Math. J. Okayama Univ. **67** (2025), 101–131

THE CHARACTERIZATIONS OF AN ALTERNATING SIGN MATRICES USING A TRIPLET

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Abstract. An alternating sign matrix (ASM for short) is a square matrix which consists of 0, 1 and −1. In this paper, we characterize an ASM by showing a bijection between alternating sign matrix and six vertex model, and a bijection between six vertex model and height function. In order to show these bijections, we define a triplet $(a_{i,j}, c_{i,j}, r_{i,j})$ for each entry of an ASM. We also define a *track* for each index of height function, and state more properties of height function.

1. Introduction

In this paper, we mainly deal with the alternating sign matricies which were adovocated by Mills-Robbins-Rumsey [7]. An alternating sign matrix is a square matrix consisting of 0, 1 and −1 which satisfies several conditions, and is often abbreviated as ASM. Among the proofs of their enumeration, the proof by Zeilberger [13] and the proof by Kuperberg [4] are famaous. In the proof by Kuperberg [4], the six vertex models introduced from the statistical physics is used. The six vertex model is described as a map which gives direction to the edges of the lattice graph. On the other hands, there is a related object called Fully pucked loop model, it is abbreviated as FPL, which is described as a map such that it associates one of two colors with each edges of a lattice graph. In order to get some characterizations of ASMs which has degree n, a graph consisting of $n \times n$ vertices called interiror vertex and $4n$ vertices called boundary vertices, where each interior vertecies is adjacent to 4 edges is used. We denote this lattice graph as L_n. Wieland [12] states a characterization using a operation which called gyration defined for FPLs. As another approach, a $(n+1) \times (n+1)$ matrix which called height function is defined for a ASM of size n, and it define partial order for ASMs of size n. In this paper, we focus on the conditions of ASM for the sum of a row (resp. column), and define a triplet $(a_{i,j}, c_{i,j}, r_{i,j})$ for each entries of a ASM. Then we describe a correspondence ASMs, six vertex models and height functions by using the triplets $(a_{i,j}, c_{i,j}, r_{i,j})$. The fact that the state of each interior vertex (i, j) of L_n corresponds to the (i, j)-entry of ASM is well known, but the main purpose is to describe that correspondence more clearly by using the triplet $(a_{i,j}, c_{i,j}, r_{i,j})$. The proof of bijection between ASM and six vertex model, and between six vertex model

Mathematics Subject Classification. Primary 05A19; Secondary 05A05, 05E10.
Key words and phrases. Alternating sign matrix, six vertex model, height function.

and height functions are our own original proof. Moreover, we state the properties of height functions about range of possible values of their entries. From section 4 onwards, we introduce the characterizations of FPL by using gyration which stated in [3], [12].

2. Alternating sign matrix

In this section we define the alternating sign matrices and related objects called six vertex model and fully packed loop model. There are bijections between these objects, and we explain those bijections in this section.

2.1. Alternating sign matrix (ASM). As it is well-known, each element σ in $\mathfrak{S}_n$ corresponds to its permutation matrix $P_\sigma = (\delta_{i,\sigma(j)})_{1 \le i,j \le n}$. For example, the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \end{pmatrix} \in \mathfrak{S}_5$ corresponds to the following square matrix:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

A permutation matrix $A = (a_{i,j})_{1 \le i,j \le n}$ of size n is characterized by the properties (i) $a_{i,j} \in \{0, 1\}$ for $1 \le i, j \le n$ and $\sum_{i=1}^{n} a_{i,j} = \sum_{j=1}^{n} a_{i,j} = 1$ for each i, j. Now we define the alternating sign matrices whose entries consist of $0, 1$ or -1 and regarded as an extension of the permutation matrices.

Definition 1. A square matrix $A = (a_{i,j})_{1 \le i,j \le n}$ of size n is called *alternating sign matrix* (or *ASM* shortly) if it satisfies the following conditions:

$$a_{i,j} \in \{0, 1, -1\} \qquad (1 \le i, j \le n), \tag{2.1a}$$

$$\sum_{k=1}^{j} a_{i,k}, \quad \sum_{k=1}^{i} a_{k,j} \in \{0, 1\} \qquad (1 \le i, j \le n), \tag{2.1b}$$

$$\sum_{i=1}^{n} a_{i,j} = \sum_{j=1}^{n} a_{i,j} = 1 \qquad (1 \le i, j \le n). \tag{2.1c}$$

Let $\mathcal{A}_n$ denote the set of all ASM's of size n.

The following is an example of ASM of size 4:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

.

2.2. **Six vertex model and Fully packed loop model.** Now we define the six vertex model and fully packed loop model for the sake we provide a rigorous approach. First we give the definition of a planary (simple and finite) graph which we name $L_{m,n} = (V(L_{m,n}), E(L_{m,n}))$. The vertex set $V(L_{m,n}) = V_0(m,n) \sqcup V_1(m,n)$ and the edge set $E(L_{m,n}) = E_0(m,n) \sqcup E_1(m,n)$ are composed of the two kinds of sets, respectively. Each vertex set is defined as

$$V_0(m,n) := \left\{(i,j) \in \mathbb{Z}^2 \,\middle|\, 1 \le i \le m, 1 \le j \le n\right\},$$

which is called the set of *interior vertices* and

$$V_1(m,n) := \{(i,j) \,|\, i \in [m],\ j \in \{0, n+1\}\} \sqcup \{(i,j) \,|\, i \in \{0, m+1\},\ j \in [n]\},$$

which is called the set of *boundary vertices*. In graph theory a pair of vertices is called an edge. Here we call a pair of the form $\{(i,j),(i,j+1)\}$ with $(i,j),(i,j+1) \in V(L_{m,n})$ a *horizontal edge*, and a pair of the form $\{(i,j),(i+1,j)\}$ with $(i,j),(i+1,j) \in V(L_{m,n})$ a *vertical edge*. The edge set $E(L_{m,n})$ of the graph $L_{m,n}$ is, by definition, the set of all horizontal and vertical edges. We also define $E_0(m,n)$ as the set of all edges, whose endpoints are both interior vertices, and $E_1(m,n)$ as the set of edges such that one of the endpoints is a boundary vertex. More precisely

$$\begin{aligned} E_0(m,n) := &\{\{(i,j),(i,j+1)\} \,|\, 1 \le i \le m,\ 1 \le j \le n-1\} \\ &\sqcup \{\{(i,j),(i+1,j)\} \,|\, 1 \le i \le m-1,\ 1 \le j \le n\}, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} E_1(m,n) := &\{\{(i,j),(i,j+1)\} \,|\, 1 \le i \le m,\ j \in \{0,n\}\} \\ &\sqcup \{\{(i,j),(i+1,j)\} \,|\, i \in \{0,m\},\ 1 \le j \le n\}. \end{aligned} \tag{2.3}$$

We call an element of $E_1(m,n)$ a *boundary edge*. Especially, we denote $L_{n,n}$ as L_n. We can regard the graph L_n as the subset of xy-plane as in Figure 1. We label the boundary edges counterclockwise with $e_1 = \{(1,0),(1,1)\}, e_2, \ldots$ starting from e_1. For example, Figure 1 is the graph L_3, in which the double circle dots are the boundary vetices.

2.2.1. *Six vertex model.* A six vertex model is a type of statistical mechanics model in which the Boltzmann weights are associated with each vertex in the model. If this model has six possible states for each vertex, we call it a *six vertex model*. A state of our model is obtained by giving a direction to each edge of $L_{m,n}$. Recall that an edge of $L_{m,n}$ is denoted by an unorderd pair $\{u,v\}$ of vertices. We use ordered pair (u,v) to denoted the directed edge from u to v, and (v,u) to denote the opposite one. We call a map which associate (u,v) or (v,u) to $\{u,v\}$ for each edge $\{u,v\} \in E(L_{m,n})$ *orientation of* $L_{m,n}$. Here, if the orientation φ satisfies $\varphi(\{u,v\}) = (u,v)$,

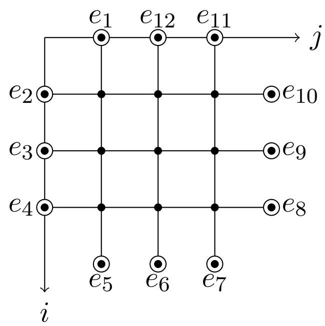


FIGURE 1. L_3

we say u is a *source* or the directed edge (u, v) goes *out* of u. On the other hand, we say v is a *sink* or (u, v) comes *in* v. A *state* is, by definition, a way asigning a direction to each edge. Since each interior vertex v has exactly 4 adjacent edges, there are 2^4 ways to orient these edges, We say a state *2-in-2-out* if there are 2 edges in and 2 edges out for every vertex. Then we call a orientation φ a *state* of *six vertex model* on $L_{m,n}$ if each interior vertex is 2-in-2-out. When φ is a state of six vertex model on $L_{m,n}$, we call $\varphi|_{E_1(m,n)}$ *boundary condition* of φ. We usually fix a boundary condition. Then the boundary condition on L_n like Figure 2 is called *open boundary condition*. We denote the set of all six vertex model on L_n which has open boundary condition as $\mathcal{SV}(n)$.

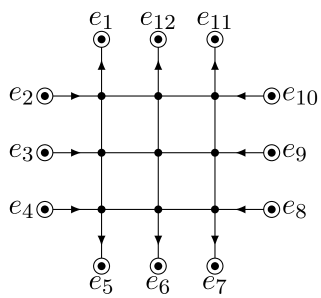


FIGURE 2. An example of boundary condition

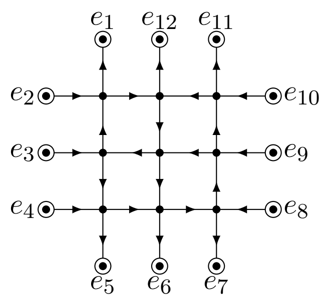


FIGURE 3. An example of a state of Six vertex model

2.2.2. *Fully packed loop model.* We define the following map $\psi \colon E(L_{m,n}) \to \{b, w\}$. Here, b comes from black and w comes from white. Let v be an interior vertex, then we say v is *2-2-colored* if 2 out of 4 edges which is incident to v are b, and the rest are w. We also call a ψ *fully packed loop model* on $L_{m,n}$ if each interior vertex is 2-2-colored, and it is abbreviated as FPL. Figure 4 is an example of FPL on L_3. In Figure 4, we draw edge e with solid line (resp. dashed line) when e has color b (resp. w) for each edge $e \in E(L_3)$.

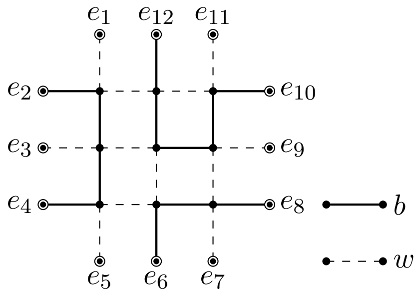


FIGURE 4. an example of FPL on L_3

Here, let ψ be a FPL on $L_{m,n}$ and e_i be the i-th boundary edge, then

$$\tau = (\psi(e_1), \psi(e_2), \ldots, \psi(e_{2m+2n}))$$

is called boundary condition of ψ. We also denote $\mathfrak{fpl}(n, \tau)$ the set of all FPL on L_n with a fixed boundary condition τ, when $\tau \in \{b, w\}^{4n}$ is given. Now, we denote the following boundary conditions on L_n as τ_+ and τ_- respectively:

$$\tau_+ := (b, w, b, w, \cdots, b, w)\,, \tag{2.4a}$$

$$\tau_- := (w, b, w, b, \cdots, w, b)\,. \tag{2.4b}$$

These conditions have b and w arranged alternately. We usually consider $\mathfrak{fpl}(n, \tau_-)$. It is well-known that there is the correopondence between the six vertex model on L_n which has the open boundary condition and the FPL model on L_n with τ_-. Here, we explain the well-known, stated in [12], correspondence from a state φ to a FPL ψ. Firstly, we shall define parity of vertex. When $(i,j) \in V(L_n)$, we call (i,j) odd if $i+j$ is odd, even otherwise. Notice that each edge of L_n is incident to an odd vertex and an even vertex. For each edge $\{u,v\} \in E(L_n)$ which goes out of u (i.e., $\varphi(\{u,v\}) = (u,v)$), we set $\psi(\{u,v\}) = b$ (resp. w) if u is odd vertex (resp. even). In Figure 7, we draw odd vertex (resp. even) as $\circ$ (resp. $\bullet$).

