SCHWARZ MAPS FOR THE HYPERGEOMETRIC DIFFERENTIAL EQUATION

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ABSTRACT. We introduce the de Sitter Schwarz map for the hypergeometric differential equation as a variant of the classical Schwarz map. This map turns out to be the dual of the hyperbolic Schwarz map, and it unifies the various Schwarz maps studied before. an example is also studied.

Introduction

The (*original*) *Schwarz map* for a differential equation of the form

$$u'' + q_1(x)u' + q_2(x)u = 0$$

with independent complex variable x is defined as

$$\mathcal{S}^{\mathrm{ori}}: \ X \ni x \longmapsto u_1(x) : u_2(x) \in \mathbb{P}^1,$$

where X is a domain that the equation is defined, u_1 and u_2 are linearly independent solutions of the equation, and $\mathbb{P}^1$ is the complex projective line. The map is multi-valued, and the map coincides with the Schwarz map of its SL normal form (obtained by multiplying a function to the unknown u so that the coefficient of the first derivative may vanish).

The monodromy group of the equation is a subgroup of $\mathrm{PGL}(2,\mathbb{C})$, which acts on the target $\mathbb{P}^1$. In this paper, we modify the Schwarz map of an equation in SL-form

$$u'' - q(x)u = 0 \tag{0.1}$$

to those having the hyperbolic 3-space and the de Sitter 3-space as targets. These spaces have $\mathbb{P}^1$ in their boundaries and enjoy the natural $\mathrm{PGL}(2,\mathbb{C})$-action. We study relations among those Schwarz maps, and singularities of those maps.

As a typical example, we treat the differential equation $E(\mu_0, \mu_1, \mu_\infty)$:

$$u'' - q(x)u = 0, \quad q = -\frac{1}{4}\left(\frac{1-\mu_0^2}{x^2} + \frac{1-\mu_1^2}{(1-x)^2} + \frac{1+\mu_\infty^2-\mu_0^2-\mu_1^2}{x(1-x)}\right), \tag{0.2}$$

which is the SL-form of the hypergeometric equation

$$x(1-x)u'' + \{c-(a+b+1)x\}u' - abu = 0,$$

where

$$\mu_0 = 1-c, \quad \mu_1 = c-a-b, \quad \mu_\infty = b-a;$$

refer to [14]. We make a detailed global study about the various Schwarz maps for $E(1/2, 1/2, 1/3)$, which has a dihedral group as its monodromy group.

The *Lorentz* space $\mathbb{L}^4$ is a 4-dimensional space with norm

$$\langle t, t\rangle = -t_0^2 + t_1^2 + t_2^2 + t_3^2, \quad t = (t_0, t_1, t_2, t_3) \in \mathbb{L}^4.$$

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To the memory of Professor Shoshichi Kobayashi.

The quotient space $(\mathbb{L}^4-\{0\})/\mathbb{R}_{>0}$ $(\cong S^3)$ is divided into five parts:

$$\begin{aligned} H^3_\pm &:= \{t\in\mathbb{L}^4 \mid \langle t,t\rangle<0,\ \pm t_0>0\}/\mathbb{R}_{>0},\\ S^2_\pm &:= \{t\in\mathbb{L}^4 \mid \langle t,t\rangle=0,\ \pm t_0>0\}/\mathbb{R}_{>0},\\ S^3_1 &:= \{t\in\mathbb{L}^4 \mid \langle t,t\rangle>0\}/\mathbb{R}_{>0}, \end{aligned}$$

hyperbolic 3-spaces, 2-spheres, and *de Sitter 3-space*, respectively. We can consider $\mathbb{L}^4$ to be the space Herm of 2×2 self-adjoint matrices $(h^*=h)$ by the identification

$$\mathbb{L}^4 \ni t=(t_0,t_1,t_2,t_3)\leftrightarrow h=\begin{pmatrix} t_0+t_3 & t_1+it_2\\ t_1-it_2 & t_0-t_3\end{pmatrix}\in \mathrm{Herm}.$$

Since $\langle t,t\rangle=-\det h$ holds, we have

$$\begin{aligned} H^3_\pm &= \{h\in\mathrm{Herm} \mid \det h>0,\ \pm\mathrm{trace}\ h>0\}/\mathbb{R}_{>0},\\ S^2_\pm &= \{h\in\mathrm{Herm} \mid \det h=0,\ \pm\mathrm{trace}\ h>0\}/\mathbb{R}_{>0},\\ S^3_1 &= \{h\in\mathrm{Herm} \mid \det h<0\}/\mathbb{R}_{>0}. \end{aligned}$$

These hyperbolic spaces and the de Sitter 3-space can be identified with

$$\begin{aligned} \mathbb{H}^3_\pm &:= \{\pm UU^* \mid U\in\mathrm{SL}(2,\mathbb{C})\}\cong \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2),\\ \mathbb{S}^3_1 &:= \{UKU^* \mid U\in\mathrm{SL}(2,\mathbb{C})\}\cong \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(1,1), \end{aligned}$$

respectively, where

$$K=\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$

and the 2-spheres can be identified with $\mathbb{P}^1$ by

$$S^2_\pm \ni \begin{pmatrix}\sigma&\zeta\\ \bar\zeta&\tau\end{pmatrix}\longleftrightarrow \frac{\zeta}{\tau}\in\mathbb{P}^1=\mathbb{C}\cup\{\infty\};$$

refer to Section 1. For linearly independent solutions $u_1(x)$ and $u_2(x)$ of (0.1), we set

$$U(x)=\begin{pmatrix}u_1&u_1'\\u_2&u_2'\end{pmatrix}(x)\in\mathrm{SL}(2,\mathbb{C}),\tag{0.3}$$

and define two *hyperbolic* Schwarz maps and two *de Sitter* Schwarz maps as

$$\begin{aligned} \mathcal{S}^{\mathrm{hyp}}_\pm &: X\ni x\longmapsto \pm U(x)U(x)^*\in\mathbb{H}^3_\pm,\\ \mathcal{S}^{\mathrm{deS}}_\pm &: X\ni x\longmapsto \pm U(x)KU(x)^*\in\mathbb{S}^3_1. \end{aligned}$$

We see that the four maps are (locally) flat fronts in the sense of [7] (see §1.3). So we can discuss normals of the image surfaces even at a singular point of these maps. For a point of the hyperbolic Schwarz image surface $\mathcal{S}^{\mathrm{hyp}}_+(X)$, we correspond unit normals to this surface. The unit normal vectors are space-like, and they turn out to be the de Sitter Schwarz maps. The unit normals of the other hyperbolic Schwarz image surface $\mathcal{S}^{\mathrm{hyp}}_-(X)$ defines also the de Sitter Schwarz maps. Conversely, the unit normals of de Sitter Schwarz maps define the hyperbolic Schwarz maps. In this sense, the hyperbolic ones and de Sitter ones are dual to each other (see §2.2).

Recall that the *derived* Schwarz map (c.f. [15]) is defined as

$$\mathcal{S}^{\mathrm{der}}:\ X:=\mathbb{C}-\{0,1\}\ni x\longmapsto u_1'(x):u_2'(x)\in\mathbb{P}^1.$$

If we continue geodesically the said normals, they hit the ideal boundaries $S^2_\pm(\cong \mathbb{P}^1)$, and recover the original and the derived Schwarz maps (see §2.2).

The hyperbolic and the de Sitter Schwarz maps are singular along the curve

$$C : |q(x)| = 1;$$

refer to [4]. Along the non-singular parts of the image of C, the image surfaces have cuspidal edge singularities; so we call this curve C the *cusp-line*. At a singular point of the image of C, the surfaces have swallowtail singularities or worse. Criteria of the singularities in terms of $q(x)$ are given in §3.2.

For a hyperbolic (as well as de Sitter) Schwarz image surface, we consider the family of *parallel surfaces*. One extreme of the family is the original Schwarz image and the other is the derived Schwarz image. The union of the singular points of members of this family form the *caustic*, which is also a flat front. Regular points of the caustic correspond to cuspidal edges of the parallel surface. Cuspidal edges of the caustic correspond to swallowtails of the parallel surfaces; refer to [5].

As a typical example, we study the hypergeometric equation whose monodromy group is isomorphic to the dihedral group of order 6; this is given by

$$\mu_0 = \mu_1 = \frac{1}{2}, \quad \mu_\infty = \frac{1}{3}.$$

We study the singularities of the de Sitter Schwarz image and its parallel surfaces, and the caustic of the parallel family. Those of the hyperbolic one were studied in [14, 15, 13].

As a dessert, some pictures of the de Sitter Schwarz map of the Airy equation are shown. The hyperbolic counterpart is shown in [12]; please compare and enjoy the difference/similarity.

Part 1. Various Schwarz maps

1. Lorentz, hyperbolic and de Sitter spaces

The *Lorentz* space $\mathbb{L}^4$ is a 4-dimensional space with norm

$$\langle t, t\rangle = -t_0^2 + t_1^2 + t_2^2 + t_3^2, \quad t = (t_0, t_1, t_2, t_3) \in \mathbb{L}^4.$$

Though in the introduction we defined the hyperbolic 3-space as well as the de Sitter space as quotients of subdomains by $\mathbb{R}_{>0}$, living in the quotient space $(\mathbb{L}^4 - \{0\})/\mathbb{R}_{>0}(\cong S^3)$, in this section we treat these spaces as 3-dimensional subsets lying in the space $\mathbb{L}^4$: In the space $\mathbb{L}^4$ live the hyperbolic 3-space

$$\mathbb{H}^3_+ = \{t \in \mathbb{L}^4 \mid \langle t, t\rangle = -1,\ t_0 > 0\} \cong H^3_+$$

and the de Sitter 3-space

$$\mathbb{S}^3_1 = \{t \in \mathbb{L}^4 \mid \langle t, t\rangle = 1\} \cong S^3_1,$$

with the metric induced from $\mathbb{L}^4$. The former is a simply connected Riemannian manifold of sectional curvature -1, and the latter is a simply connected Lorentzian manifold of sectional curvature 1. We understand that the spaces $H^3_\pm$ and S^3_1 bear the metric obtained through the above isomorphisms.

1.1. Geodesics in $\mathbb{S}^3_1$ and $\mathbb{H}^3_+$. Let $I \subset \mathbb{R}$ and $\gamma : I \ni s \mapsto \gamma(s) \in \mathbb{S}^3_1$ be a curve in $\mathbb{S}^3_1$. We have the following:

Proposition 1. *Let γ be a geodesic in $\mathbb{S}^3_1$. Set $\gamma(0) = p$ and $\dot\gamma(0) = v \in T_p\mathbb{S}^3_1$.*

(1) *If γ is a space-like geodesic with $\langle v, v\rangle = 1$, then γ is given by*

$$\gamma(s) = (\cos s)p + (\sin s)v.$$

(2) *If γ is a time-like geodesic with $\langle v, v\rangle = -1$, then γ is given by*

$$\gamma(s) = (\cosh s)p + (\sinh s)v.$$

(3) *If γ is a light-like geodesic with $\langle v, v\rangle = 0$, then γ is given by*

$$\gamma(s) = p + sv.$$

Proposition 2. *Let $I \subset \mathbb{R}$ and $\gamma : I \ni s \mapsto \gamma(s) \in \mathbb{H}^3_+$ be a geodesic in $\mathbb{H}^3_+$ such that $\gamma(0) = p$ and $\dot\gamma(0) = v \in T_p\mathbb{H}^3_+$ with $\langle v, v\rangle = 1$. Then γ is given by*

$$\gamma(s) = (\cosh s)p + (\sinh s)v.$$

Note that each geodesic is complete.

