant sectional curvature 1 by

$$\mathbb{S}^{2,1} = \{(t,x,y,z) \in \mathbb{R}^{3,1} \mid -t^2+x^2+y^2+z^2=1\}\ .$$

We will use the following standard 2×2-matrix model of $\mathbb{S}^{2,1}$ (see for example [FRUYY]):

$$\mathbb{S}^{2,1} = \{Xe_3X^* | X \in \mathrm{SL}(2,\mathbb{C})\} = \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(1,1),$$

where

$$e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In this model the metric on $\mathbb{S}^{2,1}$ is determined by

$$\langle Y, Y\rangle = -\det(Y),$$

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for $Y \in T_p\mathbb{S}^{2,1}$.

Let Σ be a Riemann surface. For $c \in \mathbb{R}^\times$, let $F_c : \Sigma \to \mathrm{SL}(2,\mathbb{C})$ be a solution of

$$dF_c \cdot F_c^{-1} = c \begin{pmatrix} G & -G^2 \\ 1 & -G \end{pmatrix} \omega \ , \tag{2.1}$$

where $G : \Sigma \to \mathbb{C}$ is a holomorphic function and $\omega \in \Omega^1(\Sigma)$ is a holomorphic 1-form independent of c. Then using the Weierstass-type representation for $\mathbb{S}^{2,1}$ (see for example [FRUYY]) we have that

$$f_c := F_c e_3 F_c^*,$$

is a CMC 1 surface in $\mathbb{S}^{2,1}$.

Remark 2.1. We have that $\omega = \frac{Q}{dG}$, where cQ is the Hopf differential of f_c.

3. The [RUY] method in de Sitter space

Let D be a simply-connected region in Σ with local coordinate z, bounded by a finite number of smooth arcs. Let G and ω be Weierstrass data on D producing a (possibly branched) maxface [UY2]

$$f_0 = \mathrm{Re} \int_{z_0}^{z} (-2G, 1 + G^2, i - iG^2)\omega$$

in $\mathbb{R}^{2,1}$ (with metric of signature $(-,+,+)$) bounded by planar geodesic curvature lines lying in either of two given timelike planes P_1 and P_2, and suppose that P_1 and P_2 are not parallel to each other. We allow that these geodesics might be defined only in the interiors of the corresponding smooth arcs in ∂D, creating the possibility of ends of f_0 at the endpoints of the smooth arcs. The Hopf differential $Q = \omega dG$ is real when restricted to ∂D. Suppose that repeated inclusion of reflected copies of f_0 across P_1 and P_2 (and their images under reflections) extends f_0 to a (possibly branched) maxface $\hat{f}_0$ of finite topology and no boundary.

Label the smooth boundary arcs of D as $S_{1,1}$, $S_{1,2}$, ..., S_{1,k_1}, $S_{2,1}$, $S_{2,2}$, ..., S_{2,k_2}, where each $S_{i,j}$ has image under f_0 in the plane P_i, for $i = 1, 2$. For technical reasons (in the proof of Theorem 3.2), we also make the following further assumption: at least one endpoint of one smooth arc of ∂D is mapped by f_0 to a finite point in $\partial f_0 \subset \mathbb{R}^{2,1}$, i.e., is not an end of f_0.

Let $f_0(\lambda)$ be a smooth family of maxfaces in $\mathbb{R}^{2,1}$ depending on a parameter λ, where λ is contained in an open subset N of $\mathbb{R}^{k_1+k_2-2}$ such that $f_0(\lambda_0) = f_0$ for some $\lambda_0 \in N$. Thus, for each λ, $f_0(\lambda)$ is determined by Weierstrass data $G(\lambda), \omega(\lambda)$ and domain $D(\lambda)$ depending smoothly on λ. Assume that for each λ, we can identify the boundary arcs $S_{i,j}(\lambda)$ of $D(\lambda)$ with $S_{i,j}$ and that $f_0(\lambda)|_{S_{i,j}(\lambda)}$ is a planar geodesic in a plane $P_{i,j}(\lambda)$ parallel to P_i.

