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ON GROUPS OF UNITS OF COMMUTATIVE FINITE RINGS

Chiteng'a John Chikunji

Abstract. Let R be a commutative completely primary finite ring with unique maximal ideal $\mathcal{J}$ such that $\mathcal{J}^3 = \{0\}$ and $\mathcal{J}^2 \neq \{0\}$. In this paper, we investigate and determine the structure of the group of units $U(R)$ of the ring R of characteristic p^3 without giving any conditions on the generators for the maximal ideal $\mathcal{J}$ or on the invariants of the ring R.

1. Introduction

A finite ring R with identity $1 \neq 0$ is called *completely primary* if all its zero divisors including the zero element form the unique maximal ideal $\mathcal{J}$. Completely primary finite rings are local rings with unique maximal ideals.

Throughout this paper, all rings are finite and commutative (unless otherwise stated) with identity element $1 \neq 0$, subrings have the same identity, ring homomorphisms preserve 1 and modules are unital. For a given completely primary finite ring R, $\mathcal{J}$ will denote the Jacobson radical of R. We denote the group of units of R by $U(R)$; if g is an element of $U(R)$, then $o(g)$ denotes its order, and $< g >$ denotes the cyclic group generated by g. Further, for a subset A of R or $U(R)$, $|A|$ will denote the number of elements in A. The ring of integers modulo the number n will be denoted by $\mathbb{Z}_n$, and the characteristic of R will be denoted by char$R$. We denote a direct product of r cyclic groups of $\mathbb{Z}_m$ by $\mathbb{Z}_m^r$ or by $\underbrace{\mathbb{Z}_m \times \cdots \times \mathbb{Z}_m}_{r}$. The character p represents a prime.

Although finite rings have been studied extensively in recent years ([6], [7]) and the tools necessary for describing completely primary finite fings have been available for some time, their classification into well known structures (which are necessarily given in [1], [6], [7]) is not complete.

Much of the recent work on completely primary rings has demonstrated fundamental importance of these rings in the structure theory of finite rings with identity. Let R be a finite ring. It turns out that R has a unique maximal ideal if and only if it is a full matrix ring over a completely primary ring. In particular, rings with a unique maximal ideal are not necessarily completely primary. Therefore, the study of rings with a unique maximal ideal (i.e. local rings) reduces to the study of completely primary rings.

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More evidence for the importance of completely primary rings comes from the fact that any commutative finite ring is a direct sum of completely primary rings. Moreover, any finite ring R is of the form $S+N$, where $S \cap N = (0)$ with N a subgroup of the Jacobson radical of R and S a direct sum as an additive abelian group of certain matrix rings over completely primary rings.

Because of the feeling that completely primary rings play an important role in the classification of all finite rings with identity, they have been the subject of a good deal of research in recent years. Perhaps the most important results here are theorems of Raghavendran [6], which show that the zero-divisors of any completely primary finite ring R form the Jacobson radical $\mathcal{J}$ of R, and that $|R| = p^{nr}$, $R/\mathcal{J} \cong GF(p^r)$, $\text{char}(R) = p^k$ with $(1 \leqslant k \leqslant n)$, and $\mathcal{J}^n = (0)$ for some prime p and positive integers $n,\ r,\ k$. The group of units $U(R)$ of R contains a cyclic subgroup $< b >$ of order $p^r - 1$, and it is a direct (semi-direct in case R is non-commutative) product of $1+\mathcal{J}$ by $< b >$ (see Section 2).

We now assume that R is a completely primary finite ring with unique maximal ideal $\mathcal{J}$ such that $\mathcal{J}^3 = (0)$, $\mathcal{J}^2 \neq (0)$, $R/\mathcal{J} \cong GF(p^r)$. Then following Raghavedran's work [6], these rings are either of characteristic p, p^2 or p^3. In [1], the author gave constructions and treated the isomorphism problem of these rings according to their characteristics. In [2] through [5], we obtained the structure of the group of units of these rings under various conditions. After noticing that for such a ring, $U(R)$ is a direct product of a cyclic group by a p-group $1+\mathcal{J}$, we attempted to solve the problem by concentrating on determining the structure of $1+\mathcal{J}$. This problem is not completely solved because the structure of the p-subgroup $1+\mathcal{J}$ of $U(R)$ may be quite involving.