1.2. **Hermitian matrix models.** We can consider $\mathbb{L}^4$ to be the space Herm of 2×2 self-adjoint matrices by the identification

$$\mathbb{L}^4 \ni t \leftrightarrow h = \begin{pmatrix} t_0 + t_3 & t_1 + it_2 \\ t_1 - it_2 & t_0 - t_3 \end{pmatrix} \in \mathrm{Herm}.$$

Since

$$\langle s, t\rangle = -\frac{1}{2}\mathrm{trace}(k\tilde{h}), \quad \mathbb{L}^4 \ni s \leftrightarrow k \in \mathrm{Herm},$$

where $\tilde{h}$ is the cofactor matrix of h, and in particular, $\langle t, t\rangle = -\det h$, we have

$$\begin{aligned} \mathbb{H}^3_+ \quad &= \{h \in \mathrm{Herm} \mid \det h = 1, \text{positive definite}\}, \\ \mathbb{S}^3_1 \quad &= \{h \in \mathrm{Herm} \mid \det h = -1\}. \end{aligned}$$

1.3. **Flat fronts.** Let $D \subset \mathbb{C}$ be a simply connected domain (since we only study a local theory in this section, we always assume that D is simply-connected). Let $f : D \to \mathbb{S}^3_1$ be a *space-like* immersion, that is, let f be an immersion with the induced metric $\mathrm{I}_f = \langle df, df\rangle$ which is positive definite, and let $g : D \to \mathbb{H}^3_+$ be also an immersion. Assume for each $p \in D$, a unit normal vector of f at p is given by $g(p)$. Then the first, the second and the third fundamental forms are

$$\mathrm{I}_f = \langle df, df\rangle, \qquad \mathrm{I\!I} = -\langle df, dg\rangle \ (= \mathrm{I\!I}_f = \mathrm{I\!I}_g), \qquad \mathrm{I\!I\!I}_f = \langle dg, dg\rangle \ (= \mathrm{I}_g).$$

The Gauss curvatures K_f and K_g of f and g are given respectively as

$$K_f = -\frac{\det \mathrm{I\!I}}{\det \mathrm{I}_f} + 1, \qquad K_g = \frac{\det \mathrm{I\!I}}{\det \mathrm{I}_g} - 1. \tag{1.1}$$

A space-like immersion f is said to be *flat* if K_f vanishes identically, and an immersion g is said to be *flat* if K_g vanishes identically. For not necessarily regular maps, we make the following definition.

Definition 1. *(flat front)*

(1) *$f : D \to \mathbb{S}^3_1$ is called a* flat front *if f is a space-like flat immersion on the set of regular points, there exists a map $g : D \to \mathbb{H}^3_+$ such that the pair $(f, g) : D \to \mathbb{S}^3_1 \times \mathbb{H}^3_+$ is an immersion, and for any $X \in T_pD$,*

$$\langle df(X), g(p)\rangle = 0$$

holds for any $p \in D$. We call g a unit normal *of f.*

(2) *$g : D \to \mathbb{H}^3_+$ is called a* flat front *if g is a flat immersion on the set of regular points, there exists a map $f : D \to \mathbb{S}^3_1$ such that the pair $(g, f) : D \to \mathbb{H}^3_+ \times \mathbb{S}^3_1$ is an immersion, and for any $X \in T_pD$,*

$$\langle dg(X), f(p)\rangle = 0$$

holds for any $p \in D$ We call f a unit normal *of g.*

Remark 1. In general, a map f from D to a 3-manifold M is called a *front* if there exists an immersion $L_f : D \to T_1M$ such that $\pi \circ L_f = f$ holds and L_f is integrable with respect to the canonical contact structure on the unit tangent bundle $\pi : T_1M \to M$. Since $T_1\mathbb{S}^3_1$ is canonically identified with $\mathbb{S}^3_1 \times \mathbb{H}^3_+$, we defined flat fronts as above. See [4, 7] for details.

Now we assume that f is a flat front as in the above definition; I_f and II respectively defines a positive semi-definite metric on D, and $\mathrm{I}_f + \mathrm{III}_f$ is positive definite. There exists a complex coordinate x such that

$$\mathrm{II} = \lambda dx d\bar{x},$$

where $\lambda : D \to \mathbb{R}_{\geq 0}$ is a non-negative function. Set

$$\alpha := \langle f_x, f_x \rangle, \qquad \beta := \langle f_x, f_{\bar{x}} \rangle.$$

Then the first fundamental form can be written as

$$\mathrm{I}_f = \alpha dx^2 + 2\beta dx d\bar{x} + \bar{\alpha} d\bar{x}^2.$$

Since $\det \mathrm{I}_f = \det \mathrm{II}$ by (1.1), we have

$$\lambda = 2\sqrt{\beta^2 - |\alpha|^2}.$$

By direct computations, we have the following Weingarten formula:

$$g_x = -\frac{\beta}{\lambda} f_x + \frac{\alpha}{\lambda} f_{\bar{x}}, \qquad g_{\bar{x}} = \frac{\bar{\alpha}}{\lambda} f_x - \frac{\beta}{\lambda} f_{\bar{x}}.$$

Thus the third fundamental form of f (the first fundamental form of g) is written as

$$\mathrm{III}_f = \mathrm{I}_g = -\alpha dx^2 + 2\beta dx d\bar{x} - \bar{\alpha} d\bar{x}^2.$$

This yields that $f : D \to \mathbb{S}^3_1$ is a (space-like) flat front if and only if a unit normal vector field $g : D \to \mathbb{H}^3_+$ is a flat front, and likewise $g : D \to \mathbb{H}^3_+$ is a flat front if and only if a unit normal vector field $f : D \to \mathbb{S}^3_1$ is a flat front.

We note that for a flat front f, there are two normals: $g_+ : D \to \mathbb{H}^3_+$, which is said to be *future pointing*, and $g_- = -g_+ : D \to \mathbb{H}^3_-$, which is said to be *past pointing*.

More details can be found in [1, 7].

2. Schwarz maps

2.1. **Definition of Schwarz maps.** Consider a differential equation (0.1)

$$u'' - q(x)u = 0,$$

where q is holomorphic in a simply connected domain D with variable x; the notation $'$ means the derivation relative to x. We define various Schwarz maps: the original one

$$\mathcal{S}^{\mathrm{ori}} : D \ni x \longmapsto u_1(x) : u_2(x) \in \mathbb{P}^1,$$

the derived one

$$\mathcal{S}^{\mathrm{der}} : D \ni x \longmapsto u_1'(x) : u_2'(x) \in \mathbb{P}^1,$$

the hyperbolic ones

$$\mathcal{S}^{\mathrm{hyp}}_{\pm} : D \ni x \longmapsto \pm U(x)U(x)^* \in \mathbb{H}^3_{\pm},$$

and the de Sitter ones

$$\mathcal{S}^{\mathrm{deS}}_{\pm} : D \ni x \longmapsto \pm U(x)KU(x)^* \in \mathbb{S}^3_1,$$

where

$$U(x) = \begin{pmatrix} u_1 & u_1' \\ u_2 & u_2' \end{pmatrix}(x) \in \mathrm{SL}(2, \mathbb{C}), \quad K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

as in Introduction. Note that U solves the matrix-equation

$$\frac{dU}{dx} = UA, \quad \text{where} \quad A = \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix},$$

and that other solutions are given by MU for certain $M \in \mathrm{SL}(2, \mathbb{C})$. The correspondence $U \to MU$ induces via $\mathcal{S}^{\mathrm{ori}}/\mathcal{S}^{\mathrm{der}}$ a conformal automorphism of $\mathbb{P}^1$, and via $\mathcal{S}^{\mathrm{hyp}}_{\pm}$ and $\mathcal{S}^{\mathrm{deS}}_{\pm}$ an orientation preserving isometry of $\mathbb{H}^3_{\pm}$ and $\mathbb{S}^3_1$, respectively.

Remark 2. Set $v = u'$. Since $v' = u'' = qu$ and $v'' = q'u + qu'$, v satisfies the differential equation

$$v'' - \frac{q'}{q}v' - qv = 0.$$

Thus the derived Schwarz map is the original Schwarz map of the equation above by definition. Note that this equation has a singular point at a zero point of q. Around this point, the equation has single valued independent solutions; such a singular point is often said to be *apparent*. In particular, if it is a simple zero of q, then the characteristic exponents are $\{0, 2\}$.

2.2. **Relation among the Schwarz maps.** For notational simplicity, we put $f = UKU^*$. We find the future pointing unit normal vector field of f. First, we have

$$\begin{aligned} f_x &= U'KU^* = UAKU^* = U\begin{pmatrix} 0 & -q \\ 1 & 0 \end{pmatrix}U^* \\ f_{\bar{x}} &= UK(U')^* = UKA^*U^* = U\begin{pmatrix} 0 & 1 \\ -\bar{q} & 0 \end{pmatrix}U^*. \end{aligned} \tag{2.1}$$

Next, we set $g := UU^*$. Then we see that g satisfies

$$\langle g, f \rangle = \langle g, f_x \rangle = \langle g, f_{\bar{x}} \rangle = 0, \quad \langle g, g \rangle = -1, \quad \mathrm{trace}(g) > 0.$$

In fact, since $\det U = 1$, the cofactor matrix of U is just U^{-1} and then $\tilde{f} = -(U^*)^{-1}KU^{-1}$; so for example,

$$\langle g, f \rangle = -\frac{1}{2}\mathrm{trace}(g\tilde{f}) = \frac{1}{2}\mathrm{trace}(UKU^{-1}) = 0,$$

and so forth. Thus g gives the future pointing unit normal vector field of f.

Furthermore, since

$$g_x = U'U^* = UAU^*, \qquad g_{\bar{x}} = U(U')^* = UA^*U^*,$$

we can compute the fundamental forms of g as follows.

$$\begin{aligned} \mathrm{I}_g &= q dx^2 + (1 + |q|^2) dx d\bar{x} + \bar{q} d\bar{x}^2, \\ \mathrm{II}_g &= (1 - |q|^2) dx d\bar{x}, \\ \mathrm{III}_g &= -q dx^2 + (1 + |q|^2) dx d\bar{x} - \bar{q} d\bar{x}^2. \end{aligned} \tag{2.2}$$

The singular set of f (the set of points in D where $\mathrm{I}_f = \mathrm{III}_g$ degenerates) is given by

$$\{p \in D \; ; \; |q(p)| = 1\}.$$

Similarly, the singular set of g (the set of points in D where $\mathrm{I}_g = \mathrm{III}_f$ degenerates) is also given by $\{p \in D \; ; \; |q(p)| = 1\}$.

The future/past pointing unit normal vector $\pm g(x)$ is perpendicular to the surface f in $\mathbb{S}^3_1$ and is tangent to the space $\mathbb{S}^3_1$ in $\mathbb{L}^4$ at each point $f(x) \in \mathbb{S}^3_1$ for $x \in D$. Let $P(x)$ be the unique 2-dimensional plane in $\mathbb{L}^4$ containing the three points $(0, 0, 0, 0)$ and $f(x)$ and $g(x)$, that is,

$$P(x) := \{af(x) + bg(x) \; ; \; a, b \in \mathbb{R}\}.$$

Then

$$P(x)\cap \mathbb{S}^3_1=\{af(x)+bg(x)\ ;\ a,b\in\mathbb{R},\ \langle af(x)+bg(x),af(x)+bg(x)\rangle=1\}.$$

Since

$$\begin{aligned}\langle af(x)+bg(x),af(x)+bg(x)\rangle &= a^2\langle f(x),f(x)\rangle+2ab\langle f(x),g(x)\rangle+b^2\langle g(x),g(x)\rangle\\ &= a^2-b^2,\end{aligned}$$

we have $a^2-b^2=1$ and hence

$$P(x)\cap\mathbb{S}^3_1=\{(\cosh t)f(x)+(\sinh t)g(x)\ ;\ t\in\mathbb{R}\}.$$

Thus $P(x)\cap\mathbb{S}^3_1$ consists of the geodesic γ in $\mathbb{S}^3_1$ that starts at $f(x)$ and extends in the direction of $\pm g(x)$ (cf. Proposition 1 in Section 1.1). The plane $P(x)$ also intersects the upper/lower half cone

$$N_\pm=\{(t_0,t_1,t_2,t_3)\in\mathbb{L}^4\ ;\ -t_0^2+t_1^2+t_2^2+t_3^2=0,\ \pm t_0\ge 0\}$$

of the light cone of $\mathbb{L}^4$ along two lines

$$\{\alpha(f(x)\pm g(x))\ ;\ \alpha\in\mathbb{R}_{>0}\},$$

which are asymptotic lines of the geodesic γ. Therefore the limiting direction of γ is $f(x)\pm g(x)$.

