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RETRACT PROBLEM AND A SUBSET ADMITTING COMPLETE BOOLEAN ALGEBRA STRUCTURE OF MONOIDALLY DISTRIBUTIVE POSETS

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Abstract. In [3], Hovey and Palmieri proved many interesing results around the Bousfield lattice of the stable homotopy caregory of spectra. In [4], the author, Shimomura and Tatehara defined monoidally distributive posets as a generalization of the Bousfield lattice. In this paper, we consider the retract problem and a subset admitting complete Boolean algebra structure of monoidally distributive posets. In the last section, we see that some results in [3] are given by our results in this paper.

1. Introduction

Let p be a prime number and $\mathcal{S}_p$ the stable homotopy category of spectra localized at p. For a spectrum $X \in \mathcal{S}_p$, the Bousfield class $\langle X\rangle$ is defined to be the class consisting of spectra Y such that the smash product $X \wedge Y$ is 0. We denote by $\mathbb{B}$ the collection of Bousfield classes. In [5], Ohkawa showed that the collection $\mathbb{B}$ is a set. This set also has a lattice structure, which is given by $\langle X\rangle \le \langle Y\rangle \Leftrightarrow \langle X\rangle \supset \langle Y\rangle$. Immediately, we see that the join of $\langle X\rangle$ and $\langle Y\rangle$ is the Bousfield class of the wedge sum $X \vee Y$. On the other hand, the meet of $\langle X\rangle$ and $\langle Y\rangle$ is not $\langle X \wedge Y\rangle$. In [3], Hovey and Palmieri considered a sup-sublattice $\mathbb{DL}$ of $\mathbb{B}$ (see [3, §3]). In $\mathbb{DL}$, the meet of $\langle X\rangle$ and $\langle Y\rangle$ is the same as $\langle X \wedge Y\rangle$. Furthermore, $\mathbb{DL}$ admits a (complete) distributive lattice structure. For the inclusion $\mathbb{DL} \subset \mathbb{B}$, we have its retraction $r\colon \mathbb{B} \to \mathbb{DL}$. Hovey and Palmieri noticed that the retraction r sends every "strange" Bousfield class to $\langle 0\rangle$. On these backgrounds, they proposed the *retract conjecture* on $\mathbb{B}$ ([3, Conj. 3.12], or Conjecture 5.3), which claims that the retraction r induces an isomorphism from the quotient lattice $\mathbb{B}$/(strange classes) to $\mathbb{DL}$.

In [4], the author, Shimomura and Tatehara considered *monoidally distributive posets* as a generalization of $\mathbb{B}$ (see §2), and extended some results of [3] to monoidally distributive posets. A monoidally distributive poset B is a lattice, and also it is a commutative monoid with 0. As a generalization of $\mathbb{DL}$, we consider

$$DL = \{x \in B \colon x^2 = x\},$$

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the subset consisting of idempotent elements in B. The inclusion $DL \subset B$ has a retraction $r\colon B \to DL$ (see (3.3)). In this paper, we consider the following problem (*cf.* [4, Conj. 3.18]).

Problem 1.1 (Retract problem)**.** *For a given lattice ideal I of a monoidally distributive poset B, we consider the induced mapping*

$$r_*\colon B/I \to DL; [x] \mapsto r(x).$$

Is this a well-defined isomorphism?

This problem is a generalization of the retract conjecture. We consider the subset

$$\mathfrak{I} = \{c \in B\colon r(x \vee c) = r(x) \text{ for any } x \in B\},$$

which is a lattice ideal of B (Proposition 3.10). In §3, we prove the following.

Theorem 1.2. (1) *The induced mapping r_* in Problem 1.1 is well-defined if and only if $I \subset \mathfrak{I}$.*
(2) *If the mapping r_* in Problem 1.1 is an isomorphism, then $I = \mathfrak{I}$.*

Remark 1.3. Theorem 1.2 is an improved version of [4, Th. 3.16]. In [4], the author, Shimomura and Tatehera considered the subset $A = \{x \in B\colon r(x) = 0\}$ instead of $\mathfrak{I}$. However, we don't know whether or not A is a lattice ideal of B. Theorem 1.2 settles this problem.

We turn to the Bousfield lattice $\mathbb{B}$, and consider the sup-sublattice $\mathbb{DL}$. A subset $\mathbb{BA}$ is defined to be the subset consisting of complemented Bousfield classes (see [3, §4]). Then $\mathbb{BA}$ is a sublattice of $\mathbb{DL}$, and also $\mathbb{BA}$ admits a Boolean algebra structure. For them, we have a problem that $\mathbb{DL}$ is complete and $\mathbb{BA}$ is not complete. On these backgrounds, Hovey and Palmieri constructed a subset $\mathbf{c}\mathbb{BA}$ (see [3, §6]) of $\mathbb{B}$, which satisfies that

- $\mathbb{BA} \subset \mathbf{c}\mathbb{BA} \subset \mathbb{DL}$, and
- $\mathbf{c}\mathbb{BA}$ admits a complete Boolean algebra structure.

They also proved that we can describe the structure of $\mathbf{c}\mathbb{BA}$ under some conjectures ([3, Prop. 6.13], or Theorem 5.12).

For a monoidally distributive poset B, we have an order-reversing mapping $a\colon B \to B$ (see (4.1)). For an element x of DL, we define $A(x) = r(a(x))$ (see (4.4)). As a generalization of $\mathbf{c}\mathbb{BA}$, we consider a subset

$$cBA = \{x \in DL\colon A^2(x) = x\}$$

of B. This subset has a complete Boolean algebra structure (Theorem 4.10). A nonzero element $f \in DL$ is *fieldlike* if, for any $x \in B$, we have $xf \in \{0, f\}$ (Definition 4.21). An element $d \in DL$ is *dense* if $A^2(d) = 1$ (Definition

4.11). We put $F = \{f \in DL\colon f \text{ is fieldlike}\}$, $D = \{d \in DL\colon d \text{ is dense}\}$, and $a(DL) = \{a(x)\colon x \in DL\}$. For a subset S of B, we denote by

$$\bigvee S = \bigvee_{x \in S} x \quad \text{and} \quad \bigwedge S = \bigwedge_{x \in S} x,$$

the join and the meet, respectively. In §4, we prove the following.

Theorem 1.4. *We suppose that*

(1) r_* *in Problem 1.1 at* $I = \mathfrak{I}$ *is an isomorphism,*
(2) $\mathfrak{I}$ *is a complete ideal,*
(3) $\mathfrak{I} \subset a(DL)$, *and*
(4) $\bigvee F = \bigwedge D$.

Then, cBA is isomorphic to the complete Boolean algebra generated by F, that is,

$$cBA \cong \left\{ \bigvee F_0 \colon F_0 \text{ is a subset of } F \right\}.$$

Here we remark that $\bigvee \varnothing = 0$.

In the last section, we consider the case for $B = \mathbb{B}$. Some results in [3] are obtained from the viewpoint of this paper. In particular, we see that [3, Prop. 6.13] is easily shown by Theorem 1.4 (see Theorem 5.12).

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2. Monoidally distributive posets

A *commutative monoid with* 0 is a commutative monoid M having an element 0 such that $0x = 0$ for any $x \in M$. In [4], the author, Shimomura and Tatehara defined monoidally distributive posets as a generalization of the Bousfield lattice $\mathbb{B}$.

Definition 2.1 ([4, Def. 2.4])**.** *A monoidal poset* $B = (B, \le, \cdot, 1, 0)$ *cosists of the following data.*

(1) $(B, \cdot, 0, 1)$ *is a commutative monoid with* 0.
(2) $(B, \le)$ *is a poset.*
(3) *The following are equivalent.*
  (a) $x \le y$.
  (b) *For any* $c \in B$, $yc = 0$ *implies* $xc = 0$.

For an element $x \in B$, we put

$$\langle x \rangle = \{c \in B\colon xc = 0\}. \tag{2.1}$$

Then, Definition 2.1 (3) is rewriten as

$$x \le y \Leftrightarrow \langle x \rangle \supset \langle y \rangle. \tag{2.2}$$

Remark 2.2. We immdiately see that, for a monoidal poset B, the following hold.

