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ON THE STRATIFICATION OF COMBINATORIAL SPECTRA

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Abstract. In this note, we investigate a mixture of combinatorial spectra and stratified simplicial sets, which would be thought of as a model of the spectrum objects of (∞,∞)-categories.

1. Introduction

In this note, we investigate a mixture of combinatorial spectra and stratified simplicial sets, which would be thought of as a model of the spectrum objects of (∞,∞)-categories.

In [5], Kan introduced the notion of combinatorial spectrum, which is a certain presheaf in pointed sets over a category Δ_{st} that is a stabilization of the simplex category Δ in an appropriate sense. Intuitively speaking, a combinatorial spectrum is a pointed simplicial set with $\mathbb{Z}$-graded simplices. In [6], Kan and Whitehead introduced a product for combinatorial spectra, which may not give rise to a monoidal structure on the nose, and showed that it works well up to homotopy. Later, Brown showed the category of combinatorial spectra admits a model structure in [1]. Bousfield and Friedlander showed there exists a chain of Quillen equivalences between that and another model of spectra in [2].

In [10], Ozornova and Rovelli introduced the notion of prestratified simplicial sets, which has already appeared in [15] without name and is also a presheaf in sets over a category similar to Δ, denoted by $t\Delta$. Intuitively speaking, a prestratified simplicial set is a simplicial set with two layers of simplices. Verity constructed in [16] a model structure on the category of stratified simplicial sets, which are prestratified simplicial sets satisfying a certain condition. Ozornova and Rovelli then constructed a model structure on the category of prestratified simplicial sets based on Verity's work and it is expected the model structure models (∞,∞)-categories.

In this note we investigate the presheaves in pointed sets over a stabilization of $t\Delta$. Such a presheaf thus would be viewed as a pointed simplicial set consists of two kinds of n-simplices with $n \in \mathbb{Z}$. We show the presheaf category inherits a homotopy theory and the stratified analogue of Kan-Whitehead product is compatible with that in an appropriate sense.

2. Preliminary

2.1. **Stabilization of simplicial sets.** In this section, we recall combinatorial spectra from [5], [6] and [1]. We focus on Kan-Whitehead product and Brown's model structure of them.

Let Δ denote the simplex category and $\star : \Delta \times \Delta \to \Delta$ denote the concatenation functor. More precisely, for any $[m], [n] \in \Delta$, $[m] \star [n] = [m+n+1]$, and for $\theta : [m] \to [m'], \tau : [n] \to [n'] \in \Delta$, the morphism $\theta \star \tau : [m+n+1] \to [m'+n'+1]$ is given by

$$(\theta \star \tau)(i) = \begin{cases} \theta(i) & (0 \leq i \leq m), \\ \tau(i-m-1) + m' + 1 & (m+1 \leq i \leq m+m'+1). \end{cases}$$

Remark 2.1. By definition we get the following equations:

$$d^i \star s^j = d^i \circ s^{n+j},\ s^j \star d^i = d^{m+1+i} \circ s^j, d^i \star d^{i'} = d^{n+1+i'} \circ d^i,\ \ s^j \star s^{j'} = s^{m+1+j'} \circ s^j$$

for any face operators $d^i : [n-1] \to [n]$ and $d^{i'} : [n'-1] \to [n']$, and degeneracy operators $s^j : [m+1] \to [m]$ and $s^{j'} : [m'+1] \to [m']$.

Using this concatenation functor, Chen, Kriz, and Pultr stabilized Δ in the following sense.

Definition 2.2. We define the following shift functors:

$$K : \Delta \to \Delta,\ K(\theta : [n] \to [m]) = (\theta \star [0] : [n+1] \to [m+1]),$$

The other one J is the dual of K.

$$J : \Delta \to \Delta,\ J(\theta : [n] \to [m]) = ([0] \star \theta : [n+1] \to [m+1]),$$

By using these functors we define the following categories:

$$\Delta_{st} := \mathrm{colim}(\Delta \xrightarrow{K} \Delta \xrightarrow{K} \Delta \xrightarrow{K} \cdots),$$

$$\Delta_{st'} := \mathrm{colim}(\Delta \xrightarrow{J} \Delta \xrightarrow{J} \Delta \xrightarrow{J} \cdots),$$

$$\Delta_{st^2} := \mathrm{colim}(\Delta \xrightarrow{K} \Delta \xrightarrow{J} \Delta \xrightarrow{K} \Delta \xrightarrow{J} \Delta \xrightarrow{K} \cdots).$$

Remark 2.3. (1)The first one Δ_{st} is defined in [3]. By definition, we have $J \circ K = K \circ J$. So we can change the order of the functors in the colimit diagram of Δ_{st^2}.

(2)These three categories above admit the following descriptions respectively. By abusing notation, the objects of Δ_{st} may be denoted by $[n]$ for all $n \in \mathbb{Z}$. The morphism of Δ_{st} are generated by the morphisms

$$d^i : [n-1] \to [n],\ s^j : [n+1] \to [n]$$

for all $n \in \mathbb{Z}$ and integers $i, j \geq 0$ subject to the following identities

$$d^i d^j = d^{j+1} d^i \quad (i \leq j),$$

$$s^j s^i = s^i s^{j+1} \quad (i \leq j),$$

$$s^j d^i = \begin{cases} d^i s^{j-1} & (i < j), \\ \mathrm{id} & (i \in \{j, j+1\}), \\ d^{i-1} s^j & (i > j+1). \end{cases}$$

Dually, the objects of $\Delta_{st'}$ are denoted by $[n]$ for all $n \in \mathbb{Z}$. The morphism of $\Delta_{st'}$ are generated by the morphisms

$$d^i : [n-1] \to [n], \; s^j : [n+1] \to [n]$$

for all $n \in \mathbb{Z}$ and integers $i, j \leq n$ subject to the same identities.

The objects of Δ_{st^2} are denoted by $[n]$ for all $n \in \mathbb{Z}$. The morphism of Δ_{st^2} are generated by the morphisms

$$d^i : [n-1] \to [n], \; s^j : [n+1] \to [n]$$

for all $n, i, j \in \mathbb{Z}$ subject to the same identities.

(3)There is a functor $\mathrm{rev} : \Delta \to \Delta$ defined by

$$\mathrm{rev}(d^i : [n-1] \to [n]) = (d^{n-i} : [n-1] \to [n]),$$

$$\mathrm{rev}(s^i : [n+1] \to [n]) = (s^{n-i} : [n+1] \to [n]).$$

This fits into the commutative diagram

$$\begin{array}{ccc} \Delta & \xrightarrow{K} & \Delta \\ {\scriptstyle \mathrm{rev}}\downarrow & & \downarrow{\scriptstyle \mathrm{rev}} \\ \Delta & \xrightarrow[J]{} & \Delta. \end{array}$$

This gives a functor $\Delta_{st} \to \Delta_{st'}$, which we also denote by rev abusing notation.

Definition 2.4 ([5], [3])**.** We let $\mathbf{Set}_*$ denote the category of pointed sets and pointed maps.

(1) A functor $\Delta_{st}^{op} \to \mathbf{Set}_*$ is called a stable simplicial set. A stable simplicial set X satisfying the following condition is called a combinatorial spectrum: for every $x \in X$, there exists an integer i such that $d_k x = *$ when $k > i$. We denote by $\mathbf{Set}_*^{\Delta_{st}^{op}}$ the category of stable simplicial sets and natural transformations and by **Comb** the full subcategory of combinatorial spectra.

(2) A functor $\Delta_{st^2}^{op} \to \mathbf{Set}_*$ is called a bistable simplicial set. A bistable simplicial set X satisfying the following condition is called a combinatorial bispectrum: for every $x \in X$, there exist integers i and j such that $d_k x = *$ when $k > i$ or $k < j$. We denote by $\mathbf{Set}_*^{\Delta_{st^2}^{op}}$ the category of bistable simplicial sets and natural transformations.