We call $N_\pm/\mathbb{R}_{>0}$ the *ideal boundary* of $\mathbb{S}^3_1$, which is identified with $S^2_+\cup S^2_-$. Then, the limiting direction $f(x)\pm g(x)$ can be regarded as a point in the boundary $N_\pm/\mathbb{R}_{>0}$. Thus, we have the following.

Proposition 3. *The geodesic γ in $\mathbb{S}^3_1$ starting from the point $f(x)$ in the direction $\pm g(x)$ hits the ideal boundary $N_\pm/\mathbb{R}_{>0}$ of $\mathbb{S}^3_1$ at $f(x)\pm g(x)$.*

On the other hand, let $g=UU^*:D\to\mathbb{H}^3_+$ be an immersion and $f=UKU^*:D\to\mathbb{S}^3_1$ a unit normal vector field. Then by the similar arguments, we can see the following, where the space $N_+/\mathbb{R}_{>0}$ (resp. $N_-/\mathbb{R}_{>0}$) is now the ideal boundary of $\mathbb{H}^3_+$ (resp. $\mathbb{H}^3_-$).

Proposition 4. *The geodesic in $\mathbb{H}^3_+$ starting from the point $g(x)$ in the direction $\pm f(x)$ hits the ideal boundary $N_+/\mathbb{R}_{>0}$ of $\mathbb{H}^3_+$ at the point $g(x)\pm f(x)$.*

We next state the mutual relations among several Schwarz maps as follows. Since $f=UKU^*$ and $g=UU^*$, we see

$$f+g=2U\begin{pmatrix}1&0\\0&0\end{pmatrix}U^*=2\begin{pmatrix}u_1\bar u_1&u_1\bar u_2\\ \bar u_1u_2&u_2\bar u_2\end{pmatrix}\in S^2_+,$$

$$f-g=2U\begin{pmatrix}0&0\\0&-1\end{pmatrix}U^*=-2\begin{pmatrix}u_1'\bar u_1'&u_1'\bar u_2'\\ \bar u_1'u_2'&u_2'\bar u_2'\end{pmatrix}\in S^2_-;$$

recall that the 2-spheres are identified with $\mathbb{P}^1$ as

$$S^2_\pm\ni\begin{pmatrix}\sigma&\zeta\\ \bar\zeta&\tau\end{pmatrix}\longleftrightarrow\frac{\zeta}{\tau}\in\mathbb{P}^1=\mathbb{C}\cup\{\infty\}.$$

The argument above and that in the previous subsection lead to the following proposition.

Proposition 5. *For the de Sitter Schwarz map $\mathcal{S}^{\rm deS}_+(=f)$, one of the two unit normals of the image surface at $\mathcal{S}^{\rm deS}_+(x)$ is future pointing and is given by $\mathcal{S}^{\rm hyp}_+(x)(=g(x))$, the other is past pointing and is given by $\mathcal{S}^{\rm hyp}_-(x)$.*

The geodesic curve in $\mathbb{S}^3_1$ passing $\mathcal{S}^{\rm deS}_+(x)$ with the direction $\mathcal{S}^{\rm hyp}_+(x)$ hits the ideal boundary at $\mathcal{S}^{\rm deS}_+(x)+\mathcal{S}^{\rm hyp}_+(x)\in S^2_+$, which is the image $\mathcal{S}^{\rm ori}(x)$ of the original Schwarz map.

The geodesic curve in $\mathbb{S}^3_1$ passing $\mathcal{S}^{\mathrm{deS}}_+(x)$ with the direction $\mathcal{S}^{\mathrm{hyp}}_-(x)$ hits the ideal boundary at $\mathcal{S}^{\mathrm{deS}}_+(x)+\mathcal{S}^{\mathrm{hyp}}_-(x)\in S^2_-$, which is the image $\mathcal{S}^{\mathrm{der}}(x)$ of the derived Schwarz map.

Conversely, for the hyperbolic Schwarz map $\mathcal{S}^{\mathrm{hyp}}_+$, the unit normals of the image surface at $\mathcal{S}^{\mathrm{hyp}}_+(x)$ are given by $\mathcal{S}^{\mathrm{deS}}_\pm(x)$.

The geodesic curves in $\mathbb{H}^3_+$ passing $\mathcal{S}^{\mathrm{hyp}}_+(x)$ with the directions $\mathcal{S}^{\mathrm{deS}}_\pm(x)$ hit the ideal boundaries at $\mathcal{S}^{\mathrm{hyp}}_+(x)+\mathcal{S}^{\mathrm{deS}}_\pm(x)\in S^2_+$, which are the images $\mathcal{S}^{\mathrm{ori}}(x)$ and $\mathcal{S}^{\mathrm{der}}(x)$ of the original and the derived Schwarz maps.

The above statement with every subscript $+$ and $-$ switched is also true.

We remark that $\mathcal{S}^{\mathrm{ori}}$ and $\mathcal{S}^{\mathrm{der}}$ are the hyperbolic Gauss maps for both f and g (see [7]).

From the above proposition, we have the following picutre. For each x, the eight points $\pm f(x)$, $\pm g(x)$, $f(x)\pm g(x)$, and $-f(x)\pm g(x)$ are regarded as lying in the 3-sphere $(\mathbb{L}^4-\{0\})/\mathbb{R}_{>0}\cong S^3=H^3_\pm\cup S^2_\pm\cup S^3_1$ and they are joined by the image of the geodesics that are described in the proposition. The union of these curves form a circle.

3. Singularities of Schwarz maps

3.1. **Preliminaries for surface singularities.** Two map-germs χ and $\psi:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ are said to be *right-left equivalent* if there exist diffeomorphism-germs $\sigma:(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$ and $\tau:(\mathbb{R}^3,0)\to(\mathbb{R}^3,0)$ such that $\tau\circ\chi\circ\sigma=\psi$.

A map-germ $\chi:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ is called a

- *cuspidal edge* if χ is right-left equivalent to the map-germ
$$(x_1,x_2)\mapsto(x_1,x_2^2,x_2^3).$$
- *swallowtail* if χ is right-left equivalent to
$$(x_1,x_2)\mapsto(x_1,3x_2^4+x_1x_2^2,4x_2^3+2x_1x_2).$$
- *cuspidal butterfly* (A_4*-singularity*) if χ is right-left equivalent to
$$(x_1,x_2)\mapsto(x_1,5x_2^4+2x_1x_2,4x_2^5+x_1x_2^2-x_1^2).$$
- *cuspidal lips* (resp. *cuspidal beaks*) if χ is right-left equivalent to
$$(x_1,x_2)\mapsto(x_1,4x_2^3+4ex_1^2x_2,3x_2^4+2ex_1^2x_2^2),\quad e=+1\ (\text{resp. } e=-1).$$

Let 0 be a singular point of a map-germ $\chi:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$. Assume that rank $d\chi|_0=1$, then there exists a non-zero vector field η on $(\mathbb{R}^2,0)$ such that $\eta|_p$ spans the kernel of $d\chi|_p$ if p is a singular point of χ. We call η the *null vector field* of χ.

3.2. **Criteria of singularities of de Sitter and hyperbolic Schwarz maps.** Let $f:D\to\mathbb{S}^3_1$ (resp. $g:D\to\mathbb{H}^3_+$) be a front and let $g:D\to\mathbb{H}^3_+$ (resp. $f:D\to\mathbb{S}^3_1$) denote its unit normal. Let (x_1,x_2) be a coordinate system on D. We define a function
$$\delta=\det\left(\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},g,f\right)\quad\left(\text{resp. }\delta=\det\left(\frac{\partial g}{\partial x_1},\frac{\partial g}{\partial x_2},f,g\right)\right),$$
which we call the *signed area density function* of f. Note that the singular set of f coincides with the zero set of δ.

We compute the signed area density function for the de Sitter Schwarz map $f=\mathcal{S}^{\mathrm{deS}}_+=UKU^*$. Thanks to the expressions in (2.1), we have
$$f_x\times f_{\bar x}=\frac{i}{2}\left(f_x\,f^{-1}\,f_{\bar x}-f_{\bar x}\,f^{-1}\,f_x\right)=\frac{i}{2}\,U\begin{pmatrix}-q\bar q+1 & 0\\ 0 & 1-q\bar q\end{pmatrix}U^*.\tag{3.1}$$

Thus it holds that

$$\langle f_x \times f_{\bar{x}}, g\rangle = \frac{1}{2}\operatorname{trace}\left((f_x \times f_{\bar{x}})\,\bar{g}\right) = \frac{i}{4}(q\bar{q}-1)\operatorname{trace}\left(U^*\bar{U}\overline{U^*\bar{U}}\right),$$

where $g = \mathcal{S}_+^{\mathrm{hyp}} = UU^*$. Since $\operatorname{trace}\left(U^*\bar{U}\overline{U^*\bar{U}}\right) \neq 0$ for $U \neq 0$, the signed area density function δ of f is

$$\delta = q\bar{q} - 1.$$

We next define a vector field η as

$$\eta = \partial_x + q\partial_{\bar{x}}.$$

Then $\eta f = 0$ holds on the singular set of f by (2.1) and $q\bar{q} = 1$ on that singular set. Thus we can take η as a null vector field for f. We remark that the vector field $(1-\bar{q})\partial_x - (1+q)\partial_{\bar{x}}$ also can be taken as a null vector field as in [14, 10].

Remark 3. In the case of the hyperbolic Schwarz map $g = UU^*$, the signed area density function is the same δ as above, while the null direction is $\partial_x - q\partial_{\bar{x}}$, which is not the same as above. See [14, 10].

Now we compute the criteria of singularities of f. Since

$$\eta g = g_x + q g_{\bar{x}} = U\begin{pmatrix} 0 & 2q \\ 2 & 0 \end{pmatrix}U^*$$

holds, we have $\eta g \neq 0$ on singular points. Thus the map f is always a front.