2.2.3. *Plaquette.* We define subgraph of L_n called plaquette. For $0 \le i,j \le n$, we define $\alpha_{i,j} = (V(\alpha_{i,j}), E(\alpha_{i,j}))$ as follows:

$$V(\alpha_{i,j}) := \{(i,j), (i,j+1), (i+1,j), (i+1,j+1)\} \cap V(L_n), \tag{2.5a}$$

$$E(\alpha_{i,j}) := \{e \in E(L_n) \mid e \text{ incdents } u, v \ (u, v \in V(\alpha_{i,j}))\}\,. \tag{2.5b}$$

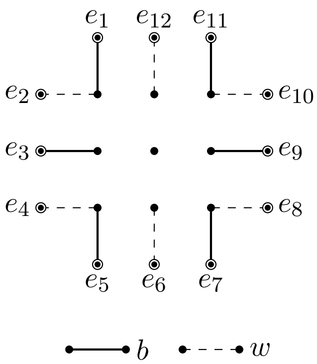


FIGURE 5. τ_+ when $n = 3$

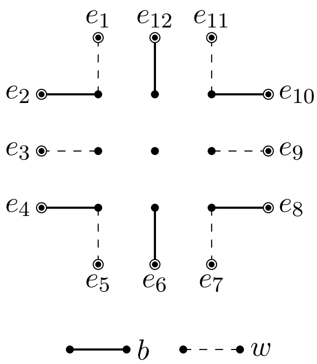


FIGURE 6. τ_- when $n = 3$

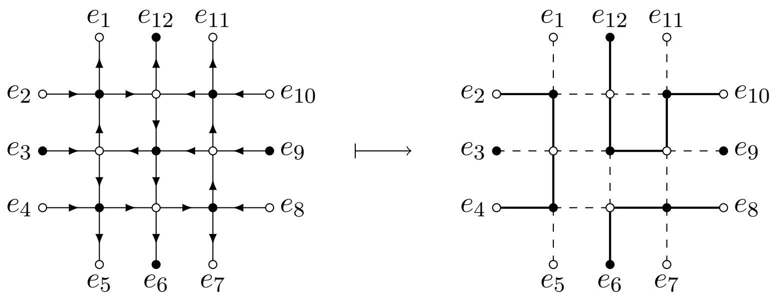


FIGURE 7. An example of assignment a six vertex model to a FPL

We call $\alpha_{i,j}$ interior plaquette if any vertex of $\alpha_{i,j}$ is interior vertex, boundary plaquette otherwise. Now, $\alpha_{i,j}$ is interior plaquette if and only if $1 \leq i, j \leq n-1$. We also define parity of plaquette $\alpha_{i,j}$. We call $\alpha_{i,j}$ odd if $i+j$ is odd, even otherwise.

2.3. **The correpondence between six vertex model and ASM.** It is well known that there is a bijection $\mathcal{SV}(n)$ to $\mathcal{A}_n$ (e.g., mentioned in [12]). We shall state the correpondence between six vertex model and ASM in this section. Notice that each interior vertex (i,j) has 6 possible choise when (i,j) is 2-in-2-out. Here, we respectively set the 4 edges which is incident to interior vertex (i,j) as $N = \{(i,j),(i-1,j)\}$, $E = \{(i,j),(i,j+1)\}$, $S = \{(i,j),(i+1,j)\}$, $W = \{(i,j),(i,j-1)\}$. Then, the 6 possible choices can be expressed as NE, NS, NW, ES, EW, SW by specifying two edges goes out of (i,j). In Figure 8, we illustrate the 6 possible choise of an interior vertex.

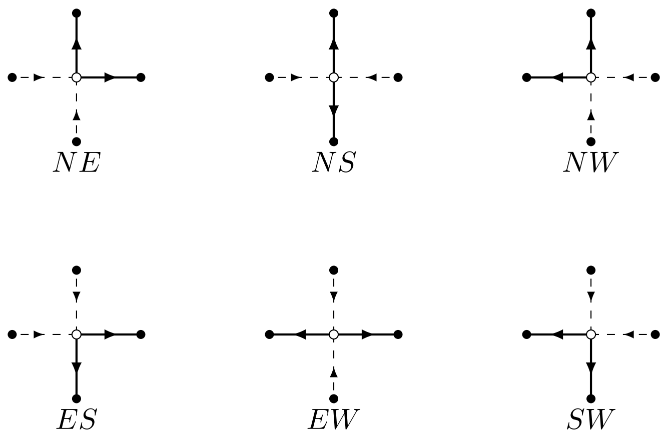


FIGURE 8. The 6 possible choise of an interior vertex

We shall asign a state of six vertex model on L_n to a square matrix of degree n by using the choise of the interiror vertex (i,j) to determine the (i,j)-entry for each i,j which satisfy $1 \le i,j \le n$.

Definition 2. Let n be an positive integer. We define a map $f\colon \mathcal{SV}(n) \to \mathcal{A}_n$. When we put $f(\varphi)$ as $(a_{i,j})_{1\le i,j\le n}$ for a $\varphi \in \mathcal{SV}(n)$, we define $a_{i,j}$ as follows: $a_{i,j} = 1$ if the choice of (i,j) is NS, $a_{i,j} = -1$ if the choice of (i,j) is EW, $a_{i,j} = 0$ otherwise for each $1 \le i,j \le n$.

Firstly, we shall show the matrix $(a_{i,j})_{1\le i,j\le n}$ which is obtained by definition 2 satisfies the conditions of ASM. Next, we introduce a triplet $(a_{i,j}, c_{i,j}, r_{i,j})$ for each $1 \le i,j \le n$, we explain the reverse map by using the triplet. It is our own original proof.

We fix $i \in [n]$ arbitraily. There exists $j \in [n]$ such that both of the horizontal edges $\{(i,j),(i,j-1)\}$ and $\{(i,j),(i,j+1)\}$ come in or go out of (i,j). In fact, if $\{(i,j),(i,j-1)\}$ come in (resp. go out of) (i,j) and $\{(i,j),(i,j+1)\}$ go out of (resp. come in) (i,j) for each $1 \le j \le n$, it contradicts the boundary condition for $\{(i,n),(i,n+1)\}$ (resp. $\{(i,0),(i,1)\}$). Moreover, both of the horizontal edge $\{(i,j),(i,j-1)\}$and $\{(i,j),(i,j+1)\}$ come in (i,j) for the smallest and the largest j such that both of the horizontal edge $\{(i,j),(i,j-1)\}$and $\{(i,j),(i,j+1)\}$ come in or go out of (i,j). Then we remark that the choise of (i,j) is NS (resp. EW) if both of the horizontal edge $\{(i,j),(i,j-1)\}$ and $\{(i,j),(i,j+1)\}$ come in (resp. go out of) (i,j), and $a_{i,j}$ is equal to 0 if either of the horizontal edge $\{(i,j),(i,j-1)\}$

and $\{(i,j),(i,j+1)\}$ comes in (i,j). Since a vertex such that both of the horizontal edges come in the vertex and a vertex such that both of the horizontal edges go out of the vertex appears except a vertices such that either of the horizontal edges come in the vertex in the i-th row, the matrix $(a_{i,j})_{1\le i,j\le n}$ satisfies $\sum_{1\le l\le n} a_{i,l} = 1$ and $\sum_{1\le l\le j} a_{i,l} \in \{0,1\}$ for each $1 \le j \le n$. Then the matrix $(a_{i,j})_{1\le i,j\le n}$ satisfies the condition on row of ASM. The same goes for columns.

Now, we denote $\sum_{1\le k\le i} a_{k,j}$ as $c_{i,j}$ and $\sum_{1\le l\le j} a_{i,l}$ as $r_{i,j}$ for each (i,j). Then, we shall show that the choice of (i,j) can be determined by the trilpet $(a_{i,j}, c_{i,j}, r_{i,j})$ for each interior veretex (i,j). First, we state that there are exactly 6 possible value for $(a_{i,j}, c_{i,j}, r_{i,j})$. When $a_{i,j} = 0$, (i,j) can be devided into following the four cases:

(i) There exists i_1 greater than i and j_1 greater than j such that $a_{i_1,j} = a_{i,j_1} = 1$, $a_{k,j} = 0$ for $i \le k < i_1$ and $a_{i,l} = 0$ for $j \le l < j_1$,
(ii) There exists i_1 greater than i and j_0 less than j such that $a_{i_1,j} = a_{i,j_0} = 1$, $a_{k,j} = 0$ for $i \le k < i_1$ and $a_{i,l} = 0$ for $j_0 < l \le j$,
(iii) There exists i_0 less than i and j_1 greater than j such that $a_{i_0,j} = a_{i,j_1} = 1$, $a_{k,j} = 0$ for $i_0 < k \le i$ and $a_{i,l} = 0$ for $j \le l < j_1$,
(iv) There exists i_0 less than i and j_0 less than j such that $a_{i_0,j} = a_{i,j_0} = 1$, $a_{k,j} = 0$ for $i_0 < k \le i$ and $a_{i,l} = 0$ for $j_0 < l \le j$.

In each four cases, the triplet$(a_{i,j}, c_{i,j}, r_{i,j})$ equals $(0,0,0)$ when the case (i), $(0,0,1)$ when the case (ii), $(0,1,0)$ when the case (iii) and $(0,1,1)$ when the case (iv). On the other hand, the triplet $(a_{i,j}, c_{i,j}, r_{i,j})$ must be $(1,1,1)$ (resp. $(-1,0,0)$) when $a_{i,j} = 1$ (resp. $a_{i,j} = -1$) because 1 apper first except 0, and 1 and -1 alternatly appear except 0 for each row and column in ASM. Then, the 6 possible value for the triplet $(a_{i,j}, c_{i,j}, r_{i,j})$ are $(0,0,0)$, $(0,0,1)$, $(0,1,0)$, $(0,1,1)$, $(1,1,1)$ and $(-1,0,0)$.