In earlier work, we approached the problem of determining the group of units by first fixing the dimensions of the vector spaces $\mathcal{J}/\mathcal{J}^2$ and $\mathcal{J}^2$ over the residue field $R/\mathcal{J}$ to 2 and 1, respectively; and when the dimension of $\mathcal{J}^2$ over $R/\mathcal{J}$ is the maximal possible, for all the characteristics. In these cases, the structure of $U(R)$ and its generators depend on the structural matrices and its parameters. We also determined $U(R)$ when the dimensions of $\mathcal{J}/\mathcal{J}^2$ and $\mathcal{J}^2$ over $R/\mathcal{J}$ are both equal to 2 when the characteristic of R is p. In these particular cases, we assumed that $\text{ann}(\mathcal{J}) = \mathcal{J}^2$. We further extended the problem to the case when $\mathcal{J}^2 \subseteq \text{ann}(\mathcal{J})$, dimension of $\mathcal{J}/\mathcal{J}^2$ is 3 and dimension of $\mathcal{J}^2$ is 1, and to when the dimensions of $\mathcal{J}/\mathcal{J}^2$ and $\mathcal{J}^2$ over $R/\mathcal{J}$ are both equal to 2, and the characteristic of R is p. In subsequent work, we determined the structure of $U(R)$ when the characteristic of R is p^2 and the dimension of $\mathcal{J}/\mathcal{J}^2$ is greater that 2 while the dimension of $\mathcal{J}^2$ is lesser than the maximum possible and the structure of $U(R)$ was determined without considering structural matrices of isomorphic classes of these types of rings. When the characteristic of R is p^2, various cases were further taken into consideration: when $p \in \mathcal{J} - \mathcal{J}^2$ and $p \in \mathcal{J}^2$, in determining the structure of $1 + \mathcal{J}$. It was clear that these cases lead to distinct structures of $1 + \mathcal{J}$.

In this paper, we extend the above problem to the case when the characteristic of R is p^3, without giving any conditions on the generators for $\mathcal{J}$ or on the invariants of the ring R. We leave the problem where we consider fixed dimensions and bases for the vector spaces $\mathcal{J}^i/\mathcal{J}^{i+1}$ $(i = 1,\ 2)$ over the residue field $R/\mathcal{J}$ and where we fix the order of the ideal $\mathcal{J}^2$, for further inverstigation.

The rest of the paper is organised as follows. In section 2, we state without proofs some of the general results on completely primary finite rings and on their groups of units which are relevant to our work. In section 3, we give an explicit description of certain completely primary finite rings with $\mathcal{J}^3 = \{0\}$, $\mathcal{J}^2 \neq \{0\}$ whose groups of units have been considered earlier in [2], [3], [4] and [5]. Finally, in section 4, we determine the structure of the group of units of the rings of characteristic p^3 for any fixed prime number p. This complements the author's earlier solution of the problem in the cases when the characteristic of R is p and p^2 studied under various conditions.

2. Completely primary finite rings

Let R be a completely primary finite ring with maximal ideal $\mathcal{J}$. Then $|R| = p^{nr}$, $\mathcal{J}$ is the Jacobson radical of R, $\mathcal{J}^m = (0)$, where $m \leqslant n$, $|\mathcal{J}| = p^{(n-1)r}$, and the residue field $R/\mathcal{J} \cong GF(p^r)$, the finite field of p^r elements, for some prime p and positive integers $n,\ r$. The characteristic of R, $\mathrm{char}R = p^k$, where $1 \leqslant k \leqslant m$. If $k = n$, then $R = \mathbb{Z}_{p^k}[b]$, where b is an element of R of order $p^r - 1$; $\mathcal{J} = pR$ and $\mathrm{Aut}(R) \cong \mathrm{Aut}(R/pR)$ (see Proposition 2 in [6]). Such a ring is called a *Galois ring*, denoted by $GR(p^{kr},\ p^k)$, and a concrete model is the quotient $\mathbb{Z}_{p^k}[X]/ < f(X) >$, where $f(X)$ is a monic polynomial of degree r, irreducible modulo p. Any such polynomial will do: the rings are all isomorphic. Trivial cases are $GR(p^n,\ p^n) = \mathbb{Z}_{p^n}$ and $GR(p^n,\ p) = \mathbf{F}_{p^n}$. The Galois ring $GR(p^{nr},\ p^n)$ is a completely primary ring of characteristic p^n with maximal ideal $pGR(p^{nr},\ p^n)$ and residue field isomorphic to $\mathbb{F}_{p^r}$. Its group of units is well known and we present it below in Theorem 2, for easy reference. Furthermore, if $k < n$ and $\mathrm{char}R = p^k$, it can be deduced from [6] and [7] that R has a coefficient subring R_o of the form $GR(p^{kr},\ p^k)$ which is clearly a maximal Galois subring of R. The maximal ideal of R_o is

$$\mathcal{J}_o = pR_o = \mathcal{J} \cap R_o, \quad \text{and} \quad R_o/\mathcal{J}_o \cong GF(p^r).$$