We construct the stable analogue of the concatenation functor as follows: Let $\theta : [n] \to [n'] \in \Delta_{st'}$ and $\tau : [m] \to [m'] \in \Delta_{st}$. By the definitions of $\Delta_{st'}$ and Δ_{st}, there exist $k, l \in \mathbb{N}$ and morphisms $\tilde{\theta}, \tilde{\tau} \in \Delta$ such that

$$\tilde{\theta} : [n + k] \to [n' + k] \in \Delta$$

represents θ and

$$\tilde{\tau} : [m + l] \to [m' + l] \in \Delta$$

represents τ. We denote by $\theta \star \tau : [n] \star [m] \to [n'] \star [m']$ the morphism in Δ_{st^2} represented by

$$\tilde{\theta} \star \tilde{\tau} : [n + m + k + l] \to [n' + m' + k + l] \in \Delta.$$

This defines the functor again denoted by

$$\star : \Delta_{st'} \times \Delta_{st} \to \Delta_{st^2}.$$

For $X, Y \in \mathbf{Set}_*^{\Delta_{st}^{op}}$, we let $X \wedge'' Y$ denote the point-wise smash product, namely $(X \wedge'' Y)([m], [n]) = X([m]) \wedge Y([n])$, where $\wedge$ denotes the smash product of pointed sets.

Let $X \wedge' Y$ denote the left Kan extension of $X \wedge'' Y$ with respect to

$$\Delta_{st}^{op} \times \Delta_{st}^{op} \xrightarrow{\text{rev} \times \text{id}} \Delta_{st'}^{op} \times \Delta_{st}^{op} \xrightarrow{\star} \Delta_{st^2}^{op}.$$

This construction defines a functor $\mathbf{Set}_*^{\Delta_{st}^{op}} \times \mathbf{Set}_*^{\Delta_{st}^{op}} \to \mathbf{Set}_*^{\Delta_{st^2}^{op}}$, which is the stable analogue of the join construction of simplicial sets. To define the product on $\mathbf{Set}_*^{\Delta_{st}^{op}}$ called reduced join in [6], we recall a functor $\mathbf{Set}_*^{\Delta_{st^2}^{op}} \to \mathbf{Set}_*^{\Delta_{st}^{op}}$.

Definition 2.5 ([5])**.** Let $V \in \mathbf{Set}_*^{\Delta_{st^2}^{op}}$. We define $V_{-1} \in \mathbf{Set}_*^{\Delta_{st}^{op}}$ as follows. For any $[n] \in \Delta_{st}$, we set

$$V_{-1}([n]) := \{x \in V([n+1]) | d_j^V x = *, j < 1\}$$

The generators of maps are given as follows:

$$d_i := d_{i+1}^V, s_i := s_{i+1}^V,$$

where d_j^V and s_j^V are the generators for V.

Definition 2.6 ([6])**.** Let $X, Y \in \mathbf{Set}_*^{\Delta_{st}^{op}}$. The Kan-Whitehead smash product $X \wedge Y$ is $(X \wedge' Y)_{-1}$.

Remark 2.7. In [6] Kan and Whitehead took a functor before applying the functor $(\text{-})_{-1} : \mathbf{Set}_*^{\Delta^{op}_{st2}} \to \mathbf{Set}_*^{\Delta^{op}_{st}}$. More precisely, by using freely generating functor $\mathrm{F} : \mathbf{Set} \to \mathbf{Grp}$ piecewise, where **Grp** denotes the category of groups and homomorphisms, they considered $(\mathrm{F}(X \wedge' Y))_{-1}$. There the celebrated fact that the underlying simplicial set of any simplicial group is a Kan complex plays a pivotal role.

Kan and Whitehead have shown this product plays a role in the stable homotopy theory. To describe that, we recall prespectra from [5].

As is pointed out in [7], Kan's suspension functor ([5, Definition 2.2]) $\mathrm{S} : \mathbf{sSet}_* \to \mathbf{sSet}_*$ is the left Kan extension along yoneda embedding of

$$\Delta \to \mathbf{sSet}_*, [n] \mapsto \Delta[n+1]_+/(\Delta[n]_+ \vee \Delta[0]_+).$$

Definition 2.8 ([5])**.** (1) A prespectrum L consists of

(i) a sequence of pointed simplicial sets L_i with $i \in \mathbb{N}$,

(ii) a sequence of monomorphisms $\lambda_i : \mathrm{S}\, L_i \to L_{i+1}$ of pointed simplicial sets with $i \in \mathbb{N}$, where S denotes the suspension functor.

(2) A morphism $\psi : \{L_i, \lambda_i\} \to \{M_i, \mu_i\}$ of prespectra is a sequence of morphisms $\psi_i : L_i \to M_i$ of pointed simplicial sets such that $\psi_{i+1} \circ \lambda_i = \mu_i \circ \mathrm{S}\, \psi_i$.

(3) A morphism $\psi : \{L_i, \lambda_i\} \to \{M_i, \mu_i\}$ of prespectra is called a weak homotopy equivalence if for every $q \in \mathbb{Z}$ the abelian group homomorphism

$$\pi_q(\psi) : \mathrm{colim}_{i\to\infty}\, \pi_{i+q}(L_i) \to \mathrm{colim}_{i\to\infty}\, \pi_{i+q}(M_i)$$

is an isomorphism.

Kan related combinatorial spectra and prespectra as follows.

Definition 2.9 ([5])**.** Let $X \in \mathbf{Set}_*^{\Delta^{op}_{st}}$. We define the corresponding prespectrum $\mathrm{Ps}(X) = \{X_i, \xi_i\}$ as follows.

For any $i \in \mathbb{N}$, the pointed set of n-simplices in X_i is given by

$$X_i([n]) := \{\alpha \in X([n-i]) | d_0 \cdots d_n \alpha = *, d_j \alpha = *, (j > n)\},$$

where $*$ denotes (the degeneracy of) the base point. Face and degeneracy operators on X_i will be induced by those of X and X_i is indeed a pointed simplicial set with them.

For any $i \in \mathbb{N}$, the monomorphism $\xi_i : \mathrm{S}\, X_i \to X_{i+1}$ is the obvious inclusion.

Brown has shown that the weak homotopy equivalences give rise to a model structure.

Theorem 2.10 ([1])**.** *The category of combinatorial spectra admits the model structure in which*

- *cofibrations are precisely monomorphims,*
- *a morphism f is a fibration if and only if* $\mathrm{Ps}(f)_i$ *is a fibration for all i in the classical model structure on pointed simplicial sets,*
- *weak equivalences are the morphisms which are weak homotopy equivalences after taking* Ps.

Kan and Whitehead has proven that the product is compatible with the weak equivalences.

Theorem 2.11 ([6]). *The functor*

$$(\mathrm{F}((\text{-}) \wedge' (\text{-})))_{-1} : \mathbf{Comb} \times \mathbf{Comb} \to \mathbf{Comb}$$

preserves weak equivalences.

Let $\mathbb{S}$ be the stable simplicial set all of whose simplices are degenerate except one simplex of degree 0 ([6, Example 2.2]). In other words, this is the stable simplicial set corresponding to the simplicial set $\Delta[0]$. It is shown there that $\mathbb{S}$ acts as the unit of the product up to homotopy.

Theorem 2.12 ([6]). *For any combinatorial spectrum X,* $(\mathrm{F}(\mathbb{S} \wedge' X))_{-1}$ *is weakly equivalent to X.*

Remark 2.13. In [5, Section 10] Kan defined the q-th homotopy group $\pi_q(X)$ of a combinatorial spectrum X to be the homotopy group $\pi_{q+i}((\mathrm{F}\,X)_i)$ with an integer $i > -q$, where F denotes the free group functor. This is well-defined since $\pi_n((\mathrm{F}\,X)_j) = \pi_{n+1}((\mathrm{F}\,X)_{j+1})$ for all n and j. It is shown there that a morphism f of combinatorial spectra is a weak equivalence if and only if the morphism $\mathrm{Ps}(f)$ of prespectra is a termwise weak homotopy equivalence.

2.2. **Stratification of simplicial sets.** In this section, we recall (pre-)stratified simplicial sets from mainly [10] and [16]. We focus on the join construction, the lax Gray-Verity tensor product and Ozornova-Rovelli model structure of them.

We recall the category $t\Delta$ from [10]. Its set of objects consists of $[n]$ with $0 \leq n \in \mathbb{Z}$ and $[m]_t$ with $1 \leq m \in \mathbb{Z}$. The morphisms in $t\Delta$ are generated by the following morphisms

$$d^i : [m-1] \to [m], \quad 0 \leq i \leq m,$$
$$s^i : [m+1] \to [m], \quad 0 \leq i \leq m,$$
$$\varphi : [m] \to [m]_t,$$
$$\zeta^i : [m+1]_t \to [m], \quad 0 \leq i \leq m,$$

subject to the usual cosimplicial identities on d^i's and s^i's, and the following additional relations

- $\zeta^i\varphi = s^i : [m+1] \to [m]$, $1 \le m$ and $0 \le i \le m$,
- $s^i\zeta^{j+1} = s^j\zeta^i : [m+2]_t \to [m]$, $0 \le i \le j \le m$.

