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WRONSKIAN THEORY REVISITED FROM A LINEAR ALGEBRAIC AND NULL SPACE VIEWPOINT

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Abstract. This paper aims at providing a new fundamental framework for the Wronskian theory by paying attention to the null space of the Wronski matrix. It is first shown that identical vanishing of the Wronskian leads to two different representations of vector-valued functions that lie in the null space of the Wronski matrix. It is further shown that they are algebraically linearly dependent, by which we are immediately led to a key identity in the Wronskian theory, derived only through very simple linear algebraic arguments. Most fundamental results on the Wronskian theory available in the literature can thus be obtained in a quite straightforward fashion with a very clear perspective, and hence the relevant arguments are believed to be of pedagogical value, too. Some further issues relevant to the null space viewpoint are also discussed, where some other results available in the literature are derived through the null space viewpoint in a somewhat strengthened form.

1. Introduction

This paper aims at revisiting the Wronskian theory from what we call a linear algebraic and null space viewpoint.

The Wronski matrix is well known to be quite important in deciding linear dependence/independence of given functions of a single real variable. More precisely, its determinant called the Wronskian is considered, and if it does not vanish identically on the domain of the given functions, they are ensured to be linearly independent. Although the converse is not true, in general, some conditions are known that together with identical vanishing of the Wronskian ensure their linear dependence. In connection with the studies on the converse, various properties have been obtained for the Wronskian. These results can be found, e.g., in [1, 2, 3] in the handbook/textbook level, in [4, 5, 6] in the recent expository articles level, and in [7, 8, 9] in the historical pioneering studies.

To the best understanding of the author, however, the literature in the Wronskian theory deals mostly with the Wronskian rather than the underlying Wronski matrix itself. The author is led to this interpretation in the sense that even when some rank properties of the Wronski or some relevant

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matrices are dealt with, little attention is paid on the vectors in the associated null space, and the fundamental tools in developing the Wronskian theory are essentially those fundamental results in calculus such as Rolle's theorem.

This paper aims at providing a new fundamental framework that focuses precisely on the null space of the Wronski matrix, with which much of the fundamental tools in the Wronskian theory is shifted to those in linear algebra. The most important feature of the present paper is to show that identical vanishing of the Wronskian leads to two different representations of vector-valued functions that, for each fixed value of the real variable, lie in the null space of the Wronski matrix evaluated at the same value of the variable. Such vector-valued functions are constructed through the determinants of some submatrices of the Wronski matrix, i.e., some minors. The mere identical vanishing of the Wronskian, however, does not seem natural to ensure the existence of two algebraically linearly independent vectors in the null space. Thus we will be interested in showing algebraic linear dependence of the two representations of vectors, and this will be established only through linear algebraic arguments. This alignment of the two representations immediately leads to an alternative identity relevant to the Wronski matrix whose determinant vanishes identically. The arguments up to this point are actually quite simple and essentially use only linear algebraic arguments, while the resulting alignment property is crucial in the arguments of the pioneering study [8] as well as the present paper. Hence, the linear algebraic and null space viewpoint developed in the present paper can be interpreted as providing very transparent and systematic treatment for the core of the Wronskian theory. As such, the arguments leading to the framework of the present paper is believed to be of pedagogical value, too.

The organization of the present paper is as follows. Section 2 describes the motivation and standpoint of the present paper after reviewing some fundamentals of the Wronskian theory. In particular, we state our key theorem (Theorem 2.3) to highlight the feature of our null space viewpoint. What is important, however, is not the mere statement of this theorem itself, but the approach and viewpoint themselves leading to this key theorem, in which attention is paid precisely on the null space of the Wronski matrix. Remarks relevant to this important viewpoint are provided and the feature of the present paper is further discussed in that section. Then, Section 3 proceeds to the proof of this key theorem after providing some preliminary fundamental linear algebraic results that play key roles in the proof. Derivation of two different representations of vector-valued functions in the null space of the Wronski matrix is one of such fundamental results. Another crucial result includes a fundamental formula about the derivative of the Wronskian. Fundamental results of the Wronskian theory for the case of two functions are also reviewed at the end of the section. They constitute a basis for completing our framework of the linear algebraic and null space viewpoint developed in Section 4, which is attained by providing the proof of the key theorem (Theorem 2.3) on the basis of the aforementioned fundamental linear algebraic results provided in Section 3. In Section 5, we further discuss relevant issues in the Wronskian theory. Subsection 5.1 shows that some well-known result for deciding linear dependence of functions can also be obtained immediately through the null space viewpoint, and further gives a simple helpful observation that follows readily from such a viewpoint. On the other hand, Subsection 5.2 is relevant to another well-known result about the Wronskian that is identically zero. More precisely, the interest there mainly lies in whether the situation could be changed if one more function is introduced to consider the associated Wronskian. We first show that the well-known result can also be derived through the null space approach developed in the present paper. We then suggest relevant interesting questions to continue our discussions. In the course of such a study, we further derive some relevant fundamental results in the literature in a strengthened form through our null space approach. Furthermore, we discuss some sort of relationship between the null space of the original Wronski matrix and that of a larger Wronski matrix for one more function. We close our paper in Section 6 by giving some concluding remarks.

2. Motivation and Standpoint of the Present Paper

Let $\mathcal{I} \subset \mathbb{R}$ be an interval and suppose that the function $f_i : \mathcal{I} \to \mathbb{R}$ be given for $i = 1, \ldots, n$. For notational simplicity, the set of these functions is denoted by $\mathcal{F}_n(\mathcal{I})$. Note, however, that the underlying n is fixed throughout the paper unless stated otherwise, and thus the relevant subscripts and/or superscripts are sometimes dropped unless some confusion could arise. The functions in $\mathcal{F}_n(\mathcal{I})$ are said to be linearly independent on $\mathcal{I}$ (or simply, $\mathcal{F}_n(\mathcal{I})$ is linearly independent) if the identity

$$\sum_{i=1}^{n} v_i f_i(t) \equiv 0 \tag{2.1}$$

on $\mathcal{I}$ with $v_i \in \mathbb{R}$ $(i = 1, \ldots, n)$ implies that $v_i = 0$ $(i = 1, \ldots, n)$; otherwise, they are said to be linearly dependent on $\mathcal{I}$ (or simply, $\mathcal{F}_n(\mathcal{I})$ is linearly dependent). To facilitate the arguments, we sometimes refer to t as time.

When all the functions in $\mathcal{F}_n(\mathcal{I})$ are $n-1$ times differentiable on $\mathcal{I}$, which is the standing assumption throughout the paper, it is well known that the

Wronski matrix defined on $\mathcal{I}$ as

$$W(t) = \begin{bmatrix} f_1(t) & f_2(t) & \dots & f_n(t) \\ f_1^{(1)}(t) & f_2^{(1)}(t) & \dots & f_n^{(1)}(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \dots & f_n^{(n-1)}(t) \end{bmatrix} \tag{2.2}$$

plays an important role in deciding linear independence of $\mathcal{F}_n(\mathcal{I})$ (see, e.g., [1, 2, 3]), where $f_i^{(k)}$ denotes the kth order derivative of f_i. More precisely, the following result is well known for the determinant of the Wronski matrix $W(t)$ denoted

$$w(t) := \det W(t) \tag{2.3}$$

and called the Wronski determinant or the Wronskian for $\mathcal{F}(\mathcal{I})$.

Proposition 2.1. *$\mathcal{F}(\mathcal{I})$ is linearly independent if $w(t) \not\equiv 0$ on $\mathcal{I}$.*

This result obviously motivates the study on the converse assertion, and very interestingly, quite famous mathematicians such as Hermite and Jordan have also tackled the issue to assert (even "prove") that the converse also holds; the historical side of the advances on the study about the converse assertion is elaborated in [5], and according to this reference as well as [4] and [3], it was Peano [7] who pointed out their wrong assertion for the first time through the following counterexample.

Example 2.1. *Suppose that $n = 2$, $\mathcal{I} = \mathbb{R}$, $f_1(t) = t^2$ and $f_2(t) = t|t|$. Then, $f_1^{(1)}(t) = 2t$ and $f_2^{(1)}(t) = 2|t|$, so that we have*

$$W(t) = \begin{bmatrix} t^2 & t|t| \\ 2t & 2|t| \end{bmatrix}, \quad w(t) = \det W(t) \equiv 0 \tag{2.4}$$

on $\mathcal{I}$. However, it is easy to confirm that $f_1(t)$ and $f_2(t)$ are linearly independent on $\mathcal{I}$.

It is surprising that the importance of this counterexample of Peano was not immediately appreciated, but it definitely played an important role in advancing the study on some form of a converse assertion of Proposition 2.1; more precisely, the counterexample suggested the importance of introducing some additional condition that, together with the condition $w(t) \equiv 0$, would ensure that $\mathcal{F}(\mathcal{I})$ is linearly dependent.

It is Bôcher who gave such an additional condition for the first time in [8] in the form of the following Proposition 2.2, who subsequently gave a further generalized additional condition in [9], where the latter covers the additional (analyticity) condition given in Proposition 2.2 as a special case.

Proposition 2.2. *If $\mathcal{F}(\mathcal{I})$ consists of analytic functions, it is linearly independent if and only if $w(t) \not\equiv 0$ on $\mathcal{I}$.*

The present paper is motivated by these studies by Bôcher, but aims at developing much more transparent arguments that give a clear perspective leading to the following key theorem about a linear algebraic property of the Wronski matrix $W(t)$.

Theorem 2.3. *If the Wronskian for $\mathcal{F}_n(\mathcal{I})$ satisfies $w(t) \equiv 0$ on $\mathcal{I}$, then there exist a subinterval $\mathcal{I}_0$ of $\mathcal{I}$ and a nonzero $v \in \mathbb{R}^n$ such that*

$$W(t)v \equiv 0 \tag{2.5}$$

on $\mathcal{I}_0$.

Even though the statement itself of the above theorem could be regarded merely as a restatement of a well-known result (used in the derivation of Proposition 2.2), we would like to reiterate that we aim at developing a new linear algebraic viewpoint/framework leading to this assertion through a clear perspective. In this sense, even though we claimed Theorem 2.3 as a key theorem of this paper, we would like to stress that the most important feature of the present paper is a unique and transparent derivation process of this key theorem, rather than its statement itself, where the process also clarifies a construction procedure for an associated $\mathcal{I}_0$ and v.

With this in mind, we start from some obvious fact and then proceed to subsequently state important suggestive remarks so that the standpoint and feature of the present paper can be clarified further and the significant viewpoint of the present paper (which we call an algebraic and null space viewpoint) can be highlighted, totally distinguishing the direction of the present approach from that in [8] and [9] and other related literature.