Let $\psi : R_o \longrightarrow R_o/\mathcal{J}_o$ be the canonical map. Since b has order $p^r - 1$ and $\mathcal{J}_o \subset \mathcal{J}$, we have that $\psi(b)$ is a primitive element of $R_o/\mathcal{J}_o$. Let $K_o =<$

$b > \cup\{0\}$ and let $R_o = \mathbb{Z}_{p^k}[b]$ be a coefficient subring of R of order p^{kr}. Then it is easy to show that every element of R_o can be written uniquely as $\sum_{i=0}^{k-1} \lambda_i p^i$, where $\lambda_i \in K_o$. Also, there exist elements $m_1,\ m_2,\ \ldots,\ m_h \in \mathcal{J}$ and automorphisms (in case R is noncommutative) $\sigma_1,\ \ldots,\ \sigma_h \in \mathrm{Aut}(R_o)$ such that

$$R = R_o \oplus \sum_{i=1}^{h} R_o m_i \quad (\text{as } R_o\text{-modules}), \quad m_i r = r^{\sigma_i} m_i,$$

for every $r \in R_o$ and any $i = 1,\ \ldots,\ h$. Further, $\sigma_1,\ \ldots,\ \sigma_h$ are uniquely determined by R and R_o. The maximal ideal of R is

$$\mathcal{J} = pR_o \oplus \sum_{i=1}^{h} R_o m_i.$$

Let R be a completely primary ring (not necessarily commutative) of order p^{nr} with unique maximal ideal $\mathcal{J}$. Then the set $R - \mathcal{J}$ consisting of invertible elements in R forms a group with respect to the multiplication defined on R, called the group of units of R. The following facts are useful (e.g. see [6, §2]): The group of units $U(R)$ of R contains a cyclic subgroup $< b >$ of order $p^r - 1$, and it is a semi-direct product of $1 + \mathcal{J}$ by $< b >$; the group $U(R)$ is solvable; if G is a subgroup of $U(R)$ of order $p^r - 1$, then G is conjugate to $< b >$ in $U(R)$; if $U(R)$ contains a normal subgroup of order $p^r - 1$, then the set $K_o = < b > \cup\{0\}$ is contained in the center of the ring R; and $(1 + \mathcal{J}^i)/(1 + \mathcal{J}^{i+1}) \cong \mathcal{J}^i/\mathcal{J}^{i+1}$ (the left hand side as a multiplicative group and the right hand side as an additive group). It is easy to check that $|U(R)| = p^{(n-1)r}(p^r - 1)$ and that $|1 + \mathcal{J}| = p^{(n-1)r}$, so that $1 + \mathcal{J}$ is a p-group.

3. Some known groups of units of finite rings with $\mathcal{J}^3 = (0)$

Now, let R be a commutative completely primary finite ring with maximal ideal $\mathcal{J}$ such that $\mathcal{J}^3 = (0)$ and $\mathcal{J}^2 \neq (0)$. The author gave constructions describing these rings for each characteristic and for details, we refer the reader to Sections 4 and 6 of [1]. Then, $R/\mathcal{J} \cong GF(p^r)$ and the char$R$ is p^k, where $1 \leqslant k \leqslant 3$. Let $R_o = GR(p^{kr},\ p^k)$ be a Galois subring of R. Let $s,\ t,\ \lambda$ be numbers in the generating sets for R_o-modules $U,\ V,\ W$, respectively. From Constructions A and B in [1], we can represent R and $\mathcal{J}$ as

$$R = R_o \oplus U \oplus V \oplus W$$

and

$$\mathcal{J} = pR_o \oplus U \oplus V \oplus W,$$

respectively.

The structure of R is characterized by the invariants $p,\ n,\ r,\ d,\ s,\ t$ and λ; and the linearly independent matrices (α_{ij}^{k}) definite in the multiplication. In [1], $d \geqslant 0$ denotes the number of the $m_i \in \{m_1,\ \ldots,\ m_h\}$ with $pm_i \neq 0$.

