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RESOLUTION THEOREM OF THE ALGEBRAIC K-THEORY AND ITS APPLICATIONS

Mariko Ohara

Abstract. The main objective is showing a variant of resolution theorem of connective Waldhausen K-theory in suitable situations. We construct two sequences of subcategories, one is consisting of finitely generated symmetric module spectra which satisfy the condition of the resolution theorem and the other is consisting of the corresponding ∞-subcategories of certain finitely generated module spectra. As an application, we construct a sequence of subcategories consisting of certain DG-modules satisfying the condition of a resolution theorem.

1. Introduction

A resolution theorem for connective K-theory, initiated by Quillen, is a powerful tool for calculating the Quillen K-theory. It is mainly applied to showing an equivalence between the Quillen K-theory of the category of finitely generated projective modules and the Quillen K-theory of the category of finite projective dimension or compact modules by constructing a sequence of subcategories between these two categories.

However, the Quillen resolution theorem only works for exact categories, with isomorphisms and admissible monomorphisms as cofibrations, and does not cover more general categories with cofibrations and weak equivalences. Therefore the several theorems, such as cofinality theorem in [34] and in [17] and a resolution theorem in [27], have ever been proposed, which take place of the Quillen resolution theorem in Waldhausen K-theory. As for cofinality theorem, cofinality condition is strong, so that, for example, the category of finitely generated modules of finite projective dimension and the full subcategory of finitely generated projective modules with arbitrary weak equivalences do not satisfy in general.

In this paper, we use a notion of left and right Waldhausen categories arising from fibrations and cofibrations and use right and left algebraic K-theory introduced in [11] and paraphrasing the notations between simplicial categories and ∞-categories. We set up some assumptions and a technique to demonstrate a variant of resolution theorem as in [27] with respect to Tor-amplitude for the left algebraic K-theory, and as a special case, construct two sequences of subcategories, one is consisting of subcategories of

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certain finitely generated symmetric module spectra which satisfy the condition of a resolution theorem of Waldhausen K-theory and the other is consisting of the corresponding ∞-subcategories of certain finitely generated module spectra. We see that the Waldhausen K-theory of each subcategory of certain symmetric module spectra is equivalent to the algebraic K-theory of the corresponding ∞-category of certain module spectra. As a consequence, we show that the algebraic K-theory of the ∞-category of finitely generated projective module spectra is equivalent to that of ∞-category of perfect module spectra as in Corollary 5.8. Also, as an application of resolution theorem, we have the following : let R be a regular $\mathbb{E}_1$-ring with only finitely many non-zero homotopy groups. Then, there is a weak equivalence $K(\mathrm{LMod}_R^{proj}) \simeq K(\mathcal{P}_{\pi_0 R})$. Here, we denote by $\mathcal{P}_{\pi_0 R}$ the category of finitely generated projective $\pi_0 R$-modules. This is a variant of a consequence in [5]. As another application for constructing such sequences, we also construct such sequences in the category of DG modules over a commutative ring with the projective model structure. As a consequence, we give the comparison of the algebraic K-theory of the category of perfect modules and finitely generated semi-projective modules as in Proposition 6.1. The projective model category of DG modules is Quillen equivalent to the stable model category of module spectra over the corresponding Eilenberg-Maclane ring spectra.

Let R be a cofibrant fibrant symmetric ring spectrum, $\mathcal{M}_R$ the category of symmetric left R-module spectra endowed with the stable model structure and $W_{\mathcal{M}_R}$ the subcategory of weak equivalences. Then, Lurie [23, Theorem 1.3.4.20, Example 4.1.7.6] shows that there is an equivalence of ∞-categories

$$\mathrm{N}((\mathcal{M}_R)^c)[W_{\mathcal{M}_R}^{-1}] \simeq \mathrm{LMod}_{R'},$$

where we denote by $(-)^c$ the subcategory of cofibrant objects, LMod_R by ∞-category of left R-modules and R' is an image in Sp corresponding to R, which becomes an $\mathbb{E}_1$-ring [23, Theorem 4.3.3.17]. As a variant of this consequence, we relate a full subcategory of the model category $\mathcal{M}_R$ with full ∞-subcategory of LMod_R as in Proposition 3.2.

If a saturated Waldhausen category $(\mathcal{C}, w)$ admits a mapping cylinder functor then the Segal-Waldhausen construction $wS_\bullet\mathcal{C}$ of $(\mathcal{C}, w)$ is homotopy equivalent to the S'-construction $wS'_\bullet\mathcal{C}$ of $(\mathcal{C}, w)$ cf. [9, Theorem 2.9].

Moreover, the K-theory given by S' construction [9] is equivalent to the algebraic K-theory of the underlying ∞-category in this case. By using this consequence, we have an equivalence between connective K-theory of suitable Waldhausen categories and that of their underlying ∞-categories.

We check that the left algebraic K-theory of each subcategory of $\mathcal{M}_R$, which is not a model category itself, which appears in the sequence is equivalent to the algebraic K-theory of the corresponding ∞-category of certain module spectra.

We show an equivalence between a sequences of certain subcategories in $\mathcal{M}_R$ and that in LMod_R, modify and apply a resolution theorem to the sequence of subcategories in $\mathcal{M}_R$.

Let R be a connective $\mathbb{E}_1$-ring and M an R-module. We say that M is finitely generated projective if it is a retract of a finitely generated free R-module [23, Proposition 7.2.2.7]. We denote by LMod_R^{proj} the ∞-category of finitely generated projective R-modules.

Let LMod_R^{perf} be the smallest stable full ∞-subcategory of ∞-category LMod_R of left R-modules which contains R and is closed under retracts [23, Definition 7.2.4.1]. We say that an R-module M in LMod_R is a perfect if it belongs to LMod_R^{perf}.

A connective ring spectrum R, or a connective $\mathbb{E}_1$-ring R, is *left coherent* if $\pi_0 R$ is left coherent (i.e. every finitely generated ideal is finitely presented as left $\pi_0 R$-module) and $\pi_n R$ is finitely presented left $\pi_0 R$-module for $n \geq 0$. If R is a left coherent $\mathbb{E}_1$-ring, by [23, Proposition 7.2.4.23 (4), Proposition 7.2.4.17], the condition of perfectness is described by the condition on homotopy groups.

Theorem 1.1. *Let R be a left coherent cofibrant $\mathbb{E}_1$-ring. Let $\mathcal{M}_R$ be the category of symmetric left R-module spectra. Let LMod_R be the ∞-category of left R-modules.*

(i) We obtain a sequence of subcategories $\mathcal{M}_R^{0,p} \subset \cdots \subset \mathcal{M}_R^{n,p} \subset \mathcal{M}_R^{n+1,p} \subset \cdots$ of $\mathcal{M}_R$ and a sequence of ∞-subcategories $\mathrm{LMod}_R^{proj} \simeq \mathrm{LMod}_R^{0,p} \to \cdots \mathrm{LMod}_R^{n,p} \to \mathrm{LMod}_R^{n+1,p} \to \cdots$ of LMod_R such that the underlying ∞-category of $\mathcal{M}_R^{n,p}$ is $\mathrm{LMod}_R^{n,p}$. Moreover, we have an equivalence $K^R((\mathcal{M}_R^{n,p})^c) \simeq K(\mathrm{LMod}_R^{n,p})$ and $K^L((\mathcal{M}_R^{n,p})^f) \simeq K(\mathrm{LMod}_R^{n,p})$ of connective right and left K-theory.

(ii) The inclusions of full subcategories induce equivalences $K(\mathcal{M}_R^{n,p}) \simeq K(\mathcal{M}_R^{n+1,p})$ and $K(\mathrm{LMod}_R^{n,p}) \simeq K(\mathrm{LMod}_R^{n+1,p})$ of right and left connective K-theory, respectively.

We remark Lurie also shows the equivalence $K(\mathrm{LMod}_R^{proj}) \simeq K(\mathrm{LMod}_R^{perf})$ of K-theory in completely ∞-categorical setting for a connective $\mathbb{E}_1$-ring R in his lecture [25]. It is a different kind of proof from ours. As an advantage of our construction, we can calculate the algebraic K-theory of other module categories by constructing a certain sequence of subcategories in other module categories and comparing our sequence of subcategories with it.

As an application, we see how things go in DG-module case. By [31], there are Quillen equivalence between the category of HR-module spectra endowed with the stable model structure and the category of DG R-modules endowed with the projective model structure [21, Theorem 3.1]. The stable model structure is the same model structure as $\mathcal{M}_R$ in this paper if $R = HA$. Here, we identified a DG algebra A with the corresponding Eilenberg-Maclane spectrum HA. The underlying ∞-category is as in [23, Definition 1.3.5.8], which includes bounded derived category as fully faithful embedding [23, Proposition 1.3.5.24]. Note that we have a canonical equivalence $K(A) \simeq K(HA)$.

The important point here is an appropriate definition of projective dimension. There are two notions of this, which are due to Yakutiel, or due to Avramov and Foxby. We adopt that by Avramov and Foxby [1], because it is suitable for extensions. We check that the sequence of subcategories obtained via the projective dimension satisfies the resolution conditions described in Definition 3.