- (i) First note the obvious relation that (2.1) holds on $\mathcal{I}$ if and only if (2.5) holds on $\mathcal{I}$ for $v = [v_1, \ldots, v_n]^T$.
- (ii-a) Hence, there exists a nonzero $v \in \mathbb{R}^n$ satisfying (2.5) on $\mathcal{I}$ if and only if $\mathcal{F}_n(\mathcal{I})$ is linearly dependent.
- (ii-b) Furthermore, the set of $v = [v_1, \ldots, v_n]^T$ satisfying (2.1) on $\mathcal{I}$ coincides with that of v satisfying (2.5) on $\mathcal{I}$.
- (iii-a) By (ii-a), Theorem 2.3 can be restated as

  Theorem 2.3' If $w(t) \equiv 0$ on $\mathcal{I}$ for $\mathcal{F}_n(\mathcal{I})$, there exists a subinterval $\mathcal{I}_0$ of $\mathcal{I}$ such that $\mathcal{F}_n(\mathcal{I}_0)$ is linearly dependent.

  and this is nothing but what has already been shown in [9] or [4]. If we were solely interested in the assertion of Theorem 2.3' itself, then the arguments in [4] would be the simplest.

In view of this observation, we continue giving important suggestive remarks on the standpoint and feature of the approach developed in the present paper. They are related to the fact that the present approach leads to a useful framework of arguments that can derive further crucial results from a linear algebraic and null space viewpoint. We further discuss such an aspect in the following.

(iii-b) The observation (ii-b) would be interesting, since it is obvious that the viewpoint about (2.5) is related to a "time-invariant null space" of the Wronski matrix $W(t)$ (rather than just looking at its determinant $w(t)$). As such, it would be worth noting that the null space of the Wronski matrix $W(t)$ for the example of Peano is time-varying on $\mathcal{I} = \mathbb{R}$; more precisely, what changes in t lies not only in the direction of the null space but also in the dimension of the null space (which depends on whether $t \neq 0$ or $t = 0$). The present approach is considered to be closely related to such changes in its ultimate root, but no existing studies focus on such a (time-invariant) null space aspect, to the best knowledge of the author. In fact., although [8, 9, 4, 6] refer to Proposition 2.2, where [8] derived it for the first time while the other three references also gave independent derivations of the same result, none of them provides a null space viewpoint as in Theorem 2.3. Through such a linear algebraic and null space viewpoint developed in the present paper, a number of existing results can actually be interpreted from a new perspective, and new arguments and insight are further provided in Section 5.

(iv) With respect to the above novel null space viewpoint for the Wronski matrix $W(t)$ in the present paper, the derivation of Theorem 2.3 will be carried out, roughly speaking, by finding two different representations of time-varying vectors in the null space, assuming that $W(t)$ always has a zero eigenvalue (i.e., $w(t) \equiv 0$). More precisely, these two time-varying vectors in the null space are further shown to be algebraically linearly dependent (i.e., always have the same direction for each $t \in \mathcal{I}$), and this in turn leads to the consequence that each of the two time-varying vector representations in the null space is actually a constant vector in $\mathbb{R}^n$ multiplied by a scalar-valued function on some subinterval $\mathcal{I}_0$ of $\mathcal{I}$.

(v) The arguments of the present paper sketched briefly in the above (iv) have a somewhat close connection with the arguments of [8] (and [9]) in the treatment of determinants, but are entirely different from the arguments available in the literature, because the present paper develops a more sophisticated linear algebraic and null space viewpoint for transparent and systematic treatment. Simply put, the present paper aims at developing a linear algebraic approach to the Wronskian theory instead of the conventional calculus approach. This is particularly true for the arguments up to Subsection 5.1, in the sense that Rolle's theorem playing a crucial role in the existing studies is not used (until the latter part of Subsection 5.2, which is devoted to more advanced issues in the Wronskian motivated by the null space viewpoint).

Having stated the above important remarks on the standpoint and novel feature of the arguments and approach in the present paper, which we hope is interesting enough to motivate the arguments in the following sections, we close this section by confirming a significant implication of Theorem 2.3; the following corollary is its immediate consequence if we note (ii-a) above and the identity theorem on analytic functions (which implies that each analytic function in $W(t)v$ defined on $\mathcal{I}$ is identically zero on $\mathcal{I}_0$ if and only if it is on $\mathcal{I}$).

Corollary 2.4. *Suppose that $\mathcal{F}(\mathcal{I})$ consists of analytic functions on $\mathcal{I}$. If $w(t) \equiv 0$ on $\mathcal{I}$, then there exists a nonzero $v \in \mathbb{R}^n$ such that (2.5) holds on $\mathcal{I}$. In other words, $\mathcal{F}(\mathcal{I})$ is linearly dependent.*

Furthermore, the contrapositive restatement of this corollary together with Proposition 2.1 immediately leads to the aforementioned pioneering result of Bôcher, i.e., Proposition 2.2 stated earlier.

3. Preliminaries

3.1. **Another Motivating Theorem.** Since the theory of the Wronskian is trivial when $n = 1$, this paper assumes, in principle, that $n \geq 2$. The purpose of this subsection is to introduce another key motivating theorem given shortly (see Theorem 3.1) that motivates the overall arguments of this paper; in particular, the relevant arguments to be motivated by Theorem 3.1 eventually lead to a transparent and systematic derivation process of our key theorem, i.e., Theorem 2.3. This second key theorem introduced in this section, which can be interpreted as lying behind the first key theorem, implies that if the Wronskian satisfies $w(t) \equiv 0$ on $\mathcal{I}$, then the Wronski matrix $W(t)$ has two different representations of time-varying vectors always contained in its null space. More precisely, each of the two vectors evaluated at $t = t_0 \in \mathcal{I}$ is contained in the null space of the singular matrix $W(t_0)$.

To precisely state the second key theorem, however, we begin by introducing relevant notions and terms.

The arguments of this paper are based on the repeated treatment of submatrices and their determinants (or minors) of a given matrix. Regarding such a situation, no established fitted terms exist, to the best knowledge of the author, that are convenient and non-confusing enough for the development of the arguments in this paper. For example, when we refer to a minor, it could be confusing with a cofactor (which could differ from each other only in their signs). Furthermore, when the (i, j) minor is referred to, how to call the associated submatrix (whose determinant gives the (i, j) minor) is not well-established. We would like to have some fitted term to call it, where it is desirable that the term is close enough to the associated term for calling its determinant so that their mutual connection is clearly and strongly suggested by the proximity itself of their terms. With this in mind, we introduce the following terms for convenience.

Definition 1. For $X = (x_{ij}) \in \mathbb{R}^{n\times n}$ with $n \geq 2$, the submatrix obtained by removing the ith row and jth column of X is called the remainder matrix of X with respect to x_{ij} (or the (i, j) remainder matrix of X) and denoted by $X^{i,j} \in \mathbb{R}^{(n-1)\times(n-1)}$. The determinant of $X^{i,j}$ is called the (i, j) remainder determinant of X and denoted by $x^{i,j}$. For $\rho_{ij} := x^{i,j}$, the matrix $R(X) := (\rho_{ij}) \in \mathbb{R}^{n\times n}$ is called the remainder determinant matrix of X.

With this notation, we define for each $i = 1, \ldots, n$ the remainder matrix of the Wronski matrix $W(t)$ with respect to $f_i^{(n-1)}(t)$, which we denote by

$$M_i(t) := W(t)^{n,i}, \quad i = 1, \ldots, n \tag{3.1}$$

Note that $M_i(t)$ is nothing but the Wronski matrix for the set of $n - 1$ functions $\mathcal{F}_n(\mathcal{I})\backslash\{f_i\}$. Furthermore, we also define the associated remainder determinant (or the Wronskian of the corresponding set of $n - 1$ functions) denoted by

$$m_i(t) := \det M_i(t), \quad i = 1, \ldots, n \tag{3.2}$$

together with

$$\sigma_i := (-1)^{n+i}, \quad i = 1, \ldots, n \tag{3.3}$$

It readily follows from the expansion of the determinant of $W(t)$ along its last row that

$$w(t) = \sigma_1 f_1^{(n-1)}(t)m_1(t) + \cdots + \sigma_n f_n^{(n-1)}(t)m_n(t) \tag{3.4}$$

We are in a position to state the following theorem, whose proof will be given in Subsection 3.3.

Theorem 3.1. *Let $n \geq 2$ and $m_\sigma : \mathcal{I} \to \mathbb{R}^n$ be*

$$m_\sigma(t) = \begin{bmatrix} \sigma_1 m_1(t) & \ldots & \sigma_n m_n(t) \end{bmatrix}^T \tag{3.5}$$

and let m'_σ be its derivative on $\mathcal{I}$. If $w(t) \equiv 0$ on $\mathcal{I}$, it follows on $\mathcal{I}$ that

$$W(t)m_\sigma(t) \equiv 0 \tag{3.6}$$

$$W(t)m'_\sigma(t) \equiv 0 \tag{3.7}$$

Remark 1. Note that the identity corresponding to the last entry in (3.6) is nothing but (3.4), since $w(t) \equiv 0$ by the assumption. Furthermore, (3.7) except for its last entry, together with (3.6) has been used in the pioneering studies [8, 9]. However, no attention has been paid on the last entry of (3.7)[1]. Roughly speaking, (3.6) and (3.7) can be interpreted as showing that when $w(t) \equiv 0$, the Wronski matrix $W(t)$ has the "two time-varying eigenvectors" $m_\sigma(t)$ and $m'_\sigma(t)$ corresponding to the zero eigenvalue of $W(t)$. However, it is unlikely that $W(t)$ has two such algebraically linearly independent vectors simply because $w(t) \equiv 0$, and thus we will be motivated to show that they are actually algebraically linearly dependent (i.e., have the same direction for each $t \in \mathcal{I}$). What plays the key role in the overall arguments in the present paper is only the relevant arguments for establishing this algebraic alignment assertion as well as (3.6). In other words, (3.7) itself will not actually be used directly in the subsequent arguments once the above theorem is established and the above alignment assertion is also suggested and then established.

Remark 2. As another important feature of the present paper, it could be seen from the proof of this theorem given later that our derivation of (3.6) and (the first $n-1$ identities of) (3.7) is more straightforward than that in [8]. For the relevance with the advanced arguments in Section 5, we remark just in case that the first $n-1$ identities in (3.6) as well as the $n-1$ identities except the last but one in (3.7) hold even without the assumption $w(t) \equiv 0$. Furthermore, the remaining identities in (3.6) and (3.7) fail if $w(t) \not\equiv 0$, but still the relevant relations

$$W(t_0)m_\sigma(t_0) = 0, \quad W(t_0)m'_\sigma(t_0) = 0 \tag{3.8}$$

hold for every t_0 such that $w(t_0) = 0$, as seen from the proof. This plays an important role in the observation in Remark 6 in Subsection 5.2.