In [2] we have determined the group of units $U(R)$ of the ring R and its generators when $s = 2,\ t = 1,\ \lambda = 0$, characteristic of R is p; and when $t = s(s+1)/2,\ \lambda = 0$, for a fixed s, for all the characteristics of R. It was noted that $U(R)$ and its generators depended on the structural matrices (a_{ij}) and on the parameters $p,\ k,\ r$ and s. In [3] we obtained the structure of $U(R)$ and its generators when $s = 2,\ t = 1,\ \lambda = 0$ and characteristic of R is p^2 and p^3; and the case when $s = 2,\ t = 2,\ \lambda = 0$ and characteristic of R is p. In both papers [2] and [3], we assumed that $\lambda = 0$ so that the annihilator of the maximal ideal $\mathcal{J}$ coincides with $\mathcal{J}^2$. It was also noted that our strategy of considering different types of symmetric matrices was not viable any more and we followed a different approach; that of considering structural matrices of isomorphic classes of these types of rings with the same invariants $p,\ r,\ k,\ s$, and t.

In [4], we proved that $1+\mathcal{J}$ is a direct product of its subgroups $1+pR_o \oplus U \oplus V$ and $1+W$ and further determined the structure of $1+W$, in general. We also determined the structure of $U(R)$ and its generators when $s = 3$, $t = 1,\ \lambda \geqslant 1$ and $\mathrm{char} R = p$. We then generalised the structure of $U(R)$ in the cases when $s = 2,\ t = 1$; $t = s(s+1)/2$ and $\mathrm{char} R = p$, for a fixed s, and for all characteristics of R; and when $s = 2,\ t = 2$ and $\mathrm{char} R = p$; determined in [2] and [3], to the general case when $\mathrm{ann}(\mathcal{J}) = \mathcal{J}^2 + W$ so that $\lambda \geqslant 1$. This complemented our earlier solution to the problem in the case when $\mathrm{ann}(\mathcal{J}) = \mathcal{J}^2$. In [5], we determined the structure of $U(R)$ when the characteristic of R is p^2, $s \geqslant 3$ and $1 \leqslant \dim_{R_o/pR_o}(\mathcal{J}^2) < s(s+1)/2$, without considering structural matrices of isomorphic classes of these types of rings.

In Section 4 of this paper, we extend the above problem to the case when the characteristic of R is p^3, without giving any restrictions to the generators or invariants of these types of rings.

4. Group of units of finite rings of characteristic p^3

We now consider the structure of the group of units of commutative completely primary finite rings with maximal ideals $\mathcal{J}$ such that $\mathcal{J}^3 = (0),\ \mathcal{J}^2 \neq (0)$, and with characteristic p^3.

4.1. **A construction of commutative rings of characteristic p^3.** Let $R_o = GR(p^{3r},\ p^3)$ be a Galois subring of characteristic p^3 and order p^{3r}. Let V be an R_o/pR_o-space, which when considered as an R_o-module, has a generating set $\{v_1,\ \ldots,\ v_t\}$ with t elements and let U be an R_o-module with an R_o-module generating set $\{u_1,\ \ldots,\ u_s\}$ of s elements; and suppose that $d \geqslant 0$ of the u_i are such that $pu_i \neq 0$. Since R_o is commutative, we can think of them as both left and right R_o-modules.

Assume $1+d+t \leqslant s(s+1)/2$. Let (a_{ij}^m) be $s \times s$ linearly independent symmetric matrices over R_o/pR_o for $1 \leqslant m \leqslant 1+d+t$.

On the additive group $R = R_o \oplus U \oplus V$, define multiplication by the following relation:

$$u_i u_j = a_{ij}^0 p^2 + \sum_{l=1}^{d} a_{ij}^l p u_l + \sum_{k=1}^{t} a_{ij}^{d+t} v_k;$$

$$u_i v_k = v_k u_i = u_i u_j u_\lambda = p^2 u_i = p v_k = v_k v_l = v_l v_k = 0;$$

$$u_i \alpha = \alpha u_i,\ v_k \alpha = \alpha v_k;\ (1 \leqslant i,\ j,\ \lambda \leqslant s;\ 1 \leqslant l \leqslant d;\ 1 \leqslant k \leqslant t);$$

where $\alpha,\ a_{ij}^0,\ a_{ij}^l,\ a_{ij}^{d+t} \in R_o/pR_o$.

By the above relations, R is a commutative completely primary finite ring of characteristic p^3 with Jacobson radical $\mathcal{J} = pR_o \oplus U \oplus V$, $\mathcal{J}^2 = p^2 R_o \oplus pU \oplus V$ and $\mathcal{J}^3 = (0)$. We call (a_{ij}^m) the *structural matrices* of the ring R and the numbers $p,\ n,\ r,\ d,\ s,\ t$ *invariants* of the ring R.