Proposition 1.2 (DG-module case Proposition 6.1)**.** *Let A be a Noetherian connective DG algebra over a field k such that its differential ∂, that decreases degree by one, sends the degree* 1 *part A_1 of A to zero.*

Let $\mathcal{C}_A$ be the category of DG A-modules and $\mathcal{C}_A^n$ the full subcategory consisting of DG A-module with the projective dimension is less than or equal to n. Then, we obtain the sequence $\mathcal{C}_A^0 \subset \cdots \mathcal{C}_A^n \subset \mathcal{C}_A^{n+1} \subset \cdots$ of full subcategories satisfies the Resolution conditions. The category $\mathcal{C}_A^0$ is equivalent to the category of finitely generated semi-projective DG A-modules.

We have a resolution theorem between the algebraic K-theory of finite semi-projective DG modules and DG modules of finite projective dimension, which is not stated explicitly ever.

Corollary 1.3. *Assume that A is Noetherian connective DG algebra over k such that its differential ∂ sends the degree* 1 *part A_1 of A to zero. The algebraic K-theory of the category of perfect A-modules is equivalent to the algebraic K-theory of the category of finitely generated semi-projective A-modules.*

2. Right and left Waldhausen category

Let $\mathsf{C} = (\mathcal{C}, \mathcal{A}_{\mathsf{C}}, w_{\mathsf{C}})$ be a triple consisting of a pointed category $\mathcal{C}$ and subcategories $\mathcal{A}_{\mathsf{C}}$ and w_{C} of $\mathcal{C}$ where both subcategories have the same objects of $\mathcal{C}$. Here, these subcategories are assumed to have a specific 0, which is preserved by inclusions of subcategories.

The definition of cofibrations and fibrations are as follows.

A class of cofibrations is a class of morphisms in $\mathcal{C}$ which satisfies the following conditions:

(i) $0 \to X$ is a cofibration for any object X, where 0 stands for a specific zero object in C.
(ii) The class $\mathcal{A}_{\mathsf{C}}$ of cofibrations includes isomorphisms,
(iii) Any composition of cofibrations is a cofibration,
(iv) For a cofibration $X \to Y$ and a morphism $X \to Z$, there exists a cocartesian square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \xrightarrow{f} & W, \end{array}$$

in which the morphism f is a cofibration.

A class of fibrations is a class of morphisms in $\mathcal{C}$ whose image in the opposite category $\mathcal{C}^{op}$ satisfies the axiom of a class of cofibrations.

According to Cisinski [11], we will introduce right and left Waldhausen categories. We say that $\mathsf{C} = (\mathcal{C}, \mathcal{A}_{\mathsf{C}}, w_{\mathsf{C}})$ is a right Waldhausen category if the triple C is a category with cofibrations and weak equivalences in the sense of Waldhausen category [34]. A morphism $f \colon x \to y$ in $\mathcal{A}_{\mathsf{C}}$ is called a cofibration and a morphism in w_{C} is called a weak equivalence. If the triple of opposite categories $\mathsf{C}^{op} = (\mathcal{C}^{op}, \mathcal{A}_{\mathsf{C}}^{op}, w_{\mathsf{C}}^{op})$ is a right Waldhausen category, we say that C is a left Waldhausen category.

For a right Waldhausen category, a sequence of composable morphism $x \overset{i}{\to} y \overset{p}{\to} z$ in C will be called a cofibration sequence if pi is the zero morphism and if i is a cofibration and the canonical morphism $y \coprod_x 0 \to z$ induced from the universal property of cofiber products is an isomorphism.

Dually, for a left Waldhausen category $\mathsf{C}' = (\mathcal{C}', \mathcal{A}_{\mathsf{C}'}, w_{\mathsf{C}'})$, a morphism $f \colon x \to y$ in $\mathcal{A}_{\mathsf{C}'}$ is called a fibration and a morphism in $w_{\mathsf{C}'}$ is called a weak equivalence and a sequence of composable morphism $x \overset{i}{\to} y \overset{p}{\to} z$ in C' will be called a fibration sequence if pi is the zero morphism and if p is a fibration and the canonical morphism $x \to 0 \times_z y$ induced from the universal property of fiber products is an isomorphism.

If C and C' essentially small and a right and left Waldhausen category, respectively, we define $K^R(\mathsf{C})$ and $K^L(\mathsf{C}')$ to be the connective right and left algebraic K-theory of C and C', by setting $K^R(\mathsf{C}) = \Omega|w_{\mathsf{C}} S_\bullet \mathcal{C}|$ and $K^L(\mathsf{C}') = \Omega|w_{\mathsf{C}'}^{op} S_\bullet \mathcal{C}'^{op}|$, which is the loop space of the geometric realization of pointed simplicial category $w_{\mathsf{C}} S_\bullet \mathcal{C}$ and $w_{\mathsf{C}'}^{op} S_\bullet \mathcal{C}'^{op}$, where S stands for Segal S-construction in [34].

We say that a full subcategory $\mathcal{D} \hookrightarrow \mathcal{C}$ is replete if any object x in C such that it is isomorphic to an object y in $\mathcal{D}$ is also in $\mathcal{D}$.

For a triple $(\mathcal{C}, \mathcal{A}_{\mathsf{C}}, w_{\mathsf{C}})$ and a full subcategory $\mathcal{D} \hookrightarrow \mathcal{C}$, we say that $\mathcal{D}$ is w_{C}-closed in $\mathcal{C}$ if, for any object C in $\mathcal{C}$, if there exists an object D in $\mathcal{D}$ and if there exists a zig-zag sequence of morphisms in w_{C} which connects C and D, then C is in $\mathcal{D}$.

Here, we give some example of cofibrations.

Example 2.1. We say that a cofibration $f\colon x \to y$ in $\mathcal{C}$ is $\mathcal{D}$-admissible if x, y and $y \coprod_x 0$ are in $\mathcal{D}$. We denote the class of all $\mathcal{D}$-admissible cofibrations by $\mathcal{C}of_{\mathsf{C},\mathcal{D}}$. Then a replete full subcategory $\mathcal{D}$ becomes a right Waldhausen category of C by setting the triple $(\mathcal{D}, \mathcal{C}of_{\mathsf{C},\mathcal{D}}, w_{\mathsf{C}} \cap \mathcal{D})$. We say this subcategory an admissible right Waldhausen subcategory.

Dually, we say that a fibration $f\colon x \to y$ in $\mathcal{C}$ is $\mathcal{D}$-admissible if x, y and $x \times_y 0t$ are in $\mathcal{D}$.

Example 2.2. Furthermore, if the subcategory $\mathcal{C}of_{\mathsf{C},\mathcal{D}}$ is just $\mathcal{C}of_{\mathsf{C}} \cap \mathcal{D}$, we say that the triple $(\mathcal{D}, \mathcal{C}of_{\mathsf{C}} \cap \mathcal{D}, w_{\mathsf{C}} \cap \mathcal{D})$ is a strictly right Waldhausen subcategory of C.

Example 2.3. We say that a cofibration $i\colon x \to y$ is split if it is of the form $x \to x \coprod y/x$ where y/x stands for $y \coprod_x 0$.

Example 2.4. Recall that a model category $\mathsf{M} = (\mathcal{M}, \mathcal{C}of_{\mathsf{M}}, \mathcal{F}ib_{\mathsf{M}}, w_{\mathsf{M}})$ is pointed if $\mathcal{M}$ has a zero object. We fix a specfic zero object 0 in $\mathcal{M}$. Let $\mathcal{C}$ be a full subcategory of $\mathcal{M}$. We write $\mathcal{C}^c$ and $\mathcal{C}^f$ for the full subcategory of $\mathcal{C}$ consisting of all cofibrant objects and fibrant objects in $\mathcal{C}$, respectively. Note that the triple $\mathsf{M}^c = (\mathcal{M}^c, \mathcal{C}of_{\mathsf{M},\mathcal{M}^c}, w_{\mathsf{M}\cap\mathcal{M}^c})$ is a right Waldhausen category and $\mathsf{M}^f = (\mathcal{M}^f, \mathcal{F}ib_{\mathsf{M},\mathcal{M}^f}, w_{\mathsf{M}\cap\mathcal{M}^f})$ is a left Waldhausen category, respectively.

Example 2.5 (cf. [30])**.** We define a cofibration and fibration in the full subcategory $\mathcal{C}$ of stable model category $\mathcal{M}$ as follows : we define a cofibration $X \to Y$ in $\mathcal{C}$ if it is a cofibration in $\mathcal{M}$ in the sense of model category and its cofiber lies in $\mathcal{C}$. Then, we have a pushout diagram for a cofibration $i : A \to B$ and an arbitrary map $A \to C$

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ \downarrow & & \downarrow \\ C & \xrightarrow{i'} & B \coprod_A C \end{array}$$

in $\mathcal{C}$ since we have an isomorphism of cokernel i and i', i.e., $(B \coprod_A C) \coprod_C 0 \cong B \coprod_A 0$. Therefore, $\mathcal{C}^c$ is a right Waldhausen category with the cofibrations.

We also define a fibration as a dual notion and the class of fibrations make $\mathcal{C}$ a left Waldhausen category.

For example, the full subcategory of finitely generated projective modules in the category of finitely generated modules becomes admissible right Waldhausen subcategory in the sense of Example 1. On the other hand, the full subcategory of finitely generated modules of finite projective dimension in the category of finitely generated modules becomes both admissible right Waldhausen subcategory and strictly Waldhausen subcategory. In this case, the algebraic K-theories of admissible and strictly right Waldhausen category are known to be equivalent [24, Theorem 10]. Lurie [25] also showed this equivalence in the setting of ∞-category in his unpublished note.