Let

$$d_{i,j} = \text{the oriented distance between } P_{i,j}(\lambda) \text{ and } P_{i,1}(\lambda)\ .$$

Thus $d_{i,j}$ changes sign when $P_{i,j}(\lambda)$ crosses from one side of $P_{i,1}(\lambda)$ to the other, and is zero if and only if $P_{i,j}(\lambda) = P_{i,1}(\lambda)$.

Definition 3.1. $f_0(\lambda)$ is said to be *non-degenerate* with respect to the parameter λ if the period map

$$\mathrm{Per} : N \to (d_{1,2}, ..., d_{1,k_1}, d_{2,2}, ..., d_{2,k_2})$$

is an open map at λ_0, i.e., there exists an open neighborhood of λ_0, $V \subset N$, such that $\mathrm{Per}(V)$ is an open neighborhood of the origin in $\mathbb{R}^{k_1+k_2-2}$.

We are now in a position to state the main theoretical tool of the paper. Note that CMC-1 faces are defined in [F].

Theorem 3.2. *If $f_0(\lambda)$ is a non-degenerate maxface, then there exists a corresponding* 1*-parameter family of CMC-*1 *faces f_c, $c \in (-\varepsilon, \varepsilon) \setminus \{0\}$, in $\mathbb{S}^{2,1}$ with no boundary and with the same topology and corresponding reflection symmetries as f_0.*

Sketch proof of Theorem 3.2: The proof of Theorem 3.2 is essentially the same as part of the proof of Theorem B in [FRUYY], and is the SU(1,1) analogue of the proof using SU(2) of Theorem 5.10 in [RUY]. In fact some of the technicalities of that proof are not needed here because Theorem B dealt with a degenerate period problem, which differs from our case. As noted before the Hopf differential $c\,Q$ satisfies a reality condition which amounts to $\overline{Q \circ \mu_{ij}} = Q$, where $\mu_{i,j}$ denotes reflection of the surface $f_0(\lambda)$ across $S_{i,j}(\lambda)$. Furthermore, as in Lemma 4.9 in [FRUYY], $\overline{G \circ \mu_{ij}} = \sigma_j \star G$, where σ_j are particular 2×2 matrices and where

$$a \star h = \frac{a_{11}h + a_{12}}{a_{21}h + a_{22}},$$

for $a \in \mathrm{SL}(2, \mathbb{C})$ (where a_{st} denote the components of a) and h a holomorphic function. In fact, without loss of generality, σ_1 is the identity matrix and σ_2 is a unitary diagonal matrix. We then have that the solution F_c of Equation (2.1) satisfies

$$\overline{F_c \circ \mu_{i,j}} = \sigma_i F_c \rho_{i,j}^{-1}\ ,$$

where $\rho_{i,j}$ is independent of z, but can depend on c and λ, and also on the initial condition used to determine the solution F_c.

We wish to transform the $\rho_{i,j}$ so that they lie in SU(1,1), because this is the condition that causes the surface f_c to have the same topology as $\hat{f}_0$. We do this as follows: we change F_c to $\hat{F}_c = F_c b$ for some $b \in \mathrm{SL}(2, \mathbb{R})$, independent of z (but allowed to depend on c and λ). Then the matrices $\rho_{i,j}$ change to $\overline{b^{-1}}\rho_{i,j}b$. We adjust b and λ until $\overline{b^{-1}}\rho_{i,j}b \in \mathrm{SU}(1,1)$ for all i, j, for any c sufficiently close to 0, by using the non-degeneracy condition and Lemma 4.4 in [FRUYY], and arguments regarding the λ dependence of the $\rho_{i,j}$ like in [RUY]. This produces a one parameter family of CMC-1 surfaces $\hat{F}_c e_3 \hat{F}_c^*$ in $\mathbb{S}^{2,1}$ with the same topology as f_0 for c sufficiently close to 0, completeing the proof of Theorem 3.2.