- $x \le y$ implies that $xz \le yz$ for any $z \in B$.
- 0 is the minimum element, and 1 is the maximum element.
- For any x and y in B, we have $xy \le x$. In particular, $x^2 \le x$.

In general, a commutative monoid (with 0) admitting an ordering is not a monoidal poset. For example, the set $\mathbb{R}$ admits the ordinary multiplication and ordering. In this case, for any nonzero $x \in \mathbb{R}$, we have $\langle x \rangle = \{0\}$. Therefore, the condition (2.2) is not satisfied. As another example, we consider the min-plus poset $([0, \infty], \ge^{op}, 0, \infty, +)$. In this case, for any $x \in [0, \infty)$, we have $\langle x \rangle = \{c \in [0, \infty] \colon x + c = \infty\} = \{\infty\}$. Therefore, the condition (2.2) is not satisfied, and hence the min-plus poset is not a monoidal poset.

Remark 2.3. In general, from a commutative monoid M with 0, we obtain the monoidal poset $\beta(M)$ (see [4, §2]). Furthermore, by [4, Prop. 2.17 (2)], M is a monoidal poset if and only if $\beta(M) = M$. A categorical argument on monoidal posets is written in [4, §2].

Definition 2.4 ([4, Def. 3.6])**.** *A monoidal poset B is a monoidally distributive poset if the following hold.*

(1) *B is a complete lattice.*
(2) *For any $x \in B$ and $\{y_\lambda\} \subset B$, we have $x(\bigvee_\lambda y_\lambda) = \bigvee_\lambda (xy_\lambda)$.*

Hereafter, throughout this paper, we assume that B is a monoidally distributive poset.

3. Retract problem

For a subset S of B, we denote

$$\bigvee S = \bigvee_{x \in S} x \quad \text{and} \quad \bigwedge S = \bigwedge_{x \in S} x.$$

We remark that $\bigvee \varnothing = 0$ and $\bigwedge \varnothing = 1$. We also consider the subset

$$DL = \{x \in B \colon x^2 = x\}. \tag{3.1}$$

It is clear that

$$x, y \in DL \Rightarrow xy \in DL. \tag{3.2}$$

Indeed, if $x, y \in DL$, then $(xy)^2 = x^2y^2 = xy$.

Lemma 3.1. *Let $w \in DL$ and $\{x, y\} \subset B$. Then, $w \le x \wedge y$ if and only if $w \le xy$.*

Proof. By Remark 2.2, we have $xy \leq x$ and $xy \leq y$, and so $xy \leq x \wedge y$. Hence, if $w \leq xy$, then $w \leq xy \leq x \wedge y$. Conversely, we assume that $w \leq x \wedge y$. This implies $w \leq x$ and $w \leq y$. Therefore, since $w \in DL$, we have $w = ww \leq xw \leq xy$. □

Lemma 3.2. *If $S \subset DL$, then $\bigvee S \in DL$.*

Proof. For any $S \subset DL$, we have

$$\bigvee S \geq \left(\bigvee S\right)^2 = \bigvee_{x,y \in S} xy \geq \bigvee_{x \in S} xx = \bigvee_{x \in S} x = \bigvee S.$$

Therefore $(\bigvee S)^2 = \bigvee S$. □

We consider the following mapping.

$$r\colon B \to DL; x \mapsto \bigvee \{w \in DL\colon w \leq x\}. \tag{3.3}$$

By Lemma 3.2, this is well-defined, that is, $r(x)$ belongs to DL for any $x \in B$. It is easy to see that

$$r(x) \leq x \text{ for any } x \in B, \tag{3.4}$$

and

$$r(0) = 0, \quad r(1) = 1. \tag{3.5}$$

Lemma 3.3. *$x \in DL$ if and only if $x = r(x)$. In particular, the mapping r is a retraction of the inclusion $DL \subset B$.*

Proof. It is clear that $x = r(x)$ implies $x \in DL$. Conversely, if $x \in DL$, then x is the maximum element of the subset $\{w \in DL\colon w \leq x\}$. Therefore, $r(x) = \bigvee\{w \in DL\colon w \leq x\} = x$. □

Lemma 3.4 ([4, Prop. 3.5])**.** *The mapping r satisfies the following.*

(1) *$x \leq y$ implies $r(x) \leq r(y)$.*
(2) *$r^2(x) = r(x)$ for any $x \in B$.*
(3) *$r(x) \leq x^n$ for any $x \in B$ and $n \geq 1$.*
(4) *$r(xy) = r(x \wedge y) = r(x)r(y)$ for any x and y in B. In particular, $r(x^2) = r(x)$ for any $x \in B$.*

Proof. (1) If $x \leq y$, then $\{w \in DL\colon w \leq x\} \subset \{w \in DL\colon w \leq y\}$. Hence $r(x) = \bigvee\{w \in DL\colon w \leq x\} \leq \bigvee\{w \in DL\colon w \leq y\} \leq r(y)$.
(2) By Lemma 3.3 and $r(x) \in DL$, we have $r(x) = r(r(x)) = r^2(x)$.
(3) Note that $r(x)^n = r(x)$ for any $n \geq 1$. Therefore, by (3.4), we have $r(x) = r(x)^n \leq x^n$.

(4) Since we have

$$\begin{aligned} r(xy) &= \bigvee\{w \in DL\colon w \le xy\} \\ &= \bigvee\{w \in DL\colon w \le x \wedge y\} \quad \text{by Lemma 3.1} \\ &= r(x \wedge y), \end{aligned}$$

the part $r(xy) = r(x \wedge y)$ is shown. We turn to the part $r(xy) = r(x)r(y)$. By (1), we have $r(xy) \le r(x)$ and $r(xy) \le r(y)$. Hence

$$r(xy) = r(xy)r(xy) \le r(x)r(y).$$

We also have

$$\begin{aligned} r(x)r(y) &= \left(\bigvee\{w \in DL\colon w \le x\}\right)\left(\bigvee\{w' \in DL\colon w' \le y\}\right) \\ &= \bigvee\{ww'\colon w, w' \in DL,\ w \le x,\ w' \le y\} \\ &\le \bigvee\{w \in DL\colon w \le xy\} \quad \text{by (3.2)} \\ &= r(xy). \end{aligned}$$

Therefore, $r(xy) = r(x)r(y)$.

□

Lemma 3.5. *For any x and y in B, we have $r(x \vee y) = r(x \vee y^n)$ for any $n \ge 1$.*

Proof. It suffices to show that $r(x \vee y) = r(x \vee y^2)$, which is given by

$$\begin{aligned} r(x \vee y) &= r((x \vee y)^2) \quad \text{by Proposition 3.4 (4)} \\ &= r(x(x \vee y) \vee y^2) \\ &\le r(x \vee y^2) \quad \text{by Proposition 3.4 (1)} \\ &\le r(x \vee y) \quad \text{by Proposition 3.4 (1).} \end{aligned}$$

□

Let B' be a subset of B. For a subset S of B', if there exists the join (resp. the meet) of S in B', then we denote it by $\bigvee^{B'} S$ (resp. $\bigwedge^{B'} S$).

Proposition 3.6. *The subset DL is complete, that is, for any subset S of DL, there exist $\bigvee^{DL} S$ and $\bigwedge^{DL} S$. More details, we have $\bigvee^{DL} S = \bigvee S$ and $\bigwedge^{DL} S = r(\bigwedge S)$.*

Proof. Let S be a subset of DL. From Lemma 3.2, it is immdiately obtained that $\bigvee^{DL} S = \bigvee S$. We turn to $\bigwedge^{DL} S = r(\bigwedge S)$. By Lemma 3.4, for any $x \in S$, we have $DL \ni r(\bigwedge S) \le r(x) = x$. If $DL \ni w \le x$ for any $x \in S$, then $w = r(w) \le r(\bigwedge S)$ by Lemma 3.4 (1). Therefore $\bigwedge^{DL} S = r(\bigwedge S)$. □

Corollary 3.7. *For any x and y in DL, we have $x \wedge^{DL} y = xy$.*

Proof. If x and y are in DL, then

$$\begin{aligned} x \wedge^{DL} y &= r(x \wedge y) \quad \text{by Proposition 3.6} \\ &= r(xy) \quad \text{by Lemma 3.4 (4)} \\ &= xy \quad \text{by (3.2) and Lemma 3.3.} \end{aligned}$$

□

Remark 3.8. By Proposition 3.6, the subset DL is a complete distributive lattice, and therefore DL admits a complete Heyting algebra structure.