For a right Waldhausen subcategory $\mathcal{D}$ in C, we denote the connective right algebraic K-theory of the triple $(\mathcal{D}, \mathcal{C}of_{\mathsf{C},\mathcal{D}}, w_{\mathsf{C}} \cap \mathcal{D})$ by $K^R(D)$ in this paper.

For a left Waldhausen category $\mathsf{C}' = (\mathcal{C}', \mathcal{F}ib_{\mathsf{C}'}, w_{\mathsf{C}'})$ and a replete full subcategory $\mathcal{D}' \hookrightarrow \mathcal{C}'$, similarly we can define the notions of $\mathcal{D}'$-admissible fibrations, left Waldhausen subcategories of $\mathcal{C}'$ and strictly left Waldhausen subcategories of $\mathcal{C}'$ and the connective left algebraic K-theory $K^L(\mathcal{D}')$.

2.1. S'**-construction.** Let $\mathcal{C}$ be a right Waldhausen category. A map $f : A \to B$ is said to be a weak cofibration if it has a zig-zag of weak equivalences to a cofibration in $\mathrm{Fun}([1], \mathcal{C})$. We denote the full subcategory of $\mathrm{Fun}([1], \mathcal{C})$ consisting of all weak cofibrations by $W - \mathcal{C}of_{\mathcal{C}}$.

Similarly, let $\mathcal{C}'$ be a pointed category endowed with fibrations. A map $f : A \to B$ in $\mathcal{C}'$ is said to be a weak fibration if it has a zig-zag of weak equivalences to a fibration in $\mathrm{Fun}([1], \mathcal{C})$. We denote the full subcategory of $\mathrm{Fun}([1], \mathcal{C})$ consisting of all weak fibrations by $W - \mathcal{F}ib_{\mathcal{C}'}$. It is the dual notion of weak cofibrations defined in [9, Definition 2.2].

From the functors $[0] \to [1]$, $0 \mapsto 0$ and $0 \mapsto 1$, we obtain the functors $s, t \colon \mathrm{Fun}([1], \mathcal{C}) \to \mathcal{C}$. Let $\mathcal{K}$ be a full subcategory of $\mathrm{Fun}([1], \mathcal{C})$. We say that $\mathcal{C}$ admits right functorial factorization for $\mathcal{K}$ if there exists a functor $T \colon \mathcal{K} \to \mathcal{C}$ and natural transformations $i \colon s \to T$ and $p \colon T \to t$ such that for any object $f \colon x \to y$ in $\mathcal{K}$, $i_f \colon x \to T(f)$ is a cofibration and $p_f \colon T(f) \to y$ is a weak equivalence and $f = p_f i_f$.

We say that $\mathcal{C}$ admits left functorial factorization for $\mathcal{K}$ if there exists a functor $T \colon \mathcal{K} \to \mathcal{C}$ and natural transformations $i \colon s \to T$ and $p \colon T \to t$ such that for any object $f \colon x \to y$ in $\mathcal{K}$, $i_f \colon x \to T(f)$ is a weak equivalence and $p_f \colon T(f) \to y$ is a fibration and $f = p_f i_f$. Especially, we say that a pointed category $\mathcal{C}$ defined above admits the left functorial factorization of weak fibrations if any weak fibration is factored functorially as a weak equivalence followed by a fibration in $\mathcal{C}$.

Definition 1 ([10], Theorem 6.4)**.** A category with weak equivalences is called saturated if a map is a weak equivalence if and only if it is an isomorphism in the homotopy category.

In the case of Waldhausen category with functorial mapping cylinders for weak cofibrations, the property of being saturated is equivalent to the condition that the weak equivalences satisfying the two out of six property.

Remark ([8] 2.4, [9] 5.5, 6.4). We note that categories with weak equivalences satisfying homotopy calculi of fractions admit coincide model of mapping spaces in the Dwyer-Kan simplicial localization. Especially, the homotopy additivity is preserved via the localiztion with respect to weak equivalences.

In full subcategory of model category cases, if we require functorial mapping cylinders for maps that are equivalent via a zig-zag of weak equivalences to a cofibration, then the category has a homotopy calculus of left fractions.

Let $\mathcal{C}$ be a Waldhausen category that admits the right functorial factorization of weak cofibrations and saturated. We have the Waldhausen S-construction $S_\bullet\mathcal{C}$, and also have $S'_\bullet$-construction $S'_\bullet\mathcal{C}$ by using weak cofibrations defined in [9, Definition 2.7] as follows.

For each integer n, let $Ar_{[n]}(\mathcal{C})$ be the full subcategory of $\mathrm{Fun}(\mathrm{N}\{(i,j) \in [n] \times [n] : i \leq j\}, \mathcal{C})$ spanned by those functors X satisfying the following three conditions:

- For each $i \leq j \leq k$, the natural map $X(i,j) \to X(i,k)$ is a weak cofibration.
- For each i, the object $X(i,i)$ is 0.
- For each $i \leq j \leq k$, the diagram

$$\begin{array}{ccc} X(i,j) & \longrightarrow & X(i,k) \\ \downarrow & & \downarrow \\ 0 & \xrightarrow[g]{} & X(j,k) \end{array}$$

  is a pushout square.

We can also define $wS'_n\mathcal{C}$ by the nerve of the category of weak equivalences in $S'_n\mathcal{C} = w_{\mathcal{C}}Ar_{[n]}(\mathcal{C})$. We denote by $wS'_\bullet\mathcal{C}$ the bisimplicial set which sends $[n] \in \Delta^{op}$ to $wS'_n\mathcal{C}$.

Proposition 2.6 ([9] Theorem 2.9)**.** *If $\mathcal{C}$ satisfies the right functorial factorization for weak cofibrations and saturated, the inclusion $wS_\bullet\mathcal{C} \to wS'_\bullet\mathcal{C}$ induces a weak equivalence $wS_\bullet\mathcal{C} \to wS'_\bullet\mathcal{C}$ of bisimplicial set. This means that we have a weak equivalence $wS_n\mathcal{C} \to wS'_n\mathcal{C}$ for each $n \geq 0$.*

□

2.2. **Notion of homotopically small model.** Since the algebraic K-theory is defined via taking simplicial set, it is basically defined for (essentially) small category with some finiteness with respect to colimits. We need to take a small model.

Note that, we take homotopically small model for just a category with weak (co)fibrations, to proceed S'-construction and compare the algebraic K-theory of ∞-category. We do not require the homotopicall small model is a Waldhausen category.

We fix two uncountable strongly inaccessible cardinals $\kappa_0 < \kappa_1$ and the corresponding universes $\mathbb{U} \in \mathbb{V}$. We say that a mathematical object T is small if all the data defining T is collected by sets isomorphic to elements of $\mathbb{U}$, and large if all the data defining T is collected by sets isomorphic to elements of $\mathbb{V}$.

Definition 2 (cf. [29])**.** We say that a replete right and left Waldhausen full subcategory $\mathcal{C}$ of a simplicial model category M is a homotopically essentially small if there exists a small $W_{\mathcal{C}}$-closed full subcategory $\mathcal{C}'$ of $\mathcal{C}$ such that

(i) the inclusion functors ${\mathcal{C}'}^c \hookrightarrow \mathcal{C}^c$ and ${\mathcal{C}'}^f \hookrightarrow \mathcal{C}^f$ induce equivalences of categories $ho({\mathcal{C}'}^c) \simeq ho(\mathcal{C}^c)$ and $ho({\mathcal{C}'}^f) \simeq ho(\mathcal{C}^f)$ as large mathematical objects,

(ii) the inclusion functor induces equivalences $\mathrm{N}_\Delta({\mathcal{C}'}^{cf}) \simeq \mathrm{N}({\mathcal{C}'}^c)[W_{\mathcal{C}'}^{-1}] \simeq \mathrm{N}(\mathcal{C}^c)[W_{\mathcal{C}}^{-1}]$ of simplicial sets and both hands sides are ∞-categories [23, Example 4.1.7.6] with weak cofibrations and weak fibrations via cofibrations and fibrations in $\mathcal{C}$.

In this case, we call $\mathcal{C}'$ a homotopically essentially small model of $\mathcal{C}$.

In this case, we can define $K^R(\mathcal{C}^c)$ to be the connective right (respectively left) algebraic K-theory of $\mathcal{C}^c$ by setting $K^R(\mathcal{C}^c) = \Omega|wS'_\bullet\mathcal{C}'|$. Similary, we can define the connective left algebraic K-theory by $K^L(\mathcal{C}^f) = \Omega|w^{op}S'_\bullet{\mathcal{C}'}^{op}|$. If the Waldhausen category $\mathcal{C}$ satisfies the conditions in Remark 2.1, by approximation theorem, it turns out that this definition does not depend upon a choice of $\mathcal{C}'$ up to homotopy equivalence.

3. Comparison with algebraic K-theory for ∞-categories

Lurie gave a definition of the algebraic K-theory for pointed ∞-category as follows.

An ∞-category with cofibrations is a pointed ∞-category $\mathcal{C}$ with a distinguished class of morphisms, which we will call ∞-cofibrations, which satisfy the following axioms:

- All equivalences are ∞-cofibrations and the collection of ∞-cofibrations is closed under composition.

- For every object X in $\mathcal{C}$, the canonical map $0 \to X$ is an ∞-cofibration.
- For an ∞-cofibration $f : X \to X'$ and an arbitrary map $X \to Y$, there exists a pushout square

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow & & \downarrow \\ Y & \xrightarrow[g]{} & Y', \end{array}$$

  and the map g is also an ∞-cofibration.