Remark 3. A further crucial feature of the present paper can be explained as follows. In the arguments of the pioneering study [8], the key issue is to derive on $\mathcal{I}$ the relation

$$m_n(t)m'_i(t) - m_i(t)m'_n(t) \equiv 0, \quad i = 1, \dots, n-1 \tag{3.9}$$

(which we also derive in the following section; see Lemma 4.2). The derivation of this key relation in the arguments of [8], however, is quite involved and actually rather hard to follow. In contrast, our arguments can be interpreted as replacing its derivation with proving that the two vectors $m_\sigma(t)$

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[1] The reason seems to be either of the following: (i) justifying this last identity was thought to require n times differentiability of the functions in $\mathcal{F}_n(\mathcal{I})$; (ii) this identity was thought to be useless.

and $m'_\sigma(t)$ are algebraically linearly dependent for each $t \in \mathcal{I}$. More precisely, an important property of the derivative $m'_i(t)$ of the (n,i) remainder determinant $m_i(t)$ of the Wronski matrix $W(t)$ is first shown: $m'_i(t)$ coincides with another (i.e., $(n-1,i)$) remainder determinant of $W(t)$. In addition, another important property between the singularity of a square matrix X and a rank property of the associated remainder determinant matrix $R(X)$ consisting of the remainder determinants of X is shown. As it turns out, combining these properties immediately leads to the key relation in the pioneering study [8], i.e., (3.9), in a quite elementary and transparent fashion in this paper. These features are the most significant advantages of the arguments of this paper. Further advanced issues are studied in Section 5 along the line of such features.

3.2. **Preliminary Results on Determinants and Remainder Determinant Matrices.** This subsection is devoted to providing some fundamental results used in the arguments of this paper. Eq. (3.11) in Lemma 3.2 below plays the most crucial role throughout the paper (for example, in the derivation of (3.7) and the proof of Lemma 4.1 given later, which are crucial results in this paper), while (3.10) and Lemma 3.3 also play a crucial role in the treatment of remainder determinant matrices.

Lemma 3.2. *Let $G : \mathcal{I} \to \mathbb{R}^{l\times l}$ and $G(t) =: [\underline{g}_1(t)^T, \ldots, \underline{g}_l(t)^T]^T$, where $\underline{g}_1(t) = \sum_{i=1}^m c_i \underline{g}_{1,i}(t)$ with $c_i \in \mathbb{R}$ $(i = 1, \ldots, m)$. Then, $g(t) = \det G(t)$ satisfies the following.*

$$g(t) = \sum_{i=1}^{m} c_i \det \begin{bmatrix} \underline{g}_{1,i}(t) \\ \underline{g}_2(t) \\ \vdots \\ \underline{g}_l(t) \end{bmatrix} \tag{3.10}$$

$$g'(t) = \det \begin{bmatrix} \underline{g}'_1(t) \\ \underline{g}_2(t) \\ \vdots \\ \underline{g}_l(t) \end{bmatrix} + \det \begin{bmatrix} \underline{g}_1(t) \\ \underline{g}'_2(t) \\ \vdots \\ \underline{g}_l(t) \end{bmatrix} + \cdots + \det \begin{bmatrix} \underline{g}_1(t) \\ \underline{g}_2(t) \\ \vdots \\ \underline{g}'_l(t) \end{bmatrix} \tag{3.11}$$

Proof. The assertions follow immediately from the definition of the determinant

$$\det G(t) = \sum_{\pi} \operatorname{sgn}(\pi) \prod_{k=1}^{l} g_{k\pi(k)}(t) \tag{3.12}$$

where $g_{ij}(t)$ denotes the (i, j) entry of $G(t)$, while π denotes a permutation on the set $\{1, \ldots, l\}$, $\mathrm{sgn}(\pi)$ denotes its sign, and the summation is taken over the set of all permutations. □

Lemma 3.3. *Let* $X \in \mathbb{R}^{n \times n}$ $(n \geq 2)$ *and consider its remainder determinant matrix* $R(X) \in \mathbb{R}^{n \times n}$. *Let* R_k *be a submatrix of* $R(X)$ *consisting of* k *different rows of* $R(X)$, *where* $2 \leq k \leq n$. *If* $\det X = 0$, *then* R_k *always fails to be of full row rank. Conversely, if* $\det X \neq 0$, *then* $R(X)$ *is nonsingular.*

Proof. For the first assertion, it suffices to give the proof for $k = 2$, and without loss of generality, we may assume that the two rows taken from $R(X)$ are its first and second rows, because singularity of X is invariant under the permutations of the rows of X. Let $\underline{x}_i$ $(i = 1, \ldots, n)$ be the ith row of X. If the $n-1$ rows $\underline{x}_i$ $(i = 2, \ldots, n)$ are linearly dependent, then the first row of $R(X)$ (and thus R_k) is zero. Hence, R_k fails to be of full row rank. Otherwise if these $n-1$ rows are linearly independent, the assumption $\det X = 0$ implies that the first row $\underline{x}_1$ can be represented as a linear combination of the remaining $n-1$ rows, i.e.,

$$\underline{x}_1 = \sum_{i=2}^{n} c_i \underline{x}_i \tag{3.13}$$

with $c_i \in \mathbb{R}$ $(i = 2, \ldots, n)$. Hence, it readily follows from (3.10) that

$$\underline{r}_2(X) = c_2 \underline{r}_1(X) \tag{3.14}$$

where $\underline{r}_i(X)$ denotes the ith row of $R(X)$. Hence, R_k (consisting of the first and second rows of $R(X)$) fails to be of full row rank.

For the second assertion, note that $DR(X)^T D = \mathrm{adj}(X)$ by the definitions of the remainder determinant matrix $R(X)$ and the adjugate matrix $\mathrm{adj}(X)$ for X, where $D = \mathrm{diag}[1, -1, 1, -1, \ldots] \in \mathbb{R}^{n \times n}$. Since $\det X \neq 0$ implies that $\mathrm{adj}(X) = (\det X) X^{-1}$ is nonsingular, so is $R(X)$. □

Remark 4. The above lemma is closely related to a special case of the results in [10], but the above proof is much more straightforward.

We also give the following result just for reference, which is somewhat relevant to our subsequent arguments.

Lemma 3.4. *Let* $X \in \mathbb{R}^{n \times n}$ $(n \geq 2)$ *and suppose that* $\det X = 0$. *Then, the remainder determinant matrix* $R(X)$ *is nonzero if and only if* $\dim(\ker(X)) = 1$.

Proof. Suppose that $\dim(\ker(X)) \geq 2$. Then, X has $n-2$ linearly independent rows at most. Hence, whatever $n-1$ rows are taken from X, they are linearly dependent, and thus the matrix $R(X)$ becomes zero. This completes the necessity proof. Conversely, if $\dim(\ker(X)) = 1$, then there exists

i such that X with the ith row removed leads to a matrix with full row rank. Hence, there exists j such that the remainder determinant $x^{i,j}$ is nonzero for some j. This completes the sufficiency proof. □

3.3. **Proof of Theorem 3.1.** This subsection is devoted to the proof of Theorem 3.1, our second key theorem, where we denote the Wronski matrix $W(t)$ as follows:

$$W(t) = \begin{bmatrix} \underline{f}_0(t) \\ \vdots \\ \underline{f}_{n-1}(t) \end{bmatrix} \tag{3.15}$$

By the hypothesis of Theorem 3.1, we have $\det W(t) \equiv 0$ on $\mathcal{I}$. Letting $\underline{h}(t) := \underline{f}_{n-1}(t)$ and expanding this determinant along the last row of $W(t)$, i.e., $\underline{h}(t)$, we readily see that

$$\sigma_1 h_1(t) m_1(t) + \cdots + \sigma_n h_n(t) m_n(t) \equiv 0 \tag{3.16}$$

where $\underline{h}(t) =: [h_1(t), \ldots, h_n(t)]$; an equivalent alternative representation is

$$\underline{h}(t) m_\sigma(t) \equiv 0 \tag{3.17}$$

We next take $\underline{h}(t) = \underline{f}_k(t)$ and consider $W(t)$ with the last row replaced by $\underline{h}(t)$, which is obviously singular for each t and every $k = 0, \ldots, n-2$ because there are two identical rows. Hence, expanding its determinant along the last row leads to (3.17) also with $\underline{h}(t) = \underline{f}_k(t)$ for $k = 0, \ldots, n-2$. Hence, we have (3.6) as claimed.

Next, we first consider how $m_i'(t)$ can be represented. Since $m_i(t)$ is the determinant of the Wronski matrix for $\mathcal{F}_n(\mathcal{I}) \backslash \{f_i\}$, we can apply Lemma 3.2 to describe $m_i'(t)$, where only one term can be nonzero in (3.11) (which is actually the last term) because the determinant of a matrix with two identical rows is zero (the same observation is found also in [9]). That is, if $G(t)$ is the Wronski matrix given by

$$G(t) = \begin{bmatrix} \underline{\gamma}_0(t) \\ \vdots \\ \underline{\gamma}_{l-1}(t) \end{bmatrix} \tag{3.18}$$

where $\underline{\gamma}_k(t)$ is the kth order derivative of $\underline{\gamma}_0(t)$ for $k = 1, \ldots, l-1$, then

$$(\det G(t))' = \det \begin{bmatrix} \underline{\gamma}_0(t) \\ \vdots \\ \underline{\gamma}_{l-2}(t) \\ \underline{\gamma}_l(t) \end{bmatrix} \tag{3.19}$$

With this in mind, let us take $\underline{h}(t) := \underline{f}_{n-2}(t)$ and consider expanding $\det W(t)$ ($\equiv 0$) along the $(n-1)$st row of $W(t)$ (i.e., $\underline{h}(t)$). Then, we immediately see that

$$\underline{h}(t) m'_\sigma(t) \equiv 0 \tag{3.20}$$

This is true also for $\underline{h}(t) = \underline{f}_k(t)$ ($k = 0, \ldots, n-3$) and $\underline{h}(t) = \underline{f}_{n-1}(t)$; this can easily be seen by considering the expansion of the determinant of $W(t)$ with the $(n-1)$st row replaced by $\underline{h}(t)$, which is nonsingular as a matrix with two identical rows. Hence, we have (3.7) as claimed.

3.4. **Preliminary Basic Results for $n = 2$.** To facilitate the following arguments by laying key preliminaries, this subsection is devoted to reviewing some basic results of the Wronskian theory for the case $n = 2$ available in the literature. The following result in [8] for $n = 2$ plays a crucial role also in the present paper, which follows readily from $(d/dt)(f_2/f_1) = w(t)/f_1^2$.

Lemma 3.5. *Let $n = 2$ and suppose that $f_1(t) \neq 0$, $\forall t \in \mathcal{I}$. If $w(t) \equiv 0$ on $\mathcal{I}$, then there exists $c_0 \in \mathbb{R}$ such that $f_2(t) = c_0 f_1(t)$, $\forall t \in \mathcal{I}$.*

The following result is crucial in the arguments of [9] and [4], which is also the case in the present paper. The proof is given just for the sake of making the paper self-contained (by which the assertion of Theorem 2.3 for $n = 2$ is implied).