Remark 3.9. We may think that B is a category such that, for any x and y in B, the set of morphisms from x to y is $\varnothing$ or $\{\leq\}$. Similarly, the subset DL becomes a category. Since B and DL are complete lattices, they are complete and cocomplete categories. Furthermore, $r\colon B \to DL$ is a right adjoint of the inclusion functor $DL \subset B$. Indeed, for any $x \in DL$ and $y \in B$, we have $x \leq y \Leftrightarrow x \leq r(y)$. From this viewpoint, Proposition 3.6 means that the right adjoint functor $r\colon B \to DL$ preserves limits.

We recall that a subset I of B is a *lattice ideal* if

- $\{x, y\} \subset I$ implies $x \vee y \in I$, and
- $x \leq y \in I$ implies $x \in I$.

For a lattice ideal I, the quotient $B/I = \{[x]\colon x \in B\}$ is defined by

- $[x] = \{x' \in B\colon x \vee i = x' \vee i$ for some $i \in I\}$, and
- $[x] \leq [y]$ iff $x \leq y \vee i$ for some $i \in I$.

We define

$$\mathfrak{I} = \{c \in B\colon r(x \vee c) = r(x) \text{ for any } x \in B\}\,. \tag{3.6}$$

Proposition 3.10. *The subset $\mathfrak{I}$ is a lattice ideal of B.*

Proof. If c and d are in $\mathfrak{I}$, then, for any $x \in B$, we have

$$\begin{aligned} r(x \vee c \vee d) &= r(x \vee c) \quad \text{by } d \in \mathfrak{I} \\ &= r(x) \quad \text{by } c \in \mathfrak{I}. \end{aligned}$$

Therefore, $c \vee d$ is in $\mathfrak{I}$.

If $c \leq d \in \mathfrak{I}$, then, for any $x \in B$, we have

$$\begin{aligned} r(x) &\leq r(x \vee c) \quad \text{by Lemma 3.4 (1)} \\ &\leq r(x \vee d) \quad \text{by Lemma 3.4 (1)} \\ &= r(x) \quad \text{by } d \in \mathfrak{I}. \end{aligned}$$

Hence $r(x) = r(x \vee c)$, and therefore $c \in \mathfrak{I}$. □

Lemma 3.11. *If $c \in \mathfrak{I}$, then $r(c) = 0$.*

Proof. If $c \in \mathfrak{I}$, then

$$\begin{aligned} r(c) &= r(0 \vee c) = r(0) \quad \text{by } c \in \mathfrak{I} \\ &= 0 \quad \text{by (3.5).} \end{aligned}$$

□

For $r\colon B \to DL$ and a lattice ideal I of B, we consider the induced mapping

$$r_*\colon B/I \to DL; [x] \mapsto r(x). \tag{3.7}$$

Proof of Theorem 1.2. (1) We assume that r_* in (3.7) is well-defined. If $c \in I$, then, for any $x \in B$, we have $[x] = [x \vee c]$ in B/I. This implies $r(x \vee c) = r_*([x \vee c]) = r_*([x]) = r(x)$, and so $c \in \mathfrak{I}$. Therefore $I \subset \mathfrak{I}$.

Conversely, we assume that $I \subset \mathfrak{I}$. If $[x] = [y]$ in B/I, then $x \vee c = y \vee c$ for some $c \in I(\subset \mathfrak{I})$. Thus, we have

$$\begin{aligned} r_*([x]) &= r(x) = r(x \vee c) \quad \text{by } c \in \mathfrak{I} \\ &= r(y \vee c) \quad \text{by } x \vee c = y \vee c \\ &= r(y) \quad \text{by } c \in \mathfrak{I} \\ &= r_*([y]). \end{aligned}$$

Therefore, r_* is well-defined.

(2) We assume that r_* is an isomorphism. By (1), it suffices to show that $\mathfrak{I} \subset I$. If $c \in \mathfrak{I}$, then $r(c) = 0 = r(0)$ by Lemma 3.11 and (3.5). Thus, $r_*([c]) = r(c) = r(0) = r_*([0])$. Since r_* is an isomorphism, we have $[c] = [0]$ in B/I, and so $c \in I$. Therefore $\mathfrak{I} \subset I$.

□

4. The subset cBA

We consider the following mapping.

$$a\colon B \to B; x \mapsto \bigvee \langle x \rangle = \bigvee \{c \in B\colon xc = 0\}. \tag{4.1}$$

Here, $\langle x \rangle$ is in (2.1). It is clear that

$$xa(x) = 0 \text{ for any } x \in B, \tag{4.2}$$

and

$$a(0) = 1, \quad a(1) = 0. \tag{4.3}$$

Indeed, we have $xa(x) = x(\bigvee_{xc=0} c) = \bigvee_{xc=0} xc = 0$, $a(0) = \bigvee \langle 0 \rangle = \bigvee B = 1$, and $a(1) = \bigvee \langle 1 \rangle = \bigvee \{0\} = 0$.

For any subset S of B, we denote

$$a(S) = \{a(x)\colon x \in S\}.$$

Lemma 4.1 (*cf.* [3, Lemma 2.3])**.** *The mapping $a\colon B \to B$ satisfies the following.*

(1) *$x \le y$ implies $a(x) \ge a(y)$.*
(2) *$xy = 0$ if and only if $x \le a(y)$.*
(3) *$a^2(x) = x$ for any $x \in B$.*
(4) *For any $S \subset B$, we have $a(\bigvee S) = \bigwedge a(S)$ and $a(\bigwedge S) = \bigvee a(S)$.*

Proof. (1) If $x \le y$, then $\langle x\rangle \supset \langle y\rangle$ by (2.2). Hence $a(x) = \bigvee\langle x\rangle \ge \bigvee\langle y\rangle = a(y)$.

(2) If $xy = 0$, then $x \in \langle y\rangle$, and so $x \le \bigvee\langle y\rangle = a(y)$. Conversely, if $x \le a(y)$, then $xy \le ya(y)$. Since $ya(y) = 0$ by (4.2), we have $xy = 0$.

(3) By (4.2) and (2), we have $x \le a^2(x)$ for any $x \in B$. This and (1) imply $a(x) \ge a^3(x)$. Furthermore, we have $a(x)a^2(x) = 0$ by (4.2), and so $a(x) \le a^3(x)$ by (2). Thus $a(x) = a^3(x)$, and so

$$\begin{aligned} cx = 0 \quad &\Leftrightarrow \quad c \le a(x) \quad \text{by (2)} \\ &\Leftrightarrow \quad c \le a^3(x) \quad \text{by } a(x) = a^3(x) \\ &\Leftrightarrow \quad ca^2(x) = 0 \quad \text{by (2).} \end{aligned}$$

Therefore $x = a^2(x)$.

(4) By (1), we have $a(\bigvee S) \le a(x)$ for any $x \in S$. If $w \le a(x)$ for any $x \in S$, then, by (1) and (3), $a(w) \ge a^2(x) = x$ for any $x \in S$. This implies $a(w) \ge \bigvee S$. Therefore, by (1) and (3), we have $w = a^2(w) \le a(S)$, and so $a(\bigvee S) = \bigwedge a(S)$. Similarly, we see that $a(\bigwedge S) = \bigvee a(S)$.

□

Remark 4.2. By Lemma 4.1 (3), we have $a(S) = \{x\colon a(x) \in S\}$.

We recall the subset DL in (3.1), and consider the following composite.