Now, the vertical edge $\{(i,j),(i-1,j)\}$ goes out of (resp. comes in) (i,j) and $\{(i,j),(i+1,j)\}$ comes in (resp. goes out of) (i,j) if there exists i_1 greater (resp. i_0 less) than i such that $a_{i_1,j} = 1$ (resp. $a_{i_0,j} = 1$) and $a_{k,j} = 0$ for $i \le k < i_1$ (resp. $i_0 < k \le i$) because the vertical edge $\{(i_1,j),(i_1-1,j)\}$ (resp. $\{(i_0,j),(i_0+1,j)\}$) goes out of (i_1,j) (resp. (i_0,j)) and either of the vertical edges $\{(k,j),(k-1,j)\}$ and $\{(k,j),(k+1,j)\}$ comes in (k,j) for $i \le k < i_1$ (resp. $i_0 < k \le i$). On the other hand, the horizontal edge $\{(i,j),(i,j-1)\}$ comes in (goes out of) of (i,j) and $\{(i,j),(i,j+1)\}$ goes out of (comes in) (i,j) if there exists j_1 greater (resp. j_0 less) than j such that $a_{i,j_1} = 1$ (resp. $a_{i,j_0} = 1$) and $a_{i,l} = 0$ for $j \le l < j_1$ (resp. $j_0 < l \le j$) because the horizontal edge $\{(i,j_1),(i,j_1-1)\}$ (resp. $\{(i,j_0),(i,j_0+1)\}$) comes in (i,j_1) (resp. (i_0,j)) and either of the vertical edges $\{(i,l),(i,l-1)\}$ and $\{(i,l),(i,l+1)\}$ comes in (i,l) for $j \le l < j_1$ (resp. $j_0 < l \le j$). Therefore the choice of (i,j) is NE when $(a_{i,j}, c_{i,j}, r_{i,j}) = (0,0,0)$, NW

when $(a_{i,j}, c_{i,j}, r_{i,j}) = (0,0,1)$, ES when $(a_{i,j}, c_{i,j}, r_{i,j}) = (0,1,0)$, SW when $(a_{i,j}, c_{i,j}, r_{i,j}) = (0,1,1)$, NS when $(a_{i,j}, c_{i,j}, r_{i,j}) = (1,1,1)$ and EW when $(a_{i,j}, c_{i,j}, r_{i,j}) = (-1,0,0)$. Since the triplet $(a_{i,j}, c_{i,j}, r_{i,j})$ are consitent for each $1 \leq i, j \leq n$ if the image are consistent, the map $f: \mathcal{SV}(n) \to \mathcal{A}_n$ is injective.

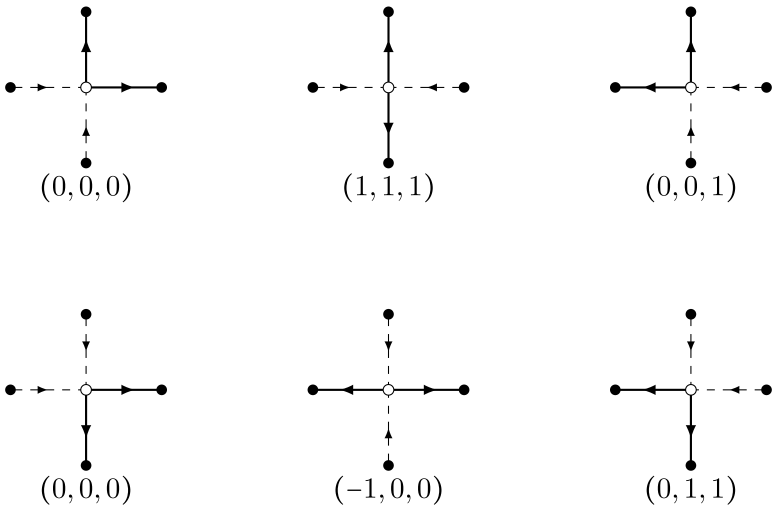


FIGURE 9. The triplet $(a_{i,j}, c_{i,j}, r_{i,j})$ which correspond the choise of (i,j)

Next, we shall show a state of six vertex model in $\mathcal{SV}(n)$ can be determined by using the triplet of each entries for any ASM. Let us take an ASM $(a_{i,j})_{1 \leq i,j \leq n}$ arbitrarily. In this manner, the direction of a horizontal (resp. vertical) edage $\{(i,j),(i,j+1)\}$ (resp. $\{(i,j),(i+1,j)\}$) can be determined in two ways from $(a_{i,j}, c_{i,j}, r_{i,j})$ and $(a_{i,j+1}, c_{i,j+1}, r_{i,j+1})$ (resp. $(a_{i,j}, c_{i,j}, r_{i,j})$ and $(a_{i+1,j}, c_{i+1,j}, r_{i+1,j})$). First, we shall show the two way determine the same direction. For the edges $W = \{(i,j),(i,j-1)\}$, $E = \{(i,j),(i,j+1)\}$, $N = \{(i,j),(i-1,j)\}$ and $S = \{(i,j),(i+1,j)\}$, the direction of them are determined by $r_{i,j} - a_{i,j}$, $r_{i,j}$, $c_{i,j} - a_{i,j}$ and $c_{i,j}$ respectively. We write the four value for 6 possible choice down in the table 1. The horizontal edge $\{(i,j),(i,j-1)\}$ comes in (i,j) if $r_{i,j} - a_{i,j} = 0$, and $\{(i,j),(i,j+1)\}$ comes in $(i,j+1)$ if $r_{i,j} = 0$. The vertical edge $\{(i,j),(i-1,j)\}$ goes out of (i,j) if $c_{i,j} - a_{i,j} = 0$, and $\{(i,j),(i+1,j)\}$ goes out of $(i+1,j)$ if $c_{i,j} = 0$. Since $r_{i,j}$ (resp. $c_{i,j}$) equals $r_{i,j+1} - a_{i,j+1}$ (resp. $c_{i+1,j} - a_{i+1,j}$), the two ways determine the same. Then, the orientation of L_n is obtaioned by $(a_{i,j})_{1 \leq i,j \leq n}$, and it satisfies 2-in-2-out. Further more, the orientation satisfies the boundary conditions for $e_2, e_3, \ldots, e_{n+1}$ (resp. $e_{3n+2}, e_{3n+3}, \ldots, e_{4n}$ and e_1) because

	NE	NS	NW	ES	EW	SW
$r_{i,j} - a_{i,j}$	0	0	1	0	1	1
$r_{i,j}$	0	1	1	0	0	1
$c_{i,j} - a_{i,j}$	0	0	0	1	1	1
$c_{i,j}$	0	1	0	1	0	1

TABLE 1. each value for the 6 possible choices

$r_{k,1}$ (resp. $c_{1,k}$) equals $a_{k,1}$ (resp. $a_{1,k}$) for $1 \le k \le n$. On the other hand, the orientation satisfies the boundary conditions for $e_{2n+2}, e_{2n+3}, \ldots, e_{3n+1}$ (resp. $e_{n+2}, e_{n+3}, \ldots, e_{2n+1}$) because $r_{k,n}$ (resp. $c_{n,k}$) equals 1 for $1 \le k \le n$. Therefore, a state of six vertex model in $\mathcal{SV}(n)$ cane be determined by the triplet of each entries for any ASM.

3. HEIGHT FUNCTION

Let m and n be positive integers. In this section, we make a state of six vertex model on $L_{m,n}$ correspond to a matrix of size $(m+1) \times (n+1)$. Especially, we introduce a square matrix of size $n+1$ which is called height function of degree n, and we shall construct a bijection between $\mathcal{SV}(n)$ and the set of all height function of degree n.

3.1. **Properties of a boundary condition of six vertex model.** As a preparation to make a state of six vertex model correspond to a matrix, we present the following lemma.

Lemma 3.1. *Let m and n be positive integers. For any state of six vertex model on $L_{m,n}$, the number of bounday edges which comes in the boundary vertex equals $m+n$.*

Let φ be a state of six vertex model on $L_{m,n}$. First, we show the claim when $m = 1$.

(I) When $n = 1$, the claim is clearly correct from the definition.
(II) When $n > 2$, we assume the claim holds up to $n-1$. If the edge $\{(1, n-1), (1, n)\}$ goes out of (resp. comes in) $(1, n-1)$, two (resp. one) of the three boundary edges $\{(1, n), (0, n)\}$, $\{(1, n), (1, n+1)\}$ and $\{(1, n), (2, n)\}$ come in the bounday vertex. Moreover, $n-1$ (resp. n) of the other $2n-1$ boundary edges come in the boundary vertex from the hypothesis of induction. Therefore, $n+1$ of $2n+2$ boundary edges comes in the boundary vertex.

Then we showed the claim when $m = 1$. Next, we assume the claim hold up to $m-1$ when $m > 2$. Suppose that exactly k out of n vertical edges $\{(m-1, j), (m, j)\}$ $(1 \le j \le n)$ come in the below vertex in φ. Applying the hypothesis of induction to the state which is obtained by restricting φ to the $(m-1)\times n$ grid which has $(1,1)$ as the leftmost interior veretx in the top row, it follows that exactly $m+n-k-1$ out of the other $2m+n-2$ boundary edges come in the boundary vertex. On the other hand, the $1\times n$ grid which has $(m,1)$ as the mostleft interior vertex implies that exactly $k+1$ out of $n+2$ boundary edges $\{(m,1),(m,0)\}$, $\{(m,n),(m,n+1)\}$ and $\{(m,j),(m+1,j)\}$ $(1\le j\le n)$ come in the boundary vertex. Therefore exactly $m+n$ boundary edges come in the boundary vertex in φ. Then we showed lemma 3.1. □

Now we have following claim as a corollary of lemma 3.1.

Corollary 3.2. *Let n be a positive integer. For any state of sixvertex model in $\mathcal{SV}(n)$ and any $1\le i\le n$, exactly i of n vertical edges $\{(i,j),(i+1,j)\}$ $(1\le j\le n)$ comes in the below vertex, and $n-i$ of n horizontal edges $\{(j,i),(j,i+1)\}$ $(1\le j\le n)$ come in the right vertex.*

3.2. **The map from six vertex model to matrices.** Let us take a state φ of six vertex model on $L_{m,n}$ arbitrary. We shall make φ of six vertex model on $L_{m,n}$ to correspond a $(m+1)\times(n+1)$ matrix $(h_{i,j})_{0\le i\le m,\,0\le j\le n}$ which satisfies following conditions:

$$|h_{i,j}-h_{i,j-1}|=1 \qquad (0\le i\le m,\ 0<j\le n)\,, \tag{3.1a}$$

$$|h_{i,j}-h_{i-1,j}|=1 \qquad (0<i\le m,\ 0\le j\le n)\,, \tag{3.1b}$$

$$h_{0,0}=0. \tag{3.1c}$$

Definition 3. Let m and n be positive integers. We define the matrix $(h_{i,j})_{0\le i\le m,\,0\le j\le n}$ by setting $h_{i,j}-h_{i,j-1}$ $(0\le i\le m,\ 0<j\le n)$ and $h_{i,j}-h_{i-1,j}$ $(0<i\le m,\ 0\le j\le n)$ as follows:

$$h_{i,j}-h_{i,j-1}=\begin{cases}1 & \text{if } \{(i,j),(i+1,j)\} \text{ comes in } (i,j),\\ -1 & \text{otherwise,}\end{cases} \tag{3.2a}$$

$$h_{i,j}-h_{i-1,j}=\begin{cases}1 & \text{if } \{(i,j),(i,j+1)\} \text{ goes out of } (i,j),\\ -1 & \text{otherwise.}\end{cases} \tag{3.2b}$$

Then we shall show the well-definedness in this way. Now, we set k_0, k_1, l_0 and l_1 as follows respectively:

$$k_0=\#\{i\in[m]\,|\,\{(i,0),(i,1)\} \text{ comes in } (i,0)\}\,, \tag{3.3a}$$

$$k_1=\#\{i\in[m]\,|\,\{(i,n),(i,n+1)\} \text{ comes in } (i,n+1)\}\,, \tag{3.3b}$$

$$l_0=\#\{j\in[n]\,|\,\{(0,j),(1,j)\} \text{ comes in } (0,j)\}\,, \tag{3.3c}$$

$$l_1=\#\{j\in[n]\,|\,\{(m,j),(m+1,j)\} \text{ comes in } (m+1,j)\}\,. \tag{3.3d}$$