The following result is proved in [1, Theorem 6.1]

Theorem 1. *Let R be a ring. Then R is a commutative completely primary finite ring of characteristic p^3 with maximal ideal $\mathcal{J}$ such that $\mathcal{J}^3 = (0)$, $\mathcal{J}^2 \neq (0)$, $\mathcal{J}^2$ lies in the annihilator of $\mathcal{J}$, if and only if R is isomorphic to one of the rings given by the above relations.*

Remark 1. *We know that $R = R_o \oplus R_o m_1 \oplus \ldots \oplus R_o m_h$, where the elements $m_i \in \mathcal{J}$; and that $\mathcal{J} = pR_o \oplus R_o m_1 \oplus \ldots \oplus R_o m_h$. Since $\mathcal{J}^3 = (0)$ and $\mathcal{J}^2 \subseteq ann(\mathcal{J})$, with $\mathcal{J}^2 \neq (0)$, we can write*

$$\{m_1,\ \ldots,\ m_h\} = \{u_1,\ \ldots,\ u_s,\ v_1,\ \ldots,\ v_t\}$$

where $u_1,\ \ldots,\ u_s \in \mathcal{J} - \mathcal{J}^2$ and $v_1,\ \ldots,\ v_t \in \mathcal{J}^2$, so that $s + t = h$.

In view of the above considerations and by [1.8] in [1], the non-zero elements in the set

$$\{1,\ p,\ p^2,\ u_1,\ \ldots,\ u_s,\ pu_1,\ \ldots,\ pu_s, v_1,\ \ldots,\ v_t\}$$

form a "basis"for R over R_o/pR_o.

Note that since characteristic of R is p^3 it is clear that $p^2 m = 0$ for every $m \in \mathcal{J}$ and $pm = 0$ for every $m \in \mathrm{ann}(\mathcal{J}) \supseteq \mathcal{J}^2$. It then follows that $p \in \mathcal{J} - \mathcal{J}^2$ and $p^2 \in \mathcal{J}^2$.

Remark 2. *Suppose that $d \geqslant 0$ is the number of the elements pu_i which are non-zero. Suppose, without lose of generality, that $pu_1,\ \ldots,\ pu_d$ are the d non-zero elements. Then, the above set becomes*

$$\{1,\ p,\ p^2,\ u_1,\ \ldots,\ u_s,\ pu_1,\ \ldots,\ pu_d, v_1,\ \ldots,\ v_t\};$$

and by Proposition 3.2 of [1], we have $1 \leqslant 1+d+t \leqslant s(s+1)/2$ *and hence, every element of* R *may be written as*

$$\lambda_o + \lambda_1 p + \lambda_2 p^2 + \sum_{i=1}^{s} \alpha_i u_i + \sum_{l=1}^{d} \beta_l p u_l + \sum_{k=1}^{t} \gamma_k u_i u_j;$$

where $\lambda_o,\ \lambda_1,\ \lambda_2,\ \alpha_i,\ \beta_l,\ \gamma_k \in R_o/pR_o$.

Clearly, the product $u_i u_j \in \mathcal{J}^2$. *Hence,*

$$u_i u_j = a_{ij}^{0} p^2 + \sum_{l=1}^{d} a_{ij}^{l} p u_l + \sum_{k=1}^{t} a_{ij}^{d+k} v_k,$$

and $dim_{R_o/pR_o}(\mathcal{J}^2) \leqslant 1+d+t$, *where in this particular case* $d \geqslant 0$ *and* $t \geqslant 0$.

Now, since $p^2,\ pu_l,\ v_k \in \mathcal{J}^2$ $(l = 1,\ \ldots,\ d;\ k = 1,\ \ldots,\ t)$, we can write them as sums of products of elements of $\mathcal{J}$. In particular, $p^2,\ pu_l,\ v_k$ can be written as linear combinations of $p^2,\ pu_i$ and $u_i u_j$ with coefficients in R_o/pR_o. Hence, since $p^2,\ pu_l,\ v_k \in \mathcal{J}^2$ $(l = 1,\ \ldots,\ d;\ k = 1,\ \ldots,\ t)$ is a basis for $\mathcal{J}^2$ over R_o/pR_o, we conclude that p^2, pu_i and $u_i u_j$ $(i,\ j = 1,\ \ldots, s)$ generate $\mathcal{J}^2$.

Clearly, $|R| = p^{3r} \cdot p^{2dr} \cdot p^{(s-d)r} \cdot p^{tr} = p^{(3+s+d+t)r}$ and $|\mathcal{J}| = p^{(2+s+d+t)r}$. (Notice that $|R_o u_i| = p^{2r}$ if $pu_i \neq 0$, and $|R_o u_i| = p^r$, if otherwise.)