By a direct calculations, we have

$$\begin{aligned}
\eta\delta &= q'\bar{q} + q^2\overline{q'} \\
\eta\eta\delta &= q''\bar{q} + 3qq'\overline{q'} + q^3\overline{q''} \\
\eta\eta\eta\delta &= \bar{q}q''' + 3(q')^2\overline{q'} + 4qq''\overline{q'} + 4q^2q'\overline{q''} + q^4\overline{q'''}.
\end{aligned}$$

Let p be a singular point of f, that is $q(p)\bar{q}(p) = 1$. Since $d\delta = (q'\bar{q}, q\overline{q'})$, we see that $d\delta(p) = 0$ is equivalent to $q'(p) = 0$. Assume that $q'(p) = 0$. Then we have

$$\operatorname{Hess}\delta = \begin{pmatrix} q''\bar{q} & 0 \\ 0 & q\overline{q''} \end{pmatrix} = q\bar{q}q''\overline{q''} = q''\overline{q''} = |q''|^2.$$

Thus we can paraphrase the criteria of singularities stated in [4, Proposition 1.3] (see also [11, Corollary 2.5]), [3, Theorem A.1] and [2, Theorem 8.2] in terms of q as follows:

Proposition 6. *Assume $q(p)\bar{q}(p) = 1$. Then the de Sitter Schwarz map $\mathcal{S}_+^{\mathrm{deS}}(= f)$ at p is*

- *cuspidal edge if*
$$q'\bar{q} + q^2\overline{q'} \neq 0.$$
- *swallowtail if $q'\bar{q} + q^2\overline{q'} = 0$, $q' \neq 0$ and*
$$q''\bar{q} + 3qq'\overline{q'} + q^3\overline{q''} \neq 0.$$
- *cuspidal butterfly if $q'\bar{q} + q^2\overline{q'} = 0$, $q' \neq 0$, $q''\bar{q} + 3qq'\overline{q'} + q^3\overline{q''} = 0$ and*
$$\bar{q}q''' + 3(q')^2\overline{q'} + 4qq''\overline{q'} + 6q^2q'\overline{q''} + q^4\overline{q'''} \neq 0.$$
- *cuspidal beaks if $q' = 0$, $|q''|^2 > 0$ and $q''\bar{q} + q^3\overline{q''} \neq 0$.*

Furthermore, no cuspidal lips does appear.

Remark 4. The hyperbolic counterpart of this proposition is known in [14, 10]. Assume $q(p)\bar{q}(p) = 1$. Then the hyperbolic Schwarz map $\mathcal{S}_+^{\mathrm{hyp}}(= g)$ at p is

- cuspidal edge if
$$q'\bar{q} - q^2\overline{q'} \neq 0.$$
- swallowtail if $q'\bar{q} - q^2\overline{q'} = 0$, $q' \neq 0$ and
$$q''\bar{q} - 3qq'\overline{q'} + q^3\overline{q''} \neq 0.$$
- cuspidal butterfly if $q'\bar{q} - q^2\overline{q'} = 0$, $q' \neq 0$, $q''\bar{q} - 3qq'\overline{q'} + q^3\overline{q''} = 0$ and
$$\bar{q}q''' - 3(q')^2\overline{q'} - 4qq''\overline{q'} + 6q^2q'\overline{q''} - q^4\overline{q'''} \neq 0.$$
- cuspidal beaks if $q' = 0$, $|q''|^2 > 0$ and $q''\bar{q} + q^3\overline{q''} \neq 0$.

Cuspidal lips does not appear. Note that the condition for cuspidal beaks is common for f and g. This means that the de Sitter Schwarz map has cuspidal beaks at a point if and only if the hyperbolic Schwarz map has cuspidal beaks at the point.

4. Parallel surfaces and caustic surfaces

The collection of points on oriented normal geodesics of a surface S with a constant distance from S is called a *parallel surface* of S. We study parallel surfaces of the image surface of the de Sitter Schwarz map. The hyperbolic counterpart is studied in [13]; to make the description complete, we repeat some of them.

4.1. **Parallel surfaces and their limits.** Parallel surfaces of the image surfaces under the de Sitter and hyperbolic Schwarz maps, $\mathcal{S}_+^{\mathrm{deS}}$ and $\mathcal{S}_+^{\mathrm{hyp}}$, for $u'' - q(x)u = 0$ are given by
$$\mathcal{S}_k^{\mathrm{deS}} : X \ni x \longmapsto U(x) \begin{pmatrix} k & 0 \\ 0 & -\dfrac{1}{k} \end{pmatrix} U^*(x) \in \mathbb{S}_1^3,$$
and
$$\mathcal{S}_k^{\mathrm{hyp}} : X \ni x \longmapsto U(x) \begin{pmatrix} k & 0 \\ 0 & +\dfrac{1}{k} \end{pmatrix} U^*(x) \in \mathbb{H}_+^3,$$
respectively, where $k \in (0, \infty)$ is a parameter. Inserting the expression of $U(x)$ in (0.3), we have
$$\mathcal{S}_k^{\mathrm{deS}}(x) = \begin{pmatrix} ku_1\overline{u_1} - \frac{1}{k}u_1'\overline{u_1'} & ku_1\overline{u_2} - \frac{1}{k}u_1'\overline{u_2'} \\ ku_2\overline{u_1} - \frac{1}{k}u_2'\overline{u_1'} & ku_2\overline{u_2} - \frac{1}{k}u_2'\overline{u_2'} \end{pmatrix},$$
and
$$\mathcal{S}_k^{\mathrm{hyp}}(x) = \begin{pmatrix} ku_1\overline{u_1} + \frac{1}{k}u_1'\overline{u_1'} & ku_1\overline{u_2} + \frac{1}{k}u_1'\overline{u_2'} \\ ku_2\overline{u_1} + \frac{1}{k}u_2'\overline{u_1'} & ku_2\overline{u_2} + \frac{1}{k}u_2'\overline{u_2'} \end{pmatrix}.$$
Then, since
$$S_\pm^2 \ni \begin{pmatrix} \sigma & \zeta \\ \bar{\zeta} & \tau \end{pmatrix} \longleftrightarrow \frac{\zeta}{\tau} \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\},$$
and
$$\lim_{k\to 0} \frac{ku_1\overline{u_2} \mp \frac{1}{k}u_1'\overline{u_2'}}{ku_2\overline{u_2} \mp \frac{1}{k}u_2'\overline{u_2'}} = \frac{u_1'}{u_2'}, \qquad \lim_{k\to\infty} \frac{ku_1\overline{u_2} \mp \frac{1}{k}u_1'\overline{u_2'}}{ku_2\overline{u_2} \mp \frac{1}{k}u_2'\overline{u_2'}} = \frac{u_1}{u_2},$$
we have the following proposition.

Proposition 7. *As $k \to 0$ both the de Sitter and hyperbolic Schwarz maps tend to the derived Schwarz map, and as $k \to \infty$, they tend to the original Schwarz map.*

4.2. **Singularities on parallel surfaces.** We characterize the singularities of the members of the parallel family of the de Sitter Schwarz map as follows (the hyperbolic counterpart is given in [13]). Since

$$dV_k = V_k \begin{pmatrix} 0 & q/k^2 \\ 1 & 0 \end{pmatrix} dy, \qquad V_k = U \begin{pmatrix} \sqrt{k} & 0 \\ 0 & \dfrac{1}{\sqrt{k}} \end{pmatrix},$$

relative to the new coordinate $y = kx$, the parallel surfaces can be seen as the de Sitter and hyperbolic Schwarz images, respectively, associated with the equation

$$\ddot{v} - Q(y)v = 0, \quad \text{where} \quad \cdot = d/dy \quad \text{and} \quad Q(y) = Q(k;y) = \frac{1}{k^2} q\left(\frac{y}{k}\right).$$

Then Proposition 6 gives the characterization of singularities of the map $\mathcal{S}_k^{\mathrm{deS}}$:

(1) The map $\mathcal{S}_k^{\mathrm{deS}}$ is singular along the curve $\{y \mid Q\overline{Q} = 1\}$,
(2) the map has cuspidal edge singularity at a point y_0 if $Q\overline{Q} = 1$ and

$$\dot{Q} \neq 0, \quad Q^3\overline{\dot{Q}} + \dot{Q} \neq 0,$$

(3) swallowtail singularity if $Q\overline{Q} = 1$ and

$$\dot{Q} \neq 0, \quad Q^3\overline{\dot{Q}} + \dot{Q} = 0, \quad \mathrm{Re}\left(\frac{\ddot{Q}}{Q^2} - \frac{3}{2}\frac{\dot{Q}^2}{Q^3}\right) \neq 0,$$

(4) cuspidal butterfly if $Q\overline{Q} = 1,\ \dot{Q} \neq 0,\ Q^3\overline{\dot{Q}} + \dot{Q} = 0$ and

$$\mathrm{Re}\left(\frac{\ddot{Q}}{Q^2} - \frac{3}{2}\frac{\dot{Q}^2}{Q^3}\right) = 0, \quad \dddot{Q}/Q + \overline{\dddot{Q}}Q^4 - 5\dot{Q}\ddot{Q}/Q^2 - 5\overline{\dot{Q}\ddot{Q}}Q^5 \neq 0,$$

(5) cuspidal beaks if $Q\overline{Q} = 1$ and

$$\dot{Q} = 0, \quad \ddot{Q} \neq 0, \quad \mathrm{Re}\left(\frac{\ddot{Q}}{Q^2} - \frac{3}{2}\frac{\dot{Q}^2}{Q^3}\right) \neq 0,$$

where $Q = Q(y_0), \dot{Q} = Q_y(y_0), \dots$

In fact, for example, if $Q\bar{Q} = 1$ and $\dot{Q}\bar{Q} + Q^2\bar{\dot{Q}} = 0$, then

$$\ddot{Q}\bar{Q} + 3Q\dot{Q}\bar{\dot{Q}} + Q^3\bar{\ddot{Q}} = Q\left(\frac{\ddot{Q}}{Q^2} - 3\frac{\dot{Q}^2}{Q^3} + Q^2\bar{\ddot{Q}}\right).$$

Remark 5. The cusp-line of the de Sitter and hyperbolic Schwarz maps for the equation $\ddot{v} - Q(y;k)v = 0$ is the curve $|Q(y;k)| = 1$ in y-plane, while this curve can be expressed by $|q(x)| = k^2$ in the x-plane. For the hypergeometric equation (0.2), the coefficient q is given as

$$-4q(x) = \frac{(1-\mu_\infty^2)x^2 + (\mu_0^2 + \mu_\infty^2 - \mu_1^2 - 1)x + 1 - \mu_0^2}{x^2(1-x)^2}.$$

We can read a rough picture of the cusp-lines when k tends to 0 and ∞ from that of the curve

$$C_k : |q(x)| = k^2$$

on the x-plane when k tends to 0 and ∞:

- When $k \to 0$, the curve C_k consists of a big circle and small circles around zero(s) of the numerator, if none of μ_0^2, μ_1^2 and μ_∞^2 is equal to 1.
- When $k \to \infty$, C_k tends to small circles around 0 and 1.

Here, a circle means a simple closed curve. Indeed, in the expression of $q(x)$ above, the denominator is of degree 4 in x, and the numerator is quadratic with roots different from 0 and 1. The cusp-lines in y-plane can be obtained from C_k by the relation $y = kx$.