Let $\mathcal{C}$ be an ∞-category with ∞-cofibrations. For each integer n, we let $Gap_{[n]}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{N}\{(i,j) \in [n] \times [n] : i \leq j\}, \mathcal{C})$ spanned by those functors X satisfying the following three conditions:

- For each $i \leq j \leq k$, the natural map $X(i,j) \to X(i,k)$ is an ∞-cofibration.
- For each i, the object $X(i,i)$ is 0.
- For each $i \leq j \leq k$, the diagram

$$\begin{array}{ccc} X(i,j) & \longrightarrow & X(i,k) \\ \downarrow & & \downarrow \\ 0 & \xrightarrow[g]{} & X(j,k) \end{array}$$

  is a pushout square.

Note that an object X of $Gap_{[n]}(\mathcal{C})$ is determined up to unique homotopy.

Let $\mathcal{C}$ be an ∞-category with ∞-cofibrations. We let $S_\bullet(\mathcal{C})$ denote the simplicial space given by the formula $S_n(\mathcal{C}) = Gap_{[n]}(\mathcal{C})^{\simeq}$. Let $K(\mathcal{C})$ denote the space given by $\Omega|S_\bullet(\mathcal{C})|$, and we refer it to the right algebraic K-theory for the ∞-category $\mathcal{C}$.

We can also consider a notion of an ∞-category with ∞-fibrations and their left algebraic K-theory for the ∞-category.

Here is a comparison theorem with the algebraic K-theory for ∞-categories as follows.

For a pair of simplicial sets X and Y, we denote the function space from X to Y by $\mathrm{Fun}(X,Y)$.

Let Z be a simplicial subset of X. We say that Z is 0-full if for any n-simplex x in X, x is in Z if and only if all vertices of x is in Z.

We say that Z is 1-full if for any n-simplex x in X $(n \geq 1)$, x is in Z if and only if all edges of x is in Z. For a pair of vertices x and y in a simplicial set X, we write $X(x,y)$ for the simplicial subset of $\mathrm{Fun}(\Delta[1], X)$ consisting of those n-simplexes $f\colon \Delta[n] \times \Delta[1] \to X$ such that $f|_{\Delta[n]\times\{0\}} = \phi_n^* x$ and

$f|_{\Delta[n]\times\{1\}} = \phi_n^* y$ where $\phi_n^*\colon X_0 \to X_1$ stands for the induced map from the canoical map $\phi_n\colon [n] \to [0]$.

Proposition 3.1 ([16] 4.18, [7] 7.11, [11] 4.10, 4.11)**.** *Let $\mathsf{C} = (\mathcal{C}, \mathcal{C}of_{\mathsf{C}}, w_{\mathsf{C}})$ be a small right Waldhausen category. Assume that there exists a model category $\mathsf{M} = (\mathcal{M}, \mathcal{C}of_{\mathsf{M}}, \mathcal{F}ib_{\mathsf{M}}, w_{\mathsf{M}})$ and $\mathcal{C}$ is a right Waldhausen full subcategory of M such that $\mathcal{C} \hookrightarrow \mathcal{M}^c$. In this case, we have $\mathrm{N}(\mathcal{C})[w_{\mathsf{C}}^{-1}] \simeq \mathrm{N}_\Delta(L^H(\mathcal{C}, w_{\mathsf{C}})^{fib})$ where $L^H(\mathcal{C}, w_{\mathsf{C}})$ is a hammock localization of $\mathcal{C}$ with respect to w_{C} and $(-)^{fib}$ stands for the fibrant replacement with respect to Bergner model structure on the category of small simplicial categories. Here, N_Δ is the homotopy coherent nerve given in* [22, Definition 1.1.5.5] *and* [23, Notation 1.3.4.11]*. We define $\mathcal{C}of_{\mathrm{N}(\mathcal{C})[w_{\mathsf{C}}^{-1}]}$ to be the smallest 1-full subcategory of $\mathrm{N}(\mathcal{C})[w_{\mathsf{C}}^{-1}]$ that is closed under homotopy relations and contains the equivalences and the image of weak cofibrations from $\mathcal{C}$.*

If C admits a functorial factorization for weak cofibrations in $\mathcal{C}$ and w_{C} satisfies two out of six property, then we define the ∞-category $\mathrm{N}(\mathsf{C})[w_{\mathsf{C}}^{-1}] = (\mathrm{N}(\mathcal{C})[w_{\mathsf{C}}^{-1}], \mathcal{C}of_{\mathrm{N}(\mathcal{C})[w_{\mathsf{C}}^{-1}]})$ with ∞-cofibrations and there exists a natural homotopy equivalence of spectra

$$K^R(\mathsf{C}) \simeq K^R(\mathrm{N}(\mathsf{C})[w_{\mathsf{C}}^{-1}])$$

□

Let $\mathcal{C}$ be a pointed ∞-category and let P be a property of objects in $\mathcal{C}$.

Assume that there exists a simplicial (small) model category $\mathcal{M}$ and a weakly categorical equivalence $f\colon \mathrm{N}_\Delta\mathcal{M}^{cf} \to \mathcal{C}$. We say that a property P is stable under equivalences if an object x in $\mathcal{C}$ has the propery P and an object y is equivalent to x, then y also has the property P. Assume that P is stable under equivalences.

We say that an object x in $\mathcal{M}^c$ has the property P via f if for a fibrant replacement x' of x, $f(x')$ has a property P. This definition does not depend upon a choice of x'. We denote the 0-full subcategory of $\mathcal{C}$ and $\mathcal{M}^c$ spanned by all objects which have the property P by $\mathcal{C}_P$ and $\mathcal{M}^c_P$ respectively.

Proposition 3.2. *Assume that we have a weakly categorical equivalence $f\colon \mathrm{N}_\Delta\mathcal{M}^{cf} \to \mathcal{C}$ and $\mathcal{M}$ be a simplicial model category. Assume that $\mathcal{M}$ admits a functorial factorization for weak cofibrations in $\mathcal{M}^c$ and $w_{\mathcal{M}}$ satisfies two out of six property. We have the following:*

(i) f induces a weakly categorical equivalence of ∞-categories $N_\Delta\mathcal{M}^{cf}_P \simeq \mathcal{C}_P$

(ii) If $\mathcal{M}^c_P$ is a right Waldhausen subcategory of $\mathcal{M}$ and $\mathcal{C}_P$ is closed under weak equivalences and finite colimits, then f induces a homotopy equivalence $K^R(\mathcal{M}^c_P) \to K^R(\mathcal{C}_P)$ on connective K-theory. Here a class of

∞-cofibrations in $\mathcal{C}_P$ is spanned by the image of weak cofibrations in $\mathcal{M}_R^{cf}$ via f.

To prove this proposition, we need some notations.

Let X and Y be a pair of simplicial sets and let f and $g\colon X \to Y$ be simplicial maps. Recall that a natural transformation from f to g is a simplicial map $\theta\colon X \times \Delta[1] \to Y$ such that $\theta|_{X\times\{0\}} = f$ and $\theta|_{X\times\{1\}} = g$. For any vertex x in X, we write θ_x for $\theta(\phi_1^*(x), id_{[1]})$ where $\phi_1^*\colon X_0 \to X_1$ stands for the induced map from the canonical map $\phi_1^*\colon [1] \to [0]$.

A natural transformation θ from f to g is a natural equivalence if θ_x is an isomorphism in $\tau^1 Y$ for any vertex x in X. Here τ^1 stands for the left adjoint functor of the classical nerve functor $N\colon \mathrm{Cat} \to \mathrm{Set}_\Delta$.

A simplicial map $f\colon X \to Y$ is a strongly categorical equivalence if there exists a simplicial map $g\colon Y \to X$ and natural equivalences $id_X \to gf$ and $fg \to id_Y$. If both X and Y are ∞-categories, it was shown by Joyal that f is a strongly categorical equivalence if and only if it is a weakly categorical equivalence.

Proof. First of all, we will show the following assertion.

Let $f\colon X \to Y$ be a strongly categorical equivalence between simplicial sets and let S be a set of vertices in Y which is stable under equivalences. Then the restriction of f to $\underline{f^{-1}(S)}$ induces a strongly categorical equivalence $\underline{f^{-1}(S)} \to \underline{S}$. Here, $\underline{f^{-1}(S)}$ and $\underline{S}$ be the 0-full simplicial subsets spanned by $f^{-1}(S)$ in X and S in Y, respectively.

For this, let $g\colon Y \to X$ be a simplicial map such that there are natural equivalences $\theta\colon id_X \to gf$ and $\eta\colon fg \to id_Y$. We claim that g induces a simplicial map $\underline{S} \to \underline{f^{-1}(S)}$. Note that for any vertex $x \in S$, x and $fg(x)$ are equivalent via η_x. Since S is stable under equivalences in Y, $fg(x)$ is in S and then $g(x)$ is in $f^{-1}(S)$, i.e., for any integer n and any n-simplex $x \in \underline{S}_n$, $g(x)$ is in $\underline{f^{-1}(S)}_n$. Since $\underline{S}$ and $\underline{f^{-1}(S)}$ is 0-full, g induces the simplicial map $\underline{S} \to \underline{f^{-1}(S)}$ which we denote by the same letter g for simplicity. Thus, the restrictions of θ and η are natural equivalences $id_X \to gf$ and $fg \to id_Y$ respectively.