4.2. **The group of units.** We know that for a commutative completely primary finite ring R,

$$U(R) = < b > \cdot (1+\mathcal{J}) \cong \mathbb{F}_{p^r}^{\times} \times (1+\mathcal{J});$$

and $\mathbb{F}_{p^r}^{\times}$ is cyclic of order $p^r - 1$, so it suffices to determine the structure of the p-group $1+\mathcal{J}$ in order to obtain the complete structure of $U(R)$.

There are many important results on the group of units of certain finite rings. For example, it is well known that the multiplicative group of the finite field $GF(p^r)$ is a cyclic group of order $p^r - 1$, and the multiplicative group of the finite ring $\mathbb{Z}/p^k\mathbb{Z}$, the ring of integers modulo p^k, for p a prime number, and k a positive integer, is a cyclic group of order $p^{k-1}(p-1)$ if p is odd, and is a direct product of a cyclic group of order 2 and a cyclic group of order 2^{k-2}, if $p = 2$.

Let $U(R_o)$ denote the group of units of the Galois ring $R_o = GR(p^{nr},\ p^n)$. Then $U(R_o)$ has the following structure [6]:

Theorem 2. $U(R_o) = < b > \times (1+pR_o)$, *where* $< b >$ *is the cyclic group of order* $p^r - 1$ *and* $1+pR_o$ *is of order* $p^{(n-1)r}$ *whose structure is described below.*

(i) If (a) p is odd, or (b) $p = 2$ and $n \leqslant 2$, then $1 + pR_o$ is the direct product of r cyclic groups each of order $p^{(n-1)}$.

(ii) When $p = 2$ and $n \geqslant 3$, the group $1 + pR_o$ is the direct product of a cyclic group of order 2, a cyclic group of order $2^{(n-2)}$ and $r-1$ cyclic groups each of order $2^{(n-1)}$.

4.3. **Structure of the group of units of characteristic p^3.** Suppose that the $1+s$ elements $p,\ u_1,\ \ldots,\ u_s$ generate $\mathcal{J}/\mathcal{J}^2$ over R_o/pR_o under the multiplication and relations defined above, and suppose that $1+d+t$ elements $p^2,\ pu_1,\ \ldots,\ pu_d, v_1,\ \ldots,\ v_t$ generate $\mathcal{J}^2$ over R_o/pR_o, where $d \geqslant 0$ and $t \geqslant 0$. We first note that in this case $1+d+t \leqslant s(s+1)/2$. The following lemma is useful.

Lemma 1. *Suppose char$R = p^3$. If $u_i \in \mathcal{J}$ and $pu_i = 0$, then $u_i \in ann(\mathcal{J})$.*

Proof. This is obvious following comments after Remark 1. □

Following the above lemma, if $pu_i = 0$, then $u_iu_j = 0$, for every $j = 1,\ \ldots, s$; and if $pu_i \neq 0$, then

$$u_iu_j = a_{ij}^0p^2 + \sum_{l=1}^{d} a_{ij}^l pu_l + \sum_{k=1}^{t} a_{ij}^{d+k} v_k,$$

where $a_{ij}^0,\ a_{ij}^l,\ a_{ij}^{d+k} \in R_o/pR_o$.

Theorem 3. *Let R be a ring of characteristic p^3 with maximal ideal $\mathcal{J}$ such that $\mathcal{J}^3 = \{0\}$, $\mathcal{J}^2 \neq \{0\}$. Suppose further that $\mathcal{J}$ consists of elements $u_1,\ \ldots,\ u_s$ such that the multiplication in R is defined by $pu_i \neq 0$, for $i = 1,\ \ldots,\ d$, $pu_i = 0$, for $i = d+1,\ \ldots,\ s$, and $u_i^2 = u_iu_j = 0$, for every $i,\ j = 1,\ \ldots,\ s$. Then*

$$U(R) \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4^{r-1} \times \underbrace{\mathbb{Z}_4^r \times \ \cdots \ \times \mathbb{Z}_4^r}_{d} & \\ \times \underbrace{\mathbb{Z}_2^r \times \ \cdots \ \times \mathbb{Z}_2^r}_{s-d} \times \underbrace{\mathbb{Z}_p^r \times \ \cdots \ \times \mathbb{Z}_p^r}_{t}, & if\ p = 2 \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^2}^r \times \underbrace{\mathbb{Z}_{p^2}^r \times \ \cdots \ \times \mathbb{Z}_{p^2}^r}_{d} & \\ \times \underbrace{\mathbb{Z}_p^r \times \ \cdots \ \times \mathbb{Z}_p^r}_{s-d} \times \underbrace{\mathbb{Z}_p^r \times \ \cdots \ \times \mathbb{Z}_p^r}_{t}, & if\ p \neq 2 \end{cases}$$