Lemma 3.6. *Let $n = 2$. If $w(t) \equiv 0$ on $\mathcal{I}$, then there exists a subinterval $\mathcal{I}_0$ of $\mathcal{I}$ such that $\mathcal{F}_2(\mathcal{I}_0)$ is linearly dependent. In particular, if $f_1(t) \not\equiv 0$ on $\mathcal{I}$, then one such $\mathcal{I}_0$ is any subinterval of $\mathcal{I}$ on which f_1 never vanishes; in that case, there exists $c_0 \in \mathbb{R}$ such that $f_2(t) = c_0 f_1(t)$, $\forall t \in \mathcal{I}_0$. Otherwise, if $f_1(t) \equiv 0$, then we can take $\mathcal{I}_0 = \mathcal{I}$.*

Proof. When $f_1(t) \equiv 0$ on $\mathcal{I}$, the assertion is obvious since $1 \cdot f_1(t) + 0 \cdot f_2(t) \equiv 0$ on $\mathcal{I}$, while if $f_1(t) \not\equiv 0$ on $\mathcal{I}$, the assertion is an immediate consequence of the continuity of $f_1(t)$ together with Lemma 3.5. □

4. Proof of Theorem 2.3

This section is devoted to showing in what a transparent and systematic fashion, the proof of the first key theorem (Theorem 2.3) can proceed on the basis of Theorem 3.1 just established. The most essential point for the proof is the derivation of the identity (3.9), as is the case with the arguments of the pioneering study [8] in the proof of Proposition 2.2. The derivation in [8] for the target identity is based on the assertion that taking an appropriate linear combination of the $n-1$ identities in (3.7) except the last one leads to the desired result. Explicit coefficients for the linear combination are also stated in [8], which we could restate as follows in the terms introduced in the present paper:

> Consider the right-bottom entry $f_n^{(n-1)}(t)$ of the Wronski matrix $W(t)$ and the associated remainder matrix $M_n(t)$. Then, the set of the coefficients should be taken as the ith row of its cofactor matrix (and thus each coefficient is equal to the remainder determinant with respect to an entry in the ith row of the remainder matrix $M_n(t)$, except for its sign).

Unfortunately, however, no detailed reason is stated why the linear combination with the coefficients constructed in this fashion indeed leads to the ith target identity of (3.9). In other words, what should be quite informative arguments on the explicit algebraic structure behind the construction of the coefficients and derivation of the target identity is totally missing very unfortunately. Thus, it is rather hard to understand the involved tedious calculations leading to the target identity. To the best understanding of the author, this is true even if we note that the relevant $n-1$ identities in (3.7) except the last one (whose linear combination is to be taken) may be restated with the aid of $M_n(t)$ (but also with another $(n-1)$-dimensional vector-valued function), which is once again the Wronski matrix for $\mathcal{F}_{n-1}(\mathcal{I}) = \mathcal{F}_n(\mathcal{I}) \setminus \{f_n\}$, so that the rationale behind the construction of the coefficients may be somewhat relevant to our preceding simple arguments in Section 3.3.

Instead, the present paper takes a completely different approach with the two vectors $m_\sigma(t)$ and $m'_\sigma(t)$. More precisely, we aim at a quite straightforward derivation of (3.9) in a generalized form by showing that $m_\sigma(t)$ and $m'_\sigma(t)$ are algebraically linearly dependent vectors for each $t \in \mathcal{I}$. We begin with the following result to this end.

Lemma 4.1. *The derivative $m'_i(t)$ of $m_i(t)$, the determinant of the Wronski matrix $M_i(t)$ for $\mathcal{F}_n(\mathcal{I}) \setminus \{f_i\}$, equals the the determinant of $M_i(t)$ with the last row replaced by its derivative. In other words,*

$$m'_i(t) = \det W(t)^{n-1,i}, \quad i = 1, \dots, n \tag{4.1}$$

Proof. The proof is quite straightforward since it proceeds essentially in the same fashion as the proof of the fact that the derivative of the Wronskian $g(t) = \det G(t)$ is given by the right-hand side of (3.19). □

On the other hand, recall that $m_i(t) = \det M_i(t) = \det W(t)^{n,i}$. This together with (4.1) leads to the following result, which is a generalized version of the key identity (3.9).

Lemma 4.2. *If $w(t) \equiv 0$ on $\mathcal{I}$, then it also follows on $\mathcal{I}$ that*

$$m_j(t)m'_i(t) - m_i(t)m'_j(t) \equiv 0, \quad i, j = 1, \dots, n \tag{4.2}$$

Proof. It follows from the preceding observation and Lemma 4.1 that the left-hand side of (4.2) can be interpreted as the determinant of a 2×2 submatrix contained in the last two rows of the remainder determinant matrix $R(W(t))$. Hence, the assertion follows readily from Lemma 3.3. $\square$

We remark just in case that since (4.2) is equivalent to $(\sigma_j m_j(t))(\sigma_i m_i(t))' - (\sigma_i m_i(t))(\sigma_j m_j(t))' \equiv 0$, it is easy to see the above proof is based on an equivalent interpretation that the two time-varying vectors $m_\sigma(t)$ and $m'_\sigma(t)$ are algebraically linearly dependent for each $t \in \mathcal{I}$.

If we note that the left-hand side of (4.2) is nothing but the Wronskian for $m_i(t)$ and $m_j(t)$, the following result is an immediate consequence of Lemma 3.6.

Lemma 4.3. *Suppose that the Wronskian for $\mathcal{F}_n(\mathcal{I})$ satisfies $w(t) \equiv 0$ on $\mathcal{I}$. Once we fix $i = 1, \dots, n$, there exists for each $j = 1, \dots, n$ a subinterval $\mathcal{I}_0$ of $\mathcal{I}$ such that the two functions $m_i(t)$ and $m_j(t)$ are linearly dependent on $\mathcal{I}_0$. In particular, if $m_i(t) \not\equiv 0$ on $\mathcal{I}$, then one such $\mathcal{I}_0$ is a subinterval of $\mathcal{I}$ on which $m_i(t)$ never vanishes; in that case, $m_j(t)$ is a scalar constant multiple of $m_i(t)$ on the subinterval for $j = 1, \dots, n$. Otherwise (i.e., if $m_i(t) \equiv 0$ on $\mathcal{I}$), we can take $\mathcal{I}_0 = \mathcal{I}$.*

Note that each $m_i(t)$ is continuous by the standing assumption on $\mathcal{F}_n(\mathcal{I})$. Hence, assuming that $w(t) \equiv 0$, it follows from the above lemma that unless $m_i(t) \equiv 0$ on $\mathcal{I}$ for all $i = 1, \dots, n$ (or equivalently, $m_\sigma(t) \equiv 0$ on $\mathcal{I}$, so that (3.6) and (3.7) are trivial), there exists i and $\mathcal{I}_0$ such that $m_j(t) = c_j m_i(t)$ with some $c_j \in \mathbb{R}$ on $\mathcal{I}_0$ for $j = 1, \dots, n$. This obviously implies that $m_\sigma(t) = m_i(t)v$ on $\mathcal{I}_0$ for some nonzero $v \in \mathbb{R}^n$. Here, we may assume without loss of generality that $\mathcal{I}_0$ is such that $m_i(t)$ never vanishes on it (see the above lemma), and thus we are immediately led to the assertion of Theorem 2.3 by (3.6) in this specific case (i.e., when $m_i(t) \not\equiv 0$ on $\mathcal{I}$ for some i).

For convenience in later reference to this fact, we state this result as follows, which is the first one of the two very important key results in the proof of Theorem 2.3 as well as our further arguments in the following section.

Corollary 4.4. *Let $n \geq 2$. Suppose that the Wronskian for $\mathcal{F}_n(\mathcal{I})$ satisfies $w(t) \equiv 0$ on $\mathcal{I}$, while $m_i(t) \not\equiv 0$ on $\mathcal{I}$ for some $i = 1, \dots, n$. Then, there exist a subinterval $\mathcal{I}_0$ of $\mathcal{I}$ and a nonzero $v \in \mathbb{R}^n$ such that (2.5) holds (i.e., $W(t)v \equiv 0$) on $\mathcal{I}_0$, where any interval on which at least one of m_i ($i = 1, \dots, n$) never vanishes can be taken as such $\mathcal{I}_0$.*

Remark 5. Actually, the associated subinterval $\mathcal{I}_0$ on which $m_i(t)$ never vanishes is virtually independent of which i to choose, in the sense that as a chosen $m_i(t)$ tends to zero, all the remaining $m_j(t)$ also tend to zero (by the preceding arguments given below Lemma 4.3). Furthermore, the corresponding nonzero vector v on $\mathcal{I}_0$ can be constructed explicitly, in principle (i.e., if we compute all $m_i(t)$ for $i = 1, \ldots, n$ and note their zeros and mutual ratios). Hence, if the zero set of $m_i(t)$ does not have an accumulating point in the closure of $\mathcal{I}$, a possible direction for confirming linear dependence of $\mathcal{F}(\mathcal{I})$ given the information $w(t) \equiv 0$ could be checking that v determined on the subinterval $\mathcal{I}_0$ defined by taking two consecutive points in the zero set of $m_i(t),\ i = 1, \ldots, n$ (and the edges of $\mathcal{I}$) does not depend on $\mathcal{I}_0$. A similar idea can be found, e.g., in [9], but not in the null space viewpoint as in (2.5) but just through the identity (2.1). In this connection, it may be interesting to study the dimension of the linear space consisting of v satisfying (2.5), because if the dimension is larger than 1, it may be easier for a nonzero v to exist such that (2.5) holds on both (small) subintervals to the left as well as to the right of a zero of $m_i(t)$. Lemma 3.4 is somewhat relevant but is too weak to address such a direction, unfortunately.

If we recall the preceding arguments, what remains about the proof of Theorem 2.3 is the treatment of the case when $m_i(t) \equiv 0$ on $\mathcal{I}$ for $i = 1, \ldots, n$. Such a case might be more or less relevant to the situation where two or more linearly independent vectors v satisfying the identity (2.5) exist, but this is not the case, in general; Lemma 3.4 cannot be applied to conclude such a property, while generalizing this lemma to a direction convenient to the situation here does not seem straightforward. In fact, a sort of counterexample can be constructed without difficulties. Hence, this case is also handled by proceeding to explicitly construct a nonzero $v \in \mathbb{R}^n$ satisfying (2.5). In this connection, since $m_i(t)$, which is assumed to be identically zero on $\mathcal{I}$, is nothing but the Wronskian for the $n-1$ functions $\mathcal{F}_n(\mathcal{I}) \setminus \{f_i\}$, we can see that dealing with this case could be relevant to some recursive treatment with respect to the underlying n.

To facilitate such arguments in a non-confusing fashion, we first introduce the following notation.

Definition 2. Let n and $\mathcal{F}_n(\mathcal{I})$ satisfying the standing assumption for n be given. For each $1 \le k \le n$, the Wronski matrix for $\mathcal{F}_k(\mathcal{I})$, the set of the first k functions in $\mathcal{F}_n(\mathcal{I})$, is denoted by $W^{[k]}(t)$, and its determinant is denoted by $w^{[k]}(t)$. Furthermore, the remainder determinant of $W^{[k]}(t)$ with respect to its right-bottom entry is denoted by $m^{[k]}(t)$ if $k \ge 2$. Obviously, $m^{[k]}(t) = \det W^{[k-1]}(t) = w^{[k-1]}(t)$, and for $k = n$, we have $W^{[n]}(t) = W(t)$, $w^{[n]}(t) = w(t)$ and $m^{[n]}(t) = m_n(t) = \det M_n(t) = \det W^{[n-1]}(t) = w^{[n-1]}(t)$.