(4.4) $$A = ra\colon DL \xrightarrow{a} B \xrightarrow{r} DL.$$

Then, we have

(4.5) $$xA(x) = 0 \text{ for any } x \in DL$$

and

(4.6) $$A(0) = 1, \quad A(1) = 0.$$

Indeed, by (3.4) and (4.2), we have $xA(x) = xr(a(x)) \le xa(x) = 0$. Furthermore, by (3.5) and (4.3), we have $A(0) = ra(0) = r(1) = 1$ and $A(1) = ra(1) = r(0) = 0$.

Lemma 4.3. *For any x and y in DL, we have $A(x \vee y) = A(x)A(y)$. In particular, $A(x \vee A(x)) = 0$.*

Proof. If $x, y \in DL$, then

$$\begin{aligned} A(x \vee y) &= ra(x \vee y) \\ &= r(a(x) \wedge a(y)) \quad \text{by Lemma 4.1 (4)} \\ &= r(a(x))r(a(y)) \quad \text{by Lemma 3.4 (4)} \\ &= A(x)A(y). \end{aligned}$$

In particular, $A(x \vee A(x)) = A(x)A^2(x) = 0$ by (4.5). □

For any subset S of DL, we denote

$$A(S) = \{A(x) \colon x \in S\}.$$

As an analogue of Lemma 4.1, we see the following.

Lemma 4.4 (*cf.* [3, Lemma 6.2]). *The mapping A satisfies the following.*

(1) *For any x and y in DL, $x \leq y$ implies $A(x) \geq A(y)$.*
(2) *For any x and y in DL, $xy = 0$ if and only if $x \leq A(y)$.*
(3) *For any x in DL, $x \leq A^2(x)$ and $A(x) = A^3(x)$.*
(4) *For any $S \subset DL$, we have $A(\bigvee^{DL} S) = \bigwedge^{DL} A(S)$.*

Proof. (1) By Lemma 3.4 (1), the mapping r is order-preserving. By Lemma 4.1 (1), the mapping a is order-reversing. Therefore, the composite $A = ra$ is order-reversing.

(2) If $xy = 0$, then $x \leq a(y)$ by Lemma 4.1 (2). Hence , by $x \in DL$ and Lemma 3.4 (1), we have $x = r(x) \leq r(a(y)) = A(y)$. Conversely, if $x \leq A(y)$, then $xy \leq yA(y) = 0$ by (4.5).

(3) By (4.5), for any $x \in DL$, we have $xA(x) = 0$. This and (2) imply $x \leq A^2(x)$. We next turn to $A(x) = A^3(x)$. By (1), $x \leq A^2(x)$ implies $A(x) \geq A^3(x)$. On the other hand, $A(x)A^2(x) = 0$ by (4.5). This and (2) imply $A(x) \leq A^3(x)$. Therefore, we have $A(x) = A^3(x)$.

(4) For any $S \subset DL$, by (1), we have $A(\bigvee^{DL} S) \leq A(x)$ for any $x \in S$. Furthermore, we have

$$\begin{aligned} DL \ni w \leq A(x) \text{ for any } x \in S \quad &\Rightarrow \quad wx = 0 \text{ for any } x \in S \quad \text{by (2)} \\ &\Rightarrow \quad w(\bigvee S) = w(\bigvee_{x \in S} x) = \bigvee_{x \in S} (wx) = 0 \\ &\Rightarrow \quad w \leq A(\bigvee S) \quad \text{by Lemma 3.2 and (2).} \end{aligned}$$

Besides, by Lemma 3.6, $A(\bigvee S) = A(\bigvee^{DL} S)$. Therefore $A(\bigvee^{DL} S) = \bigwedge^{DL} A(S)$.

□

Remark 4.5. For any x and y in B, we have

$$x \leq y \quad \Leftrightarrow \quad a(x) \geq a(y) \quad \text{by Lemma 4.1 (1)}$$

$$\Rightarrow \quad ra(x) \geq ra(y) \quad \text{by Lemma 3.4 (1)},$$

which is an analogue of Lemma 4.4 (1).

Lemma 4.6 (*cf.* [3, Lemma 6.5 (1)])**.** *For any x and y in DL, we have $A(xy) = A(A^2(x)A^2(y))$.*

Proof. Let $x, y \in DL$. By Lemma 4.4 (3), we have $x \leq A^2(x)$ and $y \leq A^2(y)$. These imply $xy \leq A^2(x)A^2(y)$. From this and Lemma 4.4 (1), we obtain $A(xy) \geq A(A^2(x)A^2(y))$. On the other hand, we have

$$\begin{aligned}
xyA(xy) = 0 \quad \text{by (4.5)} \quad &\Rightarrow \quad yA(xy) \leq A(x) \quad \text{by Lemma 4.4 (2)}\\
&\Rightarrow \quad yA(xy)A^2(x) \leq A(x)A^2(x) = 0 \quad \text{by (4.5)}\\
&\Rightarrow \quad yA(xy)A^2(x) = 0\\
&\Rightarrow \quad A(xy)A^2(x) \leq A(y) \quad \text{by Lemma 4.4 (2)}\\
&\Rightarrow \quad A(xy)A^2(x)A^2(y) \leq A(y)A^2(y) = 0 \quad \text{by (4.5)}\\
&\Rightarrow \quad A(xy)A^2(x)A^2(y) = 0\\
&\Rightarrow \quad A(xy) \leq A(A^2(x)A^2(y)) \quad \text{by Lemma 4.4 (2).}
\end{aligned}$$

Therefore $A(xy) = A(A^2(x)A^2(y))$. □

Corollary 4.7. *For any x and y in DL, we have $A^2(xy) = A^2(x)A^2(y)$.*

Proof. If x and y are in DL, then

$$\begin{aligned}
A^2(xy) &= A^2(A^2(x)A^2(y)) \quad \text{by Lemma 4.6}\\
&= A^2(A^2(x) \wedge^{DL} A^2(y)) \quad \text{by Corollary 3.7}\\
&= A^3(A(x) \vee^{DL} A(y)) \quad \text{by Lemma 4.4 (4)}\\
&= A(A(x) \vee^{DL} A(y)) \quad \text{by Lemma 4.4 (3)}\\
&= A^2(x) \wedge^{DL} A^2(y) \quad \text{by Lemma 4.4 (4)}\\
&= A^2(x)A^2(y) \quad \text{by Corollary 3.7.}
\end{aligned}$$

□

We consider the subset

$$cBA = \{x \in DL \colon A^2(x) = x\}$$

of DL.

Proposition 4.8. *The subset cBA is complete, that is, for any subset S of cBA, there exist $\bigvee^{cBA} S$ and $\bigwedge^{cBA} S$. More details, we have $\bigvee^{cBA} S = A^2(\bigvee S)$ and $\bigwedge^{cBA} S = A^2(\bigwedge S)$.*

Proof. Let S be a subset of cBA. By Lemma 4.4 (1), the mapping A is order-reversing. This implies that $A^2\colon DL \to DL$ is order-preserving. Furthermore, by Lemma 4.4 (3), $A^2(x)$ belongs to cBA for any $x \in DL$. Hence we have $cBA \ni A^2(\bigvee S) \geq A^2(x) = x$ for any $x \in S$. Assume that $cBA \ni w \geq x$ for any $x \in S$. This implies $w = A^2(w) \geq A^2(\bigvee S)$, and so $\bigvee^{cBA} S = A^2(\bigvee S)$. Similarly, we see that $\bigwedge^{cBA} S = A^2(\bigwedge S)$. □

Proposition 4.9. *For any $S \subset cBA$, we have $\bigwedge^{cBA} S = \bigwedge^{DL} S$. In particular, for any x and y in cBA, we have $x \wedge^{cBA} y = xy$.*

Proof. Let $S \subset cBA (\subset DL)$. Then, we have

$$\begin{aligned}
\bigwedge^{cBA} S &= A^2(\bigwedge S) \quad \text{by Proposition 4.8} \\
&= ra(ra(\bigwedge S)) \\
&\geq ra(a(\bigwedge S)) \quad \text{by (3.4) and Remark 4.5} \\
&= r(\bigwedge S) \quad \text{by Lemma 4.1 (3)} \\
&= \bigwedge^{DL} S \quad \text{by Proposition 3.6} \\
&\geq \bigwedge^{cBA} S \quad \text{by } cBA \subset DL.
\end{aligned}$$