4.3. **Caustic surfaces and their singularities.** The caustic $\mathcal{C}$ of a space-like surface $f : D \to S^3_1$ is the union of singular values of parallel surfaces of f. We assume that q has no zeros on D. Then by (2.2), f has no umbilic points on D, and $\mathcal{C}$ consists of two surfaces, $\mathcal{C}_1$ and $\mathcal{C}_2$, and they are parameterized as

$$\mathcal{C}_i(x) = \cosh r_i(x) f(x) + \sinh r_i(x) g(x) \quad (i = 1, 2),$$

where g is the Gauss map of f, and r_i is determined by the identity $\tanh r_i = -1/\kappa_i$ when $|\kappa_i| > 1$, where κ_i $(i = 1, 2)$ are the principal curvatures, namely, the eigenvalues of dg. Let V_i be the vector fields which give the principal directions with respect to κ_i $(i = 1, 2)$, respectively. Since $V_i g = \kappa_i V_i f$ holds, we have

$$\begin{aligned}
V_i(\mathcal{C}_i) &= V_i(r_i)\sinh r_i f + \cosh r_i V_i f + V_i(r_i)\cosh r_i g + \sinh r_i V_i g \\
&= V_i(r_i)\sinh r_i f + (\cosh r_i + \sinh r_i \kappa_i) V_i f + V_i(r_i)\cosh r_i g \\
&= V_i(r_i)\big(\sinh r_i f + \cosh r_i g\big), \\
V_{i+1}(\mathcal{C}_i) &= V_{i+1}(r_i)\sinh r_i f + \big(\cosh r_i + \sinh r_i \kappa_{i+1}\big) V_{i+1} f + V_{i+1}(r_i)\cosh r_i g,
\end{aligned}$$

for $i = 1, 2$, and $V_3 = V_1$. Since $\langle V_i f, V_{i+1} f\rangle = 0$ holds because V_1 and V_2 are the principal directions, we see that

$$\langle V_i f, V_i(\mathcal{C}_i)\rangle = \langle V_i f, V_{i+1}(\mathcal{C}_i)\rangle = \langle V_i f, \mathcal{C}_i\rangle = 0.$$

Thus the unit normal vector field of $\mathcal{C}_i$ can be taken as

$$\frac{V_i f}{|V_i f|}.$$

Then we see that the signed area density function for $\mathcal{C}_i$ can be calculated as

$$\det\left(V_i(\mathcal{C}_i),\ V_{i+1}(\mathcal{C}_i),\ \frac{V_i f}{|V_i f|},\ \mathcal{C}_i\right) = -\frac{V_i(r_i)(\cosh r_i + \kappa_{i+1}\sinh r_i)}{|V_i f|},$$

and this is proportional to $V_i(r_i)$ because $\kappa_i \neq \kappa_{i+1}$ on D. On the other hand, if $V_i(r_i) = 0$, then $V_i(\mathcal{C}_i) = 0$. This implies that the null vector field of $\mathcal{C}_i$ can be taken as V_i. Thus as we got Proposition 6 from [11, Corollary 2.5], we have:

Lemma 1. *Under the above setting, let p be a singular point of $\mathcal{C}_i$. Let us assume that $\mathcal{C}_i$ at p is a front. Then $\mathcal{C}_i$ at p is*

- *cuspidal edge if and only if $V_i V_i(r_i) \neq 0$,*
- *swallowtail if and only if $V_i V_i(r_i) = 0$, $V_{i+1} V_i(r_i) \neq 0$ and $V_i V_i V_i(r_i) = 0$.*

Now let us assume that f is a de Sitter Schwarz map: $f = UKU^*$. We rewrite the fundamental forms I_f and $\mathrm{I\!I}$ in the real variables $dx = du + idv$ and $q = \alpha + i\beta$. Then, wee see

$$\begin{aligned}
\mathrm{I}_f &= (du \ \ dv)\begin{pmatrix} -2\alpha + 1 + |q|^2 & 2\beta \\ 2\beta & 2\alpha + 1 + |q|^2 \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix}, \\
\mathrm{I\!I} &= (du \ \ dv)\begin{pmatrix} 1 - |q|^2 & 0 \\ 0 & 1 - |q|^2 \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix}.
\end{aligned}$$

Hence

$$\begin{aligned}\det(\mathrm{I\!I} - \kappa \mathrm{I}_f) &= \det \begin{pmatrix} 1-|q|^2-\kappa(-2\alpha+1+|q|^2) & -2\kappa\beta \\ -2\kappa\beta & 1-|q|^2-\kappa(2\alpha+1+|q|^2) \end{pmatrix} \\ &= (1-|q|^2-\kappa(1+|q|^2)-2\kappa|q|)(1-|q|^2-\kappa(1+|q|^2)+2\kappa|q|),\end{aligned}$$

which is zero precisely when $\kappa = \kappa_1$ or $\kappa = \kappa_2$, where

$$\kappa_1 = \frac{1+|q|}{1-|q|}, \quad \kappa_2 = \frac{1-|q|}{1+|q|}$$

are principal curvatures. Since $|\kappa_2| < 1$ holds, there is no solution for $\tanh r_2 = -1/\kappa_2$. Thus we consider only the case of $\kappa = \kappa_1$. By a computation, we see that the principal direction with respect to κ_1 is given as $V_1 = \beta\partial_u + i(-|q|^2+\alpha)\partial_v$, or equivalently, as

$$V_1 = (-\alpha+\beta+|q|^2)\partial_x + (\alpha+\beta-|q|^2)\partial_{\bar{x}}, \tag{4.1}$$

where $\partial_x = (\partial_u - i\partial_v)/2$. On the other hand, the caustic $\mathcal{C}_1$ is written as

$$\begin{aligned}\mathcal{C}_1 &= \cosh r_1 f + \sinh r_1 g \\ &= U\left(\cosh r_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \sinh r_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) U^* \\ &= U \begin{pmatrix} |q|^{1/2} & 0 \\ 0 & -|q|^{-1/2} \end{pmatrix} U^*.\end{aligned}$$

Set

$$U_c = UB, \quad B = \begin{pmatrix} q^{1/4} & 0 \\ 0 & q^{-1/4} \end{pmatrix}.$$

Then we see that $U_c \in SL(2,\mathbb{C})$ and $\mathcal{C}_1 = U_c J U_c^*$. This implies that the caustic $\mathcal{C}_1$ is flat. We set $g_c = U_c U_c^*$. Since q is non-zero on D, g_c is well-defined on D, and is a unit normal of $\mathcal{C}_1$. Moreover, we have

Lemma 2. *Under the above setting, $\mathcal{C}_1$ is a front.*

Proof. First we show that the rank of $d\mathcal{C}_1$ is at least 1, namely $\mathcal{C}_1' \neq O$. Since $\mathcal{C}_1' = (U_c)' K U_c^*$, it is enough to see $(U_c)' \neq O$. Furthermore, since $(U_c)' = U(AB+B')$ and the $(1,2)$-element of $AB+B'$ is $q^{3/4}$ as we will see, we see that $(U_c)' \neq O$ for $q \neq 0$. We next show that $dg_c(\xi) \neq 0$, for a vector $\xi = \xi_1\partial_x + \xi_2\partial_{\bar{x}}$ which satisfies $d\mathcal{C}_1(\xi) = 0$. Since

$$\begin{aligned} d\mathcal{C}_1(\xi) &= \xi_1(U_cKU_c^*)' + \xi_2(U_cKU_c^*)_{\bar{x}} = U\Big(\xi_1(AB+B')KB^* + \xi_2 BK(AB+B')^*\Big)U^*, \\ dg_c(\xi) &= \xi_1(U_cU_c^*)' + \xi_2(U_cU_c^*)_{\bar{x}} = U\Big(\xi_1(AB+B')B^* + \xi_2 B(AB+B')^*\Big)U^*,\end{aligned}$$

and $\det U \neq 0$, it is enough to show that

$$\xi_1(AB+B')B^* + \xi_2 B(AB+B')^* \neq O$$

under the assumption

$$\xi_1(AB+B')KB^* + \xi_2 BK(AB+B')^* = O.$$

Let us set

$$AB+B' = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Then we see that

$$\xi_1(AB+B')KB^* + \xi_2 BK(AB+B')^* = \begin{pmatrix} \xi_1\bar{q}^{1/4}a_{11} + \xi_2 q^{1/4}\overline{a_{11}} & -\xi_1\bar{q}^{-1/4}a_{12} + \xi_2 q^{1/4}\overline{a_{21}} \\ \xi_1\bar{q}^{1/4}a_{21} - \xi_2 q^{-1/4}\overline{a_{12}} & -\xi_1\bar{q}^{-1/4}a_{22} + \xi_2 q^{-1/4}\overline{a_{22}} \end{pmatrix}$$

and

$$\xi_1(AB+B')B^* + \xi_2 B(AB+B')^* = \begin{pmatrix} \xi_1\overline{q}^{1/4}a_{11}+\xi_2 q^{1/4}\overline{a_{11}}) & \xi_1\overline{q}^{-1/4}a_{12}+\xi_2 q^{1/4}\overline{a_{21}} \\ \xi_1\overline{q}^{1/4}a_{21}+\xi_2 q^{-1/4}\overline{a_{12}} & \xi_1\overline{q}^{-1/4}a_{22}+\xi_2 q^{-1/4}\overline{a_{22}}. \end{pmatrix}$$

Hence, under the condition $\xi_1(AB+B')KB^* + \xi_2 BK(AB+B')^* = O$, we have

$$\xi_1(AB+B')B^* + \xi_2 B(AB+B')^* \quad = \quad \begin{pmatrix} 0 & 2\xi_1\overline{q}^{-1/4}a_{12} \\ 2\xi_1\overline{q}^{1/4}a_{21} & 0 \end{pmatrix}$$

On the other hand, we have

$$AB+B' = \begin{pmatrix} \dfrac{1}{4}q^{-3/4}q' & q^{3/4} \\ q^{1/4} & -\dfrac{1}{4}q^{-5/4}q' \end{pmatrix};$$

hence, $(a_{12}, a_{21}) = (q^{3/4}, q^{1/4})$ and it cannot be $(0,0)$ because $q \neq 0$. Thus, $\xi_1(AB+B')B^* + \xi_2 B(AB+B')^*$ does not vanish, which completes the proof. □

The same assertion for flat fronts in the hyperbolic space is shown in [5, Proposition 6.1]. See also [7, Theorem 2.9]. By Lemmas 1 and 2, we have:

Proposition 8. *Under the same setting as in Lemma 2, let p be a singular point of $\mathcal{C}_1$, namely, $V_1(\kappa_1)(p) = V_1\left(\dfrac{1+|q|}{1-|q|}\right)(p) = 0$. Then $\mathcal{C}_1$ at p is*

- *cuspidal edge if and only if $V_1V_1\left(\dfrac{1+|q|}{1-|q|}\right) \neq 0$,*
- *swallowtail if and only if*

$$V_1V_1\left(\frac{1+|q|}{1-|q|}\right) = 0, \quad V_2V_1\left(\frac{1+|q|}{1-|q|}\right) \neq 0, \quad V_1V_1V_1\left(\frac{1+|q|}{1-|q|}\right) \neq 0,$$

where V_1 is the vector field as in (4.1).

See [13, §4] for the case of hyperbolic Schwarz map. For investigations about caustics of flat fronts in the hyperbolic space, see [5, 6, 8].

Part 2. Examples

5. A hypergeometric differential equation

We treat the hypergeometric differential equation (0.2) with special parameters $E(1/2, 1/2, 1/3)$, which has the dihedral group of order 6 as its monodromy group. A study of the hyperbolic Schwarz map of this equation was made in [13]. In this section we study the de Sitter Schwarz image, parallel family, the caustic surface, and especially their singularities.