With this notation, we indeed prove the following lemma by mathematical induction on k; this is the second one of the two very important key results in the proof of Theorem 2.3; note that the assertion is valid without assuming $w(t) = w^{[n]}(t) \equiv 0$ on $\mathcal{I}$.

Lemma 4.5. *Suppose that $m^{[k]}(t) \equiv 0$ on $\mathcal{I}$ for some $2 \leq k \leq n$. Then, there exists a nonzero $v^{[k]} \in \mathbb{R}^k$ such that $W^{[k]}(t)v^{[k]} \equiv 0$ on some subinterval $\mathcal{I}_0$ of $\mathcal{I}$.*

Proof. The assertion holds for $k = 2$ with $\mathcal{I}_0 = \mathcal{I}$ and $v^{[k]} = [1, 0]^T$, because the assumption on $m^{[k]}(t)$ implies $f_1(t) \equiv 0$ on $\mathcal{I}$ in that case.

Suppose that the assertion holds for $k = k_0 \geq 2$, and consider the case with $k = k_0 + 1$. Thus, we assume that $m^{[k_0+1]}$ $(= w^{[k_0]}(t)) \equiv 0$ on $\mathcal{I}$ as a hypothesis of induction.

(i) The case with $m^{[k_0]}(t) \equiv 0$ on $\mathcal{I}$

Since the assertion holds for $k = k_0$, there exists a nonzero $v^{[k_0]} \in \mathbb{R}^{k_0}$ such that $W^{[k_0]}(t)v^{[k_0]} \equiv 0$ on some subinterval $\mathcal{I}_0$ of $\mathcal{I}$. Since the (first k_0) functions in $\mathcal{F}_{k_0+1}(\mathcal{I})$ are k_0 times differentiable by the standing assumption, the left-hand side is differentiable and we have $(W^{[k_0]}(t))'v^{[k_0]} \equiv 0$ on $\mathcal{I}_0$. These two identities imply that $(W^{[k_0+1]}(t)J)v^{[k_0]} \equiv 0$ on $\mathcal{I}_0$, where J denotes the identity matrix on $\mathbb{R}^{(k_0+1)\times(k_0+1)}$ with the last column removed. Hence, $W^{[k_0+1]}(t)v^{[k_0+1]} \equiv 0$ on $\mathcal{I}_0$, where $v^{[k_0+1]} := Jv^{[k_0]} \in \mathbb{R}^{k_0+1}$ is nonzero.

(ii) The case with $m^{[k_0]}(t) \not\equiv 0$ on $\mathcal{I}$

Note that $m^{[k_0+1]}(t) \equiv 0$ on $\mathcal{I}$ by the hypothesis of induction, which is equivalent to $w^{[k_0]}(t) \equiv 0$ on $\mathcal{I}$. Hence, by the continuity of $m^{[k_0]}(t)$, Corollary 4.4 leads immediately to the existence of a subinterval $\mathcal{I}_0 \subset \mathcal{I}$ and a nonzero $v^{[k_0]} \in \mathbb{R}^{k_0}$ such that $W^{[k_0]}(t)v^{[k_0]} \equiv 0$ on $\mathcal{I}_0$. Then, $v^{[k_0+1]} := Jv^{[k_0]}$ is nonzero and satisfies $W^{[k_0+1]}(t)v^{[k_0+1]} \equiv 0$ on $\mathcal{I}_0$ as in (i).

This completes the proof of this lemma by induction. □

To summarize, we are finally in a position to finish the proof of Theorem 2.3. Indeed, the assertion follows immediately from the above lemma applied to $k = n$.

5. Further Relevant Results

This section is devoted to showing that the framework of the arguments and the results derived in the preceding sections are helpful in straightforward derivations of some relevant results on the Wronskian available in the literature, some times in a strengthened fashion. Note that even though some of the following arguments might look straightforward if Theorem 2.3 is used, they are based instead on the framework itself of the arguments used in its proof and the results derived in the preceding sections. This is because we are basically and eventually interested in the properties on the original interval $\mathcal{I}$ rather than those valid only on some subinterval $\mathcal{I}_0$ of $\mathcal{I}$.

5.1. **Showing Linear Dependence via Decreasing the Number of Functions.** We begin by showing the following result about deciding linear dependence of $\mathcal{F}(\mathcal{I})$ through the treatment of decreasing the number of functions. This result is an immediate consequence of Corollary 4.4, and is nothing but Theorem II of [9], Theorem 2 of [4] and Theorem 3.8 of [2] stated with the notation in Definition 2.

Corollary 5.1. *Let $n \geq 2$, and suppose that the Wronskian $w^{[n]}(t)$ for $\mathcal{F}_n(\mathcal{I})$ satisfies $w^{[n]}(t) \equiv 0$ on $\mathcal{I}$, while the Wronskian $w^{[n-1]}(t)$ for $\mathcal{F}_{n-1}(\mathcal{I}) := \mathcal{F}_n(\mathcal{I}) \setminus \{f_n\}$ never vanishes on $\mathcal{I}$. Then, there exists a nonzero $v \in \mathbb{R}^n$ such that (2.5) holds on $\mathcal{I}$ and thus $\mathcal{F}_n(\mathcal{I})$ is linearly dependent.*

Note that the nonzero v in the above Corollary can be constructed, in principle (see Remark 5).

It would be worth stating the following observation: suppose that $w^{[n]}(t) \equiv 0$ on $\mathcal{I}$ but $w^{[n-1]}(t)$ vanishes somewhere on $\mathcal{I}$. Then, one might consider reordering of the functions in $\mathcal{F}_n(\mathcal{I})$, so that the above corollary applied to the reordered functions could conclude that $\mathcal{F}_n(\mathcal{I})$ is linearly dependent. However, such a situation can never occur unless $w^{[n-1]}(t) \equiv 0$ on $\mathcal{I}$ in the original order of the n functions; this can readily be seen by the arguments below Lemma 4.3 or Remark 5. Some relevant observation can be found also in [9] and [11].

5.2. **Analysis on the Subinterval $\mathcal{I}_0$ and Increasing the Number of Functions.** We are finally interested in deriving the following theorem (which is nothing but Theorem VIII of [9]) in the framework of the present paper, as well as discussing relevant informative results, which are related to the treatment of increasing the number of functions.

Theorem 5.2. *Let $n \geq 1$ and suppose that the Wronskian $w^{[n]}(t)$ ($= m^{[n+1]}(t)$) for $\mathcal{F}_n(\mathcal{I}) = \mathcal{F}_{n+1}(\mathcal{I}) \setminus \{f_{n+1}\}$ satisfies $w^{[n]}(t) \equiv 0$ on $\mathcal{I}$. If the nth order derivative of each function in $\mathcal{F}_{n+1}(\mathcal{I})$ is continuous on $\mathcal{I}$, then the Wronskian for $\mathcal{F}_{n+1}(\mathcal{I})$ satisfies $w^{[n+1]}(t) \equiv 0$ on $\mathcal{I}$.*

Some of the relevant results given later correspond to existing results in the literature (e.g., Lemma II of [9]) in a strengthened form, while others are new through our null space approach. What plays a crucial role in such a direction of study is to clarify whether the subinterval $\mathcal{I}_0$ in Theorem 2.3, or more importantly in Corollary 4.4 and Lemma 4.5, can be taken so that it includes a given arbitrary $t_0 \in \mathcal{I}$. To tackle such an issue, we introduce the following definition with respect to the zero set $\mathcal{Z}^{[k]}$ of $m^{[k]}(t)$ for $k =$

$2, \dots, n+1$, where $m^{[n+1]}(t)$ is interpreted to mean $w^{[n]}(t)$ (even when we are actually interested only in $\mathcal{F}_n(\mathcal{I})$). Note that when $m^{[n+1]}(t)$ $(= w^{[n]}(t))$ is referred to in this subsection, we do not necessarily assume $w^{[n]}(t) \equiv 0$. However, we refer to $m^{[n+1]}(t)$ in a substantial fashion (i.e., we use the relevant property arising from the treatment of $\mathcal{Z}^{[n+1]}$) only when $m^{[n+1]}(t)$ is continuous on $\mathcal{I}$ so that $\mathcal{I}^{[n+1]}_{\neq 0} = \mathcal{I} \setminus \mathcal{Z}^{[n+1]}$ in the following definition is ensured to be an open set. It is indeed open (as an empty set) when $w^{[n]}(t) \equiv 0$ is actually assumed on $\mathcal{I}$. Alternatively, it is an open set also when we are actually interested in $\mathcal{F}_{n+1}(\mathcal{I})$ satisfying the standing assumption, rather than merely in $\mathcal{F}_n(\mathcal{I})$. Note that $w^{[n+1]}(t)$ is not continuous, in general, however.

Definition 3. For each $2 \leq k \leq n+1$, define the zero set of $m^{[k]}(t)$ denoted by

$$\mathcal{Z}^{[k]} := \{\tau \in \mathcal{I} \,|\, m^{[k]}(\tau) = 0\} \tag{5.1}$$

Then, four mutually disjoint subsets of $\mathcal{I}$ such that $\mathcal{I} = \mathcal{I}^{[k]}_{\neq 0} \cup \mathcal{I}^{[k]}_{=\neq 0} \cup \mathcal{I}^{[k]}_{==0} \cup \mathcal{I}^{[k]}_{\equiv 0}$ and $\mathcal{Z}^{[k]} = \mathcal{I}^{[k]}_{=\neq 0} \cup \mathcal{I}^{[k]}_{==0} \cup \mathcal{I}^{[k]}_{\equiv 0}$ are defined as follows.

(a) $\mathcal{I}^{[k]}_{\neq 0} := \{\tau \in \mathcal{I} \,|\, \tau \notin \mathcal{Z}^{[k]}\}$ $(= \mathcal{I} \setminus \mathcal{Z}^{[k]})$
(b-1) $\mathcal{I}^{[k]}_{=\neq 0} := \{\tau \in \mathcal{I} \,|\, \exists$ an interval $\mathcal{I}_\tau$ such that $\mathcal{I}_\tau \cap \mathcal{Z}^{[k]} = \{\tau\}\}$
(= the set of the isolated points of $\mathcal{Z}^{[k]}$)
(b-2) $\mathcal{I}^{[k]}_{==0} := \{\tau \in \mathcal{I} \,|\, \tau \in \mathcal{Z}^{[k]},\ \tau \notin \mathcal{I}^{[k]}_{=\neq 0}, \nexists$ an interval $\mathcal{I}_\tau$ such that $\tau \in \mathcal{I}_\tau$ and $m^{[k]}(t) \equiv 0$ on $\mathcal{I}_\tau\}$
(= the set of the accumulation points of $\mathcal{Z}^{[k]}$ whose every neighborhood contains a point in $\mathcal{I}^{[k]}_{\neq 0}$)
(c) $\mathcal{I}^{[k]}_{\equiv 0} := \{\tau \in \mathcal{I} \,|\, \exists$ an interval $\mathcal{I}_\tau$ such that $\tau \in \mathcal{I}_\tau$ and $m^{[k]}(t) \equiv 0$ on $\mathcal{I}_\tau\}$

Under this definition, the following lemma relevant to the classification into the four categories in Definition 3 is an immediate consequence of Corollary 4.4 (since $w^{[1]}(t) = W^{[1]}(t)$).