Therefore $\bigwedge^{cBA} S = \bigwedge^{DL} S$. Furthermore, if x and y are in cBA, then $x \wedge^{cBA} y = x \wedge^{DL} y = xy$ by Corollary 3.7. □

Theorem 4.10 (*cf.* [3, Th. 6.4])**.** *The subset cBA is a complete Boolean algebra.*

Proof. By Proposition 4.8, cBA is a complete lattice. We note that, for any $x \in cBA$, the element $A(x)$ belongs to cBA. Besides, we have

$$\begin{aligned}
x \wedge^{cBA} A(x) &= x \wedge^{DL} A(x) \quad \text{by Proposition 4.9} \\
&= xA(x) \quad \text{by Corollary 3.7} \\
&= 0 \quad \text{by (4.5)}
\end{aligned}$$

and

$$\begin{aligned}
x \vee^{cBA} A(x) &= A^2(x \vee A(x)) \quad \text{by Proposition 4.8} \\
&= A(A(x \vee A(x))) = A(0) \quad \text{by Lemma 4.3} \\
&= 1 \quad \text{by (4.6).}
\end{aligned}$$

Therefore, $A(x)$ is the complement of x in cBA.

Furthermore, for $x, y, z \in cBA$, we have

$$\begin{aligned} x \wedge^{cBA} \left(y \vee^{cBA} z\right) &= A^2(x) \wedge^{cBA} A^2(y \vee z) \quad \text{by } x \in cBA \text{ and Proposition 4.8} \\ &= A^2(x)A^2(y \vee z) \quad \text{by Proposition 4.9} \\ &= A^2(x(y \vee z)) \quad \text{by Corollary 4.7} \\ &= A^2(xy \vee xz) \\ &= xy \vee^{cBA} xz \quad \text{by Proposition 4.8} \\ &= (x \wedge^{cBA} y) \vee^{cBA} (x \wedge^{cBA} z) \quad \text{by Proposition 4.9.} \end{aligned}$$

Therefore $\wedge^{cBA}$ is distributive on $\vee^{cBA}$. □

Definition 4.11 (*cf.* [3, Def. 6.6])**.** *An element $d \in DL$ is dense if $A^2(d) = 1$.*

Remark 4.12. By (4.6) and Lemma 4.4 (3), for an element $d \in DL$, it is dense if and only if $A(d) = 0$.

Remark 4.13. As a typical example of a (complete) distributive lattice, we have $\Omega(X)$, the set of open subsets of a topological space X. In this case, for any $\{U_\lambda\} \subset \Omega(X)$, we have $\bigvee_\lambda U_\lambda = \bigcup_\lambda U_\lambda$ and $\bigwedge_\lambda U_\lambda = \mathrm{Int}(\bigcap_\lambda U_\lambda)$, where $\mathrm{Int}(-)$ denotes the interior. Furthermore, we have $0 = \varnothing$ in $\Omega(X)$. Hence the mapping A of (4.4) on $\Omega(X)$ is given by

$$A \colon \Omega(X) \to \Omega(X); U \mapsto \mathrm{Int}(X \setminus U).$$

Thus, in $\Omega(X)$, the equality $A(U) = 0$ means that $\mathrm{Int}(X \setminus U) = \varnothing$, that is, U is a dense subset. This is a reason for that we use the word "dense element".

We consider the following subsets.

$$\begin{aligned} D &= \{d \in DL \colon d \text{ is dense}\}, \\ a(D) &= \{a(d) \colon d \in D\} = \{x \in B \colon a(x) \in D\}. \end{aligned}$$

Lemma 4.14. *The subset $\mathfrak{I} \cap a(DL)$ is contained in $a(D)$. Here, $\mathfrak{I}$ is in (3.6).*

Proof. If $c \in \mathfrak{I} \cap a(DL)$, then $a(c) \in DL$ and

$$\begin{aligned} A(a(c)) &= ra(a(c)) = r(c) \quad \text{by Lemma 4.1 (3)} \\ &= 0 \quad \text{by Lemma 3.11.} \end{aligned}$$

By this and Remark 4.12, the element $a(c)$ is dense. Therefore $c \in a(D)$. □

Proposition 4.15. *If the mapping $r_* \colon B/\mathfrak{I} \to DL$ in Problem 1.1 at $I = \mathfrak{I}$ is an isomorphism, then $\mathfrak{I} \cap a(DL) = a(D)$.*

Proof. By Lemma 4.14, it suffices to show that $a(D) \subset \mathfrak{I} \cap a(DL)$. If $c \in a(D)$, then $a(c) \in D \subset DL$. Thus $c \in a(DL)$. Furthermore, we have

$$\begin{aligned} r_*([c]) &= r(c) = ra(a(c)) \quad \text{by Lemma 4.1 (3)} \\ &= A(a(c)) = 0 \quad \text{by } a(c) \in D \text{ and Remark 4.12} \\ &= r_*([0]). \end{aligned}$$

Since r_* is an isomorphism, we have $[c] = [0]$ in $B/\mathfrak{I}$. Therefore $c \in \mathfrak{I}$. □

Theorem 4.16 (*cf.* [3, Th. 6.7])**.** *For any x and y in DL, the following are equivalent.*

(1) $A^2(x) = A^2(y)$.
(2) *There exists a dense element d such that $xd = yd$.*

Proof. First we prove the part (2) ⇒ (1). If $xd = yd$ for some dense elment d, then

$$\begin{aligned} A^2(x) &= A^2(x) \cdot 1 = A^2(x)A^2(d) = A^2(xd) \quad \text{by Corollary 4.7} \\ &= A^2(yd) = A^2(y)A^2(d) \quad \text{by Corollary 4.7} \\ &= A^2(y) \cdot 1 = A^2(y). \end{aligned}$$

Next turn to the part (1) ⇒ (2). We assume that $A^2(x) = A^2(y)$, then

$$\begin{aligned} A(xy) &= A(A^2(x)A^2(y)) \quad \text{by Lemma 4.6} \\ &= A(A^2(x)A^2(x)) \\ &= A(A^2(x)) \quad \text{by } A^2(x) \in DL \\ &= A^3(x) = A(x) \quad \text{by Lemma 4.4 (3).} \end{aligned} \tag{4.7}$$

We put $d = xy \vee A(x) = xy \vee^{DL} A(x)$. Then, we have

$$\begin{aligned} A(d) &= A(xy \vee^{DL} A(x)) = A(xy) \wedge^{DL} A^2(x) \quad \text{by Lemma 4.4 (4)} \\ &= A(xy)A^2(x) \quad \text{by Corollary 3.7} \\ &= A(x)A^2(x) \quad \text{by (4.7)} \\ &= 0 \quad \text{by (4.5).} \end{aligned}$$

Hence, by Remark 4.12, the element d is dense. We also note that, by Lemma 4.4 (3), the condition $A^2(x) = A^2(y)$ implies $A(x) = A(y)$. Therefore, we have

$$\begin{aligned} xd &= x(xy \vee A(x)) = x^2y \vee xA(x) = xy \quad \text{by } x \in DL \text{ and (4.5)} \\ &= xy^2 \vee yA(y) \quad \text{by } y \in DL \text{ and (4.5)} \\ &= y(xy \vee A(y)) = y(xy \vee A(x)) \quad \text{by } A(x) = A(y) \\ &= yd. \end{aligned}$$

Hence we see that (1) ⇒ (2). □

For a subset S of B, we define

$$\begin{aligned} \uparrow S &= \{x \in B \colon x \geq s \text{ for some } s \in S\}, \\ \downarrow S &= \{x \in B \colon x \leq s \text{ for some } s \in S\}. \end{aligned}$$

Furthermore, for an element $x \in B$, we denote

$$\uparrow x = \uparrow \{x\} \quad \text{and} \quad \downarrow x = \downarrow \{x\}. \tag{4.8}$$

Lemma 4.17. *For any subset S of B, $\uparrow a(S) = a(\downarrow S)$ and $\downarrow a(S) = a(\uparrow S)$.*

Proof. We have

$$\begin{aligned} x \in \uparrow a(S) &\Leftrightarrow x \geq a(s) \text{ for some } s \in S \\ &\Leftrightarrow a(x) \leq s \text{ for some } s \in S \quad \text{by Lemma 4.1 (1), (3)} \\ &\Leftrightarrow a(x) \in \downarrow S \\ &\Leftrightarrow x \in a(\downarrow S). \end{aligned}$$

Therefore $\uparrow a(S) = a(\downarrow S)$. Similarly, we see that $\downarrow a(S) = a(\uparrow S)$. □

We note that a lattice ideal I is complete if and only if $I = \downarrow \bigvee I$.