First, we show that the values of $h_{m,n}$ are consistent when it is determined

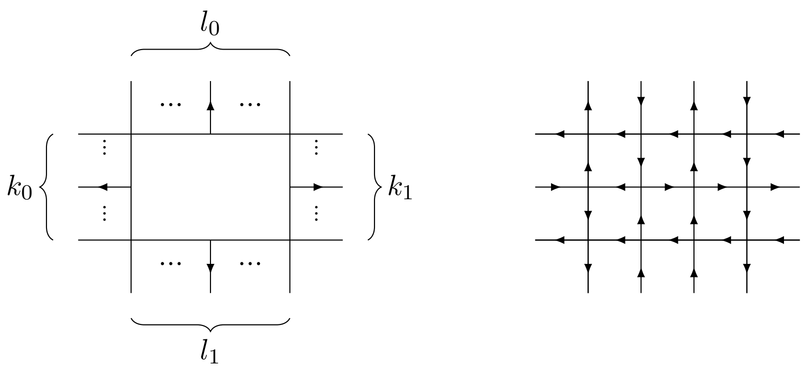


FIGURE 10. right;$k_0 = 2$, $k_1 = 1$, $l_0 = 2$, $l_1 = 2$.

clockwise by l_0 and k_1 and when it is determined counterclockwise by k_0 and l_1. If we determine $h_{m,n}$ clockwise, then we have

$$h_{m,n} = 2(k_1 + l_0) - (m + n). \tag{3.4}$$

On the other hand,

$$h_{m,n} = -2(k_0 + l_1) + (m + n) \tag{3.5}$$

if we determine counterclockwise. Since $k_0 + k_1 + l_0 + l_1 = m + n$, from the lemma 3.1, the difference between the RHS of (3.4) and the RHS of (3.5) equals 0. Then we showed the values of $h_{m,n}$ which is determined clockwise and which is determined counterclockwise are consistent. Moreover, substitute $m + n = k_0 + k_1 + l_0 + l_1$ for (3.4) yields

$$h_{m,n} = l_0 + k_1 - l_1 - k_0. \tag{3.6}$$

Next, let us take $1 \le i_0 \le m$ and $1 \le j_0 \le n$ arbitrarily. We shall show that the value of h_{i_0,j_0}, which is determined from the $i_0 \times j_0$ grid with $(1,1)$ as the leftmost interior vertex of the top row, and the value of $h_{m,n} - h_{i_0,j_0}$, which is determined from the $(m - i_0) \times (n - j_0)$ grid with (m,n) as the rightmost interior vertex of the bottom row, does not contradict each other. Now, we set k_i, l_i $(2 \le i \le 5)$ as follows respectively:

$$k_2 = \#\{i \in [i_0] \mid \{(i,0),(i,1)\} \text{ comes in } (i,0)\}, \tag{3.7a}$$

$$k_3 = \#\{i \in [i_0] \mid \{(i,j_0),(i,j_0+1)\} \text{ comes in } (i,j_0+1)\}, \tag{3.7b}$$

$$k_4 = \#\{i_0 < i \le m \mid \{(i,j_0),(i,j_0+1)\} \text{ comes in } (i,j_0)\}, \tag{3.7c}$$

$$k_5 = \#\{i_0 < i \le m \mid \{(i,n),(i,n+1)\} \text{ comes in } (i,n+1)\}, \tag{3.7d}$$

$$l_2 = \#\{j \in [j_0] \mid \{(0,j),(1,j)\} \text{ comes in } (0,j)\}, \tag{3.7e}$$

$$l_3 = \#\{j \in [j_0] \mid \{(i_0,j),(i_0+1,j)\} \text{ comes in } (i_0+1,j)\}, \tag{3.7f}$$

(3.7g) $\quad l_4 = \#\{j_0 < j \le n \,|\, \{(i_0, j), (i_0+1, j)\} \text{ comes in } (i_0, j)\},$

(3.7h) $\quad l_5 = \#\{j_0 < j \le n \,|\, \{(m, j), (m+1, j)\} \text{ comes in } (m+1, j)\}.$

Focusing on the $i_0 \times j_0$ grid, we have

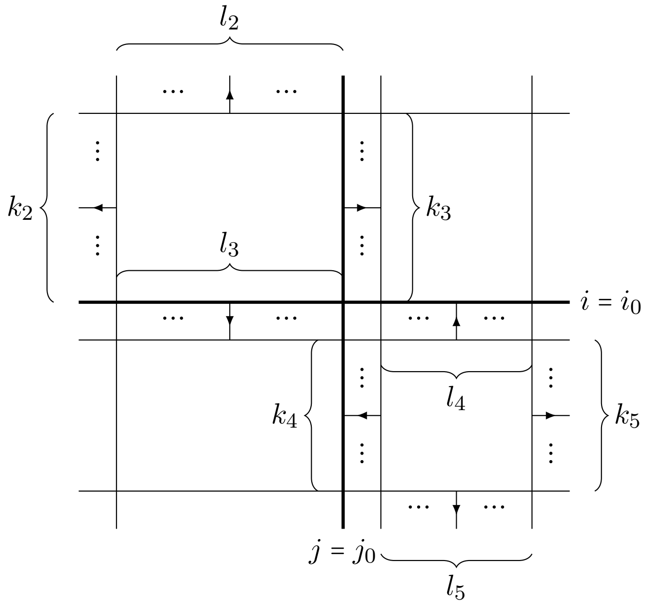


FIGURE 11

$$h_{i_0,j_0} = l_2 + k_3 - l_3 - k_2. \tag{3.8}$$

On the other hand, it follows from the $(m - i_0) \times (n - j_0)$ grid that

$$h_{m,n} - h_{i_0,j_0} = l_4 + k_5 - l_5 - k_4. \tag{3.9}$$

Then we just show the sum of the RHS of (3.8) and the RHS of (3.9) equals the RHS of (3.6). Now, from the lemma 3.1, the boundary condition of the $i_0 \times (n - j_0)$ grid with $(1, n)$ as the rightmost interior vertex of the top row implies that

$$l_0 - l_2 + k_1 - k_5 - l_4 - k_3 = 0. \tag{3.10}$$

Moreover, the boundary condition of the $(m - i_0) \times j_0$ grid with $(m, 1)$ as the leftmost interior vertex of the bottom row show that

$$-l_3 - k_4 + l_1 - l_5 + k_0 - k_2 = 0. \tag{3.11}$$

We remark that the sum of the RHS of (3.8) and the RHS of (3.9) equals (RHS of (3.6)) − (LHS of (3.10)) + (LHS of (3.11)). It follows that the sum of the RHS of (3.8) and the RHS of (3.9) equals the RHS of (3.6). Therefore, it was shown that the value of h_{i_0,j_0}, which is determined from the $i_0 \times j_0$ grid, and the value of $h_{m,n} - h_{i_0,j_0}$, which is determined from the $(m - i_0) \times (n - j_0)$ grid, does not contradict each other for any $1 \le i_0 \le m$, $1 \le j_0 \le n$.

3.3. **height function.** We define height function of degree n.

Definition 4. The $(n+1) \times (n+1)$ matrix $H = (h_{i,j})_{0 \le i,j \le n}$ is called *height function* of degree n if it satisfying the following conditions:

$$|h_{i+1,j} - h_{i,j}| = 1 \qquad (0 \le i < n,\ 0 \le j \le n), \tag{3.12a}$$

$$|h_{i,j+1} - h_{i,j}| = 1 \qquad (0 \le i \le n,\ 0 \le j < n), \tag{3.12b}$$

$$h_{i,0} = h_{0,i} = h_{n-i,n} = h_{n,n-i} = i \qquad (0 \le i \le n). \tag{3.12c}$$

We call (3.12a) and (3.12b) adjacent conditons, and (3.12c) boundary conditions of height function. We denote the set of all height functions of degree n as $\mathcal{H}_n$. A partial order on $\mathcal{H}_n$ is defined as $(h_{i,j})_{0 \le i,j \le n} \le (g_{i,j})_{0 \le i,j \le n}$ if $h_{i,j} \le g_{i,j}$ for $1 \le i, j < n$. Now, we have the following proposition.

Proposition 3.3. *Let n be a positive integer. We define a map $f\colon \mathcal{SV}(n) \to \mathcal{H}_n; \varphi \mapsto (h_{i,j})_{0 \le i,j \le n}$ by setting $h_{i,j} - h_{i,j-1}$ $(0 \le i \le n,\ 0 < j \le n)$ and $h_{i,j} - h_{i-1,j}$ $(0 < i \le n,\ 0 \le j \le n)$ as follows:*

$$h_{i,j} - h_{i,j-1} = \begin{cases} 1 & \text{if } \{(i,j),(i+1,j)\} \text{ comes in } (i,j), \\ -1 & \text{otherwise}, \end{cases} \tag{3.13a}$$

$$h_{i,j} - h_{i-1,j} = \begin{cases} 1 & \text{if } \{(i,j),(i,j+1)\} \text{ goes out of } (i,j), \\ -1 & \text{otherwise}. \end{cases} \tag{3.13b}$$

Then the map $f\colon \mathcal{SV}(n) \to \mathcal{H}_n$ is bijective.

Proof. From the definition, it is clear that f is injective. We just show surjectivity. First, we remark that any height function $(h_{i,j})_{0 \le i,j \le n}$ gives a orientation of L_n by setting the direction of $\{(i,j),(i+1,j)\}$ from the value $h_{i,j} - h_{i,j-1}$ for $0 \le i \le n$, $1 \le j \le n$, and the direction of $\{(i,j),(i,j+1)\}$ from the value $h_{i,j} - h_{i-1,j}$ for $1 \le i \le n$, $0 \le j \le n$. Moreover, it is clear that the boundary condition of the orientation is the open boundary condition. Second, the triplet $(h_{i-1,j}, h_{i,j}, h_{i,j-1})$ can be one of the following six values:

$$(h_{i-1,j-1} + 1, h_{i-1,j-1} + 2, h_{i-1,j-1} + 1), \tag{3.14a}$$

$$(h_{i-1,j-1} + 1, h_{i-1,j-1}, h_{i-1,j-1} + 1), \tag{3.14b}$$

$$(h_{i-1,j-1} + 1, h_{i-1,j-1}, h_{i-1,j-1} - 1), \tag{3.14c}$$

$$(h_{i-1,j-1} - 1, h_{i-1,j-1}, h_{i-1,j-1} + 1), \tag{3.14d}$$

$$(h_{i-1,j-1} - 1, h_{i-1,j-1}, h_{i-1,j-1} - 1), \tag{3.14e}$$

$$(3.14f) \qquad (h_{i-1,j-1} - 1, h_{i-1,j-1} - 2, h_{i-1,j-1} - 1)\,.$$

Then exactly two of the four edges which are incident to (i,j) goes out of (i,j) in the orientation which is determined by a height function. In fact, when (3.14a), a state of (i,j) is NE, when (3.14b), a state of (i,j) is NS, when (3.14c), a state of (i,j) is NW, when (3.14d), a state of (i,j) is ES, when (3.14e), a state of (i,j) is EW, and when (3.14f), a state of (i,j) is SW for any $1 \le i,j \le n$. Therefore, we obtain a state of six vertex model on L_n which has the open boundary condition from each height function of degree n. □

3.4. **The bijection between ASM and height function.** Up to here, we showed a bijection between $\mathcal{A}_n$ and $\mathcal{SV}(n)$, and a bijection between $\mathcal{SV}(n)$ and $\mathcal{H}_n$. Then we hold the following proposition from the two bijections.