Note that, for a small simplicial category $\mathcal{M}$, the essential image of nerve functor N_Δ is the 0-full simplicial subset spanned by the image of objects in $\mathcal{M}$. We apply these argument to the case $X = \mathrm{N}_\Delta\mathcal{M}^{cf}$, $Y = \mathcal{C}$ and $S = \mathcal{C}_P$. Thus, we complete the proof of the assertion (i).

The assertion (ii) in the proposition follows from the assertion (i) and Proposition 3.1. $\square$

4. Resolution theorem and comparison

Let $\mathsf{C} = (\mathcal{C}, \mathcal{F}ib_{\mathsf{C}}, w_{\mathsf{C}})$ be a left Waldhausen category. Let $\mathcal{D} \hookrightarrow \mathcal{C}$ be a replete full subcategory of $\mathcal{C}$.

Definition 3. We say that the inclusion functor $\mathcal{D} \hookrightarrow \mathcal{C}$ satisfies the resolution condition if it satisfies the following three conditions:

(Res1) $\mathcal{D}$ is closed under extensions in $\mathcal{C}$. Namely for a fibration sequence $x \to y \to z$ in $\mathcal{C}$, if x and z are in $\mathcal{D}$ respectively, then y is also in $\mathcal{D}$.

(Res2) For any object z in $\mathcal{C}$, there exists a fibration sequence $x \to y \to z$ in $\mathcal{C}$ such that y is in $\mathcal{D}$.

(Res3) For a fibration sequence $x \to y \to z$ in $\mathcal{C}$, if y and z are in $\mathcal{D}$, then x is also in $\mathcal{D}$.

Let w be a class of morphisms in a category $\mathcal{C}$. We say that w is a multiplicative system of $\mathcal{C}$ if w is closed under finite compositions and closed under isomorphisms. For a category $\mathcal{C}$ and a multiplicative system w of $\mathcal{C}$, we define the full subcategory $\mathcal{C}(m, w)$ of $\mathrm{Fun}_{\mathrm{Cat}}([m], \mathcal{C})$ of those functors which take values in w for each m.

We say that the inclusion functor $\tau\colon \mathcal{D} \hookrightarrow \mathcal{C}$ is satisfying *the strong resolution conditions* if for any non-negative integer m, $\mathcal{D}(m, w) \hookrightarrow \mathcal{C}(m, w)$ satisfies the resolution conditions.

Theorem 4.1 (cf. [27] Theorem 1.13)**.** *Let $\mathsf{C} = (\mathcal{C}, \mathcal{F}, w_{\mathsf{C}})$ be an essentially small left Waldhausen category and let $\mathcal{D} \hookrightarrow \mathcal{C}$ be a full subcategory of $\mathcal{C}$. Assume the inclusion functor $\mathcal{D} \hookrightarrow \mathcal{C}$ satisfies the strong resolution conditions. Then the inclusion functor induces a homotopy equivalence $K^L(\mathcal{D}; w_{\mathsf{C}} \cap \mathcal{D}) \to K^L(\mathsf{C})$ on connective left algebraic K-theory. Here, a morphism $f : X \to Y$ in $\mathcal{D}$ is a fibration if f is a fibration in $\mathcal{C}$ and the fiber of f is in $\mathcal{D}$.*

□

Let $\mathsf{C} = (\mathcal{C}, \mathcal{F}ib_{\mathsf{C}}, w_{\mathsf{C}})$ be an essentially small left Waldhausen category and let $\mathcal{D} \hookrightarrow \mathcal{C}$ be a replete full subcategory of $\mathcal{C}$. For a non-negative integer n and a full subcategory X of $\mathcal{C}$, we write $X(n, w)$ for the full subcategory $\mathrm{Fun}([n], X)$ the category of functors from the totally ordered set $[n] = \{k \in \mathbb{Z} | 0 \leq k \leq n\}$ to $\mathcal{C}$ consisting of those objects $x\colon [n] \to \mathcal{C}$ such that for any pair of integers $0 \leq i \leq j \leq n$, $x(i \leq j)$ is in w_{C}.

For an object x and a morphism $f\colon x \to y$ in $X(n, w)$ and for integers $0 \leq k \leq n$ and $0 \leq j \leq n-1$, we write x_k, f_k and i_j^x for $x(k)$, $f(k)$ and $x(j \leq j+1)$. If X is a left Waldhausen subcategory of C, then we can make $X(n, w)$ into a left Waldhausen category. Namely for a morphism $p\colon x \to y$ in $X(n, w)$ is a fibration (respectively weak equivalence) if and only if $p_k\colon x_k \to y_k$ is a fibration (respectively weak equivalence) in X for any $0 \le k \le n$.

Proposition 4.2. *Let $\mathsf{C} = (\mathcal{C}, \mathcal{F}ib_{\mathsf{C}}, w_{\mathsf{C}})$ be an essentially small left Waldhausen category and let $\mathcal{D} \hookrightarrow \mathcal{C}$ be a replete full subcategory of $\mathcal{C}$. we assume the following two conditions:*

(i) $\mathcal{C}$ satisfies the left functorial factorization for fibrations.
(ii) $\mathcal{D}$ is w_{C}-closed.

Then if $\mathcal{D} \hookrightarrow \mathcal{C}$ satisfies the resolution conditions in Definition 3, then $\mathcal{D} \hookrightarrow \mathcal{C}$ satisfies the strong resolution condition. In particular the inclusion functor $\mathcal{D} \hookrightarrow \mathcal{C}$ induces a homotopy equivalence $K^L(\mathcal{D}, w|_{\mathcal{D}}) \to K^L(\mathcal{C}, w)$ on connective left algebraic K-theory.

Proof. The last statement follows from [27, Theorem 1.13]. Since $K^L(\mathsf{C})$ is defined to be the opposite $\Omega|w_{\mathsf{C}}^{op} S_\bullet \mathcal{C}^{op}|$. What we need to show is that for any non-negative integer m, the inclusion functor $\mathcal{D}(m, w) \hookrightarrow \mathcal{C}(m, w)$ satisfies the resolution condition.

Take a fibration sequence $x \to y \to z$ in $\mathcal{C}(m, w)$. If x and z are objects in $\mathcal{D}(m, w)$, for each $0 \le i \le m$, x_i and z_i are in $\mathcal{D}$. By condition (Res 1), y_i is in $\mathcal{D}$ for each i. Thus, y is in $\mathcal{D}(m, w)$ and we verify condition (Res 1) for each m.

We show condition (Res3).

The condition (Res 3) is valid for $m = 0$. Take a fibration sequence $x \to y \to z \in \mathcal{C}(m, w)$.

$$\begin{array}{ccccccc}
x_0 & \longrightarrow & \cdots & \longrightarrow & x_{m-1} & \longrightarrow & x_m \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
y_0 & \longrightarrow & \cdots & \longrightarrow & y_{m-1} & \longrightarrow & y_m \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
z_0 & \longrightarrow & \cdots & \longrightarrow & z_{m-1} & \longrightarrow & z_m .
\end{array}$$

Here, the horizontal maps are weak equivalences. Assume that we have a fibration $y \to z$ in $\mathcal{D}(m, w)$, i.e., each $y_i \to z_i$ is a fibration in $\mathcal{D}$ for $1 \le i \le m$. By condition (Res 3) for $m = 0$ and the assumption (ii), we have a fibration sequence $x_i \to y_i \to z_i$ in the full subcategory $\mathcal{D}$ of $\mathcal{C}$ for each $0 \le i \le m$.

Now, we show condition (Res2). We proceed by induction on m. For $m = 0$, assertion follows from assumption that the inclusion functor $\mathcal{D} \to \mathcal{C}$ satisfies condition (Res 2). We assume that $m \ge 1$. Let x be an object in $\mathcal{C}(m, w)$ and we define x' to be an object in $\mathcal{C}(m-1, w)$ by setting $x' = x \circ \iota$ where $\iota\colon [m-1] \to [m]$ is the inclusion functor $\iota(k) = k$ for all $0 \le k \le m-1$.

By inductive hypothesis, we have the following diagram:

$$\begin{array}{ccccccc} y_0 & \longrightarrow & \cdots & \longrightarrow & y_{m-1} & & \\ \downarrow & & \downarrow & & \downarrow & & \\ x_0 & \longrightarrow & \cdots & \longrightarrow & x_{m-1} & \longrightarrow & x_m, \end{array}$$

where the vertical morphisms are fibration. Then by applying condition (i) to the composition $y_{m-1} \to x_{m-1} \to x_m$, it turns out that there exists y_m such that there is a fibration $p_m \colon y_m \to x_m$ in $\mathcal{C}$ and a week equivalence $i^y_{m-1} \colon y_{m-1} \to y_m$ such that $p_m i^y_{m-1} = i^x_{m-1} p_{m-1}$. Since y_{m-1} is in $\mathcal{D}$ and i^y_{m-1} is in w_{C}, y_m is in $\mathcal{D}$ by $w_{\mathcal{C}}$-closedness. We define y and $p : y \to x$ to be an object in $\mathcal{D}(m, w)$ and a fibration in $\mathcal{C}(m, w)$ by setting the following way:

$$y_k = \begin{cases} y_k & \text{if } 0 \le k \le m-1 \\ y_m & \text{if } k = m \end{cases} \qquad i^y_k = \begin{cases} i^y_k & \text{if } 0 \le k \le m-2 \\ i^y_m & \text{if } k = m \end{cases}$$

and

$$p_k = \begin{cases} p'_k & \text{if } 0 \le k \le m-1 \\ p_{n+1} & \text{if } k = m \end{cases}$$

Now we complete the proof. □

5. Relation between classical and ∞-categorical spectra

For a symmetric associative ring spectrum R, let $\mathcal{M}_R$ be a category which consists symmetric left R-module spectra and the morphisms compatible with the left R-module structure [19].