Proposition 4.18. *We suppose the following.*

(1) *The mapping $r_* \colon B/\mathfrak{I} \to DL$ is an isomorphism,*
(2) *$\mathfrak{I}$ is complete, and*
(3) *$\mathfrak{I} \subset a(DL)$.*

Then $\bigwedge D$ is a dense element. In particular, $\bigwedge D$ is in DL.

Proof. Under the conditions, we have

$$\begin{aligned} D &= a(\mathfrak{I} \cap a(DL)) \quad \text{by Proposition 4.15} \\ &= a(\mathfrak{I}) \quad \text{by } \mathfrak{I} \subset a(DL) \\ &= a(\downarrow \textstyle\bigvee \mathfrak{I}) \quad \text{since } \mathfrak{I} \text{ is complete} \\ &= \uparrow a(\textstyle\bigvee \mathfrak{I}) \quad \text{by Lemma 4.17} \\ &= \uparrow \textstyle\bigwedge a(\mathfrak{I}) \quad \text{by Lemma 4.1 (4)} \\ &= \uparrow \textstyle\bigwedge a(\mathfrak{I} \cap a(DL)) \quad \text{by } \mathfrak{I} \subset a(DL) \\ &= \uparrow \textstyle\bigwedge D \quad \text{by Proposition 4.15.} \end{aligned} \tag{4.9}$$

Therefore $\bigwedge D \in \uparrow \bigwedge D = D$, that is, $\bigwedge D$ is dense. □

Example 4.10. We consider the following commutative monoid M with 0:

$$M = \{0, x, y, 1\} \quad \text{with } x^2 = 0,\ xy = 0,\ y^2 = y.$$

This is a monoidally distributive poset under the ordering $0 \leq x \leq y \leq 1$. In this case, we have $r(x) = 0$, $r(y) = y$, $a(x) = y$, and $a(y) = x$. These imply

$$DL = \{0, y, 1\}, \quad \mathfrak{I} = \{0, x\}, \quad cBA = \{0, 1\}, \quad D = \{y, 1\}.$$

It is easy to see that M satisfies all conditions in Proposition 4.18. Furthermore, we have $\bigwedge D = y \in DL$.

Theorem 4.19. *We suppose that the conditions in Proposition 4.18 hold. Then, for any x and y in DL, the following are equivalent.*

(1) $A^2(x) = A^2(y)$.
(2) $x(\bigwedge D) = y(\bigwedge D)$.

Proof. First, we prove the part (2) $\Rightarrow$ (1). By Proposition 4.18, the element $\bigwedge D$ is dense. Hence, if $x(\bigwedge D) = y(\bigwedge D)$, then $A^2(x) = A^2(y)$ by Theorem 4.16.

Next turn to the part (1) $\Rightarrow$ (2). By Proposition 4.18, the element $\bigwedge D$ is dense, and so $\bigwedge D \in DL$. Therefore, for any dense element d, we have

$$\begin{array}{rcll} d(\bigwedge D) & = & d \wedge^{DL} (\bigwedge D) & \text{by Corollary 3.7} \\ & = & \bigwedge D & \text{by } d \in D. \end{array} \tag{4.11}$$

By Theorem 4.16, if (1) holds, then $xd_0 = yd_0$ for some $d_0 \in D$. Therefore, we have

$$\begin{array}{rcll} x(\bigwedge D) & = & xd_0(\bigwedge D) & \text{by (4.11)} \\ & = & yd_0(\bigwedge D) & \text{by } xd_0 = yd_0 \\ & = & y(\bigwedge D) & \text{by (4.11).} \end{array}$$

□

For an element $w \in B$, we put

$$wcBA = \{wx \colon x \in cBA\}.$$

Corollary 4.20. *We suppose that the conditions in Proposition 4.18 hold. Then, the mapping*

$$\times(\bigwedge D) \colon cBA \to (\bigwedge D)cBA; x \mapsto x(\bigwedge D)$$

is an order-preserving bijection.

Proof. It suffices to show that the mapping is an injection. If x and y in cBA satisfy $x(\bigwedge D) = y(\bigwedge D)$, then $A^2(x) = A^2(y)$ by Theorem 4.19. Since $x, y \in cBA$, we have $x = y$. □

Definition 4.21. *A nonzero element $f \in DL$ is fieldlike if, for any $x \in B$, we have $xf \in \{0, f\}$.*

Remark 4.22. We consider the case for $B = \mathbb{B}$, the Bousfield lattice of the stable homotopy category. A spectrum K is a *field spectrum* if, for any spectrum X, the smash product $K \wedge X$ is a wedge of suspensions of K. For such K, the Bousfield class $\langle K\rangle$ satisfies the condition of Definition 4.21. This is a reason for that we use the word "fieldlike element".

Lemma 4.23. *If f_0 and f_1 are distinct fieldlike elements, then $f_0f_1 = 0$.*

Proof. By definiton of fieldlike elements, if $f_0f_1 \neq 0$, then $f_0 = f_0f_1 = f_1$. This is a contradiction. □

A nonzero element $m \in B$ is *minimal* if, for any $x \in B$, the inequality $0 \leq x \leq m$ implies $x \in \{0, m\}$.

Lemma 4.24. *If m is a minimal element and $m^2 \neq 0$, then m is fieldlike.*

Proof. Since m is minimal and $0 \neq m^2 \leq m$, we have $m^2 = m$, that is, m is in DL. For an element x, since $mx \leq m$, we have $mx \in \{0, m\}$. Therefore m is fieldlike. □

We put

$$F = \{f \in DL : f \text{ is fieldlike}\}. \tag{4.12}$$

Lemma 4.25. *If two subsets F_0 and F_1 of F satisfy $\bigvee F_0 \leq \bigvee F_1$, then $F_0 \subset F_1$.*

Proof. If $F_0 \not\subset F_1$, then there exists a fieldlike element $f \in F_0 \setminus F_1$. Note that Lemma 4.23 implies $f(\bigvee F_0) = f$ and $f(\bigvee F_1) = 0$. By them and the assumption $\bigvee F_0 \leq \bigvee F_1$, we have a contradiction $f = f(\bigvee F_0) \leq f(\bigvee F_1) = 0$. □

Proposition 4.26. *Let F_0 be a subset of F. For any $f \in F_0$, we have $\bigvee F_0 \leq f \vee a(f)$.*

Proof. For $f \in F_0$, we denote $F_0^f = F_0 \setminus \{f\}$. By Lemma 4.23, we have $f(\bigvee F_0^f) = 0$, and so $\bigvee F_0^f \leq a(f)$. This implies $\bigvee F_0 = f \vee \bigvee F_0^f \leq f \vee a(f)$. □

For an element $x \in B$, we define

$$F(x) = \{f \in F : xf = f\}.$$

We then have

$$x(\bigvee F) = x(\bigvee_{f\in F} f) = \bigvee_{f\in F}(xf) = \bigvee F(x) \tag{4.13}$$

and the following.