Remark 3.3. The method in Proposition 5.4 of [UY2] shows the existence of a Chen-Gackstatter type maxface in $\mathbb{R}^{2,1}$, as a companion surface to the Chen-Gackstatter surface in $\mathbb{R}^3$. However, our result here will not apply to this and other companion surfaces in $\mathbb{R}^{2,1}$, because that method involves changing the G in the Weierstrass data to iG, and while the Hopf differential is real along boundary curves in the case of $\mathbb{R}^3$, it becomes pure imaginary in $\mathbb{R}^{2,1}$. Thus our result cannot be applied.

4. Application: Genus-s examples with two or three ends

We now seek examples of maximal surfaces in $\mathbb{R}^{2,1}$ to which we can apply Theorem 3.2. For any positive odd number s, define a Riemann surface

$$\hat{M} := \{(z, w) \in (\mathbb{C} \cup \{\infty\})^2 \mid w^{s+1} = (z - \lambda_1)(z^2 - 1)^s\}, \tag{4.1}$$

of genus s with Weierstrass data

$$G = \lambda_2 z^j w^k \quad \text{and} \quad \omega = z^l w^m dz, \tag{4.2}$$

where $\lambda := (\lambda_1, \lambda_2) \in (-1, 1) \times \mathbb{R}$ and $j, k, l, m \in \mathbb{Z}$. This determines a (possibly branched) maxface

$$f := \mathrm{Re} \int_{z_0}^{z} (-2G, 1 + G^2, i - iG^2)\omega$$

on the universal cover of $M := \hat{M}\backslash\{p_1, ..., p_r\}$ in $\mathbb{R}^{2,1}$ with metric

$$ds^2 = (1 - G\bar{G})^2\omega\bar{\omega},$$

where ds^2 blows up at the points p_j, for $j \in \{1, ..., r\}$. Consider the sheets of $\{(z, w) \in \hat{M}|\mathrm{Im}z \geq 0\}$. Let D be the sheet of $\{(z, w) \in \hat{M}|\mathrm{Im}z \geq 0\}$ such that $w((1, \infty)) \subset \mathbb{R}^+$. Label the boundary curves of D as

$$\begin{aligned}
S_{1,1} &:= \{(z, w(z)) \in D|\, z \in [1, \infty]\},\\
S_{1,2} &:= \{(z, w(z)) \in D|\, z \in [-1, \lambda_1]\},\\
S_{2,1} &:= \{(z, w(z)) \in D|\, z \in [\lambda_1, 1]\},\\
S_{2,2} &:= \{(z, w(z)) \in D|\, z \in [-\infty, -1]\}.
\end{aligned}$$

To obtain surfaces possessing the reflectional symmetry that Theorem 3.2 requires, we use the following lemma:

Lemma 4.1. *The images of the boundary arcs of D are planar geodesics if and only if $k + m$ is an integer multiple of $s + 1$.*

Proof. The result follows by checking when the Hopf differential $Q = dG\,\omega$ is real valued along the boundary of D. □

From now on we will assume that $k + m$ is an integer multiple of $s + 1$.

Lemma 4.2. *When $\lambda_1 = 0$ we have that $d_{2,2} = \pm d_{1,2}$.*

Proof. Define two curves in M by

$$\tau_1, \tau_2 : [0, \pi] \to M, \quad \tau_1(\nu) = (e^{i\nu} + \tfrac{1}{2}, w(e^{i\nu} + \tfrac{1}{2})), \text{ in } D, \text{ and}$$
$$\tau_2(\nu) = (-e^{i\nu} - \tfrac{1}{2}, (-1)^{\frac{s}{s+1}} w(e^{i\nu} + \tfrac{1}{2})).$$

Note that τ_2 is well-defined in M because s is odd. Then for $\theta = \frac{\pi k s}{s+1}$,

$$\begin{aligned}
d_{1,2} &= \mathrm{Re}\int_{\tau_1} i(1 - G^2)\omega\\
d_{2,2} &= -\sin\theta\,\mathrm{Re}\int_{\tau_2}(1 + G^2)\omega + \cos\theta\,\mathrm{Re}\int_{\tau_2} i(1 - G^2)\omega
\end{aligned}$$