Since $\mathcal{M}_R$ is built via the sequences of simplicial sets and has the set of generating cofibrations and acyclic cofibrations by [26, Theorem 14.1], it is a combinatorial model category for suitable cardinal [19, Proposition 3.2.3.13]. Therefore, we can take a cofibrant and fibrant replacement functorially [22, Proposition 1.2.5].

Let A and B be cofibrant objects in $\mathcal{M}_R$. Let $f : A \to B$ be a morphism. A mapping cylinder gives a functorial factorization for $\mathcal{M}^c_R$, i.e., every map $f : A \to B$ factors through $A \to Mf \to B$, where the map $A \to Mf$ is a cofibration and $Mf \to B$ has a natural section $B \to Mf$, which is a weak equivalence. Moreover, Mf is a cofibrant object.

Dually, a mapping path object gives a functorial factorization for the opposite category $(\mathcal{M}_R)^{op}$, i.e., every map $f : A \to B$ factors through $A \to Nf \to B$ of the map $A \to Nf$ has a natural projection $Nf \to A$, which is a weak equivalence, and $Nf \to B$ is a fibration.

5.1. **Sequence of certain subcategories of R-modules.** According to [23, Definition 7.2.4.21], we define certain (∞-) categories $\mathrm{LMod}^{n,p}_R$ and $\mathcal{M}^{n,p}_R$, respectively, as follows.

Thoroughout this section, we assume that R is cofibrant fibrant in the model category of algebra objects in the category of the symmetric spectra.

Definition 4. Let R be a connective $\mathbb{E}_1$-ring.

(i) We say that a right R-module M is a discrete R-module if its homotopy group $\pi_n M$ vanishes if n is not equal to 0.

(ii) We say that a left R-module M in LMod_R (resp. in $\mathcal{M}_R$) has Tor-amplitude less than or equal to n if, for all $i > n$, $\pi_i(N \otimes_R M) = 0$ for any discrete right R-module N (resp. any discrete cofibrant fibrant right R-module N). We say that M has finite Tor-amplitude if there exists an integer n such that M has Tor-amplitude less than or equal to n.

(iii) For a left coherent $\mathbb{E}_1$-ring R, we write $\mathrm{LMod}_R^{n,p}$ for a full ∞-subcategory of LMod_R^{perf} consisting of the objects which are connective and have Tor-amplitude less than or equal to n. Here, LMod_R^{perf} is the smallest stable full ∞-subcategory of ∞-category LMod_R of left R-modules which contains R and is closed under retracts [23, Definition 7.2.4.1].

(iv) For a left coherent ring spectrum R, a full subcategory $\mathcal{M}_R^p \subset \mathcal{M}_R$ is defined by those connective left R-modules M such that $\pi_m M$ are finitely presented $\pi_0 R$-modules for every $m \in \mathbb{Z}$ and M have finite Tor-amplitude less than or equal to n.

(v) For a left coherent ring spectrum R, we define a category $\mathcal{M}_R^{n,p} \subset \mathcal{M}_R^p$ by a full subcategory of those connective left R-modules of Tor-amplitude less than or equal to n for fixed n.

We show the first properties on $\mathcal{M}_R^{n,p}$.

Lemma 5.1. *Let $M' \to M \to M''$ be a fiber sequence of R-modules in $(\mathcal{M}_R^p)^f$.*

(i) Assume that M' and M'' have Tor-amplitude less than or equal to n, and M has Tor-amplitude less than or equal to $n-1$. Then, M' has Tor-amplitude less than or equal to $n-1$.

(ii) Assume that M' and M'' have Tor-amplitude less than or equal to n. Then, M has Tor-amplitude less than or equal to n.

(iii) $(\mathcal{M}_R^{n,p})^f$ is closed under extension in $(\mathcal{M}_R^p)^f$.

Proof. Let N be a cofibrant fibrant discrete R-module. We prove that $\pi_k(N \otimes_R M') \simeq 0$ for $k \geq n$. We have an exact sequence of homotopy groups

$$\pi_{k+1}(N \otimes_R M'') \to \pi_k(N \otimes_R M') \to \pi_k(N \otimes_R M).$$

We prove the assertion (i). If $k \geq n$, $\pi_{k+1}(N \otimes_R M'')$ and $\pi_k(N \otimes_R M)$ vanish by assumption that M'' has Tor-amplitude less than or equal to n and M has Tor-amplitude less than or equal to $n-1$.

If $k \geq n+1$, $\pi_k(N \otimes_R M')$ and $\pi_k(N \otimes_R M'')$ vanish in the above exact sequence of homotopy groups. Therefore the assertion (ii) is proved.

For (iii), the assertion (ii) shows $(\mathcal{M}_R^{n,p})^f$ is closed under extension. □

Lemma 5.2. *(i) The subcategory $(\mathcal{M}_R^{n,p})^f \subset (\mathcal{M}_R)^f$ is $w_{\mathcal{M}_R^f}$-closed in $(\mathcal{M}_R)^f$.*

(ii) $(\mathcal{M}_R^{n,p})^f$ is a left Waldhausen category with left functorial factorization for weak fibrations.

Proof. Note that the condition of Tor-amplitude less than or equal to n is stable under weak equivalences and so $(\mathcal{M}_R^{n,p})^f$ is $w_{(\mathcal{M}_R^{n,p})^f}$-closed. The assertion (ii) follows from (i) and the existence of mapping path space. Note that a mapping path space of a map between fibrant objects is also fibrant. □

Proposition 5.3. *Let R be a left coherent $\mathbb{E}_1$-ring. We have an equivalence $\mathrm{N}((\mathcal{M}_R^{n,p})^c)[W_{\mathcal{M}_R}^{-1}] \simeq \mathrm{LMod}_R^{n,p}$.*

Proof. By Lemma 5.2, the inclusion $\mathcal{M}_R^{n,p} \subset \mathcal{M}_R$ satisfies the assumption of Proposition 3.2. □

Definition 5. We define a cofibration and a fibration in $(\mathcal{M}_R^{n,p})^c$ and $(\mathcal{M}_R^{n,p})^f$ as follows.

(i) We define a cofibration $X \to Y$ in $(\mathcal{M}_R^{n,p})^c$ if it is a cofibration in $\mathcal{M}_R^c$ in the sense of model category and its cofiber lies in $(\mathcal{M}_R^{n,p})^c$.

(ii) We define a fibration $Y \to Z$ in $(\mathcal{M}_R^{n,p})^f$ if it is a fibration in $\mathcal{M}_R^f$ in the sense of model category and its fiber lies in $(\mathcal{M}_R^{n,p})^f$.

Then, $(\mathcal{M}_R^{n,p})^c$ is a right Waldhausen category with the cofibrations and $(\mathcal{M}_R^{n,p})^f$ is a left Waldhausen category with the fibrations.

Note that, once we define the cofibrations and fibrations of $(\mathcal{M}_R^{n,p})^c$ and $(\mathcal{M}_R^{n,p})^f$, respectively, the weak cofibrations and weak fibrations in $(\mathcal{M}_R^{n,p})^{cf}$ are automatically determined.

Note that the mapping cylinder and mapping path space admits the functorial factorization of cofibrations and fibrations, respectively.

Thus, we obtain a left Waldhausen category $(\mathcal{M}_R^{n,p})^f$. We proceed the resolution theorem to this Waldhausen category in the next subsection.

5.2. **Resolution theorem for left algebraic K-theory.** Let R be a left coherent $\mathbb{E}_1$-ring.

Theorem 5.4. *Let $\mathcal{C}$ denote the category $\mathcal{M}_R^p$. Let $\mathcal{M}$ denote the category $\mathcal{M}_R$. We consider the pair $(\mathcal{M}, \mathcal{C})$, i.e., the pair $(\mathcal{M}_R, \mathcal{M}_R^p)$. Then we have the following :*

(i) $\mathcal{C}$ *is a homotopically essentially small model of* $\mathcal{M}$.

(ii) $\mathcal{C}$ *is* $w_{\mathcal{M}}$*-closed in* $\mathcal{M}$.

(iii) $\mathcal{C}$ *is closed under extensions with respect to fibration sequences in* $\mathcal{M}$.

(iv) For each integer n*, the inclusion functors* $\mathcal{M}_R^{n,p} \to \mathcal{M}_R^{n+1,p}$ *satisfies the resolution conditions. In particular, it induces an equivalence* $K^L(\mathcal{M}_R^{n,p}) \simeq K^L(\mathcal{M}_R^{n+1,p})$ *on connective* K*-theory.*

(v) By colimit argument, we obtain an equivalence $K^L(\mathcal{M}_R^{0,p}) \simeq K^L(\mathcal{M}_R^{p})$ *on left connective* K*-theory.*

Proof. Let us take a full subcategory $(\mathcal{M}_R^p)^{cf}$ of $(\mathcal{M}_R^p)^f$. We would recall the inclusion $(\mathcal{M}_R^p)^{cf} \hookrightarrow (\mathcal{M}^p)_R^f$ induces an equivalence $\mathrm{N}_\Delta((\mathcal{M}_R^p)^{cf}) \simeq \mathrm{N}((\mathcal{M}_R^p)^f)[W^{-1}]$ as simplicial sets (moreover ∞-categories [23, Example 4.1.7.6]) where LHS can be regarded as a fibrant replacement of RHS with respect to Bergner model structure on the category of simplicial model categories.