Lemma 5.3. *Suppose that $t_0 \in \mathcal{I}^{[k+1]}_{\equiv 0}$, i.e., $m^{[k+1]}(t)$ $(= w^{[k]}(t)) \equiv 0$ in the neighborhood of $t_0 \in \mathcal{I}$ for some $2 \leq k \leq n$. Regarding the relation between t_0 and the pair of $m^{[k]}(t)$ $(= \det W^{[k-1]}(t))$ and $W^{[k]}(t)$, we have the following:*

(a) *If $t_0 \in \mathcal{I}^{[k]}_{\neq 0}$, then there exist an interval $\mathcal{I}_0$ satisfying $t_0 \in \mathcal{I}_0$ and a nonzero $v^{[k]} \in \mathbb{R}^k$ such that $W^{[k]}(t)v^{[k]} \equiv 0$ on $\mathcal{I}_0$.*

(b-1) *If* $t_0 \in \mathcal{I}^{[k]}_{=\neq 0}$*, then there exist an interval* $\mathcal{I}_0$ *satisfying* $t_0 \notin \mathcal{I}_0$ *and* $t_0 \in \overline{\mathcal{I}}_0$ *and a nonzero* $v^{[k]} \in \mathbb{R}^k$ *such that* $W^{[k]}(t)v^{[k]} \equiv 0$ *on* $\mathcal{I}_0$*, where* $\overline{\mathcal{I}}_0$ *denotes the closure of* $\mathcal{I}_0$*.*

(b-2) *If* $t_0 \in \mathcal{I}^{[k]}_{==0}$*, then for every* $\varepsilon > 0$*, there exist an interval* $\mathcal{I}_0$ *satisfying* $\bar{l}(\mathcal{I}_0, t_0) < \varepsilon$ *and a nonzero (but* $\mathcal{I}_0$*-dependent)* $v^{[k]} \in \mathbb{R}^k$ *such that* $W^{[k]}(t)v^{[k]} \equiv 0$ *on* $\mathcal{I}_0$*, where*

$$\bar{l}(\mathcal{I}_0, t_0) := \sup_{\tau \in \mathcal{I}_0} |\tau - t_0| \tag{5.2}$$

Alternatively, if $t_0 \in \mathcal{I}^{[2]}_{\equiv 0}$*, i.e.,* $m^{[2]}(t)$ $(= w^{[1]}(t)) \equiv 0$ *in the neighborhood* $\mathcal{I}_0$ *of* $t_0 \in \mathcal{I}$*, then* $v^{[1]} := 1$ *satisfies* $W^{[1]}(t)v^{[1]} \equiv 0$ *on* $\mathcal{I}_0$*.*

Note that if we are led to $t_0 \in \mathcal{I}^{[k]}_{\equiv 0}$ (which corresponds to category (c) of the classification in Definition 3) when $2 \le k \le n$ is such that $t_0 \in \mathcal{I}^{[k+1]}_{\equiv 0}$ in the above lemma, then we can either just decrement k by 1 to apply the same lemma or apply the last part of the lemma.

More importantly, further note that $W^{[k]}(t)v^{[k]} \equiv 0$ in the statement of the above lemma can be differentiated to have $W^{[n]}(t)v^{[n]} \equiv 0$ on $\mathcal{I}_0$ with $v^{[n]} := [(v^{[k]})^T, 0]^T \in \mathbb{R}^n \setminus \{0\}$ (without assuming the continuity of the $(n-1)$st order derivative of each function in $\mathcal{F}_n(\mathcal{I})$). Furthermore, if each function in $\mathcal{F}_k(\mathcal{I})$ is actually l times differentiable for $l \ge n$, then we readily see on $\mathcal{I}_0$ that

$$F^{[k]}_l(t)v^{[k]} \equiv 0 \tag{5.3}$$

for

$$F^{[k]}_l(t) := \begin{bmatrix} \underline{f}^{[k]}_0(t) \\ \vdots \\ \underline{f}^{[k]}_l(t) \end{bmatrix} \tag{5.4}$$

where $\underline{f}^{[k]}_j(t)$ denotes the jth order derivative of $\underline{f}^{[k]}_0(t)$ defined as the first row of $W^{[k]}(t)$ (see (3.15) for a relevant notation). If we apply this observation to the case $k = n$ and category (a) in the above lemma, i.e., to $t_0 \in \mathcal{I}^{[n]}_{\neq 0}$, then the corresponding consequence immediately leads to the assertion of Lemma II in [9], i.e., the rank deficiency property of $F^{[n]}_l(t_0)$ when $w^{[n]}(t) \equiv 0$ on $\mathcal{I}$ and $t_0 \in \mathcal{I}^{[n]}_{\neq 0}$; more strongly, we see that if $t_0 \in \mathcal{I}^{[n]}_{\neq 0}$, there exists $\mathcal{I}_0$ such that $t_0 \in \mathcal{I}_0$ and $F^{[n]}_l(t)v^{[n]} \equiv 0$ on $\mathcal{I}_0$ for the aforementioned $v^{[n]}$.

We can readily generalize the arguments to cover category (b-1) as well, i.e., also when $t_0 \in \mathcal{I}^{[n]}_{=\neq 0}$, because of the continuity of $\underline{f}_0(t)v^{[n]}$ on $\overline{\mathcal{I}}_0$, where the continuity immediately follows from that of the functions in $\mathcal{F}_n(\mathcal{I})$. More precisely (in the case of $k = n$), the identity $\underline{f}_0(t)v^{[n]} \equiv 0$ on $\mathcal{I}_0$ means the same identity also on $\overline{\mathcal{I}}_0$, which can be differentiated there up to l times to have $F_l^{[n]}(t)v^{[n]} \equiv 0$ on $\overline{\mathcal{I}}_0$, where $t_0 \in \overline{\mathcal{I}}_0$ by $t_0 \in \mathcal{I}^{[n]}_{=\neq 0}$.

To summarize the arguments up to this point, we see that these two categories correspond to the case where $F_l^{[n]}(t)v^{[n]} \equiv 0$ holds on an interval containing a given arbitrary $t_0 \in \mathcal{I}$.

Although this summary does not cover category (b-2), i.e., the case of $t_0 \in \mathcal{I}^{[n]}_{==0}$, the relevant part of the above lemma implies that the interval $\mathcal{I}_0$ in Theorem 2.3 and Corollary 4.4 can be chosen arbitrarily close to the given arbitrary $t_0 \in \mathcal{I}^{[n]}_{==0}$ in the sense of $\bar{l}(\mathcal{I}_0, t_0)$. With this observation taken into account, we are now in a suitable position to give the proof of Theorem 5.2.

Proof of Theorem 5.2. When we take an arbitrary $t_0 \in \mathcal{I}$, it follows from Lemma 5.3 together with the associated above observations that except for the situation

($\star$) t_0 is such that $t_0 \in \mathcal{I}^{[k]}_{\equiv 0}$ for $k = \kappa + 1, \ldots, n + 1$ and $t_0 \in \mathcal{I}^{[\kappa]}_{==0}$ for some $2 \le \kappa \le n$,

we have shown the existence of a subinterval $\mathcal{I}_0 \subset \mathcal{I}$ containing t_0 and a nonzero $v^{[n]} \in \mathbb{R}^n$ such that $F_n^{[n]}(t)v^{[n]} \equiv 0$ on $\mathcal{I}_0$. This implies $w^{[n+1]}(t_0) = 0$ because $W^{[n+1]}(t)v^{[n+1]} \equiv 0$ on $\mathcal{I}_0$ for $v^{[n+1]} := [(v^{[n]})^T, 0]^T \in \mathbb{R}^{n+1} \setminus \{0\}$, and this is nothing but the assertion of this theorem (when t_0 is not in the above situation ($\star$)).

In the remaining situation ($\star$), on the other hand, the same observations (above this proof) imply that for any $\varepsilon > 0$, there exists an interval $\mathcal{I}_0$ such that $\bar{l}(\mathcal{I}_0, t_0) < \varepsilon$ and a nonzero (but $\mathcal{I}_0$-dependent) $v^{[n]} \in \mathbb{R}^n$ satisfying $F_n^{[n]}(t)v^{[n]} \equiv 0$ on $\mathcal{I}_0$. Since this implies $W^{[n+1]}(t)v^{[n+1]} \equiv 0$ on $\mathcal{I}_0$ for the aforementioned $v^{[n+1]} \in \mathbb{R}^{n+1} \setminus \{0\}$, we have $w^{[n+1]}(t) \equiv 0$ on $\mathcal{I}_0$. Since $\varepsilon > 0$ is arbitrary and $w^{[n+1]}(t)$ is continuous on $\mathcal{I}$ by the assumption of this theorem, we are led to $w^{[n+1]}(t_0) = 0$. This completes the proof. □

Even though the established theorem (i.e., Theorem 5.2) asserts that $w^{[n+1]}(t) \equiv 0$ on $\mathcal{I}$, however, its proof does not provide any information about a t_0-dependent nonzero $v^{[n+1]} \in \mathbb{R}^{n+1}$ such that $W^{[n+1]}(t_0)v^{[n+1]} = 0$, when t_0 is in the situation ($\star$). It would thus be an interesting topic to study how we could somehow continue our arguments toward the direction of clarifying such $v^{[n+1]}$.

As a preliminary to a lemma relevant to such an issue (see Lemma 5.5 below), let us first consider the following lemma *about the case when* t_0 *is not in the situation* ($\star$). It corresponds to an alternative summary of the observations given below Lemma 5.3 (applied to a general k)[2], where a nonzero $v^{[\kappa]} \in \mathbb{R}^\kappa$ on $\mathcal{I}_0$ (with κ defined essentially by the process with which it is determined that t_0 *is not in the situation* $(\star)$) can be constructed explicitly, in principle, for the same reason as stated in Remark 5.

Lemma 5.4. *Given $t_0 \in \mathcal{I}$, suppose that either of the following conditions holds:*
(i) $t_0 \in \mathcal{I}_{\equiv 0}^{[\kappa+1]}$ and $t_0 \in \mathcal{I}_{\not\equiv 0}^{[\kappa]} \cup \mathcal{I}_{=\not\equiv 0}^{[\kappa]}$ for some $2 \le \kappa \le n$.
(ii) $t_0 \in \mathcal{I}_{\equiv 0}^{[\kappa]}$ for $\kappa = 2$.
Then, there exist a subinterval $\mathcal{I}_0$ of $\mathcal{I}$ and a nonzero $v^{[\kappa]} \in \mathbb{R}^\kappa$ such that $t_0 \in \mathcal{I}_0$ and $W^{[\kappa]}(t)v^{[\kappa]} \equiv 0$ (and thus $W^{[n+1]}(t)v^{[n+1]} \equiv 0$ for $v^{[n+1]} := [(v^{[\kappa]})^T, 0]^T \in \mathbb{R}^{n+1} \setminus \{0\}$) on $\mathcal{I}_0$.