Proposition 3.4. *Let n be a positive integer. For any ASM $(a_{i,j})_{1\le i,j\le n} \in \mathcal{A}_n$, we set a matrix $(h_{i,j})_{0\le i,j\le n}$ as following:*

$$(3.15a) \qquad h_{i,j} := i + j - 2 \sum_{1\le k\le i} \sum_{1\le l\le j} a_{i,j} \qquad (0 \le i,j \le n)\,.$$

Then a bijection between $\mathcal{A}_n$ and $\mathcal{H}_n$ is given in this way. Here, the inverse map is given as following:

$$(3.15b) \qquad a_{i,j} := -\frac{1}{2}\left(h_{i-1,j-1} - h_{i-1,j} + h_{i,j} - h_{i,j-1}\right) \qquad (1 \le i,j \le n)\,.$$

Proof. First, we recall that a vertical edge $\{(i,j),(i+1,j)\}$ goes out of (i,j) if and only if $c_{i,j} = \sum_{1\le k\le i} a_{i,j} = 1$ for $1 \le i,j \le n$. Then we have

$$(3.16) \qquad h_{i,j} = i + j - 2\#\left\{k \in [j] \,\middle|\, c_{i,j} = 1\right\} \qquad (1 \le i \le n, 0 \le j \le n)\,.$$

Since $\#\left\{k \in [j] \,\middle|\, c_{i,j} = 1\right\} = \sum_{1\le l\le j} c_{i,l}$, the equation (3.15a) follows. □

3.5. **Properties of height functions.** In this section, we shall state more properties of height functions. Especially, we focus on the possible values of each entry of a height function. First, we denote $\mathcal{I}(n)$ as the set $\left\{(i,j) \in \mathbb{Z}^2 \,\middle|\, 1 \le i,j < n\right\}$. Now, we define a positive integer-valued function $\mathrm{trc}\colon \mathcal{I}(n) \to \mathbb{Z}$.

Definition 5. Let n be a positive integer. We define $\mathrm{trc}\colon \mathcal{I}(n) \to \mathbb{Z}$ as follows:

$$(3.17) \qquad \mathrm{trc}(i,j) := \min\{i, n-i, j, n-j\} \qquad (1 \le i,j < n)\,.$$

Then we call $\mathrm{trc}(i,j)$ the *track* of (i,j) for each $(i,j) \in \mathcal{I}(n)$.

We remark that $\mathcal{I}(n)$ is decomposed into $\lfloor n/2 \rfloor$ disjoint union by the track (i.e., $\mathcal{I}(n) := \bigsqcup_{1\le l\le \lfloor \frac{n}{2} \rfloor} \left\{(i,j) \in \mathcal{I}(n) \,\middle|\, \mathrm{trc}(i,j) = l\right\}$).

Now, we hold following propisition.

Proposition 3.5. *Let n be a positive integer, and $(h_{i,j})_{0\le i,j\le n}$ a height function of degree n. For $(i,j)\in\mathcal{I}(n)$, there are $(l+1)$s possible values for the (i,j)-entry $h_{i,j}$ if track of (i,j) equals l. The possible values are as follows:*

(i) when $i=l$ and $l\le j\le n-l$ (resp. $l\le i\le n-l$ and $j=l$), $h_{i,j}=(j-l)+2k$ (resp. $(i-l)+2k$) $(0\le k\le l)$,

(ii) when $l\le i\le n-l$ and $j=n-l$ (resp. $i=n-l$ and $l\le j\le n-l$), $h_{i,j}=(n-l-i)+2k$ (resp. $(n-l-j)+2k$) $(0\le k\le l)$.

Proof. Now, we denote $\sum_{1\le i\le i,1\le l\le j}a_{i,j}$ as $s_{i,j}$ $(0\le i,j\le n)$ for $(a_{i,j})_{1\le i,j\le n}\in\mathcal{A}_n$. Here, we remark that $s_{i,0}=s_{0,i}=0$ for $0\le i\le n$. Since the total sum of a row of ASM equals 1, for $1\le i\le n$, we have the following equations:

$$0\le s_{i,1}\le s_{i,2}\le\cdots\le s_{i,n}=i, \tag{3.18a}$$

$$s_{i,j}-s_{i,j-1}=0 \text{ or } 1 \qquad (1\le j\le n)\,. \tag{3.18b}$$

Then it follows that

$$\max\{0,i+j-n\}\le s_{i,j}\le\min\{i,j\} \qquad (1\le i,j\le n)\,. \tag{3.19}$$

In the same way for a column, exactly the same equation (3.19) holds. Moreover, the equation (3.15a) show that

(i) when $i+j\le n$ and $i\le j$, $j-i\le h_{i,j}\le i+j$,
(ii) when $i+j\ge n$ and $i\le j$, $j-i\le h_{i,j}\le 2n-(i+j)$,
(iii) when $i+j\ge n$ and $i\ge j$, $i-j\le h_{i,j}\le 2n-(i+j)$,
(iv) when $i+j\le n$ and $i\ge j$, $i-j\le h_{i,j}\le i+j$.

We remark that the track of (i,j) equals i when (i), $(n-j)$ when (ii), $(n-i)$ when (iii), j when (iv). It follows that the difference between the RHS and the LHS is equal to twice the track of (i,j) for each case. Since $h_{i,j}$ decrease 2 every time $s_{i,j}$ increase 1, proposition 3.5 follows. □

Now, we define $\mathbb{P}_n$ as the set $\left\{(i,j,k)\in\mathbb{Z}^3\,\middle|\,1\le i,j<n \text{ and } 1\le k\le i,j,n-i,n-j\right\}$. Let us set $\iota\colon\mathcal{H}_n\to 2^{\mathbb{P}_n}$ as a map such that $(i,j,k)\in\iota\left((h_{i,j})_{0\le i,j\le n}\right)$ if and only if $h_{i,j}\ge|i-j|+2k$. Then, we have $\iota\left((h_{i,j})_{0\le i,j\le n}\right)\subset\iota\left((g_{i,j})_{0\le i,j\le n}\right)$ if $(h_{i,j})_{0\le i,j\le n}\le(g_{i,j})_{0\le i,j\le n}$. Here, we define a partially order on $\mathbb{P}_n$ by a cover relation. The element (i,j,k) covers (i',j',k') if it satisfies the following conditions:

$$|i-j|+2k=\left|i'-j'\right|+2k'+1, \tag{3.20a}$$

$$\left|i-i'\right|+\left|j-j'\right|=1. \tag{3.20b}$$

As a observation, $\iota\colon\mathcal{H}_n\to 2^{\mathbb{P}_n}$ gives a order ideal of $\mathbb{P}_n$. Moreover, the poset $\mathbb{P}_n$ is graded, the rank of $(i,j,k)\in\mathbb{P}_n$ is expressed as $|i-j|+2k-2$, and the generating funcition is $\sum_{0\le r\le n-2}(n-r-1)(r+1)q^r$.

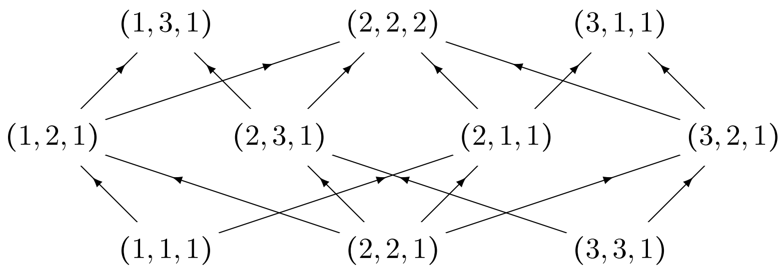


FIGURE 12. The Hasse diagram of $\mathbb{P}_3$.

4. Perfect matching and link pattern

Let n be a positive integer, and L_n the graph which is introduced in section 2.2. When a FPL on L_n is given, we can obtain a pairing of boundary vertices if we focus only on edges with either black or white color. In this sectetion, we define monochromatic path and perfect matching as a preparation for describing the behavior of FPLs using that pairing. Let $\boldsymbol{p} = (v_0, v_1, \ldots, v_m)$ be a path in L_n where $v_0, v_1, \ldots, v_m$ are all distinct, and ψ a FPL. . We call $\boldsymbol{p}$ a *monochromatic path* of ψ if all edges have the same color (i.e., $\psi(\{v_{i-1}, v_i\}) = \psi(\{v_0, v_1\})$ for $1 \leq i \leq m$). Further, we call $\boldsymbol{p}$ a *black path* (resp. *white path*) or we say $\boldsymbol{p}$ has color b (resp. w) if all edges have color b (resp. w). In graph theory, a path $\boldsymbol{p}$ is called *cycle* if the two end vertices v_0 and v_m coincide. When a monochromatic path $\boldsymbol{p}$ is a cycle, we call $\boldsymbol{p}$ a *monochromatic cycle*. Next, we define matching on $[n]$. Here, we remark that $[n]$ is the set $\{1, 2, \ldots, n\}$.

Definition 6. Let n, p be positive integers which satisfy $0 \leq 2p \leq n$, and $\mu = ([n], E(\mu))$ be a graph. We say μ is a *p-matching on* $[n]$ if it satisfies the following conditions:

(i) $u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_p$ are elements of $[n]$ which are all distinct,
(ii) $E(\mu) = .\{\{u_i, v_i\} \mid 1 \leq i \leq p\}$.

We say a vertex $j \in V(\mu)$ is *single* if $j \notin \{u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_p\}$. Especially, μ is called *perfect* if $n = 2p$.

Note that for each boundary vertex $v \in V_1(n)$, there uniquely exists a distinct boundary vertex $w \in V_1(n)$ such that there is a monochromatic path $\boldsymbol{p}$ which has v and w as its end vertices. Then, ψ determines 2 kind of perfect matchings on $[2n]$ by focusing only on the black pathes or focusing only on the white pathes. In addition, any two monochromatic paths of ψ which have the same color and start from and end at boundary vertices does not have a common vertex. We are ready to explain perfect matchings of boundary vertices determined by a FPL, we define non-crossing matching.

Definition 7. Let n, p be positive integers, and μ be p-matching on $[n]$. We say μ is *non-crossing* if neitheir of the followings happens:

(4.1a)
μ contains a pair $\{u_1, v_1\}, \{u_2, v_2\}$ of edges such that $u_1 < u_2 < v_1 < v_2$,

(4.1b)
μ contains an edge $\{u_1, v_1\}$ and a single vertex j such that $u_1 < j < v_1$.

(A) The case (4.1a) (B) The case (4.1b)

FIGURE 13. The cases which never happen in a non-crossing matching

4.1. **Link patterns.** Now, we shall introduce a link pattern. Let us write non-crossing perfect matching on $[2n]$ with $2n$ vertices on the circumference and n edges drawn by arcs inside the circle, and call this diagram a *link pattern of size* n. we denote $\mathcal{F}(2n)$ as the set of all link patterns of size n.

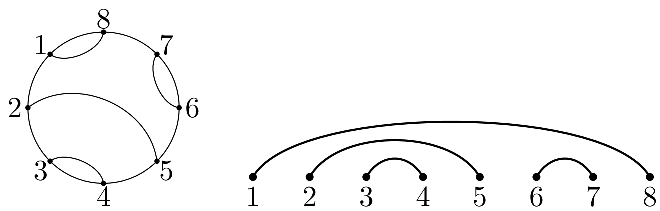


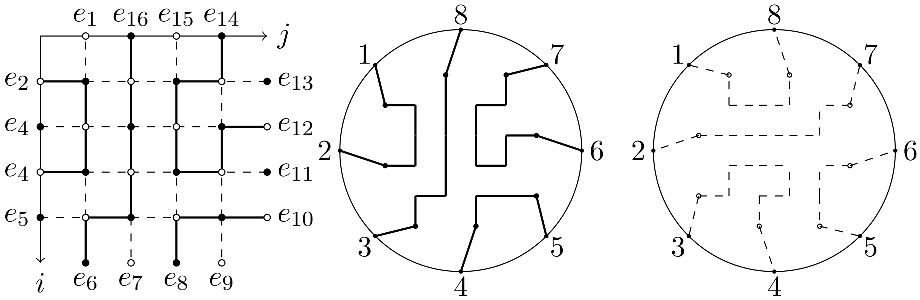
FIGURE 14. An example of link pattern of size 4

Then, we have $\#\mathcal{F}(2n) = \frac{1}{n+1}\binom{2n}{n}$, we denote $C_n = \frac{1}{n+1}\binom{2n}{n}$ which is called the nth *Catalan number*.