We may view Δ as a subcategory of $t\Delta$ in the evident way.

Definition 2.14 ([10])**.** We let **Set** denote the category of sets and maps. A prestratified simplicial set X is a functor $X : t\Delta^{op} \to \mathbf{Set}$. A stratified simplicial set X is a prestratified simplicial set such that the maps

$$\varphi^* : X([m]_t) \to X([m])$$

are injective for all $m \ge 1$. We let $\mathbf{Set}^{t\Delta^{op}}$ denote the category of prestratified simplicial sets and natural transformations and let **msSet** denote the full subcategory of stratified simplicial sets. We call an element in $X([n]_t)$ a marked n-simplex for any $X \in \mathbf{Set}^{t\Delta^{op}}$ and $n > 0$.

For any $X \in \mathbf{Set}^{t\Delta^{op}}$, We denote by mX the set of marked simplices of X, and by dX the set of degenerate simplices of X.

Definition 2.15 ([10])**.** Let n be a natural number and $k \in [n]$.

- *The standard thin n-simplex* $\Delta[n]_t$ is the simplicial set with marking whose underlying simplicial set is the standard simplicial set $\Delta[n]$ and
$$m\Delta[n]_t = \begin{cases} d\Delta[n] \cup \{\mathrm{Id}_{[n]}\} & (n \neq 0), \\ d\Delta[n] & (n = 0). \end{cases}$$
- *The k-admissible n-simplex* $\Delta^k[n]$ with $n \ge 1$[1] is the simplicial set with marking whose underlying simplicial set is the standard simplicial set $\Delta[n]$ and
$$m\Delta^k[n] = d\Delta[n] \cup \{\alpha \in \Delta[n] | \{k-1, k, k+1\} \cap [n] \subset \mathrm{Im}(\alpha)\}.$$
- *The $(n-1)$-dimensional k-admissible horn* $\Lambda^k[n]$ with $n \ge 1$ is the regular simplicial subset with marking of $\Delta^k[n]$ whose underlying simplicial set is the usual simplicial k-th horn.
- $\Delta^k[n]''$ (respectively, $\Lambda^k[n]'$) is the simplicial set with marking whose underlying simplicial set is the same as that of $\Delta^k[n]$ (respectively, $\Lambda^k[n]$) and its marked simplices are $m\Delta^k[n]$ (respectively, $m\Lambda^k[n]$) with all its $(n-1)$-simplices.
- $\Delta^k[n]' := \Delta^k[n] \cup \Lambda^k[n]'$.
- $\Delta[3]_{eq}$ is the simplicial set with marking whose underlying simplicial set is $\Delta[3]$ and all n-simplices for $n \ge 2$ and the non-degenerate two 1-simplices $\overline{02}$ and $\overline{13}$ are marked, where
$$\overline{02} : [1] \to [3], \quad \overline{02}(0) = 0, \ \overline{02}(1) = 2,$$

[1]In the case $n = 0$, hence $k = 0$, we define $\Delta^0[0]$ to be the simplicial set with marking whose underlying simplicial set is $\Delta[0]$ and $m\Delta^0[0] = d\Delta[0]$.

$$\overline{13}:[1]\to[3],\ \ \overline{13}(0)=1,\ \overline{13}(1)=3.$$

- $\Delta[3]^\sharp$ is the simplicial set with marking whose underlying simplicial set is $\Delta[3]$ with

$$m\Delta[3]^\sharp=\bigcup_{n\geq 1}\Delta[3]_n.$$

Remark 2.16. (1)Note that the functor K preserves admissible simplices. More precisely, for any morphism $\alpha:[r]\to[n]$ with $\{k-1,k,k+1\}\cap[n]\subset\mathrm{Im}(\alpha)$, the morphism $K(\alpha):[r+1]\to[n+1]$ satisfies that $\{k-1,k,k+1\}\cap[n+1]\subset\mathrm{Im}(K(\alpha))$. The same holds for J.

(2)There are evident inclusions $\Lambda^k[n]\to\Delta^k[n]$, $\Delta^k[n]'\to\Delta^k[n]''$, $\Delta[3]_{eq}\to\Delta[3]^\sharp$, and $\Delta[n]\star\Delta[3]_{eq}\to\Delta[n]\star\Delta[3]^\sharp$ for all k,n. We may call them elementary anodyne extensions. Note that by definition the inclusions $\Delta[0]\to\Delta[1]_t$ are the elementary anodyne extensions $\Lambda^k[1]\to\Delta^k[1]$ with $k\in\{0,1\}$.

We call prestratified simplicial sets having the right lifting property with respect to elementary anodyne extensions precomplicial sets ([10]).

The following model structures are expected to model (∞,∞)-categories.

Theorem 2.17 ([10])**.** *The category* $\mathbf{Set}^{t\Delta^{op}}$ *of prestratified simplicial sets admits a model structure in which*

- *the cofibrations are precisely the monomorphisms,*
- *the fibrant objects are precisely the precomplicial sets.*

These classes of morphisms give rise to a cofibrantly generated model structure on the category **msSet** *of stratified simplicial sets. These two model structures are Quillen equivalent.*

We may call these model structures Ozornova-Rovelli model structures.

The following simple morphisms in Δ (or in $t\Delta$) are used to define the lax Gray-Verity product.

Definition 2.18 ([16])**.** For any $(p,q)\in\mathbb{N}^2$, there are four maps in Δ:

$$\perp\!\!\!\perp_1^{p,q}:[p]\to[p+q],\ \perp\!\!\!\perp_1^{p,q}(i)=i,$$

$$\perp\!\!\!\perp_2^{p,q}:[q]\to[p+q],\ \perp\!\!\!\perp_2^{p,q}(i)=i+p,$$

$$\top\!\!\!\top_1^{p,q}:[p+q]\to[p],\ \top\!\!\!\top_1^{p,q}(i)=\begin{cases} i & (0\leq i\leq p),\\ p & (p<i\leq p+q),\end{cases}$$

$$\top\!\!\!\top_2^{p,q}:[p+q]\to[q],\ \top\!\!\!\top_1^{p,q}(i)=\begin{cases} 0 & (0\leq i<p),\\ i-p & (p\leq i\leq p+q).\end{cases}$$

Remark 2.19. Let $p, q \in \mathbb{N}$. We note that the morphisms $\perp\!\!\!\perp_1^{p,q} : [p] \to [p+q]$ (resp. $\perp\!\!\!\perp_2^{p,q} : [q] \to [p+q]$) in Δ is compatible with $J : \Delta \to \Delta$ (resp. $K : \Delta \to \Delta$). More precisely,

$$J(\perp\!\!\!\perp_1^{p,q}) = \perp\!\!\!\perp_1^{p+1,q}, K(\perp\!\!\!\perp_2^{p,q}) = \perp\!\!\!\perp_2^{p,q+1}$$

hold. Similarly, we also have

$$J(\top\!\!\!\top_1^{p,q}) = \top\!\!\!\top_1^{p+1,q}, K(\top\!\!\!\top_2^{p,q}) = \perp\!\!\!\perp_2^{p,q+1}.$$

In addition, the morphisms $\perp\!\!\!\perp_1^{p,q}$ and $\perp\!\!\!\perp_2^{p,q}$ in Δ correspond each other via the endofunctor $\mathrm{rev} : \Delta \to \Delta$. Since by definition $\perp\!\!\!\perp_1^{p,q} = s^p \circ s^{p+1} \circ \cdots \circ s^{p+q-1}$ and $\perp\!\!\!\perp_2^{q,p} = s^0 \circ s^0 \circ \cdots \circ s^0$ hold, so by the definition of rev we have

$$\mathrm{rev}(\perp\!\!\!\perp_1^{p,q}) = \perp\!\!\!\perp_2^{q,p}, \ \mathrm{rev}(\perp\!\!\!\perp_2^{p,q}) = \perp\!\!\!\perp_1^{q,p}.$$

Also, we have the equations below

$$\top\!\!\!\top_1^{p,q} \circ \perp\!\!\!\perp_1^{p,q} = \mathrm{id}, \quad \top\!\!\!\top_2^{p,q} \circ \perp\!\!\!\perp_2^{p,q} = \mathrm{id}\,.$$