Let us take the full subcategory $\mathcal{M}$ of $(\mathcal{M}_R^p)^{cf}$ consisting of equivalence classes. Since the category of finitely generated $\pi_0(R)$-modules is essentially small, since the class of objects of $ho(\mathcal{M}_R^p)$ is small and since a family indexed by a small set is also small, $\mathcal{M}$ becomes a homotopically essentially small model of $(\mathcal{M}_R^p)^{cf}$ by [22, Proposition 5.4.1.2]. Thus, the Waldhausen categories $\mathcal{M}_R^p$ can be replaced by homotopically small Waldhausen category $\mathcal{M}_R^{p'}$ which has the K-theory equivalent to that of $\mathcal{M}_R^p$.

Since $(\mathcal{M}_R^{n,p})^f \subset (\mathcal{M}_R^p)^f \subset (\mathcal{M}_R)^f$ is a replete left Waldhausen full subcategories which are closed under weak equivalences and admit the left functorial factorizations for weak fibrations, respectively.

Therefore, it suffices to check that the inclusion $(\mathcal{M}_R^{n,p})^f \to (\mathcal{M}_R^{n+1,p})^f$ satisfies the resolution conditions in Definition 3. It follows from Proposition 4.2 and Lemma 5.2. □

By Proposition 3.1, we have the following.

Corollary 5.5. *Let* R *be a left coherent* $\mathbb{E}_1$*-ring. Let* $\mathrm{LMod}_R^{n,p} \subset \mathrm{LMod}_R^{n+1,p}$ *be the inclusion of* ∞*-categories. Then, the induced map* $K^L(\mathrm{LMod}_R^{n,p}) \to K^L(\mathrm{LMod}_R^{n+1,p})$ *is an equivalence.*

□

Lemma 5.6. *(i) We have* $K^L(\mathrm{LMod}_R^{perf}) \simeq K^R(\mathrm{LMod}_R^{perf})$*. The left algebraic* K*-theory* $K^L(\mathcal{M}_R^p)$ *is homotopy equivalent to the right algebraic* K*-theory* $K^R(\mathcal{M}_R^p)$.

(ii) We have an equivalence $K^R(\mathrm{LMod}_R^{proj}) \simeq K^L(\mathrm{LMod}_R^{proj})$*. The left algebraic* K*-theory* $K^L(\mathcal{M}_R^{0,p})$ *is homotopy equivalent to the right algebraic* K*-theory* $K^R(\mathcal{M}_R^{0,p})$.

Proof. The homotopy equivalence $K^L(\mathrm{LMod}_R^{perf}) \simeq K^R(\mathrm{LMod}_R^{perf})$ and $K^R(\mathrm{LMod}_R^{proj}) \simeq K^L(\mathrm{LMod}_R^{proj})$ may be known as the theory of exact ∞-category.

Since $\mathcal{M}_R$ is a stable model category and $\mathcal{M}_R^{0,p}$ and $\mathcal{M}_R^p$ is the $w_{\mathcal{M}_R}$-closed homotopy additive full subcategories, a square in $\mathcal{M}_R^{0,p}$ and $\mathcal{M}_R^p$ is homotopy cartesian if and only if it is homotopy cocartesian, respectively. The class of weak cofibrations in simplicial nerve of them are sent to the class of ∞-cofibrations in their underlying ∞-categories under the equivalences in Proposition 3.1.

Note that we took admissible cofibrations and fibrations. Since an ∞-cofibration in a full ∞-subcategory X of a stable ∞-category with cofibrations is a morphism $A \to B$ of the square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & C \end{array}$$

where A, B and C are also in X and the square is homotopy cocartesian. Dually, an ∞-fibration in a full ∞-subcategory X of a stable ∞-category with fibrations is a morphism $B \to C$ of the square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & C \end{array}$$

where A, B and C are also in X and the square is homotopy cartesian. Since $\mathcal{M}_R^{0,p}$ and $\mathcal{M}_R^p$ are homotopy additive categories, we have $K^L(\mathrm{LMod}_R^{perf}) \simeq K^R(\mathrm{LMod}_R^{perf})$ and $K^L(\mathrm{LMod}_R^{proj}) \simeq K^R(\mathrm{LMod}_R^{proj})$.

Note that $\mathcal{M}_R^p$ is $w_{\mathcal{M}_R}$-closed and a pushout of cofibrant objects is homotopy pushout. We also note that a square consisting of cofibrant fibrant object is homotopy pushout if and only if it is homotopy pullback since its homotopy category is stable.

By [13, Proposition 5.2], we have equivalences

$$L^H((\mathcal{M}_R^{0,p})^c) \simeq L^H((\mathcal{M}_R^{0,p})^f) \simeq L^H(\mathcal{M}_R^{0,p}) \simeq L^H((\mathcal{M}_R^{0,p})^{cf})$$

and

$$L^H((\mathcal{M}_R^{p})^c) \simeq L^H((\mathcal{M}_R^{p})^f) \simeq L^H(\mathcal{M}_R^{p}) \simeq L^H((\mathcal{M}_R^{p})^{cf}).$$

Here, $L^H((\mathcal{M}_R^{0,p})^{cf})$ and $L^H((\mathcal{M}_R^{p})^{cf})$ are fibrant replacements with respect to Bergner model structure on the category of simplicial categories, so that we can take theb algebraic K-theory of ∞-category. This implies that, on $\mathcal{M}_R^{0,p}$ and $\mathcal{M}_R^p$, S'-constructions of weak cofibrations and weak fibrations are weak equivalence, respectively. Thus, we obtain $K^L(\mathcal{M}_R^p) \simeq K^R(\mathcal{M}_R^p)$

and $K^L(\mathcal{M}_R^{0,p}) \simeq K^R(\mathcal{M}_R^{0,p})$ by Proposition 3.1 together with left and right functorial factorizations. □

By proceeding the same proof for $\mathcal{M}_R^{n,p}$ and $\mathrm{LMod}_R^{n,p}$, we have the following.

Corollary 5.7. *We have equivalences of K-theory $K^R(\mathcal{M}_R^{n,p}) \simeq K^L(\mathcal{M}_R^{n,p})$ and $K^R(\mathrm{LMod}_R^{n,p}) \simeq K^L(\mathrm{LMod}_R^{n,p})$ for $n \geq 0$.*

□

Corollary 5.8. *For a left coherent $\mathbb{E}_1$-ring R, $K(\mathrm{LMod}_R^{proj}) \simeq K(\mathrm{LMod}_R^{perf})$.*

Proof. Note that $\mathrm{LMod}_R^{0,p} \simeq \mathrm{LMod}_R^{proj}$ by [23, Proposition 7.2.2.6 (3), Remark 7.2.2.20]. By Proposition 3.2 (ii), we obtain an equivalence $K(\mathcal{M}_R^p) \simeq K(\mathrm{LMod}_R^{perf})$ on the K-theory. By Proposition 3.1 and Proposition 3.2 (i), there exists a natural equivalence of spectra $K(\mathcal{M}_R^{0,p}) \simeq K(\mathrm{LMod}_R^{proj})$ on the K-theory. Then by Theorem 5.4, we obtain a natural equivalence of spectra $K(\mathrm{LMod}_R^{perf}) \sim K(\mathrm{LMod}_R^{proj})$ on the K-theory.

Let $(\mathrm{LMod}_R^{perf})^{cn}$ be the ∞-category of connective perfect R-modules. Then, we have an equivalence $\mathrm{colim}_n \mathrm{LMod}_R^{n,p} \simeq (\mathrm{LMod}_R^{perf})^{cn}$ from Definition 4. Since the K-theory commutes with filterd colimits, we have an equivalence $K(\mathrm{LMod}_R^{proj}) \simeq K((\mathrm{LMod}_R^{perf})^{cn})$. Thus, it suffices to show that $K((\mathrm{LMod}_R^{perf})^{cn}) \simeq K(\mathrm{LMod}_R^{perf})$. Note that the Spanier-Whitehead category of $(\mathrm{LMod}_R^{perf})^{cn}$ is LMod_R^{perf} and to take Spanier-Whitehead category does not change the K-theory.

Indeed, consider the following colimit

$$(\mathrm{LMod}_R^{perf})^{cn} \xrightarrow{\Sigma} \quad \cdots \xrightarrow{\Sigma} \quad (\mathrm{LMod}_R^{perf})^{cn} \xrightarrow{\Sigma} \quad \cdots .$$

This filterd colimit exists as an ∞-category with weak cofibrations. From this filtered colimit, $\mathrm{colim}_\Sigma K((\mathrm{LMod}_R^{perf})^{cn}) \simeq K(\mathrm{colim}_\Sigma(\mathrm{LMod}_R^{perf})^{cn})$.