Here, we explain the 4 maps from the set of FPLs to the set of link patterns. We define the following 4 maps: $\pi_{b,\tau_-}, \pi_{w,\tau_-} : \mathfrak{fpl}(n, \tau_-) \to \mathcal{F}(2n)$ and $\pi_{b,\tau_+}, \pi_{w,\tau_+} : \mathfrak{fpl}(n, \tau_+) \to \mathcal{F}(2n)$.

- The map π_{b,τ_-} (resp. π_{w,τ_-}): $\mathfrak{fpl}(n, \tau_-) \to \mathcal{F}(2n)$ associates a link pattern with a given FPL as follows: if there is a black (resp. white) path which connects e_{2i} with e_{2j} (resp. e_{2i-1} with e_{2j-1}), then we draw an edge between i and j.

The map π_{b,τ_+} (resp. π_{w,τ_+}) : $\mathfrak{fpl}(n,\tau_+) \to \mathcal{F}(2n)$ is obtained in the same way, by replacing e_{2i} (resp. e_{2i-1}) with e_{2i-1} (resp. e_{2i}) and e_{2j} (resp. e_{2j-1}) with e_{2j-1} (resp. e_{2j}). In Figure 15, the black (resp. white) path of the FPL ψ in (a) gives the link pattern (b) (resp. (c)). Let $\mu \in \mathcal{F}(2n)$. Let



(A) $\psi \in \mathfrak{fpl}(4,\tau_-)$ (B) $\pi_{b,\tau_-}(\psi)$ (C) $\pi_{w,\tau_-}(\psi)$

FIGURE 15. An example of π_{b,τ_-} and π_{w,τ_-}

$\Psi_{n,-}(\mu)$ (resp. $\Psi_{n,+}(\mu)$) denote the cardinality of the set of FPLs with the boundary condition τ_- (resp. τ_+) and whose link pattern is μ (i.e., $\Psi_{n,-}(\mu) = \#\left\{\psi \in \mathfrak{fpl}(n,\tau_-) \,\middle|\, \pi_{b,-}(\psi) = \mu\right\}$, $\Psi_{n,+}(\mu) = \#\left\{\psi \in \mathfrak{fpl}(n,\tau_+) \,\middle|\, \pi_{b,+}(\psi) = \mu\right\}$).

4.2. **Operators on link patterns.** Here, we define operators on link patterns which are useful to examine the behavior of the $\Psi_{n,-}$.

Definition 8. For a positive integer n and a positive integer j which satisfies $1 \le j \le 2n$, we define an operator $e_j{:}\,\mathcal{F}(2n) \to \mathcal{F}(2n)$ which is called *matchmaker* ([3]). For $\mu \in \mathcal{F}(2n)$, we define $e_j\mu \in \mathcal{F}(2n)$ as follows:

(i) If $\{j, j+1\} \in E(\mu)$, $e_j\mu := \mu$.
(ii) If $\{j, j+1\} \notin E(\mu)$, the edge set of $e_j\mu$ is defined as $\{\eta_1, \eta_2, \cdots, \eta_{2n-2}, \{j, j+1\}, \{u, v\}\}$ when the edge set of μ equals $\{\eta_1, \eta_2, \cdots, \eta_{2n-2}, \{j, u\}, \{j+1, v\}\}$.

Figure 16 illustlates the operator e_2.

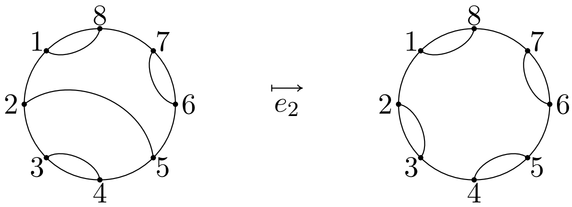


Figure 16. Example in the case when $\{j, j+1\} \notin E(\mu)$

Next, we define the rotation operator $\mathrm{R}\colon \mathcal{F}(2n) \to \mathcal{F}(2n)$.

Definition 9. For $\mu \in \mathcal{F}(2n)$, we define $\mathrm{R}\,\mu \in \mathcal{F}(2n)$ as follows: The edge set of $\mathrm{R}\,\mu$ is defined as $\{\{u_1 - 1, v_1 - 1\}, \{u_2 - 1, v_2 - 1\}, \cdots, \{u_n - 1, v_n - 1\}\}$ when the edge set of μ equals $\{\{u_1, v_1\}, \{u_2, v_2\}, \cdots, \{u_n, v_n\}\}$.

Figure 17 illustrates the rotation operator R.

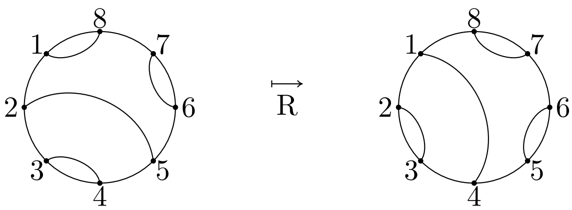


Figure 17. The example of R

5. Dihedral symmetry and Gylation

Let n, l be positive integers. For $\mu_b, \mu_w \in \mathcal{F}(2n)$, we define a $\Psi_{n,-}(\mu_b, \mu_w; l)$ (resp. $\Psi_{n,+}(\mu_b, \mu_w; l)$) to refine $\Psi_{n,-}(\mu_b)$ (resp. $\Psi_{n,+}(\mu_b)$) as follows. Let $\Psi_{n,-}(\mu_b, \mu_w; l)$ (resp. $\Psi_{n,+}(\mu_b, \mu_w; l)$) denote the cardinality of the FPLs in $\mathfrak{fpl}(n, \tau_-)$ (resp. $\mathfrak{fpl}(n, \tau_+)$) whose link patterns determined by black (resp. white) paths equals μ_b (resp. μ_w), and have l monochromatic cycles.

5.1. Wieland's dihedral symmetry theorem.

Now, the following proposition which is called *Wieland's dihedral symmetry theorem*, stated in [12], we have as a characterization of $\Psi_{n,-}$.

Proposition 5.1. *Let n, l be positive integers. For $\mu_b, \mu_w \in \mathcal{F}(2n)$, we hold*

$$\Psi_{n,-}(\mu_b, \mu_w; l) = \Psi_{n,-}\left(\mathrm{R}^{-1}\mu_b, \mathrm{R}\,\mu_w; l\right). \tag{5.1}$$

The proposition 5.1 is proven by constructing the bijection which is called gyration. Next we explain the gyration.

5.2. **Gyration.** We construct the gyration $\mathrm{G}\colon \mathfrak{fpl}(n,\tau_-) \to \mathfrak{fpl}(n,\tau_-)$ in two ways. First, we define two maps $\mathrm{G}_0\colon \mathfrak{fpl}(n,\tau_-) \to \mathfrak{fpl}(n,\tau_-)$ and $\mathrm{G}_1\colon \mathfrak{fpl}(n,\tau_-) \to \mathfrak{fpl}(n,\tau_-)$, then we construct G as $\mathrm{G}_0\,\mathrm{G}_1$. Second, we define two maps $\mathrm{H}_0\colon \mathfrak{fpl}(n,\tau_+) \to \mathfrak{fpl}(n,\tau_-)$ and $\mathrm{H}_1\colon \mathfrak{fpl}(n,\tau_-) \to \mathfrak{fpl}(n,\tau_+)$, then we construct G as $\mathrm{H}_0\,\mathrm{H}_1$.

Let α be a plaquette. We define a map $\mathrm{G}_\alpha\colon \mathfrak{fpl}(n,\tau_-) \to \mathfrak{fpl}(n,\tau_-)$ which affects only the colors of edges in α and leaves the colors of other edges invariant.

Let α be an interior plaquette. First of all, we define a map $\mathcal{N}_\alpha\colon \mathfrak{fpl}(n,\tau_-) \to \{0,1,-1\}$ to represent the state of α as follows.

Definition 10. Let n be a positive integer, and $\alpha_{i,j}$ an interior plaquette. Here, we label four edges of α as $\eta_1 = \{\{i,j\},\{i+1,j\}\}$, $\eta_2 = \{\{i+1,j\},\{i+1,j+1\}\}$, $\eta_3 = \{\{i,j+1\},\{i+1,j+1\}\}$ and $\eta_4 = \{\{i,j\},\{i,j+1\}\}$. For $\psi \in \mathfrak{fpl}(n,\tau_-)$, we define $\mathcal{N}_{\alpha_{i,j}}(\psi) \in \{0,1,-1\}$ as follows:

$$\mathcal{N}_{\alpha_{i,j}}(\psi) := \begin{cases} 1 & \text{if } \psi(\eta_1) = \psi(\eta_3) = w \text{ and } \psi(\eta_2) = \psi(\eta_4) = b, \\ -1 & \text{if } \psi(\eta_1) = \psi(\eta_3) = b \text{ and } \psi(\eta_2) = \psi(\eta_4) = w, \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

(A) $\mathcal{N}_{\alpha_{i,j}} = 1$. (B) $\mathcal{N}_{\alpha_{i,j}} = -1$.

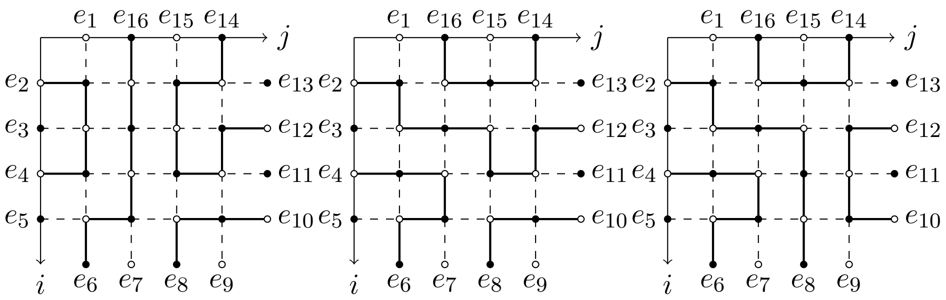
FIGURE 18. The states of $\alpha_{i,j}$ which satisfies $\mathcal{N}_{\alpha_{i,j}} \neq 0$.

Then we define an operator $\mathrm{G}_\alpha\colon \mathfrak{fpl}(n,\tau_-) \to \mathfrak{fpl}(n,\tau_-)$ as follows. We remark that G_α is defined for not only interior plaquettes but also boundary plaquettes.

Definition 11. Let n be a positive integer, and α a plaquette. For $\psi \in \mathfrak{fpl}(n,\tau_-)$, we define $\mathrm{G}_\alpha\,\psi \in \mathfrak{fpl}(n,\tau_-)$ as follows:

(i) G_α does not affect the colors of edges not in α (i.e., $\mathrm{G}_\alpha\,\psi(e) = \psi(e)$ for $e \in E(L_n) \setminus E(\alpha)$).
(ii) If α is an interior plaquette and $\mathcal{N}_\alpha(\psi) \neq 0$ (resp. $\mathcal{N}_\alpha(\psi) = 0$), G_α reverses (resp. keeps) the colors of all edges in α .
(iii) If α is a boundary plaquette, G_α keeps the colors of all edges in α.