Thus,

$$\begin{aligned}
d_{2,2} &= \mathrm{Re}\int_{\tau_2}(ie^{i\theta}\omega - ie^{-i\theta}G^2\omega)\\
&= \mathrm{Re}\int_{\tau_1}(ie^{i\theta}(-1)^{\frac{ms}{s+1}+l+1}\omega - ie^{-i\theta}(-1)^{\frac{(2k+m)s}{s+1}+l+1}G^2\omega)\\
&= (-1)^{\frac{(k+m)s}{s+1}+l+1} d_{1,2}\,,
\end{aligned}$$

since $k + m$ is an integer multiple of $s + 1$. □

In light of Lemma 4.2, we set $\lambda_1 = 0$ and our goal is to find a value of λ_2 so that $d_{1,2} = 0$, i.e., so that our surface is well-defined on M and thus has finite topology. Viewing $d_{1,2}$ as a function of λ_2, one arrives at the following lemma:

Lemma 4.3. *If*

$$\lambda_2^{\pm} := \pm\sqrt{\frac{\mathrm{Im}\int_\tau z^l w^m dz}{\mathrm{Im}\int_\tau z^{2j+l} w^{2k+m} dz}}$$

are real and non-zero, where

$$\tau : [0, \pi] \to \Sigma, \quad \nu \mapsto (e^{i\nu} + \tfrac{1}{2}, w(e^{i\nu} + \tfrac{1}{2})),$$

then the maximal surfaces determined by (4.1) and (4.2) with $\lambda = \lambda_0^{\pm} := (0, \lambda_2^{\pm})$ are well-defined on M.

Therefore let us assume that $\lambda_0 := (0, \lambda_2^0)$ determines a maximal surface that is well defined on M. We want the surface to be a maxface, i.e., we want to allow the surface to admit singularities but have no branch points and to have complete ends. Away from points of M where z is not a local coordinate, i.e., when $z \in \{0, 1, -1, \infty\}$, we have that df is non-zero, and thus, the surface is not branched. Furthermore, the surface is a maxface away from $z \in \{0, 1, -1, \infty\}$, which follows from Fact 1.1 in [FRUYY], considered on local simply connected open subsets of M.

To ensure that the surface does not have branch points when $z \in \{0, 1, -1, \infty\}$, we will require that the metric

$$ds^2 = (1 - G\bar{G})^2 \omega\bar{\omega} = (1 - \lambda_2^2 |z^j w^k|^2)^2 |z^l w^m|^2 |dz|^2$$

is either non-singular or blows up at these points:

At $z = 0$, assuming that $j(s+1) + k \neq 0$, the metric is non-singular or blows up if and only if

$$\begin{aligned} l(s+1) + m + s \leq 0 \quad &\text{when} \quad j(s+1) + k > 0\,, \\ (2j + l + 1)(s+1) + 2k + m - 1 \leq 0 \quad &\text{when} \quad j(s+1) + k < 0. \end{aligned}$$

Equality on the left hand side in either case means that f has a finite point at $z = 0$. Otherwise, f admits a complete end at $z = 0$.

At $z = \pm 1$, assuming $k \neq 0$, the metric is non-singular or blows up if and only if

$$\begin{aligned} m + 1 \leq 0 \quad &\text{when} \quad k > 0\,, \\ 2k + m + 1 \leq 0 \quad &\text{when} \quad k < 0. \end{aligned}$$

If we have equality on the left hand side in either case then f has finite points at $z = \pm 1$. Otherwise, $z = \pm 1$ are both complete ends of the surface.

At $z = \infty$, assuming $(2k + j)s + j + k \neq 0$, the metric is non-singular or blows up if and only if

$$\begin{aligned} (l + 2m + 1)s + l + m + 2 \geq 0 \quad &\text{when} \quad (2k + j)s + j + k < 0\,, \\ (4k + 2j + l + 2m + 1)s + 2j + 2k + l + m + 2 \geq 0 \quad &\text{when} \quad (2k + j)s + j + k > 0. \end{aligned}$$

If we have equality on the left hand side in either case then f has a finite point at $z = \infty$. Otherwise, f has a complete end at $z = \infty$.