Lemma 4.27. *For any x and y in B, we have $F(x \vee y) = F(x) \cup F(y)$ and $F(xy) = F(x) \cap F(y)$.*

Proof. If $f \in F(x \vee y)$, then $fx \vee fy = f(x \vee y) = f$. Thus $fx \neq 0$ or $fy \neq 0$, and so $f \in F(x) \cup F(y)$. Conversely, if $f \in F(x) \cup F(y)$, then $fx = f$ or $fy = f$. This implies $f(x \vee y) = fx \vee fy = f$, and so $f \in F(x \vee y)$.

We turn to the second claim. If $f \in F(xy)$, then $fxy = f \neq 0$. This implies $fx \neq 0$ and $fy \neq 0$, and so $f \in F(x) \cap F(y)$. Conversely, if $f \in F(x) \cap F(y)$, then $fx = f$ and $fy = f$. Hence $fxy = (fx)y = fy = f$, and so $f \in F(xy)$. □

Lemma 4.28. *For any $x \in DL$, we have $F(A^2(x)) = F(x)$. In particular, for any subset F_0 of F, we have $F(A^2(\bigvee F_0)) = F_0$.*

Proof. Let $f \in F$. From

$$\begin{aligned} fA^2(x) = 0 \quad &\Leftrightarrow \quad f \leq A^3(x) \quad \text{by Lemma 4.4 (2)} \\ &\Leftrightarrow \quad f \leq A(x) \quad \text{by Lemma 4.4 (3)} \\ &\Leftrightarrow \quad fx = 0 \quad \text{by Lemma 4.4 (2),} \end{aligned}$$

we obtain the lemma. In particular, for any subset F_0 of F, we have $F(A^2(\bigvee F_0)) = F(\bigvee F_0) = F_0$ by Lemma 4.23. □

Corollary 4.29. *For any $d \in D$, we have $F(d) = F$.*

Proof. By Lemma 4.28, we have $F(d) = F(A^2(d)) = F(1) = F$. □

Corollary 4.30. $\bigvee F \leq \bigwedge D$.

Proof. Let d be a dense element. By Corollary 4.29 and (4.13), we have $\bigvee F = \bigvee F(d) = d(\bigvee F) \leq d$ for any $d \in D$, and so $\bigvee F \leq \bigwedge D$. □

Remark 4.31. By Corollary 4.30, for the condition (4) of Theorem 1.4, the half of the equality holds for any monoidally distributive poset.

Proof of Theorem 1.4. Since $x(\bigvee F) = \bigvee F(x)$ by (4.13), we have $(\bigvee F)cBA \subset \{\bigvee F_0 \colon F_0 \text{ is a subset of } F\}$. On the other hand, for any subset F_0 of F, we have $A^2(\bigvee F_0) \in cBA$ by Lemma 4.4 (3). Thus

$$\begin{aligned} \bigvee F_0 \quad &= \quad \bigvee F(A^2(\bigvee F_0)) \quad \text{by Lemma 4.28} \\ &= \quad (\bigvee F)A^2(\bigvee F_0) \quad \text{by (4.13)} \\ &\in \quad (\bigvee F)cBA \quad \text{by } A^2(\bigvee F_0) \in cBA, \end{aligned}$$

and so $\{\bigvee F_0 \colon F_0 \text{ is a subset of } F\} \subset (\bigvee F)cBA$. Therefore

$$(\bigvee F)cBA = \left\{\bigvee F_0 \colon F_0 \text{ is a subset of } F\right\}. \tag{4.14}$$

Since we suppose that $\bigwedge D = \bigvee F$, by Corollary 4.20 and (4.14), we have an order-preserving bijection

$$cBA \xrightarrow[\sim]{\times(\bigvee F)} \left\{\bigvee F_0 \colon F_0 \text{ is a subset of } F\right\}.$$

At last, we show that this is a lattice isomorphism, that is, this bijection preserves joins and meets. For any x and y in $cBA(\subset DL)$, we have

$$\begin{aligned}
(\bigvee F)(x \vee^{cBA} y) &= (\bigvee F)A^2(x \vee y) \quad \text{by Proposition 4.8} \\
&= \bigvee F(A^2(x \vee y)) \quad \text{by (4.13)} \\
&= \bigvee F(x \vee y) \quad \text{by Lemma 4.28} \\
&= \bigvee (F(x) \cup F(y)) \quad \text{by Lemma 4.27} \\
&= \bigvee F(x) \vee^{(\bigvee F)cBA} \bigvee F(y) \quad \text{by (4.14)} \\
&= (\bigvee F)x \vee^{(\bigvee F)cBA} (\bigvee F)y \quad \text{by (4.13)},
\end{aligned}$$

and

$$\begin{aligned}
(\bigvee F)(x \wedge^{cBA} y) &= (\bigvee F)xy \quad \text{by Proposition 4.9} \\
&= \bigvee F(xy) \quad \text{by (4.13)} \\
&= \bigvee (F(x) \cap F(y)) \quad \text{by Lemma 4.27} \\
&= \bigvee F(x) \wedge^{(\bigvee F)cBA} \bigvee F(y) \quad \text{by (4.14)} \\
&= (\bigvee F)x \wedge^{(\bigvee F)cBA} (\bigvee F)y \quad \text{by (4.13)}.
\end{aligned}$$

□

Example 4.15. For the monoidally distributive poset M in Example 4.10, we have $F = \{y\}$. (Remark that the element x satisfies $xw \in \{0, x\}$ for any $w \in M$, and however $x \notin DL$. Hence x is not fieldlike.) Thus $\bigvee F = y = \bigwedge D (= \bigwedge\{y, 1\})$, and therefore M satisfies all conditions in Theorem 1.4. Indeed, the mapping

$$\begin{aligned}
cBA = \{0,1\} \xrightarrow{\times \bigvee F = \times y} (\bigvee F)cBA &= ycBA = y\{0,1\} = \{0, y\} \\
&= \left\{\bigvee F_0 \colon F_0 \text{ is a subset of } F = \{y\}\right\}
\end{aligned}$$

is an isomorphism.

5. The case for $B = \mathbb{B}$

The Bousfield lattice $\mathbb{B}$ of the stable homotopy category of (p-local) spectra is a monoidally distributive poset. Indeed, we have $0 = \langle 0 \rangle$, $1 = \langle S^0 \rangle$, where S^0 is the sphere spectrum, and also, for any $x = \langle X \rangle$ and $y = \langle Y \rangle$, we have $x \vee y = \langle X \vee Y \rangle$ and $xy = \langle X \wedge Y \rangle$. In this section, we consider the case for $B = \mathbb{B}$, and we see that some results in [3] are obtained from the viewpoint of this paper. We remark that we use the notations DL and cBA, instead of $\mathbb{DL}$ and $\mathbf{c}\mathbb{BA}$ in [3], respectively.

We recall [3, Conj. 3.12]. Let

$$h = \langle H\mathbb{F}_p \rangle,$$

the Bousfield class of the mod p Eilenberg-MacLane specrum. We recall that

$$\text{(5.1)} \qquad \text{The class } h \text{ is fieldlike, (particularly, } h \in DL)$$

in the sense of Definition 4.21. We consider the subset

$$J = \{c \in \mathbb{B} \colon c < h\}.$$

A class $c \in \mathbb{B}$ is *strange* if $c \in J$ [3, Def. 3.9]. By use of the notation in (4.8), we see the following.

Lemma 5.1 ([3, Lemma 3.10]). *We have $J = \downarrow (h \wedge a(h))$. In particular, J is a complete ideal of $\mathbb{B}$.*

Proof. First we see that $\downarrow (h \wedge a(h)) \subset J$. If $c \in \downarrow (h \wedge a(h))$, then $c \leq h \wedge a(h) \leq h$. Hence it suffices to show that $c \neq h$. If $c = h$, then $hc = h^2 = h \neq 0$ by (5.1). However, we have $0 \neq h = hc \leq h(h \wedge a(h)) \leq ha(h) = 0$ by (4.2), which is a contradiction. Therefore $c \neq h$, and so $\downarrow (h \wedge a(h)) \subset J$.