For two distinct plaquettes α and β which have the same parity, the composition of G_α and G_β commutes with each other because α and β don't have a common edge (i.e., $\mathrm{G}_\alpha\,\mathrm{G}_\beta = \mathrm{G}_\beta\,\mathrm{G}_\alpha$). Then we can define



(A) $\psi \in \mathfrak{fpl}(n, \tau_-)$ (B) $\mathrm{G}_- \psi$ (C) $\mathrm{G}_+ \mathrm{G}_- \psi$

FIGURE 19. The process of the gyration

G_0 (resp. G_1): $\mathfrak{fpl}(n, \tau_-) \to \mathfrak{fpl}(n, \tau_-)$ as $\prod_{\alpha:\text{even}} \mathrm{G}_\alpha$ (resp. $\prod_{\alpha:\text{odd}} \mathrm{G}_\alpha$) without ambiguity. Up to here, we have constructed G_0 and G_1 by repeating the operations to FPL. But in the following, we construct H_0 and H_1 which repeat the operations to a coloring which is not a FPL. First we denote the set of all maps from $E(L_n)$ to $\{b, w\}$ by $\mathrm{Map}(E(L_n), \{b, w\})$. For any map $f: E(L_n) \to \{b, w\}$, we also call f a *coloring of* $E(L_n)$.

Now we define $\mathrm{C}_\alpha: \mathrm{Map}(E(L_n), \{b, w\}) \to \mathrm{Map}(E(L_n), \{b, w\})$ as the map which reverse the colors of all edges in α and leaves the colors of other edges invariant.

Note that $\mathcal{N}_\alpha$ can be extended to a map from $\mathrm{Map}(E(L_n), \{b, w\})$ to $\{0, 1, -1\}$ when α is an interior plaquette, and G_α to a map from $\mathrm{Map}(E(L_n), \{b, w\})$ to $\mathrm{Map}(E(L_n), \{b, w\})$. Then we define an operator $\mathrm{H}_\alpha: \mathrm{Map}(E(L_n), \{b, w\}) \to \mathrm{Map}(E(L_n), \{b, w\})$ as $\mathrm{G}_\alpha \mathrm{C}_\alpha$.

Let α and β be plaquettes which have the same parity. Here α and β do not have to be distinct. Notice that the composition of G_α and C_β is commutative (i.e., $\mathrm{G}_\alpha \mathrm{C}_\beta = \mathrm{C}_\beta \mathrm{G}_\alpha$). Hence, the composition of H_α and H_β is commutative when α and β is distinct. Therefore we can define $\prod_{\alpha:\text{even}} \mathrm{H}_\alpha$ and $\prod_{\alpha:\text{odd}} \mathrm{H}_\alpha$ without ambiguity. Moreover, $\prod_{\alpha:\text{even}} \mathrm{H}_\alpha$ (resp. $\prod_{\alpha:\text{odd}} \mathrm{H}_\alpha$) equals $\prod_{\alpha:\text{even}} \mathrm{G}_\alpha \prod_{\alpha:\text{even}} \mathrm{C}_\alpha$ (resp. $\prod_{\alpha:\text{odd}} \mathrm{G}_\alpha \prod_{\alpha:\text{odd}} \mathrm{C}_\alpha$).

Note that G_α, C_α and H_α are involutions. Since G_0, G_1, $\prod_{\alpha:\text{even}} \mathrm{H}_\alpha$ and $\prod_{\alpha:\text{odd}} \mathrm{H}_\alpha$ are compositions of these commutative involutions, they are involutions.

We remark that $E(L_n)$ can be expressed $\bigsqcup_{\alpha:\text{even}} E(\alpha)$ or $\bigsqcup_{\alpha:\text{odd}} E(\alpha)$. Therefore $\bigsqcup_{\alpha:\text{even}} \mathrm{C}_\alpha$ (resp. $\bigsqcup_{\alpha:\text{odd}} \mathrm{C}_\alpha$) reverse the colors of all edges in L_n. Hence we can define a bijection $\mathrm{H}_0: \mathfrak{fpl}(n, \tau_+) \to \mathfrak{fpl}(n, \tau_-)$ (resp. $\mathrm{H}_1: \mathfrak{fpl}(n, \tau_-) \to \mathfrak{fpl}(n, \tau_+)$) by restricting $\prod_{\alpha:\text{even}} \mathrm{H}_\alpha$ (resp. $\prod_{\alpha:\text{odd}} \mathrm{H}_\alpha$) to $\mathfrak{fpl}(n, \tau_+)$ (resp. $\mathfrak{fpl}(n, \tau_-)$). We use H_0 and H_1 to prove the following lemma.

Lemma 5.2. *Let n be a positive integer. For $\mu_b, \mu_w \in \mathcal{F}(2n)$, we have following two equations:*

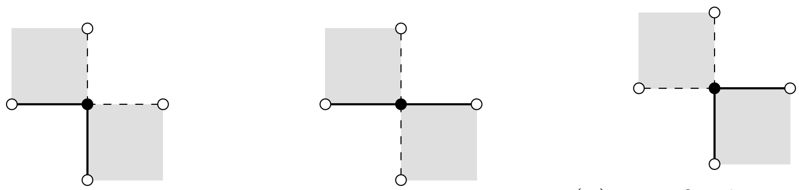
$$\Psi_{n,-}(\mu_b, \mu_w; l) = \Psi_{n,+}\left(\mathrm{R}^{-1}\mu_b, \mathrm{R}\,\mu_w; l\right), \tag{5.3a}$$

$$\Psi_{n,+}(\mu_b, \mu_w; l) = \Psi_{n,-}(\mu_b, \mu_w; l). \tag{5.3b}$$

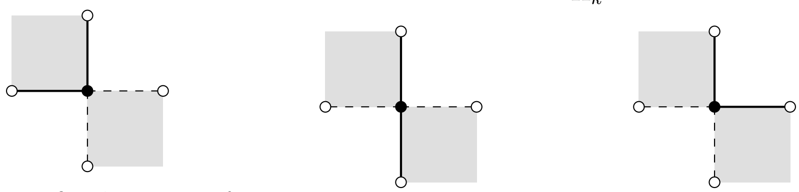
We take several steps to prove lemma 5.2. First, we define vertices which are called fixed vertice of H_1 (resp. H_0) to prove (5.3a) (resp. (5.3b)). Second, we divide a monochromatic path at each fixed vertex into short pathes. Then we show how the short pathes are affected by H_1 (resp. H_0).

Let ψ be a FPL in $\mathfrak{fpl}(n, \tau_-)$ and v an interior vertex. We label the two edges which are adjacent to v and have color b in ψ as e and e'. Now, we remark that there are two plaquettes which contain v and whose parity is odd (resp. even). Then we call v a *fixed vertex of* H_1 *(resp.* H_0*) in* ψ if e and e' are the edges of distinct plaquettes whose parity is odd (resp. even) and which contain v. For any $\psi' \in \mathfrak{fpl}(n, \tau_+)$, we also define a *fixed vertex of* H_1 *(resp.* H_0*) in* ψ' in the same way.

Let $v = (i, j)$ be an interior veretex, and $k \in \{0, 1\}$. Figure 20 (resp. Figure 21) shows whether v is fixed vertex or not if $i + j$ and k have the same parity (resp. don't have the same parity). We illustrate v as $\bullet$ and we color the plaquette which has the same parity of k in Figure 20 (resp. Figure 21).



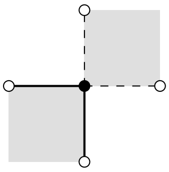
(A) fixed vertex of H_k (B) fixed vertex of H_k (C) not fixed vertex of H_k



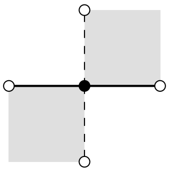
(D) not fixed vertex of H_k (E) fixed vertex of H_k (F) fixed vertex of H_k

Figure 20. If $i + j$ and k have the same parity

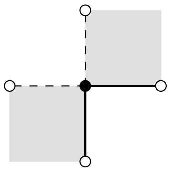
In a FPL, the state of the interior plaquette $\alpha_{i,j}$ can be in 16 different situations. Figure 22 and 23 illustrates the 16 different situations and their fixed vertex of H_k when $i + j$ and k have the same parity as $\circ$.



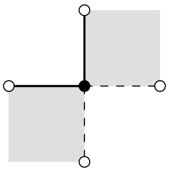
(A) not fixed vertex of H_k



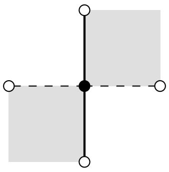
(B) fixed vertex of H_k



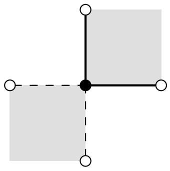
(C) fixed vertex of H_k



(D) fixed vertex of H_k



(E) fixed vertex of H_k



(F) not fixed vertex of H_k

FIGURE 21. If $i+j$ and k do not have the same parity

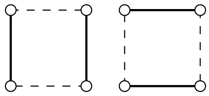


FIGURE 22. $\mathcal{N}_\alpha \neq 0$

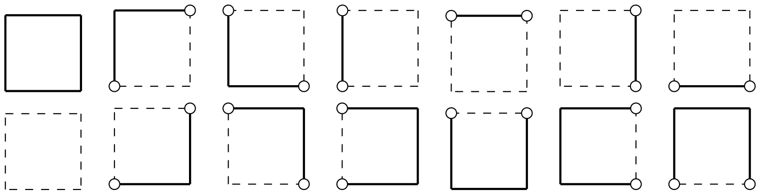


FIGURE 23. $\mathcal{N}_\alpha = 0$

Let $\alpha_{i,j}$ be an odd interior plaquette, and $\psi \in \mathfrak{fpl}(n, \tau_-)$. If $\alpha_{i,j}$ has a fixed vertex v of H_1 in ψ, there uniquely exists a vertex $w \in V(\alpha)$ such that w is a fixed vertex of H_1 in ψ, and it is connected to v by black (resp. white) path which does not pass through other fixed veretices of H_1 in ψ. Even after we operate H_1, v and w are fixed vertices of H_1 in $\mathrm{H}_1\,\psi$, and they are connected by black (resp. white) path which does not pass through other fixed veretices of H_1 in $\mathrm{H}_1\,\psi$.

Now, let $\boldsymbol{p} = (v_0, v_1, \ldots, v_m)$ be a black (resp. white) path in ψ, and v_0, v_m fixed vertices of H_1 in ψ. We divide $\boldsymbol{p}$ at each fixed vertices of H_1 in ψ into short black (resp. white) pathes. When $\boldsymbol{p}$ is divided into k pathes, we set v_0 as w_0, v_m as w_k, and the k short pathes as $(w_0, \dots, w_1)$, $(w_1, \dots, w_2)$, $\dots$, $(w_{k-1}, \dots, w_k)$. Since w_i and w_{i+1} are connected by balck (resp. white) path in $\mathrm{H}_1\,\psi$ for $0 \le i < k$, there is a black (resp. white) path which start from v_0 to v_m and pass through $w_1, w_2, \dots, w_{k-1}$ in $\mathrm{H}_1\,\psi$. Moreover such a black (resp. white) path does not pass through any other fixed vertices of H_1.

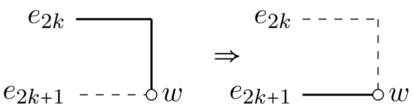
Next we focus on boundary plaquette. Let $\alpha_{i,j}$ be an odd boundary plaquette, and $\psi \in \mathfrak{fpl}(n, \tau_-)$. In the plaquette $\alpha_{i,j}$, there is exactly one fixed vertex of H_1 in ψ. Now we label the fixed vertex as w, and the two boundary edges in $\alpha_{i,j}$ as e_{2k} and e_{2k+1}. Here, e_{4n+1} means e_1. There is a black (resp. white) path which connects w and the boundary vertex that is adjacent to e_{2k} (resp. e_{2k+1}), and this path does not pass through a fixed vertex of H_1 in ψ other than w. Even after we operate H_1, w is a fixed vertex of H_1 in $\mathrm{H}_1\,\psi$, and the other vertices in $\alpha_{i,j}$ are not fixed vertices. Moreover there is a black (resp. white) path which connects w and the boundary vertex that is adjacent to e_{2k+1} (resp. e_{2k}), and this path does not pass through a fixed vertex of H_1 in $\mathrm{H}_1\,\psi$ other than w. Figure 24 illustrates the state of $\alpha_{i,j}$ in $\psi \in \mathfrak{fpl}(n, \tau_-)$ before and after we operate H_1.