So far we have shown how to construct maxfaces in $\mathbb{R}^{2,1}$ with finite topology equal to that of M. We would now like to use Theorem 3.2 to obtain CMC 1 faces in $\mathbb{S}^{2,1}$ from these examples. To do this we create a period problem by viewing $\lambda = (\lambda_1, \lambda_2)$ in (4.1) and (4.2) as a parameter in the domain $(-1, 1) \times \mathbb{R}$. Then to check that the period problem is non-degenerate, it suffices to check that the map

$$(\lambda_1, \lambda_2) \mapsto (d_{1,2}, d_{2,2})$$

is immersed at the solution point λ_0 of the period problem, i.e., the determinant of the Jacobian at λ_0,

$$\left.\begin{pmatrix} \frac{\partial}{\partial \lambda_1} d_{1,2} & \frac{\partial}{\partial \lambda_2} d_{1,2} \\ \frac{\partial}{\partial \lambda_1} d_{2,2} & \frac{\partial}{\partial \lambda_2} d_{2,2} \end{pmatrix}\right|_{\lambda = \lambda_0},$$

is non-zero. One can check that

$$\begin{aligned} \frac{\partial}{\partial \lambda_1} d_{2,2}|_{\lambda=\lambda_0} &= (-1)^{l + \frac{(k+m)s}{s+1}} \frac{\partial}{\partial \lambda_1} d_{1,2}|_{\lambda=\lambda_0}, \text{ and} \\ \frac{\partial}{\partial \lambda_2} d_{2,2}|_{\lambda=\lambda_0} &= (-1)^{l + 1 + \frac{(k+m)s}{s+1}} \frac{\partial}{\partial \lambda_2} d_{1,2}|_{\lambda=\lambda_0}. \end{aligned}$$

Thus, the non-degeneracy condition reduces to both $\frac{\partial}{\partial \lambda_1} d_{1,2}|_{\lambda=\lambda_0}$ and $\frac{\partial}{\partial \lambda_2} d_{1,2}|_{\lambda=\lambda_0}$ being non-zero. This is the case in all the examples we are about to consider.

FIGURE 4.1. Genus 1 example with $j = -1$, $k = 1$, $l = 0$, $m = -1$, see [FRUYY].

FIGURE 4.2. Genus 1 example with two complete ends with $j = 0$, $k = 1$, $l = -1$, $m = -1$. $(z, w) = (0, 0)$ corresponds to an embedded end, and $(z, w) = (\infty, \infty)$ corresponds to a non-embedded end.

Remark 4.4. If introducing parameters λ_j, like above, in a certain way does not yield a non-degenerate period problem for a particular example, then there may be other ways that parameters can be introduced into the data so that a non-degenerate period problem is obtained.

4.1. **Two-ended examples.** If we consider the case where

$$j = -1,\ k = 1,\ l = 0,\ m = -1,$$

then we obtain the genus s surfaces with two complete ends in $\mathbb{R}^{2,1}$ given in [FRUYY]. $(z, w) = (0, 0)$ and $(z, w) = (\infty, \infty)$ correspond to the ends. When $s = 1$, these ends are embedded. Figure 4.1 shows the genus 1 case. In [FRUYY], a period problem that is not non-degenerate is considered for these surfaces. However, if we introduce parameters (λ_1, λ_2) as above, the period problem becomes non-degenerate and we can apply Theorem 3.2.

We also give two new examples of genus s maxfaces with two complete ends in $\mathbb{R}^{2,1}$ which when parameters are introduced as above have non-degenerate period problems, see Figures 4.2 and 4.3.