Proof. For the rest of this paper, we shall take r elements $\varepsilon_1,\ \ldots,\ \varepsilon_r$ in R_o with $\varepsilon_1 = 1$ such that $\{\bar{\varepsilon}_1,\ \ldots,\ \bar{\varepsilon}_r\}$ is a basis for the quotient ring R_o/pR_o regarded as a vector space over its prime subfield $GF(p)$. Suppose without generality, that $pu_j \neq 0$ for at least one $j = 1,\ \ldots,\ s$. Let $a = 1 + x$ be an element of $1+\mathcal{J}$ with the highest possible order and assume that $x \in \mathcal{J}-\mathcal{J}^2$. Then $o(a)=p^2$. This is true because, for any ε_i $(i=1,\ \ldots,\ r)$,

$$(1+\varepsilon_i x)^p = 1 + p\varepsilon_i x + \frac{p(p-1)}{2}(\varepsilon_i x)^2 \quad (\text{since } x^3=0).$$

If p is odd, then $(1+\varepsilon_i x)^p = 1+p\varepsilon_i x$, since $px^2=0$.

Now,

$$\begin{aligned}(1+p\varepsilon_i x)^p &= 1+p^2\varepsilon_i x + \frac{p(p-1)}{2}(p\varepsilon_i x)^2 \\ &= 1, \quad \text{since } p^2x=0.\end{aligned}$$

Hence, $(1+\varepsilon_i x)^{p^2}=1$.

For any odd prime number p and for each $i=1,\ \ldots,\ r$, we see that $(1+\varepsilon_i p)^{p^2}=1$, $(1+\varepsilon_i u_j)^{p^2}=1$, for $j=1,\ \ldots,\ d$, while $(1+\varepsilon_i u_l)^p=1$, for $l=d+1,\ \ldots,\ s$, $(1+\varepsilon_i v_k)^p=1$.

For integers $l_i \leqslant p^2$, $n_{ij}\leqslant p^2$, $m_{il}\leqslant p$ and $q_{ik}\leqslant p$, we assert that

$$\prod_{i=1}^{r}\{(1+\varepsilon_i p)^{l_i}\}\cdot\prod_{i=1}^{r}\prod_{j=1}^{d}\{(1+\varepsilon_i u_j)^{n_{ij}}\}\cdot\prod_{i=1}^{r}\prod_{l=d+1}^{s}\{(1+\varepsilon_i u_l)^{m_{il}}\}\cdot$$
$$\prod_{i=1}^{r}\prod_{k=1}^{t}\{(1+\varepsilon_i v_k)^{q_{ik}}\}=1,$$

will imply $l_i=p^2$, $n_{ij}=p^2$, $m_{il}\leqslant p$ and $q_{ik}\leqslant p$, for all $i=1,\ \ldots,\ r$, $j=1,\ \ldots,\ d$, $l=d+1,\ \ldots,\ s$ and for all $k=1,\ \ldots,\ t$. (Notice that we may as well include the case when $d=0$ and $t=0$.

If we set

$$\begin{aligned}D_i &= \{(1+\varepsilon_i p)^{l_i} : l_i=1,\ \ldots,\ p^2\},\\ E_{ij} &= \{(1+\varepsilon_i u_j)^{n_{ij}} : n_{ij}=1,\ \ldots,\ p^2\},\\ F_{il} &= \{(1+\varepsilon_i u_l)^{m_{il}} : m_{il}=1,\ \ldots,\ p\},\\ G_{ik} &= \{(1+\varepsilon_i v_k)^{q_{ik}} : q_{ik}=1,\ \ldots,\ p\},\end{aligned}$$

for all $i=1,\ \ldots,\ r$, we see that D_i, E_{ij}, F_{il} and G_{ik} are all subgroups of $1+pR_o\oplus\sum R_o u_i\oplus\sum R_o v_k$ and that D_i and E_{ij} are both of order p^2 and the others are all of order p as indicated in their definition. Also, pairwise intersection of these subgroups is trivial.