The reason why we state the above lemma is that we should highlight the difficulty associated with category (b-2) in Lemma 5.3 (i.e., when $t_0 \in \mathcal{I}_{\equiv 0}^{[\kappa+1]}$ and $t_0 \in \mathcal{I}_{==0}^{[\kappa]}$ as in the situation $(\star)$). In this case, we encounter the difficulty that we cannot take a subinterval $\mathcal{I}_0$ such that $t_0 \in \mathcal{I}_0$, and it is not clear how the corresponding $v^{[\kappa]} \neq 0$ on $\mathcal{I}_0$ changes as $\mathcal{I}_0$ is chosen closer to t_0, i.e., as i tends to ∞ even in the following obvious result corresponding to a special case of category (b-2).

Lemma 5.5. *Suppose that $t_0 \in \mathcal{I}_{\equiv 0}^{[\kappa+1]}$ and $t_0 \in \mathcal{I}_{==0}^{[\kappa]}$ for some $2 \le \kappa \le n$. Further suppose that $t_0 \in \mathcal{I}_{==0}^{[\kappa]}$ is an isolated point of $\mathcal{I}_{==0}^{[\kappa]} \cup \mathcal{I}_{\equiv 0}^{[\kappa]}$ (i.e., the set of the accumulation points of $\mathcal{Z}^{[\kappa]}$), and let the sequence of τ_i ($i = 0, 1, \ldots$) be either of the following:*
(i) the unique decreasing sequence tending to t_0 such that the set $\{\tau_i\}_{i=0}^\infty$ equals $\mathcal{Z}^{[\kappa]} \cap (t_0, \tau_0]$;
(ii) the unique increasing sequence tending to t_0 such that the set $\{\tau_i\}_{i=0}^\infty$ equals $\mathcal{Z}^{[\kappa]} \cap [\tau_0, t_0)$.
The intervals $\mathcal{I}_{\tau_i}$ ($i = 0, 1, \ldots$) are then defined as (τ_{i+1}, τ_i) and (τ_i, τ_{i+1}) for (i) and (ii), respectively. Then, for each $i = 0, 1, \ldots$, there exists a nonzero $v_i^{[\kappa]} \in \mathbb{R}^\kappa$ such that $W^{[\kappa]}(t)v_i^{[\kappa]} \equiv 0$ (and thus $W^{[n+1]}(t)v_i^{[n+1]} \equiv 0$ for $v_i^{[n+1]} := [(v_i^{[\kappa]})^T, 0]^T \in \mathbb{R}^{n+1} \setminus \{0\}$) on $\mathcal{I}_{\tau_i}$.

In particular, it would be an interesting topic to study whether/when the sequence of the nonzero $v_i^{[\kappa]}$ ($i = 0, 1, \ldots$) in the above lemma can be taken in such a way that it converges to a nonzero $v^{[\kappa]} \in \mathbb{R}^\kappa$ such that $W^{[n+1]}(t_0)v^{[n+1]} = 0$ for $v^{[n+1]} := [(v^{[\kappa]})^T, 0]^T$. Such a topic is indeed quite relevant to the aforementioned deficient side of the proof of Theorem 5.2,

[2]No assumption is required about the continuity of the $(n-1)$st order derivative of each function in $\mathcal{F}_n(\mathcal{I})$.

which unfortunately is the only part in the present paper that is not fully along the null space viewpoint.

Before proceeding to such a topic, we give the following two results about nonzero $v^{[n+1]}$ such that $W^{[n+1]}(t_0)v^{[n+1]} = 0$ when $t_0 \in \mathcal{I}$ is in (another special case, other than that in the above lemma, of) the situation ($\star$); note as in the above lemma that their statements do not assume that $w^{[n]}(t) \equiv 0$ on $\mathcal{I}$. The first result deals with the case $\kappa = 2$.

Proposition 5.6. *Suppose that* $t_0 \in \mathcal{I}^{[2]}_{==0}$. *Then,* $W^{[n]}(t_0)v^{[n]} = 0$ *for* $v^{[n]} = [v^{[1]}, 0, \ldots, 0] \in \mathbb{R}^n$ *with* $v^{[1]} = 1$. *In addition, if each function in* $\mathcal{F}_n(\mathcal{I})$ *is* n *times differentiable together with another function* $f_{n+1}(t)$ *on* $\mathcal{I}$, *then* $W^{[n+1]}(t_0)v^{[n+1]} = 0$ *for* $v^{[n+1]} = [v^{[1]}, 0, \ldots, 0] = [1, 0, \ldots, 0] \in \mathbb{R}^{n+1}$.

Proof. It suffices to show that $f_1^{(k)}(t_0) = 0$ for $k = 0, \ldots, n-1$ (and also $f_1^{(n)}(t_0) = 0$, when $f_1(t)$ is n times differentiable). To this end, let $\mathcal{Z}_1^{(0)}$ be the zero set of $f_1(t)$ in the neighborhood of t_0. Since $m^{[2]}(t) = f_1(t)$, however, $\mathcal{Z}_1^{(0)}$ is nothing but the intersection of $\mathcal{Z}^{[2]}$ and the neighborhood. Hence, by the assumption $t_0 \in \mathcal{I}^{[2]}_{==0}$, it follows that t_0 is an accumulation point of $\mathcal{Z}_1^{(0)}$ such that $t_0 \in \mathcal{Z}_1^{(0)}$. Let $\tau_i^{(0)}$ $(i = 0, 1, \ldots)$ be a monotonically decreasing or increase sequence in $\mathcal{Z}_1^{(0)}$ converging to t_0. Since $f_1(\tau_i^{(0)}) = 0$ $(i = 0, 1, \ldots)$, it follows from the continuity of $f_1(t)$ that $f_1(t_0) = 0$. Furthermore, since $f_1(t)$ is differentiable, we have

$$f_1'(t_0) = \lim_{i \to \infty} \frac{f_1(\tau_i^{(0)}) - f_1(t_0)}{\tau_i^{(0)} - t_0} = 0 \tag{5.5}$$

since $f_1(t_0) = f_1(\tau_i^{(0)}) = 0$, and it also follows from Rolle's theorem (together with $f_1(\tau_i^{(0)}) = 0$ $(i = 0, 1, \ldots)$) that there exists a monotonically decreasing or increasing sequence of the zeros of $f_1'(t)$ converging to t_0, which we denote by $\tau_i^{(1)}$ $(i = 0, 1, \ldots)$; they together with t_0 constitutes a zero set $\mathcal{Z}_1^{(1)}$ of $f_1'(t)$, where $t_0 \in \mathcal{Z}_1^{(1)}$ is an accumulation point of this set. Hence, if $f_1'(t)$ is differentiable, then essentially the same arguments as above leads to $f_1^{(2)}(t_0) = 0$. We can further apply Rolle's theorem to $f_1'(t)$, if necessary, to consider a zero set $\mathcal{Z}_1^{(2)}$ of $f_1^{(2)}(t)$, and repeating the same arguments leads to $f_1^{(l)}(t_0) = 0$ $(l = 0, \ldots, n-1)$ under the $n-1$ times differentiability assumption, and also to $f_1^{(n)}(t_0) = 0$ under the n times differentiability assumption. This completes the proof. $\square$

We further have the following result for the remaining case of $3 \le \kappa \le n$ (as well as $\kappa = n + 1$), whose proof relies on a key lemma given soon after this proposition.

Proposition 5.7. *Suppose that* $t_0 \in \mathcal{I}^{[\kappa]}_{==0}$*, where* $3 \le \kappa \le n+1$*. If* $t_0 \in \mathcal{I}^{[\kappa-1]}_{\neq 0}$*, then there exists a nonzero* $v^{[\kappa-1]} \in \mathbb{R}^{\kappa-1}$ *such that* $W^{[\kappa]}(t_0)v^{[\kappa]} = 0$ *for* $v^{[\kappa]} = [(v^{[\kappa-1]})^T, 0]^T \in \mathbb{R}^\kappa$ *and* $W^{[n]}(t_0)v^{[n]} = 0$ *for* $v^{[n]} = [(v^{[\kappa-1]})^T, 0]^T \in \mathbb{R}^n$*. In addition, if each function in* $\mathcal{F}_n(\mathcal{I})$ *is* n *times differentiable together with another function* $f_{n+1}(t)$ *on* $\mathcal{I}$*, then* $W^{[n+1]}(t_0)v^{[n+1]} = 0$ *for* $v^{[n+1]} = [(v^{[\kappa-1]})^T, 0]^T \in \mathbb{R}^{n+1}$*.*

Proof. The assumption $t_0 \in \mathcal{I}^{[\kappa]}_{==0}$ implies that t_0 is an accumulation point of $\mathcal{Z}^{[\kappa]}$ (i.e., the zero set of $m^{[\kappa]}(t)$) and $t_0 \in \mathcal{Z}^{[\kappa]}$. It then follows from Rolle's theorem that t_0 is also an accumulation point of the zero set of $(m^{[\kappa]})'(t)$. In particular, we have $(m^{[\kappa]})'(t_0) = 0$ by essentially the same arguments as the proof of Proposition 5.6, which further leads to $(m^{[\kappa]})^{(\lambda)}(t_0) = 0$ (again by repeating the same arguments) as long as the λth order derivative exists (i.e., up to $\lambda = n - \kappa + 1$, or $\lambda = n - \kappa + 2$ under the additional n times differentiability assumption).

On the other hand, the second assumption $t_0 \in \mathcal{I}^{[\kappa-1]}_{\neq 0}$ implies that $m^{[\kappa-1]}(t_0) = w^{[\kappa-2]}(t_0) = \det W^{[\kappa-2]}(t_0) \neq 0$. Hence, the first $\kappa - 2$ rows of $W^{[\kappa-1]}(t_0)$ are linearly independent. Here, note that $\det W^{[\kappa-1]}(t_0) = w^{[\kappa-1]}(t_0) = m^{[\kappa]}(t_0) = 0$ since $t_0 \in \mathcal{Z}^{[\kappa]}$. Hence, the last row of $W^{[\kappa-1]}(t_0)$ (i.e., $\underline{f}^{[\kappa-1]}_{\kappa-2}(t_0)$) is a linear combination of its first $\kappa - 2$ rows (i.e. the rows of $F^{[\kappa-1]}_{\kappa-3}(t_0)$). Furthermore, since $(m^{[\kappa]})'(t) = (w^{[\kappa-1]}(t))' = (\det W^{[\kappa-1]}(t))'$, we see from (3.19) that $(m^{[\kappa]})'(t_0) = 0$ implies that $\underline{f}^{[\kappa-1]}_{\kappa-1}(t_0)$ is also a linear combination of the aforementioned $\kappa - 2$ rows, the rows of $F^{[\kappa-1]}_{\kappa-3}(t_0)$.