Remark 2.20. As is demonstrated in [15], the non-degenerate $(p+q)$-simplices in the simplicial set $\Delta[p] \times \Delta[q]$ correspond to the shortest paths from the left bottom corner $(0,0)$ to the right upper corner (p,q) in the figure below.

$$\begin{array}{ccccccc}
(0,q) & \longrightarrow & (1,q) & \longrightarrow & \cdots & \longrightarrow & (p,q) \\
\uparrow & & \uparrow & & & & \uparrow \\
(0,q-1) & \longrightarrow & (1,q-1) & \longrightarrow & \cdots & \longrightarrow & (p,q-1) \\
\uparrow & & \uparrow & & & & \uparrow \\
\cdots & & \cdots & & \cdots & & \cdots \\
\uparrow & & \uparrow & & & & \uparrow \\
(0,1) & \longrightarrow & (1,1) & \longrightarrow & \cdots & \longrightarrow & (p,1) \\
\uparrow & & \uparrow & & & & \uparrow \\
(0,0) & \longrightarrow & (1,0) & \longrightarrow & \cdots & \longrightarrow & (p,0)
\end{array}$$

For instance, the $(p+q)$-simplex $(\top\!\!\!\top_1^{p,q}, \top\!\!\!\top_2^{p,q}) \in (\Delta[p] \times \Delta[q])([p+q])$ corresponds to the shortest path turning the right bottom corner $(p,0)$.

Definition 2.21 ([16])**.** Let $X, Y \in \mathbf{msSet}$. Their lax Gray-Verity product $X \otimes Y$ is the following stratified simplicial set: the underlying simplicial set is the cartesian product $X \times Y$ and an n-simplex $x \otimes y \in X \otimes Y$ is marked if and only if for any $(p,q) \in \mathbb{N}^2$ with $p+q = n$, $x \cdot \perp\!\!\!\perp_1^{p,q} \in X([p]_t)$ or $y \cdot \perp\!\!\!\perp_2^{p,q} \in Y([q]_t)$.

Remark 2.22. Let $\Delta[l]$ denote the prestratified simplicial set represented by $[l] \in t\Delta$, which is a stratified simplicial set. Note that there is only one unmarked $(m+n)$-simplex $(\top\!\!\top_1^{m,n}, \top\!\!\top_2^{m,n})$ in $\Delta[m] \otimes \Delta[n]$. Any other $(m+n)$-simplex is marked by definiton. Intuitively speaking, discarding marked simplices, $\Delta[m] \otimes \Delta[n]$ is similar to $\Delta[m+n]$, while the join of $\Delta[m]$ and $\Delta[n]$ is $\Delta[m+n+1]$. Note that $\Delta[m] \otimes \Delta[n]$ has one or more unmarked k-simplices with $k \leq m+n$ when $m, n \geq 1$.

We will extend the join construction of simplicial sets recalled in the last section to prestratified simplicial sets. To do so, abusing the notation, we now extend the concatenation functor $\star : \Delta \times \Delta \to \Delta$ to $\star : t\Delta \times t\Delta \to t\Delta$. Roughly speaking, viewing $\Delta \subset t\Delta$ and $\star|_{\Delta \times \Delta} = \star$, ζ^i's and φ's behave like s^i's and the identity morphisms respectively.

Definition 2.23. We define the concatenation functor $\star : t\Delta \times t\Delta \to t\Delta$ as follows.

(1) On the objects in $t\Delta$, $\star$ acts as follows:
For $m, n \geq 1$,
$$[m]_t \star [n] = [m] \star [n]_t = [m]_t \star [n]_t = [m+n+1]_t,$$
$$[m] \star [n] = [m+n+1].$$
For $m \geq 0$,
$$[0] \star [m] = [m] \star [0] = [m+1].$$
For $m \geq 1$,
$$[0] \star [m]_t = [m]_t \star [0] = [m+1]_t.$$

(2) On the generators of morphisms in $t\Delta$, $\star$ acts as follows:
Viewing d^i's and s^j's in $t\Delta$ as in Δ, their concatenations are defined as in Section 2.1.
For $\zeta^j : [m+1]_t \to [m]$ and $d^i : [n-1] \to [n]$,
$$d^i \star \zeta^j = d^i \circ \zeta^{n+j}, \ \zeta^j \star d^i = d^{m+1+i} \circ \zeta^j.$$
For $\zeta^j : [m+1] \to [m]_t$ and $s^i : [n+1] \to [n]$,
$$s^i \star \zeta^j = s^i \circ \zeta^{n+1+j}, \ \zeta^j \star s^i = s^{m+1+i} \circ \zeta^j.$$
For $\zeta^j : [m+1] \to [m]_t$ and $\zeta^i : [n+1] \to [n]_t$,
$$\zeta^i \star \zeta^j = \zeta^i \circ \zeta^{n+1+j}, \ \zeta^j \star s^i = s^{m+1+i} \circ \zeta^j.$$
For $\varphi : [n] \to [n]_t$ and $d^i : [m-1] \to [m]$,
$$\varphi \star d^i = \varphi \circ d^{n+1+i}, \ d^i \star \varphi = \varphi \circ d^i.$$
For $\varphi : [n] \to [n]_t$ and $s^i : [m+1] \to [m]$,
$$\varphi \star s^i = \varphi \circ s^{n+1+i}, \ s^i \star \varphi = \varphi \circ s^i.$$

For $\varphi : [n] \to [n]_t$ and $\zeta^i : [m+1]_t \to [m]$,

$$\varphi \star \zeta^i = \varphi \circ \zeta^{n+1+i},\ \zeta^i \star \varphi = \varphi \circ \zeta^i.$$

For $\varphi_n : [n] \to [n]_t$ and $\varphi_m : [m] \to [m]_t$, where we put subscripts for convenience,

$$\varphi_n \star \varphi_m = \varphi_m \star \varphi_n = \varphi_{m+n}.$$

Remark 2.24. One might define $[m] \star [n]_t$ to be $[m+n+1]$ for $m, n \geq 1$, namely the concatenation of an unmarked simplex and a marked simplex should be unmarked. But it and $[0] \star [n]_t = [n+1]_t$ would imply $[1] \star [n]_t = ([0] \star [0]) \star [n]_t \neq [0] \star ([0] \star [n]_t) = [0] \star [n+1]_t$.

We define the join functor $(\text{-}) \oplus (\text{-}) : \mathbf{Set}^{t\Delta^{op}} \times \mathbf{Set}^{t\Delta^{op}} \to \mathbf{Set}^{t\Delta^{op}}$ to be the Day convolution of $\star : t\Delta \times t\Delta \to t\Delta$. Note that our definition is compatible with the join of stratified simplicial sets [16, Definition 33].

Proposition 2.25. *For any $X \in$ **msSet**, the functors $(\text{-}) \oplus X, X \oplus (\text{-})$: **msSet** $\to$ **msSet** are left Quillen functors with respect to Ozornova-Rovelli model structure.*

Proof. In [16, Chapter 6], it has been proven that for any stratified simplicial set X, $(\text{-}) \oplus X$ and $X \oplus (\text{-})$ are left Quillen functors with respect to the model structure for non-saturated weak complicial sets on **msSet**.

It is enough to show that the morphism $\Delta[3]_{eq} \oplus X \to \Delta[3]^\sharp \oplus X$ is a trivial cofibration with respect to Ozornova-Rovelli model structure. Since $\Delta[3]_{eq} \oplus \Delta[n]_? \to \Delta[3]^\sharp \oplus \Delta[n]_?$ is a trivial cofibration for any $n \in \mathbb{N}$, where $[n]_?$ denotes $[n]$ or $[n]_t$, and the join construction is compatible with colimits, the morphism $\Delta[3]_{eq} \oplus X \to \Delta[3]^\sharp \oplus X$ is a trivial cofibration. □

Since every stratified simplicial set is cofibrant, we obtain the following.

Corollary 2.26. *Let $X, X', Y, Y' \in$ **msSet**. Suppose we have weak equivalences $f : X \to X'$ and $g : Y \to Y'$. Then the morphism $f \oplus g : X \oplus Y \to X' \oplus Y'$ is also a weak equivalence.*

Remark 2.27. In [8, §1.2.8], it is proven that the join of two quasi-categories is a quasi-category. The same argument there shows that the join of two precomplicial sets has the right lifting property with respect to the inclusions $\Lambda^k[n] \to \Delta^k[n]$, but it may not show that the join has the right lifting property with respect to the other anodyne extensions.