The argument above will show that the product of the r subgroups D_i, dr subgroups E_{ij}, $(s-d)r$ subgroups F_{il}, and the tr subgroups G_{ik} is direct. Thus, their product will exhaust $1+pR_o\oplus\sum R_o u_i\oplus\sum R_o v_k$, and we see that the proof for the case when p is odd is complete.

Now assume that $p = 2$. Then $(1+2\varepsilon_1)^2 = 1$, $(1+4\varepsilon_1)^2 = 1$ and $(1+2\varepsilon_i)^4 = 1$ for $i > 1$. Next, for each $i = 1, \dots, r$, we see that if $2u_j = 0$, then $(1+\varepsilon_i u_j)^2 = 0$ and if $2u_j \neq 0$, then $(1+2\varepsilon_i u_j)^2 = 1 + 2\varepsilon_i u_j + \varepsilon_i^2 u_j^2$, and $(1+\varepsilon_i u_j)^4 = 1$. Also, $(1+\varepsilon_i v_k)^2 = 0$, for every $j+1, \dots, s$ and every $k = 1, \dots, t$. (This includes the case when $d = 0$ and $t = 0$)

For integers $l_1 \leqslant 2$, $k_1 \leqslant 2$, $l_i \leqslant 4$, $m_{ij} \leqslant 4$, $n_{il} \leqslant 2$ and $q_{ik} \leqslant 2$, we assert that

$$(1+2\varepsilon_1)^{l_1} \cdot (1+4\varepsilon_1)^{k_1} \cdot \prod_{i=2}^{r}\{(1+2\varepsilon_i)^{l_i}\} \cdot \prod_{i=1}^{r}\prod_{j=1}^{d}\{(1+\varepsilon_i u_j)^{n_{ij}}\}$$

$$\cdot \prod_{i=1}^{r}\prod_{l=d+1}^{s}\{(1+\varepsilon_i u_l)^{m_{il}}\} \cdot \prod_{i=1}^{r}\prod_{k=1}^{t}\{(1+\varepsilon_i v_k)^{q_{ik}}\} = 1,$$

will imply $l_1 = 2$, $k_1 = 2$, $l_i = 4$, for all $i = 2, \dots, r$; $m_{il} = 2$, for all $l = d+1, \dots, s$; $n_{ij} = 4$ for all $j = 1, \dots, d$; and $q_{ik} = 2$, for all $k = 1, \dots, t$ and for all $i = 1, \dots, r$.

If we set

$$\begin{aligned}
A_1 &= \{(1+2\varepsilon_1)^{l_1} : l_1 = 1,\ 2\}\\
B_1 &= \{(1+4\varepsilon_1)^{k_1} : k_1 = 1,\ 2\}\\
C_{l_i} &= \{(1+2\varepsilon_i)^{l_i} : l_i = 1, \dots, 4;\ i = 2, \dots, r\}\\
D_{ij} &= \{(1+\varepsilon_i u_j)^{n_{ij}} : n_{ij} = 1, \dots, 4;\ i = 1, \dots, r\}\\
E_{il} &= \{(1+\varepsilon_i u_l)^{m_{il}} : m_{il} = 1,\ 2;\ i = 1, \dots, r\}\\
G_{ik} &= \{(1+\varepsilon_i v_k)^{q_{ik}} : q_{il} = 1,\ 2;\ i = 1, \dots, r\},
\end{aligned}$$

we see that A_1, B_1, C_{l_i}, D_{ij}, E_{il} and G_{ik} are all subgroups of $1 + pR_o \oplus \sum R_o u_i \oplus \sum R_o v_k$ and that C_{l_i} and D_{ij} are of order 4 and the rest are of order 2, respectively, as indicated in their definition. Moreover, pairwise intersection of these subgroups is trivial.

The argument above will show that the product of A_1, B_1, the $r-1$ subgroups C_{l_i}, the dr subgroups D_{ij}, $(s-d)r$ subgroups E_{il} and tr subgroups G_{ik} is direct. Thus, their product will exhaust $1 + pR_o \oplus \sum R_o u_i \oplus \sum R_o v_k$, and this completes the proof. □

This completes our investigation of the structure of the group of units of commutative completely primary finite rings of characteristic p^3 with unique maximal ideals $\mathcal{J}$ such that $\mathcal{J}^3 = (0)$, $\mathcal{J}^2 \neq (0)$ without constraints on minimal generators for $\mathcal{J}$ or on invariants in the definition of R.

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Department of Biometry and Mathematics, Botswana University of Agriculture and Natural Resources, Private Bag 0027, Gaborone, BOTSWANA

e-mail address: jchikunj@buan.ac.bw

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