The coefficient $q(x)$ of the equation is

$$q(x) = -\frac{1}{4}\left(\frac{3}{4x^2} + \frac{3}{4(1-x)^2} + \frac{11}{18x(1-x)}\right),$$

and the coefficient of the parallel family is

$$Q = Q(y;k) = -\frac{1}{4}\left(\frac{3}{4y^2} + \frac{3}{4(k-y)^2} + \frac{11}{18y(k-y)}\right).$$

In the following, we express the complex variable y as $s + it$, where s and t are real variables. We define several polynomials to study the cusp-line and singularities. We first denote the numerator of the rational polynomial $Q\overline{Q} - 1$ by C, which turns out to be

$$\begin{aligned} C \;=\; & -704t^2k^2 + 729k^4 + 1024t^4 - 1728k^3s + 2752k^2s^2 - 2048ks^3 + 2048s^2t^2 \\ & -124416s^4t^4 - 20736k^4s^4 + 82944k^3s^5 - 124416k^2s^6 + 82944ks^7 - 82944s^2t^6 \\ & -82944s^6t^2 - 20736k^4t^4 - 41472k^2t^6 + 1024s^4 - 20736s^8 - 20736t^8 + 248832s^3t^4k \\ & -290304k^2s^4t^2 + 248832ks^5t^2 - 207360s^2t^4k^2 + 82944k^3t^4s + 82944t^6ks \\ & -41472k^4s^2t^2 + 165888k^3s^3t^2 - 2048kst^2. \end{aligned}$$

Then the condition $Q\overline{Q} = 1$ is equivalent to

$$C = 0.$$

We next denote by Q_R the real part of the numerator of $\dot{Q}$ and by Q_I the imaginary part. Then, the condition $\dot{Q} \neq 0$ is

$$Q_R \neq 0 \quad \text{or} \quad Q_I \neq 0.$$

Similarly, we denote by S_R the real part of the numerator of the expression $Q^3\,\overline{\dot{Q}} + \dot{Q}$ and by S_I the imaginary part. Then the condition $Q^3\,\overline{\dot{Q}} + \dot{Q} = 0$ is

$$S_R = S_I = 0.$$

Furthermore, let R denote the numerator of the real part of the expression $(\ddot{Q}/Q^2 - 3/2\;\dot{Q}^2/Q^3)$. Then, the condition that appears in the characterizations (3) and (4) is nothing but

$$R = 0 \quad \text{or} \quad R \neq 0.$$

We omit the concrete expressions of these polynomials as they are directly computable from the expression of the rational polynomial Q.

5.1. **Positions of swallowtail singularities.** The swallowtail singularities, or simply swallowtails, are obtained by solving the system of equations

$$C = S_R = S_I = 0,$$

and by checking the conditions $Q_R \neq 0$ or $Q_I \neq 0$, and $R \neq 0$. To get the solutions of the above system, we make the prime decomposition of the radical $\sqrt{I}$ of the ideal $I = \langle C, S_R, S_I\rangle$. The result is $\sqrt{I} = C_1 \cap C_2 \cap C_3 \cap C_4 \cap C_5$ where C_i are given as follows:

$$C_1 = \langle P_1, t\rangle, \quad C_2 = \langle P_2, 2s - k\rangle, \quad C_3 = \langle F_1, F_2, F_3, F_4\rangle,$$

$$C_4 = \langle 9s^2 + 9t^2 - 2, k\rangle, \quad C_5 = \langle s^2 + t^2, k\rangle;$$

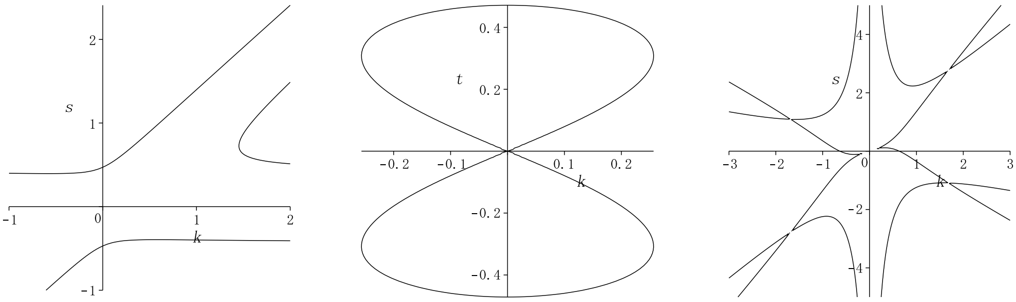
where

$$\begin{aligned} P_1 &= (-144s^2 + 27)k^2 + (288s^3 - 32s)k - 144s^4 + 32s^2, \\ P_2 &= 9k^4 + (72t^2 + 19)k^2 + 144t^4 - 32t^2, \end{aligned}$$

and the polynomials F_1 and F_2 are given in the appendix; for the polynomials F_3 and F_4, we refer to the remark below.[1]. Since k is positive and $k \in C_4$ and $k \in C_5$, we only have to consider real zeros of C_1, C_2 and C_3.

In Figure 1, we show the shape of curves defined by the polynomials P_1, P_2, and F_1, respectively.

[1]Prime decompositions in this paper are all done by the mathematical software Risa/Asir (Kobe distribution) and `noro_pd.prime_dec()`.



$\{P_1(k,s)=0\}$ $\{P_2(k,t)=0\}$ $\{F_1(k,s)=0\}$

FIGURE 1. Curves defined by the polynomials P_1, P_2 and F_1

Remark 6. The polynomial F_1 is a polynomial of k and s; A rough shape of the curve $\{F_1(k,s)=0\}$ is given in Figure 1. The polynomial F_2 is written in the form $P(k)t^2 + B(k,s)$, where

$$\begin{aligned} P(k) &= 8159999356734888960 0k^{14} - 1368335505576424550 40k^{12} \\ &\quad -2260491020098900968 96k^{10} - 1084592310855114424 32k^{8} \\ &\quad -1988937635011441459 2k^{6} - 2607350356416607027 2k^{4} \\ &\quad -2883573225000796160k^{2} - 76773399409459200. \end{aligned}$$

and B is a polynomial of k and s. The polynomials F_3 and F_4 are not independent of F_1 and F_2; we have the relations

$$P(k)F_3 = aF_2 + bF_1, \qquad P(k)^3F_4 = cF_2^2 + dF_2 + eF_1, \tag{5.1}$$

where a, b, c, d and e are polynomials of k and s; more precisely, c is a constant, a and b are polynomials of k, and d and e are polynomials of k and s.

5.1.1. *Swallowtails from the ideal* C_1. Since the swallowtails from the ideal $C_1 = \langle P_1, t\rangle$ are given by solving the equation $t = P_1(k,s) = 0$, they lie on the s-axis. Note that a rough shape of the curve defined by $P_1(k,s) = 0$ is given in the first figure of the Figure 1. Since the degree of P_1 relative to s is four, the number of the solutions $\{s; P_1(k,s) = 0\}$ is at most four for each k. We then compute the resultant of $P_1(k,s)$ as an s-polynomial, which is

$$-1358954496k^2(9k^2-19)(243k^2+16)^2.$$

This vanishes only when k is equal to

$$k_1 = \frac{1}{3}\sqrt{19} = 1.452966314.$$

This means that the number of the solutions $\{s; P_1(k,s) = 0\}$ is two, three, and four, according to $k < k_1$, $k = k_1$, and $k > k_1$.

We now check whether the solution of the equation is really a swallowtail, namely check whether $Q_R \neq 0$ or $Q_I \neq 0$, and $R \neq 0$. First we exclude the common zeros of the ideal $C_1 + \langle Q_R, Q_I\rangle$. Computation shows

$$\sqrt{C_1 + \langle Q_R, Q_I\rangle} = \langle k,s,t\rangle \cap \langle 9k^2-19, 2s-k, t\rangle \cap \langle 243k^2+16, t, 9s^2-9ks-1\rangle.$$

Since k,s,t are real and k is positive, the common solution is $(k,s,t) = (\frac{1}{3}\sqrt{19}, \frac{1}{6}\sqrt{19}, 0)$. Thus, the point p_1 with coordinates $(s,t) = (\frac{1}{6}\sqrt{19}, 0)$, which is on the cusp-line with $k = \frac{1}{3}\sqrt{19}$, is not a swallowtail.

We next exclude the common zeros of the ideal $C_1 + \langle R\rangle$. We have

$$\sqrt{C_1 + \langle R\rangle} = \langle k,s,t\rangle \cap \langle k, t, 9s^2-2\rangle \cap \langle 9801k^2-10640, t, 33s^2-33ks-19\rangle.$$

Since $k > 0$,

$$k_3 = \sqrt{10640/9801} = 1.041922986,$$

is found to be an exceptional value, for which the point $p_3^{\pm} : (s,t) = (a, 0)$, where $a = \frac{2}{99}\sqrt{665} \pm \frac{19}{99}\sqrt{23}$ is the solution of $33s^2 - 33ks - 19 = 0$, is not a swallowtail. Note that this point satisfies the condition $\dot{Q} \neq 0$, because $Q_R \neq 0$ or $Q_I \neq 0$ as seen in the above.

5.1.2. *Swallowtails from the ideal C_2.* The swallowtails given by $C_2 = \langle P_2, 2s - k \rangle$ lie on the line $s = k/2$. The polynomial $P_2(k,t)$ is even and of degree four relative to t and the curve $\{P_2(k,t) = 0\}$ has a shape given in the middle figure of Figure 1. The resultant of $P_2(k,t)$ as a t-polynomial is

$$1358954496k^2(9k^2 + 19)(243k^2 - 16)^2,$$

which vanishes only when k is equal to

$$k_2 = \frac{4}{\sqrt{243}} = 0.2566001196.$$

As seen in the figure, the number of solutions of $P_2 = 0$ is four, two, and zero according to $0 < k < k_2$, $k = k_2$, and $k > k_2$.

In order to exclude non-swallowtail solutions, we consider the zeros of $C_2 + \langle Q_R, Q_I \rangle$. We have

$$\sqrt{C_2 + \langle Q_R, Q_I \rangle} = \langle k, s, t \rangle \cap \langle 9k^2 + 19, 2s - k, t \rangle \cap \langle 243k^2 - 16, 2s - k, 243t^2 - 23 \rangle,$$

and only the last one has real zeros. Hence, we see that k_2 is an exceptional value and the point $p_2^{\pm} : (s,t) = (\frac{2}{\sqrt{243}}, \pm\frac{\sqrt{69}}{27})$ are not swallowtails.

We next consider the zeros of $C_2 + \langle R \rangle$. We have

$$\sqrt{C_2 + \langle R \rangle} = \langle k, s, t \rangle \cap \langle k, s, 9t^2 - 2 \rangle \cap \langle 9801k^2 + 10640, 2s - k, 9801t^2 - 8303 \rangle.$$

Since any component has no real solution for $k > 0$, we have no exceptional value in this case, *i.e.* the solutions of $P_2 = 2s - k = 0$ when $0 < k < k_2$ are all swallowtails.

5.1.3. *Swallowtails from the ideal C_3.* We first see that any common root of $F_1 = F_2 = F_3 = F_4 = 0$ satisfies the condition $\dot{Q} \neq 0$. In order to see this, we consider $C_3 + \langle Q_R, Q_I \rangle$.

$$\begin{aligned}\sqrt{C_3 + \langle Q_R, Q_I \rangle} &= \langle k, s, t \rangle \cap \langle 243k^2 - 16, t, 9s^2 - 9ks + 1 \rangle \\ &\quad \cap \langle 243k^2 + 16, 2s - k, 243t^2 + 23 \rangle \\ &\quad \cap \langle 2187k^4 + 304, 64s^2 - 64ks + 39k^2, 64t^2 - 23k^2 \rangle\end{aligned}$$

and see that only the second component has real solutions for which k_2 is exceptional. However, we have no real solution of $9s^2 - 9ks + 1 = 0$ for this k. In other words, the condition $\dot{Q} \neq 0$ is always satisfied.