We will show that the following equivalences

$$\mathrm{colim}_\Sigma K((\mathrm{LMod}_R^{perf})^{cn}) \simeq K((\mathrm{LMod}_R^{perf})^{cn})$$

and

$$K(\mathrm{colim}_\Sigma(\mathrm{LMod}_R^{perf})^{cn}) \simeq K(\mathrm{LMod}_R^{perf}).$$

Since we have the following cofiber sequence in $(\mathrm{LMod}_R^{perf})^{cn}$

$$\begin{array}{ccc} id & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma, \end{array}$$

and Σ induces $-id$ on K-theory, $\mathrm{colim}_\Sigma K((\mathrm{LMod}_R^{perf})^{cn})$ is equivalent to $K((\mathrm{LMod}_R^{perf})^{cn})$. We will show that $\mathrm{colim}_\Sigma(\mathrm{LMod}_R^{perf})^{cn} \simeq \mathrm{LMod}_R^{perf}$. It suffices to show that $\mathrm{colim}_\Sigma(\mathrm{LMod}_R^{perf})^{cn}$ is a stable ∞-category. Indeed, it has cofibers, and the endofunctor Σ is an equivalence. By [23, Lemma 1.1.3.3], $\mathrm{colim}_\Sigma(\mathrm{LMod}_R^{perf})^{cn}$ is stable. □

Let R be a coherent $\mathbb{E}_1$-ring. For a left coherent $\mathbb{E}_1$-ring R, an R-module M is *left coherent* R-module if $\pi_n M = 0$ for sufficiently small n and large n, and $\pi_n M$ is finitely presented left $\pi_0 R$-module.

Blumberg and Mandell show that the algebraic G-theory of the ring $\pi_0 R$ is equivalent to the algebraic K-theory of the category of coherent R-module spectra which have only finitely many non-zero homotopy groups [9].

Let R be a left coherent $\mathbb{E}_1$-ring. R is said to be *regular* if any left coherent R-module has Tor-amplitude less than or equal to n for some $n \in \mathbb{Z}_{\geq 0}$ and $\pi_0 R$ is regular [5], [9].

Together with the regularity, we have the following as an immediate corollary.

Corollary 5.9. *Let R be a regular $\mathbb{E}_1$-ring with only finitely many non-zero homotopy groups. Let $\mathcal{P}_{\pi_0 R}$ be an ordinary category of finitely generated projective $\pi_0 R$-modules. Then, $K(\mathrm{LMod}_R^{proj}) \simeq K(\mathcal{P}_{\pi_0 R})$.*

Proof. Since finitely generated projective R-modules are obtained by retracts and finite direct sums of the truncated spectrum R, by Corollary 5.8 and the main theorem of [5] with the regularity of R, we obtain an equivalence $K(\mathrm{LMod}_R^{proj}) \simeq K(\mathrm{LMod}_R^{perf}) \simeq K(\mathcal{P}_{\pi_0 R})$. □

6. Application to semi-projective and compact DG-modules case

By proving the similar results as Theorem 5.4 in DG module case, we are going to show an equivalence between the algebraic K-theory of the category of perfect DG A-modules and that of the category consisting of semi-projective DG A-modules for a Noetherian connective differential graded algebra A over a field k such that its differential ∂ sends the degree 1 part A_1 of A to zero.

First of all, we recall some notation for DG-modules as follows.

Definition 6 (loc.sit. [1], [2]). Let A be a DG algebra.

(i) A DG A-module M is semi-projective if $\mathrm{Hom}_A(M,)$ preserves surjective quasi-isomorphisms.

(ii) A DG A-module M is semi-free if it has a filtration of the form :

$$0 = F^{-1} \subset \cdots F^n \subset F^{n+1} \subset \cdots$$

where F^{n+1}/F^n is isomorphic to a direct sum of suspensions of A and $\cup_n F^n = F$. When each subquotient F_{n+1}/F_n is isomorphic to a finite direct sum of suspensions of A and there exists n such that $F_n = M$, we say that M is finite semi-free of length n.

(iii) A DG A-module M is perfect if M is isomorphic to a compact object in the derived category of DG A-modules.

The key fact is that the definition of perfectness is equivalent to that M is a retract of finite semi-free of finite length if A is a Noetherian connective over a field k such that its differential ∂ sends the degree 1 part A_1 of A to zero [2, Theorem 4.2, Theorem 4.8] and in this case, perfectness is also equivalent to finite homological dimension that relates with projective dimension [1, Theorem 2.4.P (i), (iv)].

Let A be a DG algebra. Let $\mathcal{C}_A$ be the category of DG A-modules. Although there are many model structures on the category of DG modules [3], when we take the algebraic K-theory of a certain full subcategory of DG modules, we adopt the projective model structure , i.e., q-model structure in the sense of [3]. Note that every object is fibrant in q-model structure.

We denote by $\mathcal{C}_A^p$ the full subcategory consisting of perfect DG A-modules. I would like to take a proper sequence of subcategories of $\mathcal{C}_A$ as in Theorem 5.4. The most non-trivial point is wheather each subcategory is closed under extensions. To see this, we need to choose proper notion of projective dimension which should be related to homological dimension.

The important issue is the appropriate notion of projective dimension. Let us take projective dimension in the sense of Avramov and Foxby [1, Definition 2.1.P], which is given by $pd_A M = \inf_P \sup\{n | P_n \neq 0\}$, where $P_\bullet$ is a DG-projective complex equivalent to M.

Then, by [1, Theorem 2.4.P], we can control the projective dimension by homology groups.

Definition 7. Assume that A is a Noetherian connective DG algebra over k such that its differential ∂ sends the degree 1 part A_1 of A to zero. We define a subcategory $\mathcal{C}_A^n \subset \mathcal{C}_A$ consisting of finitely generated DG A-modules M with projective dimension $pd_A M \leq n$. We define cofibrations and fibrations in $\mathcal{C}_A^n$ to be cofibrations with respect to the projective model structure whose cofiber is in $\mathcal{C}_A^n$ and fibrations with respect to the projective model structure whose fiber is in $\mathcal{C}_A^n$, respectively.

Then, we will take $\mathcal{C}$ and $\mathcal{M}$ in Theorem 5.4 as the category $\mathcal{C}_A^p$ and $\mathcal{C}_A$, respectively, and show the following.

Proposition 6.1. *The inclusion $\mathcal{C}_A^n \subset \mathcal{C}_A^{n+1}$ satisfies the assumptions of Theorem 5.4. The category $\mathcal{C}_A^0$ is equivalent to the category of finitely generated semi-projective DG A-modules.*

Proof. By taking a full subcategory $(\mathcal{C}_A)^{cf}$ of $\mathcal{C}_A$ consisting of cofibrant (fibrant) objects with respect to the projective model structure, we can assume that $\mathcal{C}_A^p$ is homotopycally essentially small by the same argument as in the proof of Theorem 3 for $\mathcal{M}_R^p$.

By virtue of [2, Theorem 4.2 (i), (ii)] and [1, Theorem 2.4.P (i), (iv)], a DG A-module M is $pd_A M \leq n$ if and only if it has Tor-amplitude $n+1$ and $(n+1)$th Ext group $Ext(M, N)$ is 0 for any DG A-module N. Then, we can control the projective dimensions by homology groups. Especially, closed under extensions and fibration sequences induce the long exact sequences of Tor and Ext. Therefore, we can do the same proof as in Lemma 5.1 as follows.

Let $M' \to M \to M''$ be a fiber sequence of DG A-modules in $\mathcal{C}_A^n$.

We have an exact sequence of homology groups

$$H_{k+1}(M'') \to H_k(M') \to H_k(M) \to H_k(M'')$$

and Ext groups

$$Ext_A^k(M', N) \to Ext_A^{k+1}(M'', N) \to Ext_A^{k+1}(M, N) \to Ext_A^{k+1}(M', N).$$

We prove the assertion (i) of Theorem 5.4. Assume that M' and M'' have Tor-amplitude less than or equal to n and their $(n-1)$th Ext group are vanish for any DG A-module N. Then, obviously M has Tor-amplitude less than or equal to n and $Ext_A^{n-1}(M, N) = 0$ for all DG A-module N.

We prove the assertion (iii) of Theorem 5.4. Assume that M' and M'' have Tor-amplitude less than or equal to n and their $(n-1)$th Ext group are vanish for any DG A-module N, and M has Tor-amplitude less than or equal to $n-1$ and $Ext_A^{n-2}(M, N) = 0$. If $k \geq n$, $H_{k+1}(M'')$ and $H_k(M)$ vanish by assumption that M'' has Tor-amplitude less than or equal to n and M has Tor-amplitude less than or equal to $n-1$. Then, M' has Tor-amplitude less than or equal to $n-1$ and $Ext_A^{n-2}(M', N) = 0$ for all DG A-module N.

Especially, we can see that $\mathcal{C}_A^n$ is closed under extension, has a direct sum.

For the assertion (ii) of Theorem 5.4, we can take a surjective quasi-isomorphism from a finitely generated semi-free DG A modules. Note that such surjection is known as a cofibrant replacement, whose existence is always assured in DG case. Since semi-free is semi-projective, semi-free A-modules are in $\mathcal{C}_A^0$. □

Corollary 6.2. *Assume that A is a Noetherian connective DG algebra over k such that its differential ∂, that decreases degree by one, sends the degree 1 part A_1 of A to zero. The algebraic K-theory of the category of perfect A-modules is equivalent to the algebraic K-theory of the category of finitely generated semi-projective A-modules.*

We remark that there is a zig-zag of Quillen equivalence between the model category of HA-modules and the model category of DG A-modules by [31]. Although the Quillen equivalence is zig-zag, the sequence in Proposition 6.1 is corresponding to the sequence constructed in Definition 4 in the sense that it is completely determined by the homology groups.

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Mariko Ohara
Center for Liberal Arts and Sciences
Faculty of Engineering
Toyama Prefectural University
5180, Kurokawa, Imizu, Toyama, JAPAN, 939-0398
e-mail address: ohara@pu-toyama.ac.jp

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