Next we show that $J \subset \downarrow (h \wedge a(h))$. If $c \in J$, then $c < h$. Hence it suffices to show that $c \leq a(h)$. If $c \not\leq a(h)$, then $ch \neq 0$ by Lemma 4.1 (2). This implies that $ch = h$ by (5.1). However, this implies $h = ch \leq c < h$, which is a contradiction. Therefore $c \leq a(h)$. □

We note that

$$\text{(5.2)} \qquad (h \wedge a(h))^2 = (h \wedge a(h))(h \wedge a(h)) \leq ha(h) = 0.$$

Proposition 5.2 ([3, Prop. 3.11]). *The induced mapping $r_* \colon \mathbb{B}/J \to DL$, the mapping in Problem 1.1 at $(B, I) = (\mathbb{B}, J)$, is well-defined.*

Proof. By Theorem 1.2 (1), it suffices to show that $J \subset \mathfrak{I}$. If $c \in J$, then, for any $x \in \mathbb{B}$, we have

$$\begin{aligned} r(x) &\leq r(x \vee c) \quad \text{by Lemma 3.4 (1)} \\ &\leq r(x \vee (h \wedge a(h))) \quad \text{by Lemma 5.1} \\ &= r(x \vee (h \wedge a(h))^2) \quad \text{by Lemma 3.5} \\ &= r(x) \quad \text{by (5.2).} \end{aligned}$$

Hence $r(x) = r(x \vee c)$, and therefore $c \in \mathfrak{I}$. □

Conjecture 5.3 (Retract conjecture [3, Conj. 3.12])**.** *The well-defined mapping*

$$r_*\colon \mathbb{B}/J \to DL$$

in Proposition 5.2 is an isomorphism.

We remark that Problem 1.1 is a generalization of this conjecture.

Lemma 5.4. *If Conjecture 5.3 holds, then $J \subset a(DL)$.*

Proof. First, we show that $\uparrow(a(h) \vee h) \subset DL$. If $x \in \uparrow(a(h) \vee h)$, then

$$x \geq a(h) \vee h \geq h \tag{5.3}$$

and

$$r(x) \geq r(a(h) \vee h) \geq r(h) = h. \tag{5.4}$$

Since r_* is an isomorphism and $r_*([x]) = r(x) = r^2(x) = r_*([r(x)])$ by Lemma 3.4 (2), we have $[x] = [r(x)]$ in $\mathbb{B}/J$. This and Lemma 5.1 imply

$$x \vee (h \wedge a(h)) = r(x) \vee (h \wedge a(h)). \tag{5.5}$$

Hence

$$\begin{aligned} x &= x \vee h \quad \text{by (5.3)} \\ &= x \vee h \vee (h \wedge a(h)) \\ &= r(x) \vee h \vee (h \wedge a(h)) \quad \text{by (5.5)} \\ &= r(x) \vee h \\ &= r(x) \quad \text{by (5.4)}. \end{aligned}$$

Therefore $x \in DL$, and so $\uparrow(a(h) \vee h) \subset DL$. This implies

$$\begin{aligned} J &= \downarrow(h \wedge a(h)) \quad \text{by Lemma 5.1} \\ &= \downarrow a(a(h) \vee h) \quad \text{by Lemma 4.1 (3), (4)} \\ &= a(\uparrow(a(h) \vee h)) \quad \text{by Lemma 4.17} \\ &\subset a(DL), \end{aligned}$$

and our claim is shown. □

Let $F(n)$ be a finite spectrum of type n, and $K(n)$ the nth Morava K-theory spectrum. We also consider the v_n-telescope $T(n)$ of $F(n)$.

Conjecture 5.5 (Telescope conjecture [6, 10.5])**.** *The Bousfield class $\langle K(n)\rangle$ is the same as $\langle T(n)\rangle$ for any $n \geq 0$.*

The spectrum $A(n)$ is defined by the cofiber sequence

$$F(n) \to L_{K(n)}F(x) \to A(n) \to \Sigma F(n),$$

where $L_{K(n)}$ is the Bousfield localization functor with respect to $K(n)$. In [2, Prop. 1.6], Hovey proved that

- $\langle T(n)\rangle = \langle K(n)\rangle \vee \langle A(n)\rangle$ for any $n \geq 0$, and
- The smash product $A(n) \wedge K(m)$ is 0 for any n and m.

Furthermore, we have

$$A(n) \wedge A(n) = A(n) \tag{5.6}$$

for any $n \geq 0$ (see [2, §5]). By these facts, Conjecture 5.5 holds if and only if $\langle A(n)\rangle = 0$. On these backgrounds, Hovey and Palmieri proposed the following conjecture as a weak version of Conjecture 5.5.

Conjecture 5.6 ([3, Conj. 5.1])**.** *For any $n \geq 0$, the Bousfield class $\langle A(n)\rangle$ is 0 or minimal.*

Remark 5.7. In 2023, Burkland, Hahn, Levy, and Schlank announced that Conjecture 5.5 doesn't hold [1].

Hereafter, for the sake of simplicity, we denote

$$k_n = \langle K(n)\rangle \quad \text{and} \quad a_n = \langle A(n)\rangle.$$

Corollary 5.8. *If Conjecture 5.6 holds, then any nonzero a_n is a fieldlike element.*

Proof. The class a_n is in DL by (5.6). Hence, if a_n is minimal, then it is filedlike by Lemma 4.24. □

We recall the subset F in (4.12). We put

$$F_{kah} = \{k_m, a_n, h \colon m \geq 0,\ a_n \neq 0\}.$$

By Corollary 5.8,

(5.7) If Conjecture 5.6 is true, then $F_{kah} \subset F$.

We also recall that, by Proposition 4.26, we have $\bigvee F_{kah} \leq f \vee a(f)$ for any $f \in F_{kah}$.

Conjecture 5.9 ([3, Conj. 6.12])**.** $\bigvee F_{kah} = h \vee a(h)$.

Proposition 5.10. *If Conjecture 5.6 and Conjecture 5.9 are true, then $F_{kah} = F$, that is, any fieldlike element is one of k_n's, a_n's, and h.*

Proof. By (5.7), it suffices to show that $F \subset F_{kah}$. By Proposition 4.26 and the assumption, we have $\bigvee F \leq h \vee a(h) = \bigvee F_{kah}$. From this and Lemma 4.25, we obtain $F \subset F_{kah}$. □

Remark 5.11. For a general monoidally distributive poset, we have an analogue of Conjecture 5.9, which claims that there exists a fieldlike element $f \in F$ such that $\bigvee F = f \vee a(f)$. For example, on the monoidally distributive poset M in Example 4.10 (or Example 4.15), this claim holds. Indeed, for this M, we have $x \leq y$, $a(y) = x$, and $F = \{y\}$. Thus $\bigvee F = y = y \vee x = y \vee a(y)$.

Theorem 5.12 ([3, Prop. 6.13]). *If Conjecture 5.3, Conjecture 5.6 and Conjecture 5.9 hold, then cBA is isomorphic to a complete Boolean algebra generated by F_{kah}, that is,*

$$cBA \cong \left\{ \bigvee F_0 \colon F_0 \text{ is a subset of } F_{kah} \right\}.$$

Proof. We note that the ideal J is complete by Lemma 5.1. If Conjecture 5.3 is true, then $J = \mathfrak{J}$ by Theorem 1.2. Hence, by Lemma 5.4, all conditions of Proposition 4.18 hold. We also have

$$\begin{aligned}
\bigwedge D &= \bigwedge \left(\uparrow a(\bigvee \mathfrak{J}) \right) \quad \text{by (4.9)} \\
&= \bigwedge \left(\uparrow a(\bigvee J) \right) \quad \text{by } J = \mathfrak{J} \\
&= a\left(\bigvee J \right) \\
&= a(h \wedge a(h)) \quad \text{by Lemma 5.1} \\
&= a(h) \vee h \quad \text{by Lemma 4.1 (3), (4)} \\
&= \bigvee F_{kah} \quad \text{since we assume that Conjecture 5.9 holds.}
\end{aligned}$$

We note that $F_{kah} = F$ by Proposition 5.10. Therefore, by Theorem 1.4, we have $cBA \cong \{\bigvee F_0 \colon F_0$ is a subset of $F_{kah}\}$. □

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