3. Results

3.1. **Stratified stabilization of simplicial sets.** In this section, by using the functor $\star : t\Delta \times t\Delta \to t\Delta$, we define stratified analogue of Δ_{st}, $\Delta_{st'}$ and Δ_{st^2}. To do that, by abusing notation, we define the following shift functors.

Definition 3.1. We denote by $K : t\Delta \to t\Delta$ the shift functor $(\text{-}) \star [0] : t\Delta \to t\Delta$ and by $J : t\Delta \to t\Delta$ the other shift functor $[0] \star (\text{-}) : t\Delta \to t\Delta$

More explicitly, these functors act on the generators of morphisms as follows:

$$\begin{gathered}
K(d^i : [n-1] \to [n]) = d^i : [n] \to [n+1], \\
K(s^i : [n+1] \to [n]) = s^i : [n+2] \to [n+1], \\
K(\varphi : [n] \to [n]_t) = \varphi : [n+1] \to [n+1]_t, \\
K(\zeta^i : [n+1]_t \to [n]) = \zeta^i : [n+2] \to [n+1]_t, \\
J(d^i : [n-1] \to [n]) = d^{i+1} : [n] \to [n+1], \\
J(s^i : [n+1] \to [n]) = s^{i+1} : [n+2] \to [n+1], \\
J(\varphi : [n] \to [n]_t) = \varphi : [n+1] \to [n+1]_t, \\
J(\zeta^i : [n+1]_t \to [n]) = \zeta^{i+1} : [n+2] \to [n+1]_t.
\end{gathered}$$

By using these functors we obtain the following categories:

$$\begin{gathered}
t\Delta_{st} := \operatorname{colim}(t\Delta \xrightarrow{K} t\Delta \xrightarrow{K} t\Delta \xrightarrow{K} \cdots), \\
t\Delta_{st'} := \operatorname{colim}(t\Delta \xrightarrow{J} t\Delta \xrightarrow{J} t\Delta \xrightarrow{J} \cdots), \\
t\Delta_{st^2} := \operatorname{colim}(t\Delta \xrightarrow{K} t\Delta \xrightarrow{J} t\Delta \xrightarrow{K} t\Delta \xrightarrow{J} t\Delta \xrightarrow{K} \cdots).
\end{gathered}$$

Remark 3.2. (1)By definition, we have $J \circ K = K \circ J$. So we can change the order of the functors in the colimit diagram of $t\Delta_{st^2}$.

(2)These three categories admit the following descriptions:

By abusing notation, the objects of $t\Delta_{st}$ may be denoted by $[n]$ or $[n]_t$ for all $n \in \mathbb{Z}$. The morphism of $t\Delta_{st}$ are generated by the morphisms

$$d^i : [n-1] \to [n],\ s^j : [n+1] \to [n],\ \zeta^k : [n+1]_t \to [n],\ \varphi : [n] \to [n]_t$$

for all $n \in \mathbb{Z}$ and integers $i, j, k \geq 0$ subject to the stratified version of the identities in Remark 2.3. Note that there is $[0]_t \in t\Delta_{st}$

Dually, the objects of $t\Delta_{st'}$ are denoted by $[n]$ or $[n]_t$ for all $n \in \mathbb{Z}$. The morphism of $t\Delta_{st'}$ are generated by the morphisms

$$d^i : [n-1] \to [n],\ s^j : [n+1] \to [n],\ \zeta^k : [n+1]_t \to [n],\ \varphi : [n] \to [n]_t$$

for all $n \in \mathbb{Z}$ and integers $i, j, k \leq n$ subject to the same identities.

The objects of $t\Delta_{st^2}$ are denoted by $[n]$ or $[n]_t$ for all $n \in \mathbb{Z}$. The morphism of $t\Delta_{st^2}$ are generated by the morphisms

$$d^i : [n-1] \to [n],\ s^j : [n+1] \to [n],\ \zeta^k : [n+1]_t \to [n],\ \varphi : [n] \to [n]_t$$

for all $n, i, j \in \mathbb{Z}$ subject to the same identities.

We may view in the evident way the categories Δ_{st}, $\Delta_{st'}$, and Δ_{st^2} as subcategories of $t\Delta_{st}$, $t\Delta_{st'}$, and $t\Delta_{st^2}$ respectively.

(3)Abusing notation, we define the stratified analogue of $\mathrm{rev} : \Delta_{st} \to \Delta_{st'}$. To do that, we define $\mathrm{rev} : t\Delta \to t\Delta$ as follows:

On the objects, $\mathrm{rev}([n]) = [n]$ and $\mathrm{rev}([n]_t) = [n]_t$. On the generators of morphisms,

$$\mathrm{rev}(d^i : [n-1] \to [n]) = d^{n-i} : [n-1] \to [n],$$
$$\mathrm{rev}(s^i : [n+1] \to [n]) = s^{n-i} : [n+1] \to [n],$$
$$\mathrm{rev}(\varphi : [n] \to [n]_t) = \varphi : [n] \to [n]_t,$$
$$\mathrm{rev}(\zeta^i : [n+1]_t \to [n]) = \zeta^{n-i} : [n+1] \to [n]_t.$$

This fits into the commutative diagram

$$\begin{array}{ccc} t\Delta & \xrightarrow{K} & t\Delta \\ {\scriptstyle \mathrm{rev}}\downarrow & & \downarrow{\scriptstyle \mathrm{rev}} \\ t\Delta & \xrightarrow[J]{} & t\Delta. \end{array}$$

This defines a functor $t\Delta_{st} \to t\Delta_{st'}$, which we also denote by rev abusing notation.

By using the functor $\star : t\Delta \times t\Delta \to t\Delta$, we extend the functor $\star : \Delta_{st'} \times \Delta_{st} \to \Delta_{st^2}$ to $\star : t\Delta_{st'} \times t\Delta_{st} \to t\Delta_{st^2}$, again abusing notation.

Let $\theta : [n]_? \to [n']_? \in t\Delta_{st'}$ and $\tau : [m]_? \to [m']_? \in t\Delta_{st}$, where $[n]_?$ denotes $[n]$ or $[n]_t$. By the definitions of $t\Delta_{st'}$ and $t\Delta_{st}$, there exist $k, l \in \mathbb{N}$ and morphisms $\tilde{\theta}, \tilde{\tau}$ in $t\Delta$ such that

$$\tilde{\theta} : [n+k]_? \to [n'+k]_? \in t\Delta$$

represents θ and

$$\tilde{\tau} : [m+l]_? \to [m'+l]_? \in t\Delta$$

represents τ. Denote $\theta \star \tau : [n]_? \star [m]_? \to [n']_? \star [m']_?$ the map in $t\Delta_{st^2}$ represented by

$$\tilde{\theta} \star \tilde{\tau} : [n+m+k+l]_? \to [n'+m'+k+l]_? \in t\Delta.$$

This construction defines a functor $t\Delta_{st'} \times t\Delta_{st} \to t\Delta_{st^2}$.

Definition 3.3. We let $\mathbf{Set}_*^{t\Delta_{st}^{op}}$ denote the category of functors $t\Delta_{st}^{op} \to \mathbf{Set}_*$ and call the objects prestratified stable simplicial sets. For any $X \in \mathbf{Set}_*^{t\Delta_{st}^{op}}$ and $[n]_t \in t\Delta_{st}$, we call elements in $X([n]_t)$ marked n-simplices. A stratified stable simplicial set X is a prestratified stable simplicial set such that the maps

$$\varphi^* : X([m]_t) \to X([m])$$

are injective for all $m \geq 1$.

Remark 3.4. For any $A \in \mathbf{Set}^{t\Delta^{op}}$, we may write $\Sigma_+^\infty A$ for the corresponding prestratified stable simplicial set

$$\Sigma_+^\infty A([n]) = \begin{cases} A([n]) \coprod \{*\} & (n \geq 0), \\ \{*\} & (n < 0), \end{cases}$$

and $d_j \alpha = *$ for any $\alpha \in \Sigma_+^\infty A([m])$ and $j > m$.