We can summarize these arguments as follows: each of the κ rows of $F^{[\kappa-1]}_{\kappa-1}(t_0) \in \mathbb{R}^{\kappa \times (\kappa-1)}$ is a linear combination of its first $\kappa - 2$ rows. Since $(m^{[\kappa]})^{(\lambda)}(t_0) = (\det W^{[\kappa-1]})^{(\lambda)}(t_0) = 0$ for $\lambda = 1, \ldots, n - \kappa + 1$ (and also $\lambda = n - \kappa + 2$ under the additional n times differentiability assumption) as stated in the first paragraph, it follows immediately from Lemma 5.8 given below (with $k = \kappa - 1$ and $l = n - 1$, or $l = n$ under the additional n times differentiability assumption) that there exists a nonzero $v^{[\kappa-1]} \in \mathbb{R}^{\kappa-1}$ such that $F^{[\kappa-1]}_{n-1}(t_0)v^{[\kappa-1]} = 0$ (and $F^{[\kappa-1]}_{n}(t_0)v^{[\kappa-1]} = 0$ under the additional n times differentiability assumption). Hence, the assertion follows immediately. □

Remark 6. It is obvious from the above proof that the nonzero $v^{[\kappa-1]} \in \mathbb{R}^{\kappa-1}$ in the statement of the above proposition is such that $F^{[\kappa-1]}_{\kappa-3}(t_0)v^{[\kappa-1]} = 0$, where $F^{[\kappa-1]}_{\kappa-3}(t_0) \in \mathbb{R}^{(\kappa-2)\times(\kappa-1)}$. This condition is equivalent to $F^{[\kappa-1]}_{\kappa-2}(t_0)v^{[\kappa-1]} = 0$ because of the situation where $\underline{f}^{[\kappa-1]}_{\kappa-2}(t_0)$, the last row of $F^{[\kappa-1]}_{\kappa-2}(t_0)$, is a linear combination of the rows of $F^{[\kappa-1]}_{\kappa-3}(t_0)$. Here, note that $F^{[\kappa-1]}_{\kappa-2}(t) = W^{[\kappa-1]}(t)$ and thus the condition is $W^{[\kappa-1]}(t_0)v^{[\kappa-1]} = 0$. This determines $v^{[\kappa-1]}$ (up to a scalar factor) because the rank deficiency of $W^{[\kappa-1]}(t_0)$ is 1 by the assumption that $t_0 \in \mathcal{I}^{[\kappa-1]}_{\neq 0}$ (which means that $m^{[\kappa-1]}(t_0) \neq 0$ and thus $W^{[\kappa-2]}(t_0)$ is nonsingular). By the observations in Remark 2, particularly by the arguments around (3.8), such $v^{[\kappa-1]} \neq 0$ can be regarded as the vector $m^{[\kappa-1]}_{\sigma}(t_0)$, where the vector-valued function $m^{[\kappa-1]}_{\sigma}(t)$ is defined as $m_{\sigma}(t)$ in (3.5) corresponding to the Wronski matrix $W^{[\kappa-1]}(t)$. Note that the continuity of the nth order derivatives of the functions in $\mathcal{F}_{n+1}(\mathcal{I})$ is not used in these arguments, unlike in the statement of Theorem 5.2.

A key role was played by the following lemma in the above proof; note that $w^{[n]}(t) = \det W^{[n]}(t) \equiv 0$ on $\mathcal{I}$ is not assumed in its statement.

Lemma 5.8. *Let $k \geq 2$ and suppose that each function in $\mathcal{F}_k(\mathcal{I})$ is l times differentiable at $t = t_0 \in \mathcal{I}$, where $l \geq k$. If each of the $k+1$ rows of $F^{[k]}_k(t_0) \in \mathbb{R}^{(k+1)\times k}$ is a linear combination of its first $k-1$ rows (i.e., the rows of $F^{[k]}_{k-2}(t_0)$) and if $(\det W^{[k]})^{(\lambda)}(t_0) = 0$ $(\lambda = 1, \ldots, l-k+1)$, then* $\operatorname{rank} F^{[k]}_l(t_0) < k$.

Proof. Assuming that each of the $k+1$ rows of $F^{[k]}_k(t_0) \in \mathbb{R}^{(k+1)\times k}$ is a linear combination of its first $k-1$ rows, it suffices to show that every row of $F^{[k]}_l(t_0)$ is also a linear combination of these $k-1$ rows if $(\det W^{[k]})^{(\lambda)}(t_0) = 0$ $(\lambda = 1, \ldots, l-k+1)$. We prove it through the induction arguments on l.

First, the assertion holds obviously when $l = l_0 = k$.

Next, suppose that the assertion holds for $l = l_0$; that is, we assume that $(\det W^{[k]})^{(\lambda)}(t_0) = 0$ $(\lambda = 1, \ldots, l_0 - k + 1)$ and thus each of the $l_0 + 1$ rows of $F^{[k]}_{l_0}(t_0)$ is a linear combination of the aforementioned $k-1$ rows. Then, to complete the proof by induction, we aim at showing the claim that $\underline{f}^{[k]}_{l_0+1}(t_0)$ is also a linear combination of these $k-1$ rows by assuming that $(\det W^{[\kappa]})^{(\lambda)}(t_0) = 0$ also for $\lambda = l_0 - k + 2$.

To show this claim, note that the functions in $\underline{f}^{[k]}_{l_0+1}(t)$ arise in the derivative $(\det W^{[k]})^{(\lambda)}(t)$ for the first time when $\lambda = l_0 - k + 2$. Since $F^{[k]}_{l_0}(t_0)$ fails to be of full column rank by the aforementioned hypothesis of the induction arguments, we can see by the repeated applications of (3.11) that

$$(\det W^{[k]})^{(l_0-k+2)}(t_0) = \det \begin{bmatrix} F^{[k]}_{k-2}(t_0) \\ \underline{f}^{[k]}_{l_0+1}(t_0) \end{bmatrix} \tag{5.6}$$

Since the left-hand side is zero by the aforementioned hypothesis of the induction arguments, the claim has been established. This completes the proof. □

To summarize, Propositions 5.6 and 5.7 have successfully characterized nonzero $v^{[n+1]} \in \mathbb{R}^{n+1}$ such that $W^{[n+1]}(t_0)v^{[n+1]} = 0$ (through an underlying nonzero $v^{[\kappa-1]} \in \mathbb{R}^{\kappa-1}$ and $v^{[n+1]} := [(v^{[\kappa-1]})^T, 0]^T$) when t_0 is in a special case of the situation ($\star$), i.e., when either of the following conditions is further satisfied:
(i) $\kappa = 2$.
(ii) $3 \leq \kappa \leq n+1$ and $t_0 \in \mathcal{I}_{\neq 0}^{[\kappa-1]}$.
Even in such a special case, unfortunately however, it does not seem straightforward to relate the above $v^{[\kappa-1]} \in \mathbb{R}^{\kappa-1}$ or $v^{[n+1]}$ with those vectors $v_i^{[\kappa]} \in \mathbb{R}^{\kappa}$ or $v_i^{[n+1]}$ in Lemma 5.5. Furthermore, for the remaining case of the situation ($\star$), i.e., when $3 \leq \kappa \leq n+1$ and $m^{[\kappa-1]}(t_0) = 0$, characterizing nonzero $v^{[n+1]}$ such that $W^{[n+1]}(t_0)v^{[n+1]} = 0$ does not seem straightforward.

These observations might suggest a sort of limit in the algebraic arguments through the null space viewpoint developed in the present paper, but a further elaborated study would be quite interesting, possibly with much more emphasis on the treatment of the calculus side of the underlying subtle issues; such a topic may be beyond the scope of this paper aiming at stressing the clear perspective (particularly in the arguments up to Subsection 5.1) and new insight offered by the linear algebraic and null space viewpoint.

In connection with the above special case (ii) under the situation ($\star$), we close this section by finally giving, just in case, the following example to show that such a case is not vacuous.

Example 5.1. *Let $h(t)$ defined on $\mathcal{I}$ be a differentiable function such that $t = t_0 \in \mathcal{I}$ is an accumulation point of its zero set while it is not identically zero in the neighborhood of t_0. Then, for $\kappa = 3$, $f_1(t) = 1$ and $f_2(t) = h(t)$, we readily see that $t_0 \in \mathcal{I}_{==0}^{[\kappa]}$ while $t_0 \in \mathcal{I}_{\neq 0}^{[\kappa-1]}$. Obviously, taking $f_3(t) = f_1(t)$ further leads to $w^{[\kappa]}(t) \equiv 0$ on $\mathcal{I}$ (if $h(t)$ is twice differentiable).*

We also give the following example just in case to show that $m^{[\kappa-1]}(t_0) = 0$ is also possible.

Example 5.2. *Let $h(t)$ be as in the above example. Then, for $\kappa = 3$, $f_1(t) = t - t_0$ and $f_2(t) = (t - t_0)h(t)$, we readily see that $t_0 \in \mathcal{I}_{==0}^{[\kappa]}$ while $m^{[\kappa-1]}(t_0) = 0$.*

6. Conclusion

The present paper aimed at developing a new framework for the Wronskian theory through what we call a linear algebraic and null space viewpoint. The arguments can be summarized briefly as follows.

We first showed that when the Wronskian is identically zero, we can find two representations of vector-valued functions contained in the null space of the Wronski matrix. We then showed only through very simple linear algebraic arguments that the two representations are algebraically linearly dependent. This alignment property immediately led to a key identity relevant to the Wronski matrix when its determinant vanishes identically. This identity is a key also in the pioneering study on the Wronskian in [9], where the feature of its derivation in the present paper is that it was carried out only through a very straightforward linear algebraic and null space viewpoint and thus offers a very clear perspective. Combining these discussions with very fundamental results for the case of two functions leads readily to those results available in the literature, as discussed in the arguments up to Subsection 5.1. Due to the very clear perspective as well as straightforward treatment only through a linear algebraic viewpoint, these discussions are believed to be of pedagogical value, too.

In the last part of the paper, i.e., Subsection 5.2, we discussed further relevant issues on the Wronskian through the null space viewpoint. In the course of such a study, some relevant results in the literature were derived through this specific viewpoint and in a strengthened form. Furthermore, some interesting problems were suggested relevant to the interest in that subsection, which is directly related to the null space viewpoint. Some partial answers were given but it would be an interesting future topic to tackle unresolved issues so that a much clearer view as well as insight can be obtained about the properties of the null space of the identically vanishing Wronski matrix. Such an extended study might actually go more or less beyond the linear algebraic aspect and thus is beyond the scope of the present paper, possibly requiring more emphasis recovered on the calculus side behind the unsolved issues. Such a direction might include brigding some gap between the studies in [11] and [12].

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