We next see whether a solution of $F_1 = F_2 = F_3 = F_4 = 0$ satisfies the condition $R \neq 0$. For this purpose we consider the ideal $C_3 + \langle R \rangle$. We have

$$\begin{aligned}\sqrt{C_3 + \langle R \rangle} &= \langle s, t, k \rangle \cap \langle 9801t^2 - 8303, 9801k^2 + 10640, k - 2s \rangle \\ &\quad \cap \langle 33sk - 33s^2 + 19, 9801k^2 - 10640, t \rangle \\ &\quad \cap \langle -435848050125k^8 - 119481222825k^4 + 609206272, \\ &\qquad -106540634475k^6 + 18854345700k^4 + 14097778650k^2 \\ &\qquad -26246410560t^2 - 1108945792, \\ &\qquad 106540634475k^6 + 18854345700k^4 - 20659381290k^2 \\ &\qquad +26246410560sk - 26246410560s^2 - 1108945792 \rangle.\end{aligned}$$

Since k is real and positive, we do not need to consider the first and the second components. The third component, which is already appeared in C_1, shows that the value k_3 is exceptional. For the last component, the real positive root of the first polynomial only of k is

$$(-8993/65610 + 3277\sqrt{16385}/2952450)^{1/4}.$$

However, for this value of k, the second polynomial has no real root. Hence, the last component is also out of consideration.

Therefore, when $k \neq k_3$, any common root of $F_1 = F_2 = F_3 = F_4 = 0$ is a swallowtail. Let us give an instruction of solving this system of equations. Because of the equation (5.1), for a k with $P(k) \neq 0$, any root of $F_1 = F_2 = 0$ automatically satisfies $F_3 = F_4 = 0$; hence, it is enough to solve $F_1(k, s) = 0$ for s and then $F_2(k, s, t) = 0$ for t.

On the other hand, the system of equations $P = F_1 = F_2 = F_3 = F_4 = 0$ has no common real root: We have

$$C_3 + \langle P(k)\rangle = \langle P(k), t^4 + a(k)t^2 + b(k), c(s, k))\rangle \cap \langle P(k), t^4 + d(k)t^2 + e(k), f(s, t, k)\rangle,$$

where a, b, c, d, e, f are polynomials. Note that the polynomial $P(k)$ has only one real positive root $k_4 = 1.68379\ldots$. By evaluating k_4 with a sufficiently high precision, we find that both $t^4 + a(k_4)t^2 + b(k_4)$ and $t^4 + d(k_4)t^2 + e(k_4)$ have no real root.

5.1.4. *Summary.* We have found three special values of k:

$$\begin{aligned} k_1 &= \sqrt{19/9} = 1.452966314, \\ k_2 &= \sqrt{16/243} = 0.2566001196, \\ k_3 &= \sqrt{10640/9801} = 1.0419229865. \end{aligned}$$

We have seen that

- The number of swallowtails by C_1 is two or four, according to $k < k_1$ or $k > k_1$. When $k = k_1$, the solutions of $P_1 = 0$ are $\frac{1}{6}\sqrt{19}, \frac{1}{6}\sqrt{19}, \frac{1}{6}(\sqrt{19} \pm \sqrt{46})$. We have seen $\dot{Q} = 0$ for the point $p_1 = (\frac{1}{6}\sqrt{19}, 0)$. Hence, we really have two swallowtails at $(s, t) = (\frac{1}{6}(\sqrt{19} \pm \sqrt{46}), 0)$. Furthermore, for k_1, the point $(\frac{1}{6}\sqrt{19}, 0)$ satisfies $\dot{Q} = 0$ and we can see $R \neq 0$ at this point; namely, this point is a singularity called *cuspidal beaks* because of the characterization of singularities in Section 4.2.
- The number of swallowtails by C_2 is four or zero according to $0 < k < k_2$ or $k > k_2$. When $k = k_2$, the points $p_2^{\pm} = (s = 2/\sqrt{243}, t = \pm\sqrt{69}/27)$ solve $P_2 = 0$, for which we saw that $\dot{Q} = 0$; hence, those are not swallowtails. Moreover, we can see $R \neq 0$ at these points; hence, these are cuspidal beaks.
- When $k = k_3$, we have two singularities from both C_3 and C_1: $p_3^{\pm} : (s, t) = (a, 0)$, where $a = \frac{2}{99}\sqrt{665} \pm \frac{19}{99}\sqrt{23}$. However, we have seen that those are not swallowtails. Furthermore, at these points, $\dddot{Q}/Q + \overline{\dddot{Q}}Q^4 - 5\dot{Q}\ddot{Q}/Q^2 - 5\overline{\dot{Q}\ddot{Q}}Q^5$ happens to be zero; hence, it is neither a cuspidal butterfly.
- A numerical experiment shows that the number of real solutions of the equation $F_1 = F_2 = F_3 = F_4 = 0$ is four when $k < k_3$.

5.2. **cusp-lines and swallowtails of parallel surfaces of the de Sitter Schwarz map.** Parallel surfaces of the de Sitter Schwarz image of the equation $u'' - q(x)u = 0$ is the de Sitter Schwarz image of

$$\ddot{v} - Q(y; k)v = 0.$$

We draw the figures of the cusp-lines on y-plane for several values of k in Figure 2.

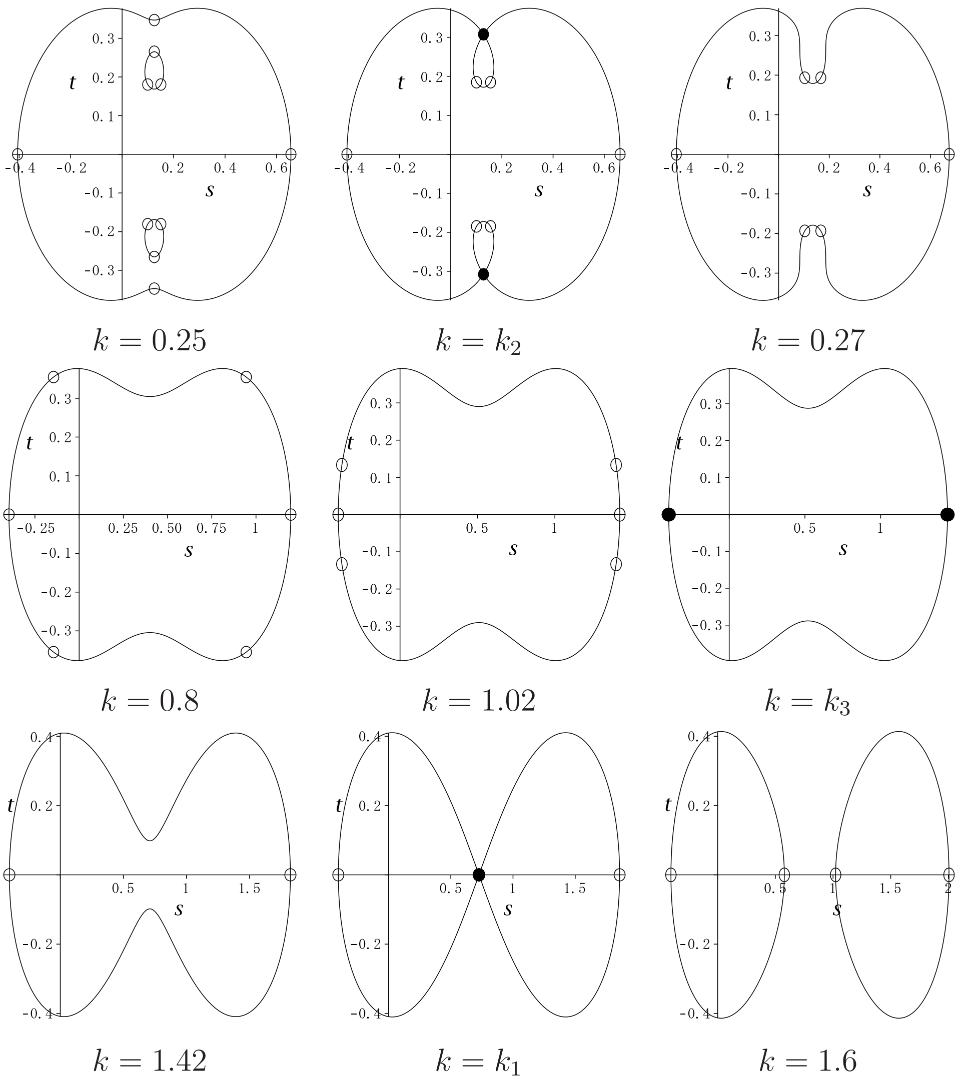


FIGURE 2. Swallowtail points on the curve $|Q(y;k)| = 1$

White balls represent swallowtails and the black balls represent points which are limits of swallowtails and are not themselves swallowtails. These are p_1 for $k = k_1$, $p_2^{\pm}$ for $k = k_2$, and $p_3^{\pm}$ for $k = k_3$ as was summarized in Section 5.1.4. Among swallowtails, those on the s-axis come from the ideal C_1, those on the axis $s = k/2$, namely the symmetry axis of the figure from the ideal C_2, and the other four white balls given for the value $k < k_3$ are the swallowtails from the ideal C_3.

See Remark 8 in the end of Section 6.4.

6. FIGURES OF IMAGE SURFACES AND THE IMAGE OF THE CUSP-LINE

6.1. **The hollow ball model of the de Sitter space.** We realize the de Sitter space S^3_1 in the 3-space by the stereographic projection from the south pole $(-1,0,0,0) \in S^3$:

$$S^3 = (\mathbb{L}^4 - \{0\})/\mathbb{R}_{>0} \ni (t_0,t_1,t_2,t_3) \longmapsto (x_1,x_2,x_3) \in \mathbb{R}^3$$

where

$$x_j = \frac{t_j}{t_0 + \|t\|}, \quad j = 1,2,3, \quad \|t\| = \sqrt{t_0^2+t_1^2+t_2^2+t_3^2}$$

If $-t_0^2+t_1^2+t_2^2+t_3^2=0$, we have

$$x_1^2+x_2^2+x_3^2=\frac{t_0^2}{3t_0^2+2t_0\sqrt{2}|t_0|}=(\sqrt{2}\mp 1)^2,$$

as t_0 positive or negative. Thus we have the correspondence

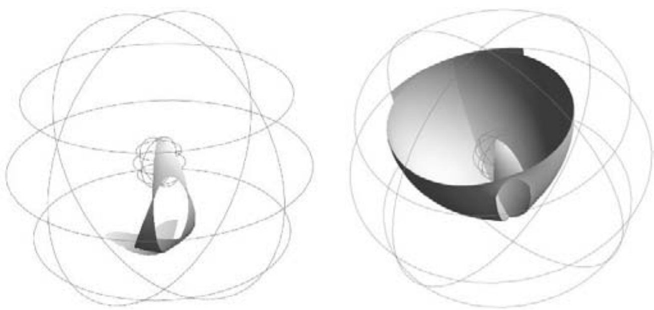
hyperbolic space H^3_+	$\longleftrightarrow$	ball of radius $\sqrt{2}-1$
sphere S^2_+	$\longleftrightarrow$	sphere of radius $\sqrt{2}-1$
de Sitter space S^3_1	$\longleftrightarrow$	hollow ball bounded by balls of radii $\sqrt{2}\pm 1$
sphere S^2_-	$\longleftrightarrow$	sphere of radius $\sqrt{2}+1$
hyperbolic space H^3_-	$\longleftrightarrow$	outside of the ball of radius $\sqrt{2}+1$
South Pole in H^3_-	$\longleftrightarrow$	the point at infinity

6.2. **Drawing dS-surfaces.** In order to draw the total image of the map, it is convenient to consider the map as defined on the image of the usual Schwarz image. In the present case, the coordinate z on the Schwarz image is related with the original coordinate x by

$$x=(z^3+1)^2/(4z^3).$$

We refer to [14] for this coordination. The coordinate z varies on the complex plane and the fundamental domain relative to the monodromy group is a fan $\{z=re^{i\theta};0\le\theta<\pi/3\}$.