As we have defined the join product on $\mathbf{Set}_*^{\Delta_{st}^{op}}$, we here define an analogous product on $\mathbf{Set}_*^{t\Delta_{st}^{op}}$. For $X, Y \in \mathbf{Set}_*^{t\Delta_{st}^{op}}$, we let $X \wedge'' Y$ denote the point-wise smash product, namely $(X \wedge'' Y)([m],[n]) = X([m]) \wedge Y([n])$, where $\wedge$ denotes the smash product of pointed sets.

Let $X \wedge' Y$ denote the left Kan extension of $X \wedge'' Y$ with respect to

$$t\Delta_{st}^{op} \times t\Delta_{st}^{op} \xrightarrow{\text{rev} \times \text{id}} t\Delta_{st'}^{op} \times t\Delta_{st}^{op} \xrightarrow{\star} t\Delta_{st^2}^{op}.$$

This construction defines a functor (-) $\wedge'$ (-) : $\mathbf{Set}_*^{t\Delta_{st}^{op}} \times \mathbf{Set}_*^{t\Delta_{st}^{op}} \to \mathbf{Set}_*^{t\Delta_{st^2}^{op}}$, which is the staratified analogue of the join construction of stable simplicial sets. To define a product on $\mathbf{Set}_*^{t\Delta_{st}^{op}}$, the stratified analogue of the Kan-Whitehead product on stable simplicial sets, we introduce a functor $\mathbf{Set}_*^{t\Delta_{st^2}^{op}} \to \mathbf{Set}_*^{t\Delta_{st}^{op}}$.

Definition 3.5. Let $V \in \mathbf{Set}_*^{t\Delta_{st^2}^{op}}$. We define $V_{-1} \in \mathbf{Set}_*^{t\Delta_{st}^{op}}$ as follows. For any $[n], [m]_t \in t\Delta_{st}^{op}$, we set

$$V_{-1}([n]) := \{x \in V([n+1]) | d_j^V x = *, j < 1\},$$

$$V_{-1}([m]_t) := \{x \in V([n+1]_t) | d_j^V \varphi^V x = *, j < 1\}.$$

The operators are given as follows:

$$d_i := d_{i+1}^V,\ s_i := s_{i+1}^V,\ \varphi := \varphi^V,\ \zeta_i := \zeta_{i+1}^V,$$

where d_j^V, s_j^V, φ^V and ζ_j^V are the operators for V.

Definition 3.6. Let $X, Y \in \mathbf{Set}_*^{t\Delta_{st}^{op}}$. The stratified Kan-Whitehead smash product $X \tilde{\wedge} Y$ is $(X \wedge' Y)_{-1}$.

By Remark 2.19, the following is well defined.

Definition 3.7. Let $p, q \in \mathbb{Z}$.

(1) We denote by $\perp\!\!\!\perp_1^{p,q} : [p] \to [p+q]$ the morphism in $\Delta_{st'}$ represented by a morphism $\perp\!\!\!\perp_1^{p+k,q} : [p+k] \to [p+q+k]$ in Δ for some $k \in \mathbb{N}$. Similarly, we denote by $\perp\!\!\!\perp_2^{p,q} : [q] \to [p+q]$ the morphism in Δ_{st} represented by a morphism $\perp\!\!\!\perp_2^{p,q+k} : [q+k] \to [p+q+k]$ in Δ for some $k \in \mathbb{N}$.

(2) We denote the morphism $\mathrm{rev}(\perp\!\!\!\perp_1^{p,q}) : [p] \to [p+q]$ in $\Delta_{st'}$ by $\perp\!\!\!\perp_2^{q,p}$: $[p] \to [p+q]$. Similarly, we denote the morphism $\mathrm{rev}(\perp\!\!\!\perp_2^{p,q}) : [q] \to [p+q]$ in Δ_{st} by $\perp\!\!\!\perp_1^{q,p} : [q] \to [p+q]$.

We may view these morphisms as those in $t\Delta_{st}$ and $t\Delta_{st'}$ respectively.

Definition 3.8. Let $X, Y \in \mathbf{Set}_*^{t\Delta_{st}^{op}}$. The stable analogue of lax Gray-Verity product $X\tilde{\otimes}Y$ is defined as follows:

- its underlying stable simplicial set is the point-wise smash product $X \wedge Y$
- an n-simplex $x\tilde{\otimes}y \in X\tilde{\otimes}Y$ is marked if and only if for any $(p,q) \in \mathbb{Z}^2$ with $p+q=n$, $x \circ \perp\!\!\!\perp_1^{p,q} \in X([p]_t)$ or $y \circ \perp\!\!\!\perp_2^{p,q} \in Y([q]_t)$.

This product $\tilde{\otimes}$ is a straightforward analogue of lax Gray tensor product for stratified simplicial sets. The lax Gray-Verity product plays a pivotal role in weak ω-category theory ([17] and [12]). Thus, it would be fair to expect that its stable analogue also would play a role in stable objects. We below construct a natural morphism between it and stratified Kan-Whitehead product.

Proposition 3.9. *There exists a natural morphism $X\tilde{\wedge}Y \to X\tilde{\otimes}Y$.*

Proof. Let $X, Y \in \mathbf{Set}_*^{t\Delta_{st}^{op}}$ and $x\tilde{\wedge}y$ be an unmarked n-simplex of $X\tilde{\wedge}Y$. Then by the definition of $X\tilde{\wedge}Y$ there exit $p, q \in \mathbb{Z}$ with $p+q=n$ such that $x \in X([p])$ and $y \in Y([q])$. We assign to it the n-simplex $(x \circ \top\!\!\!\top_1^{p,q}, y \circ \top\!\!\!\top_2^{p,q})$ of $X\tilde{\otimes}Y$.

Let $x\tilde{\wedge}y$ be a marked n-simplex of $X\tilde{\wedge}Y$. Then there exist $p, q \in \mathbb{Z}$ with $p+q=n$ such that

(1) $x \in X([p]_t)$ and $y \in Y([q])$ or
(2) $x \in X([p])$ and $y \in Y([q]_t)$ or
(3) $x \in X([p]_t)$ and $y \in Y([q]_t)$.

For each case, we again consider $(x \circ \top\!\!\!\top_1^{p,q}, y \circ \top\!\!\!\top_2^{p,q})$ of $X\tilde{\otimes}Y$. We need to show that $(x \circ \top\!\!\!\top_1^{p,q}, y \circ \top\!\!\!\top_2^{p,q})$ is marked. It is enough to see the first case, the others are the same. Since $(x \circ \top\!\!\!\top_1^{p,q}) \circ \perp\!\!\!\perp_1^{p,q} = x$ and $x \in X$ is marked, $(x \circ \top\!\!\!\top_1^{p,q}, y \circ \top\!\!\!\top_2^{p,q}) \in X\tilde{\otimes}Y$ is also marked by definition. This defines a natural map $X\tilde{\wedge}Y \to X\tilde{\otimes}Y$. □

Remark 3.10. As we have seen in Remark 2.20, any non-degenerate $(p+q)$-simplex in $\Delta[p] \times \Delta[q]$ can be expressed as a shortest path in the ordered set $[p] \times [q]$. The simplex $(\top\!\!\!\top_1^{p,q}, \top\!\!\!\top_2^{p,q})$ corresponds to the most right-lower path.

We could take another pair of surjections $\theta : [p+q] \to [p]$ and $\tau : [p+q] \to [q]$ and consider the map $X([p]) \times Y([q]) \to (X\tilde{\otimes}Y)_{p+q}$, $(x,y) \mapsto (x \circ \theta, y \circ \tau)$.

But by definition, the simplices $x \circ \theta \circ \amalg_1^{p,q}$ and $y \circ \tau \circ \amalg_2^{p,q}$ are degenerate, hence marked. Thus, $(x \circ \theta, y \circ \tau) \in X \tilde{\otimes} Y$ is always marked.

In this sense, the map we constructed in the proof above may be the only suitable one.

3.2. **A homotopy theory for stable precomplicial sets.** In this section, we introduce a homotopy theory on prestratified stable simplicial sets, which is a straight forward analogue of the structure of a category of fibrant objects on combinatorial spectra given by Brown [1].

Mimicking Kan's suspension functor, we introduce the following functor.

Definition 3.11. The suspension functor $\mathrm{S} : \mathbf{Set}_*^{t\Delta^{op}} \to \mathbf{Set}_*^{t\Delta^{op}}$ is the left Kan extension along yoneda embedding of

$$t\Delta \to \mathbf{Set}_*^{t\Delta^{op}}, \quad [n]_? \mapsto (\Delta[n+1]_?)_+/(\Delta[n]_?)_+ \vee \Delta[0]_+),$$

where $(\text{-})_+$ denotes the functor $\mathbf{Set}^{t\Delta^{op}} \to \mathbf{Set}_*^{t\Delta^{op}}$ adding the base points.