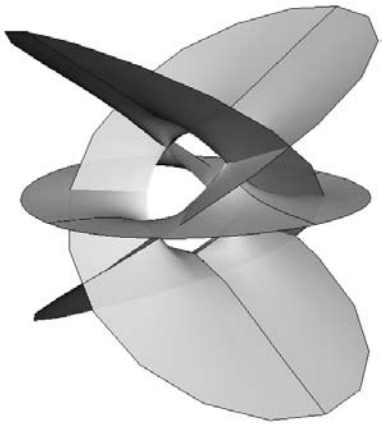

FIGURE 4.3. Genus 1 example with two complete ends with $j = 0$, $k = 1$, $l = -2$, $m = -1$. $(z, w) = (0, 0)$ and $(z, w) = (\infty, \infty)$ correspond to the ends. When $s = 1$, the end corresponding to $(z, w) = (\infty, \infty)$ is embedded.

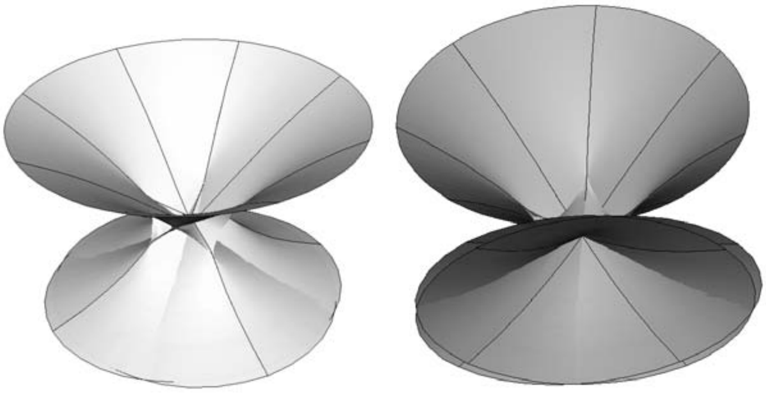

FIGURE 4.4. Two different views of the genus 1 example with three complete embedded ends with $j = 2$, $k = -1$, $l = 0$, $m = -1$. $(z, w) = (\infty, \infty)$ corresponds to the end on the top of the figure, and $(z, w) = (\pm 1, 0)$ correspond to the ends on the bottom of the figure.

4.2. **Three-ended example.** Problem 2 in [FRUYY] asked whether there are maxfaces of positive genus in $\mathbb{R}^{2,1}$ and $\mathbb{S}^{2,1}$ with more than two complete ends. We give an affirmative answer to this question by taking $s = 1$ and

$$j = 2,\ k = -1,\ l = 0,\ m = -1.$$

This determines a genus 1 maxface with three complete embedded ends in $\mathbb{R}^{2,1}$, see Figure 4.4, and by introducing parameters as above we have a non-degenerate period problem and can thus obtain a corresponding one-parameter family of surfaces in $\mathbb{S}^{2,1}$.

We also give another example of genus $s = 1$ maxface with three complete ends in $\mathbb{R}^{2,1}$ which when parameters are introduced as above has a non-degenerate period problems, see Figure 4.5.

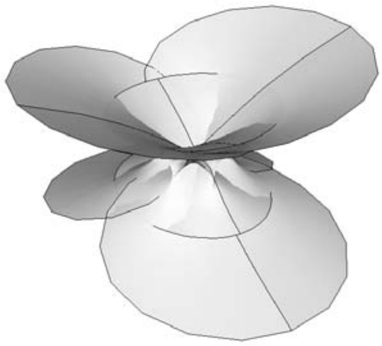

FIGURE 4.5. Genus 1 example with three complete ends with $j = 3$, $k = -1$, $l = 0$, $m = -1$. $(z, w) = (\pm 1, 0)$ correspond to embedded ends and $(z, w) = (\infty, \infty)$ corresponds to non-embedded ends.

Remark 4.5. In this paper we have only given examples of maxfaces of odd genus, but using the framework of Section 4, we believe it should be possible to construct examples of maxfaces in $\mathbb{R}^{2,1}$ (and CMC-1 faces in $\mathbb{S}^{2,1}$) with arbitrary genus and two or three ends.

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