Figure 3 visualizes the images of a part of the fan $\{z=re^{i\theta};0<r<r_1,0<\theta<\pi/3\}$, where $r_1=0.75$ and $r_1=1$. The outer curves show some circles of radius $\sqrt{2}+1$ and the inner curves are those of radius $\sqrt{2}-1$ The figure changes drastically depending on the range of the radius r.



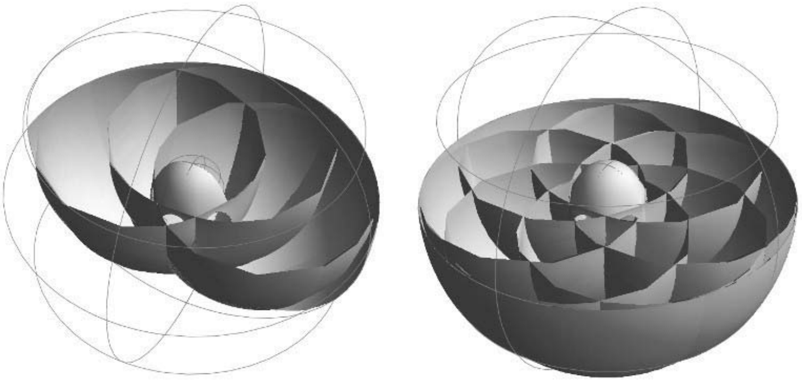
$0<r<0.75$ $\qquad$ $0<r<1$

FIGURE 3. Images of one fan

Figure 4 draws the image of the three fans $\{z=re^{i\theta};0<r<1,0<\theta<\pi\}$ and the image of the six fans $\{z=re^{i\theta};0<r<1,0<\theta<2\pi\}$.

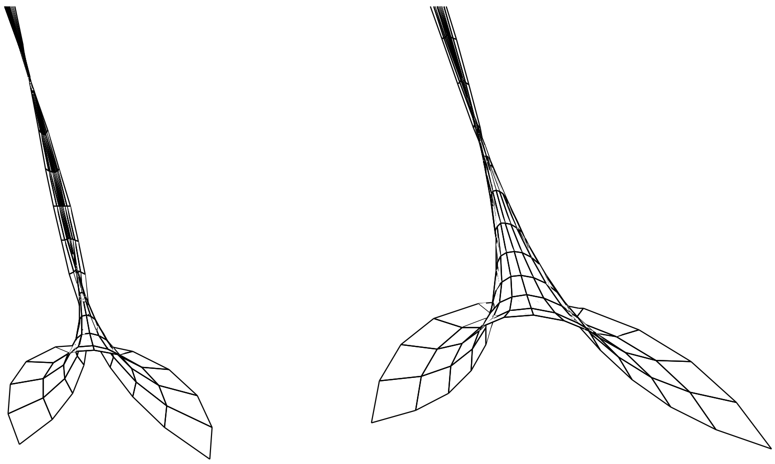
6.3. **Image around a raindrop.** In Figure 2 ($k=0.25$), showing the cusp-line, we find a small closed curve in the upper/lower half y-plane, call it a *rain drop*, carrying three swallowtail points. We draw in Figure 5 the images of the area around this curve; here $y=s+it$. Please find four swallowtails in the left figure and three in the right figure. The image of the rain drop is a triangle with three swallowtails as its vertices.

6.4. **A triangular horn on the caustic surface.** When $0<k<k_2$, the cusp-line $|Q(y)|=1$ has a shape as in Figure 2 ($k=0.25$). We are interested in the move of the rain drop in the upper half plane. By remark 6, it tends to the origin in y-plane and to the zero of the rational function $q(x)$ in the x-plane. Since the numerator of $q(x)$ is $32(x-\frac{1}{2})^2+19$, the limit point is $x_0=\frac{1}{2}+\sqrt{\frac{19}{32}}i$. As we saw in Section 4.2, the map



three fans six fans

FIGURE 4. Images of several fans



$(s,t) \in [0.05, 0.20] \times [0.15, 0.40]$ $(s,t) \in [0.075, 0.175] \times [0.15, 0.30]$

FIGURE 5. Images around a raindrop: $k = 0.25$

$\mathcal{S}_k^{\mathrm{deS}}$ tends to the derived Schwarz map when k tends to zero. The union of the image of the rain drop for k small enough form a (singular) surface, which we call a *triangular caustic horn* after the shape given in Figure 6. The apex of the horn is the image under

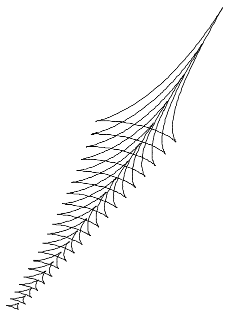


FIGURE 6. Triangular caustic horn

the derived Schwarz map of the point x_0. This figure is compared with the triangluar caustic horn in $\mathbb{H}^3$ drawn in Figure 7 of [13, pp 369–385].

Remark 7. The union of the triangular caustic horn in $S^3_1 \subset S^3$ and the triangular caustic horn in $H^3_+ \subset S^3$ would form an irreducible surface in S^3 with D_4-singularity at the common apex; though, we have no proof. We refer to [9] for D_4-singularity.

Remark 8. In Part 2, we treated the specific hypergeometric differential equation $E(1/2, 1/2, 1/3)$ only. But we believe that the move of the cusp-lines and creation/extinction of the swallowtails stated in Section 5.1.4 and Section 5.2 would be valid for the hypergeometric differential equation (0.2): $E(\mu_0, \mu_1, \mu_\infty)$ with general parameters μ_0, μ_1, $\mu_\infty \in \mathbb{R}$, if the numerator of the coefficient $q(x)$ has two imaginary (conjugate) roots; of course the values k_1, k_2, k_3 change. Moreover, the description above of the triangular horn on the caustic would be also true, though of course the value of the root x_0 changes.

6.5. Appendix: the polynomials F_1 and F_2.

$$
\begin{aligned}
F_1 \quad = \quad & 2006122600857600s^8k^4 - 8024490403430400s^7k^5 + (9654465016627200k^6 + 1203673560514560k^4 \\
& -3329851637366784k^2)s^6 + (9989554912100352k^3 - 877678637875200k^7 - 3611020681543680k^5)s^5 \\
& +(-11424613007360 - 1446156635524 3008k^4 - 6850791135682560k^8 + 6676626780979200k^6 \\
& -200777373057024k^2)s^4 + (5802474530488320k^9 - 7334885759385600k^7 \\
& +401554746114048k^3 + 22849226014720k + 12273874523652096k^5)s^3 \\
& +(-6493402335707136k^6 - 438677645819904k^4 - 23920283484160k^2 - 1417356718283520k^{10} \\
& +5509803968630784k^8)s^2 + (12495670476800k^3 + 237900272762880k^5 \\
& -2444197869195264k^9 - 292745252701440k^{11} + 2021390892564480k^7)s \\
& -255891118429440k^8 - 469264823746 56k^6 + 442574057234529k^{12} + 792402203507904k^{10} \\
& -2409879306240k^4, \\
F_2 \quad = \quad & (8159999356734 8889600k^{14} - 13683355055764 2455040k^{12} - 226049102009890096896k^{10} \\
& -108459231085511442432k^8 - 198893763501144 14592k^6 - 26073503564166070272k^4 \\
& -2883573225000796160k^2 - 767733994094592 00)t^2 \\
& +(-26962287755526144000k^4 - 175838369976778752000k^{10} + 154471440266035 20000k^6 \\
& +218204105901008486400k^8)s^6 + (80886863266578432000k^5 + 52751510993033 6256000k^{11} \\
& -463414320798105 60000k^7 - 654612317703025459200k^9)s^5 \\
& +(4475320600620957 6960k^2 - 43495952185766707200k^4 + 956988760284804 09600k^{10} \\
& -348929265422670336000k^{12} + 409164684465249976320k^8 - 421071612653680459776k^6)s^4 \\
& +(707331786529730199552k^7 - 181333319038553088000k^{13} + 86991904371533414400k^5 \\
& +899622777448081612800k^{11} - 89506412012419153920k^3 - 741093648797482352640k^9)s^3 \\
& +(105334554440364982272k^4 + 359172193908134059200k^{14} - 114103212523105812480k^6 \\
& +2469337802693672960k^2 - 994615273356217738560k^{12} - 543300569293458309120k^8 \\
& +767733994094592 00 + 790225006991311413504k^{10})s^2 \\
& +(-180586349400468139200k^{15} + 435701831681672688960k^{13} - 427401754605871997184k^{11} \\
& +203115819906356281344k^9 - 767733994094592 00k - 60581348434155405312k^5 \\
& +706072603373391052 80k^7 - 2469337802693672960k^3)s \\
& +39803903112198017925k^{16} + 23812565279422021632k^6 - 82976036869957840965k^{14} \\
& +2206479912020213760k^4 - 233803722885783183 36k^{10} + 179290388626816013280k^{12} \\
& -27724519559976910848k^8 + 647775557517 31200k^2,
\end{aligned}
$$

7. Airy equation

The most confluent hypergeometric equation is the Airy equation:

$$u'' - xu = 0.$$

Since $q = x$, it is quite easy to study the singularities of the de Sitter and hyperbolic Schwarz maps: They are singular along the unit circle $|x| = 1$, and has swallowtails at the cubic roots of ∓ 1, respectively. The hyperbolic one is studied in [12]. We show several pictures of the images under the de Sitter Schwarz map of the discs with radii r centered at the origin.

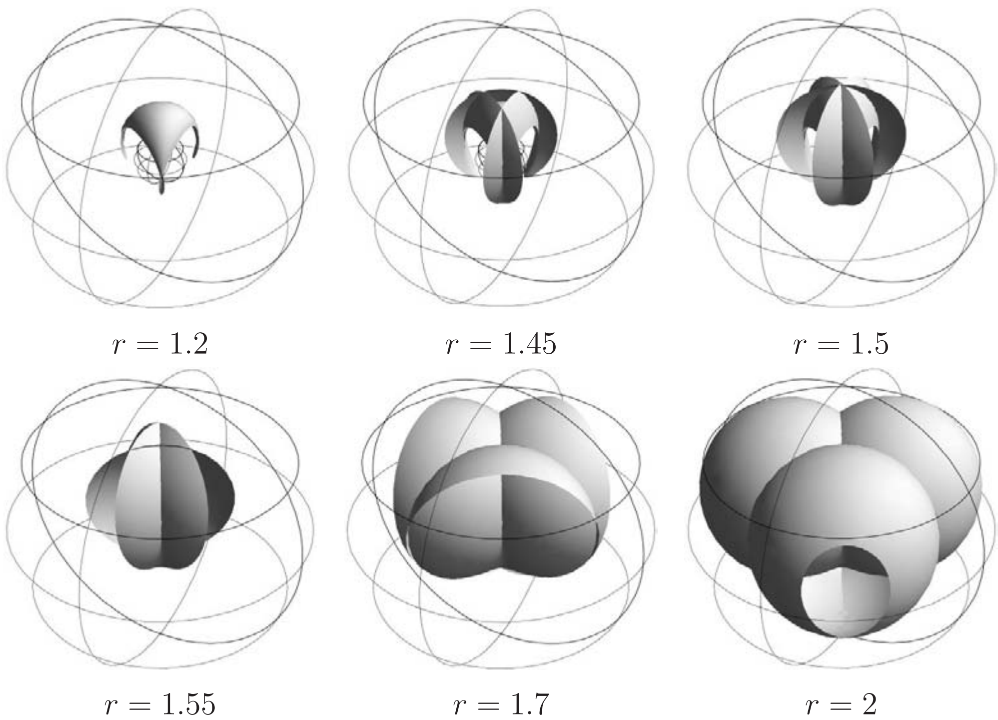


FIGURE 7. Image by de Sitter Schwarz map of the disc $|x| < r$

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