Definition 3.12. (1) A prestratified prespectrum L consists of

(i) a sequence of pointed prestratified simplicial sets L_i with $i \in \mathbb{N}$,

(ii) a sequence of monomorphisms $\lambda_i : \mathrm{S}\, L_i \to L_{i+1}$ of pointed prestratified simplicial sets with $i \in \mathbb{N}$.

(2) A morphism $\psi : \{L_i, \lambda_i\} \to \{M_i, \mu_i\}$ of prestratified prespectra is a sequence of morphisms $\psi_i : L_i \to M_i$ of pointed prestratified simplicial sets such that $\psi_{i+1} \circ \lambda_i = \mu_i \circ \mathrm{S}\, \psi_i$.

(3) A morphism $\psi : \{L_i, \lambda_i\} \to \{M_i, \mu_i\}$ of prestratified prespectra is called a weak equivalence if for every $i \in \mathbb{N}$, ψ_i is a weak equivalence for the pointed Ozornova-Rovelli model structure.

Definition 3.13. Let $X \in \mathbf{Set}_*^{t\Delta^{op}_{st}}$. We define the corresponding prestratified prespectrum $\mathrm{Ps}(X) = \{X_i, \xi_i\}$ as follows.

For any $i \in \mathbb{N}$, the pointed set of n-simplices in X_i is given by

$$X_i([n]) := \{\alpha \in X([n-i]) | d_0 \cdots d_n \alpha = *, d_j \alpha = *, (j > n)\},$$

where $*$ denotes (the degeneracy of) the base point. Similarly, the pointed set of marked n-simplices in X_i is given by

$$X_i([n]_t) := \{\alpha \in X([n-i]_t) | d_0 \cdots d_n \varphi^* \alpha = *, d_j \varphi^* \alpha = *, (j > n)\},$$

The structure morphisms on X_i will be induced by those of X and X_i is indeed a pointed prestratified simplicial set with them.

For any $i \in \mathbb{N}$, the monomorphism $\xi_i : \mathrm{S}\, X_i \to X_{i+1}$ is the obvious inclusion.

Remark 3.14. This defines a functor $\mathrm{Ps} : \mathbf{Set}_*^{t\Delta^{op}_{st}} \to \mathbf{ppSp}$, by letting **ppSp** denote the category of prestratified prespectra.. By construction, for any $i \in \mathbb{N}$ X_i is a stratified simplicial set if X is a stratified stable simplicial set.

Brown has introduced the following notion in the study of shaves valued in simplicial sets.

Definition 3.15 ([1])**.** Let $\mathcal{C}$ be a category with finite products and a final object denoted by $*$. Assume that $\mathcal{C}$ has two distinguished classes of morphisms, called the weak equivalences and the fibrations. A morphism in $\mathcal{C}$ will be called an aspherical fibration if it is both a weak equivalence and a fibration. We define a path object for an object B in $\mathcal{C}$ to be an object B^I together with morphisms

$$B \xrightarrow{s} B^I \xrightarrow{(d_0,d_1)} B \times B,$$

where s is a weak equivalence, (d_0, d_1) is a fibration, and the composite is the diagonal morphism.

We call $\mathcal{C}$ a category of fibrant objects if the following are satisfied

(A) Let f and g be morphisms such that gf is defined. If two of f, g, gf are weak equivalences then so is the third. Any isomorphism is a weak equivalence.

(B) The composite of two fibrations is a fibration. Any isomorphism is a fibration.

(C) For any morphism $A \to C$ and a fibration (resp. aspherical fibration) $B \to C$, the pullback $A \times_C B$ exists and the projection $A \times_C B \to A$ is a fibration (resp. aspherical fibration).

(D) For any object, there exists at least one path object.

(E) For any object B the morphism $B \to *$ is a fibration.

We define weak equivalences and fibrations in $\mathbf{Set}_*^{t\Delta_{st}^{op}}$.

Definition 3.16. A morphism f in $\mathbf{Set}_*^{t\Delta_{st}^{op}}$ is

1. a weak equivalence if for any $i \in \mathbb{N}$ the morphism $\mathrm{Ps}(f)_i$ of pointed prestratified simplicial sets is a weak equivalence for Ozornova-Rovelli model structure,
2. a fibration if for any $i \in \mathbb{N}$ the morphism $\mathrm{Ps}(f)_i$ of pointed prestratified simplicial sets is a fibration for Ozornova-Rovelli model structure.

We call a prestratified stable simplicial set X is fibrant if the unique morphism $X \to *$ is a fibration.

Proposition 3.17. *For any fibrant stratified stable simplicial set B, there exists the diagram of fibrant objects*

$$B \xrightarrow{s} \mathrm{Path}(B) \xrightarrow{(d_0,d_1)} B \times B,$$

where s is a weak equivalence, (d_0, d_1) is a fibration, and the composite is the diagonal morphism.

Proof. Let B be a fibrant stratified stable simplicial set. Then we have a prestratified prespectrum $\mathrm{Ps}(B) = \{B_i, \beta_i\}$, where B_i is a pointed stratified simplicial set for any $i \in \mathbb{N}$. For any $i \in \mathbb{N}$, we have the mapping stratified simplicial set $(B_i)^{\Delta[1]_t}$. There is an evident injection $\mathrm{S}((B_i)^{\Delta[1]_t}) \to (\mathrm{S}(B_i))^{\Delta[1]_t}$. By composing this and the morphism $(\mathrm{S}(B_i))^{\Delta[1]_t} \to (B_{i+1})^{\Delta[1]_t}$ induced by β_i, we obtain $\tilde{\beta}_i : \mathrm{S}((B_i)^{\Delta[1]_t}) \to (B_{i+1})^{\Delta[1]_t}$, which defines a prestratified prespectrum $\{(B_i)^{\Delta[1]_t}, \tilde{\beta}_i\}$.

Since the morphisms $\Delta[0] \to \Delta[1]_t$ are elementary anodyne extensions and B_i is fibrant, the induced morphism $(B_i)^{\Delta[1]_t} \to B_i \times B_i$ is an acyclic fibration.

We put $\mathrm{Path}(B)$ to be the mapping stratified stable simplicial set $B^{\Sigma^\infty_+ \Delta[1]_t}$, where $\Sigma^\infty_+ \Delta[1]_t$ is the stratified stable simplicial set corresponding to $\Delta[1]_t$. Then by construction, we have $\mathrm{Ps}(\mathrm{Path}(B)) = \{(B_i)^{\Delta[1]_t}, \tilde{\beta}_i\}$. The morphism $s : B \to \mathrm{Path}(B)$ corresponds to the constant path and it is a weak equivalence by the two out of three axiom. □

Proposition 3.18. *Let A, B, C be fibrant stratified stable simplicial sets. For any morphism $A \to C$ and a fibration (resp. aspherical fibration) $B \to C$, the pullback $A \times_C B$ is again a fibrant stratified stable simplicial set and the projection $A \times_C B \to A$ is a fibration (resp. aspherical fibration).*

Proof. By the definition of the functor Ps, we have

$$(A \times_C B)_i = A_i \times_{C_i} B_i$$

for any $i \in \mathbb{N}$. Since any fibration (resp. aspherical fibration) in Ozornova-Rovelli model structure is preserved by pullback, $(A \times_C B)_i \to A_i$ is a fibration for any $i \in \mathbb{N}$. This completes the proof. □

Corollary 3.19. *The full subcategory of fibrant stratified stable simplicial sets in $\mathbf{Set}_*^{t\Delta^{op}_{st}}$ admits a structure of a category of fibrant objects with the weak equivalences and fibrations in Definition 3.16.*

Remark 3.20. As is mentioned above, it is shown that the category of combinatorial spectra admits a model structure in [1] by using the cofibrantly generated classical model structure on simplicial sets. It might be possible to obtain the stratified analogue by using the cofibrantly generated model structure on **msSet** given in [10].

Theorem 3.21. *Suppose we have weak equivalences $f : X \to X'$ and $g : Y \to Y'$ of stratified stable simplicial sets. Then the morphism $f \tilde{\wedge} g : X \tilde{\wedge} Y \to X' \tilde{\wedge} Y'$ is also a weak equivalence.*

Proof. By definition, for $i, n \in \mathbb{N}$, we observe that $(X\tilde{\wedge}Y)_i([n]) = \{(x,y) \in (X\tilde{\wedge}Y)([n-i+1]) | d_1 \cdots d_{n+1}(x,y) = *, d_j(x,y) = *(j \leq 0 \text{ or } n+2 \leq j)\}$, where d's are the structure morphisms of $X\tilde{\wedge}Y$. Thus, $(X\tilde{\wedge}Y)_i$ is the join of two pointed stratified simplicial sets. Combining this observation and Corollary 2.26, we obtain the desired result. ☐

Remark 3.22. (1) We do not know whether $\tilde{\wedge}$ is a product on the full subcategory of fibrant stratified stable simplicial sets.

Let $\mathrm{F} : \mathbf{Set}_*^{t\Delta^{op}} \to \mathbf{Set}_*^{t\Delta^{op}}$ denote a fibrant replacement functor with respect to the pointed Ozornova-Rovelli model structure. For any prestratified prespectrum $\{X_i, \xi_i\}$, we have fibrant pointed prestratified set $\mathrm{F}(X_i)$ for all $i \in \mathbb{N}$. However there may not exsist the structure morphism $\mathrm{S} \wedge \mathrm{F}(X_i) \to \mathrm{F}(X_{i+1})$, which exists up to homotopy.

(2) Masuda constructed in [9] a monoidal structure for categorical spectra, which is the stabilaization of (∞,∞)-categories studied in [14], by using Steiner's theory of augmented directed complexes. It would be worth comparing that with our monoidal structure (in ∞-categorical setting).

3.3. **Reedy-like structure.** In this article, we only investigate prestratified stable simplicial objects in the category of (pointed) sets. It would be worth studying such objects in other (∞-)categories equipped with own homotopy theories.

It is well known that Δ is a Reedy category and is shown in [10] $t\Delta$ is also a Reedy category. Moreover, it is shown there that these two categories have better properties, that is to say, they are regular skeletal. In this section, we observe Δ_{st} and $t\Delta_{st}$ also have similar properties.

Definition 3.23 ([4])**.** A Reedy category is a category $\mathcal{R}$ equipped with two subcategories $\mathcal{R}_+$ and $\mathcal{R}_-$, both of which contain all the objects, and a total ordering on the set $\mathrm{ob}(\mathcal{R})$ of objects, defined by a degree function $\deg : \mathrm{ob}(\mathcal{R}) \to \mathbb{N}$ such that:

- Every nonidentity morphism in $\mathcal{R}_+$ raises degree,
- Every nonidentity morphism in $\mathcal{R}_-$ lowers degree, and
- Every morphism f in $\mathcal{R}$ factors uniquely as a morphism in $\mathcal{R}_-$ followed by a morphism in $\mathcal{R}_+$.

As is well known, Δ is a Reedy category with the following structure:

$$\deg : \mathrm{ob}(\Delta) \to \mathbb{N}, \quad [n] \mapsto n.$$

The subcategories Δ_+ and Δ_- consist of injective maps and surjective maps respectively.

Theorem 3.24 ([10])**.** *The category $t\Delta$ is a Reedy category with the following structure: The degree map* $\deg : \mathrm{ob}(t\Delta) \to \mathbb{N}$ *is given by*

$$\deg([0]) = 0,\ \deg([k]) = 2k-1,\ \deg([k]_t) = 2k,\ k \geq 1.$$

The subcategory $t\Delta_+$ is generated by Δ_+ and morphisms $\varphi : [n] \to [n]_t$ for all $n \geq 1$ and $t\Delta_-$ is generated by Δ_- and morphisms $\zeta^i : [n+1]_t \to [n]$ for all $n \geq 1$ and $0 \leq i \leq n$.

Definition 3.25 ([10])**.** A Reedy category $\mathcal{R}$ is regular skeletal if the following conditions hold.

(1) Every morphism in $\mathcal{R}_-$ admits a section.

(2) Two parallel morphisms of $\mathcal{R}_-$ are equal if and only if they admit the same set of sections.

(3) Every morphism of $\mathcal{R}_+$ is a monomorphism.

It is easy to show that Δ is regular skeletal. Furthermore, Ozornova and Rovelli have shown the following.

Theorem 3.26 ([10])**.** *The Reedy category $t\Delta$ is regular skeletal.*

In [10], the fact that $t\Delta$ is a regular skeletal Reedy category with this structure plays a pivotal role. We will show that $t\Delta_{st}$ is *almost* a regular skeletal Reedy category with an analogous structure. To do that, we show that Δ_{st} is also almost regular skeletal.

We consider the following structure on Δ_{st}: $\deg : \mathrm{ob}(\Delta_{st}) \to \mathbb{Z}, [n] \mapsto n$ The subcategories $(\Delta_{st})_-$ (resp. $(\Delta_{st})_+$) is generated by identity morphisms and s^i's (resp. identity morphisms and d^i's).

Lemma 3.27. *(1) For any i and n, the morphism $s^i : [n+1] \to [n]$ in Δ_{st} is an epimorphism.*

(2) For any i and n, the morphism $d^i : [n-1] \to [n]$ in Δ_{st} is an monomorphism.

Proof. We prove (1). (2) can be proven by the same argument. Assume that $\alpha s^i = \beta s^i$ for some morphisms $\alpha, \beta : [n] \to [m]$ in Δ_{st}. Then these morphisms s^i, α, and β are represented respectively by morphisms $s^i : [n+1+k] \to [n+k]$ and $\tilde{\alpha}, \tilde{\beta} : [n+k] \to [m+k]$ for some $k \in \mathbb{N}$ and $\tilde{\alpha} s^i = \tilde{\beta} s^i$ holds. Since s^i in Δ is an epimorphism, $\tilde{\alpha} = \tilde{\beta}$ in Δ. This shows that $\alpha = \beta$ in Δ_{st}. □

Lemma 3.28. *Every morphism in Δ_{st} factors uniquely as a morphism in $(\Delta_{st})_-$ followed by a mophism in $(\Delta_{st})_+$.*

Proof. Recall that every morphism $f : [i] \to [j]$ in Δ is uniquely decomposed as $g \circ h : [i] \to [im(f) - 1] \to [j]$ with g a monomorphism and h an epimorphism, where $im(f)$ denotes the cardinarity of the image of f.

Let $\theta : [m] \to [n]$ be a morphism in Δ_{st}. Then there exists a morphism $\tilde{\theta} : [m+k] \to [n+k]$ in Δ with a natural number $k \in \mathbb{N}$, which represents θ. The morphism $\tilde{\theta}$ in Δ can be written uniquely as $\tilde{\alpha} \circ \tilde{\beta}$ with $\tilde{\alpha}$ a monomorphism and $\tilde{\beta}$ an epimorphism in Δ. Then the morphisms α and β in Δ_{st} represented by $\tilde{\alpha}$ and $\tilde{\beta}$ respectively are monomorphism and epimorphism respectively by the lemma above, and we have $\theta = \alpha \circ \beta$. □

Expect for the fact that the codomain of the degree map is $\mathbb{Z}$, Δ_{st} satisfies the all requirements for being a regular skeletal Reedy category.

Lemma 3.29. *For each $n \in \mathbb{Z}$, the morphism $\varphi : [n] \to [n]_t$ in $t\Delta_{st}$ is a monomorphism and an epimorphism.*

Proof. This follows from that $\varphi : [n] \to [n]_t$ in $t\Delta$ is a monomorphism and an epimorphism proven in [10].

□

We consider the following structure on $t\Delta_{st}$: A map $\deg : \mathrm{ob}(t\Delta) \to \mathbb{Z}$ given by

$$\deg([0]) = 0, \deg([k]) = 2k - 1, \deg([k]_t) = 2k$$

for $k \neq 0$. The subcategory $(t\Delta_{st})_+$ is generated by $(\Delta_{st})_+$ and morphisms $\varphi : [n] \to [n]_t$ for all $n \geq 1$ and $t\Delta_-$ is generated by Δ_- and morphisms $\zeta^i : [n+1]_t \to [n]$ for all $n \geq 1$ and $0 \leq i \leq n$.

The arguments in [10, Proposition C.4] and above show the following.

Proposition 3.30. *$t\Delta_{st}$ satisfy the conditions (1), (2) and (3) in Definition 3.25.*

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