Let v_i be the boundary vertex which is adjacent to boundary edge e_i for $1 \le i \le 2n$, and $\psi \in \mathfrak{fpl}(n, \tau_-)$. We set $\boldsymbol{p}$ as a black (resp. white) path in ψ which start from v_{2k} (resp. v_{2k+1}) and end at v_{2l} (resp. v_{2l+1}). We also set w and w' as the vertices in $\boldsymbol{p}$ which satisfy following condition: when we divide $\boldsymbol{p}$ at each fixed vertices of H_1 in ψ into short black (resp. white) pathes, the short path which strat from v_{2k} (resp. v_{2k+1}) is ended at w, and the short path which end at v_{2l} (resp. v_{2l+1}) is started from w'. From the above, we have a black (resp. white) path $\boldsymbol{q}$ in $\mathrm{H}_1\,\psi$ which start from w and end at w'. Now, we glue the following three black (resp. white) pathes in $\mathrm{H}_1\,\psi$: the path which strat from v_{2k+1} (resp. v_{2k}) and end at w, $\boldsymbol{q}$, and the path which strat from w' and end at v_{2l+1} (resp. v_{2l}). Then we get the black (resp. white) path in $\mathrm{H}_1\,\psi$ which start from v_{2k+1} (resp. v_{2k}), pass through w and w', and end at v_{2l+1} (resp. v_{2l}). Therefore, $\{k+1, l+1\} \in \pi_{b,+}(\mathrm{H}_1\,\psi)$ (resp. $\{k, l\} \in \pi_{w,+}(\mathrm{H}_1\,\psi)$) if $\{k, l\} \in \pi_{b,-}(\psi)$ (resp. $\{k+1, l+1\} \in \pi_{w,-}(\psi)$). Then we have two following equations:

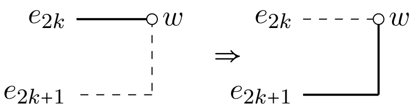
$$\pi_{b,+}(\mathrm{H}_1\,\psi) = \mathrm{R}^{-1}\,\pi_{b,-}(\psi), \tag{5.4a}$$

$$\pi_{w,+}(\mathrm{H}_1\,\psi) = \mathrm{R}\,\pi_{w,-}(\psi). \tag{5.4b}$$

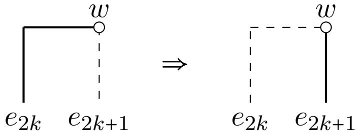
Next, we focus on monochromatic cycles. Let ψ be a FPL in $\mathfrak{fpl}(n, \tau_-)$, and $\boldsymbol{c}$ a black (resp. white) cycle in ψ. Note that $\boldsymbol{c}$ has length at least 4. First we consider the case when the length of $\boldsymbol{c}$ is 4. Now there is an interior plaquette $\alpha_{i,j}$ such that $\boldsymbol{c}$ is consist of 4 edges in $\alpha_{i,j}$ (i.e., $\boldsymbol{c} = ((i,j), (i+1,j), (i+1,j+1), (i,j+1), (i,j)))$. If the plaquette $\alpha_{i,j}$ is even,



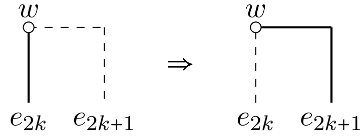
(A) When $1 \le i < n$ and $j = 0$



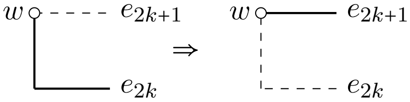
(B) When $1 \le i < n$ and $j = 0$



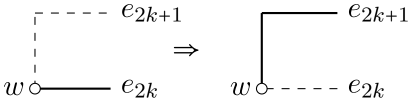
(C) When $i = n$ and $1 \le j < n$.



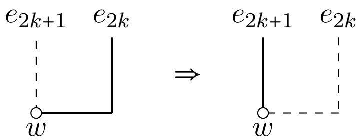
(D) When $i = n$ and $1 \le j < n$.



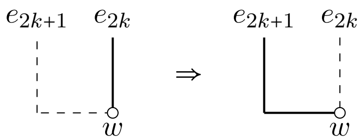
(E) When $1 \le i < n$ and $j = n$.



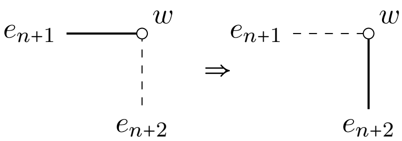
(F) When $1 \le i < n$ and $j = n$.



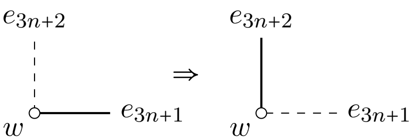
(G) When $i = 0$ and $1 \le j < n$.



(H) When $i = 0$ and $1 \le j < n$.



(I) When $i = n$, $j = 0$ and n is odd.



(J) When $i = 0$, $j = n$ and n is odd.

FIGURE 24. left: the state of an odd boundary plaquette $\alpha_{i,j}$ in $\psi \in \mathfrak{fpl}(n, \tau_-)$, right: the state of $\alpha_{i,j}$ in $\mathrm{H}_1 \psi \in \mathfrak{fpl}(n, \tau_+)$.

the 4 vertices are fixed vertices of H_1 in ψ, and thus there is a black (resp. white) cycle $\boldsymbol{c}'$ in $\mathrm{H}_1 \psi$ such that $\boldsymbol{c}'$ pass through the 4 verices of $\alpha_{i,j}$. If the plaquette $\alpha_{i,j}$ is odd, since the color of the all edges in $\alpha_{i,j}$ is reversed, there is a white (resp. black) cycle $\boldsymbol{c}'$in $\mathrm{H}_1 \psi$ such that $\boldsymbol{c}'$ is consist of 4 edges in $\alpha_{i,j}$.

Second we consider the case when the length of $\boldsymbol{c}$ is greater than 4. Then the cycle $\boldsymbol{c}$ run across multiple plaquettes, and pass through some fixed vertices of H_1 in ψ. Therefore there is a black (resp. white) cycle $\boldsymbol{c}'$ in $\mathrm{H}_1 \psi$ such that $\boldsymbol{c}$ and $\boldsymbol{c}'$ pass through the same fixed vertices of H_1.

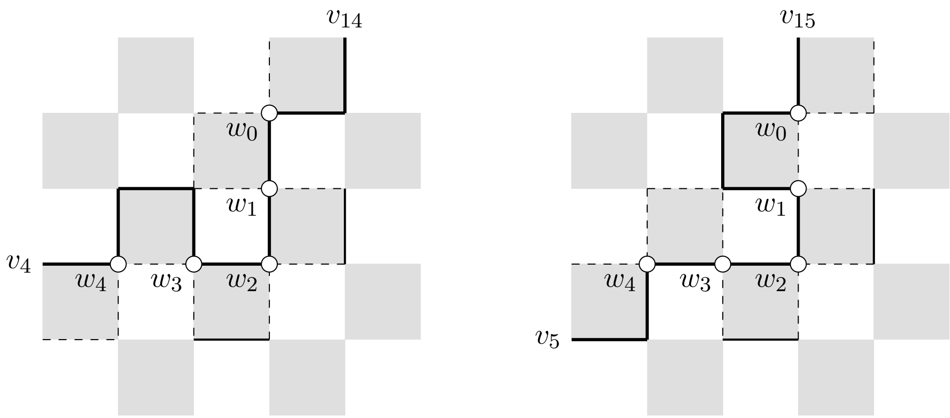


FIGURE 25. left:the black path in $\psi \in \mathfrak{fpl}(4, \tau_-)$, right:the black path in $\mathrm{H}_1\,\psi \in \mathfrak{fpl}(4, \tau_+)$.

From the above, we can construct the bijection between the set of monochromatic cycles in ψ and the set of monochromatic cycles in $\mathrm{H}_1\,\psi$. Together with (5.4a) and (5.4b), we showed (5.3a).

In the same way, each black (resp. white) path which pass through some fixed vertices of H_0 in $\mathrm{H}_1\,\psi$ also pass through same fixed vertices of H_0 after we operate H_0. Moreover, each even boundary plaquette has exactly one fixed vertex of H_0. If we label the two boundary edge in the plaquette as e_{2k} and e_{2k-1}, then the fixed vertex of H_0 is connected to the boundary vertex v_{2k-1} (resp. v_{2k}) by black (resp. white) path in $\mathrm{H}_1\,\psi$. After we operate H_0, the fixed vertex of H_0 is connected to the boundary vertex v_{2k} (resp. v_{2k-1}) by black (resp. white) path in $\mathrm{H}_0\,\mathrm{H}_1\,\psi$. Hence we have two following equations:

$$\pi_{b,-}\left(\mathrm{H}_0\,\mathrm{H}_1\,\psi\right) = \pi_{b,+}\left(\mathrm{H}_1\,\psi\right) \tag{5.5a}$$

$$\pi_{w,-}\left(\mathrm{H}_0\,\mathrm{H}_1\,\psi\right) = \pi_{w,+}\left(\mathrm{H}_1\,\psi\right) \tag{5.5b}$$

On the other hand, $\mathrm{H}_0\,\mathrm{H}_1\,\psi$ have the same number of monochromatic cycles as $\mathrm{H}_1\,\psi$. Therefore we hold (5.3b), and lemma 5.2 is prooved. Then proposition 5.1 follows from lemma 5.2. □

5.3. **Periodicity of the gyration.** We remark that the operation of repeating the gyration is periodic. Let ψ be a FPL in $\mathfrak{fpl}(n, \tau_-)$. If we repeat the gyration to ψ n times, we will get ψ at least once. Now we set $\mathfrak{fpl}\left(n, \tau_-; \mathcal{O}(\psi)\right) := \left\{\mathrm{G}^k\,\psi \,\middle|\, 0 \le k < n\right\}$. Then we hold the following proposition(stated in [3]).

Proposition 5.3. *Let ψ be a FPL in $\mathfrak{fpl}(n,\tau_-)$ and α an interior plaquette. Then we have a following equation:*

$$\sum_{\varphi\in\mathfrak{fpl}(n,\tau_-;\mathcal{O}(\psi))} \mathcal{N}_\alpha(\varphi) = 0. \tag{5.6}$$

We also take several steps to prove proposition 5.3. First, we focus on the case when α has an edge in common with a boundary plaquette. Let α be an odd plaquette, and β the boundary plaquette which has an edge e in common with α. For each $k \geq 0$, $\mathrm{G}^k\,\psi(e) \neq \mathrm{G}^{k+1}\,\psi(e)$ if and only if G_1 reverse the color of e in $\mathrm{G}^k\,\psi$ because G_0 does not reverse the color of e in $\mathrm{G}_1\,\mathrm{G}^k\,\psi$. We remark that if ψ satisfies the following conditions: $\mathrm{G}^k\,\psi(e) \neq \mathrm{G}^{k+1}\,\psi(e)$, $\mathrm{G}^{k+1}\,\psi(e) = \mathrm{G}^{k+2}\,\psi(e) = \cdots = \mathrm{G}^l\,\psi(e)$, and $\mathrm{G}^l\,\psi(e) \neq \mathrm{G}^{l+1}\,\psi(e)$, then $\mathcal{N}_\alpha\left(\mathrm{G}^k\,\psi\right) \neq 0$, $\mathcal{N}_\alpha\left(\mathrm{G}^{k+1}\,\psi\right) = \mathcal{N}_\alpha\left(\mathrm{G}^{k+2}\,\psi\right) = \cdots = \mathcal{N}_\alpha\left(\mathrm{G}^{l-1}\,\psi\right) = 0$, and $\mathcal{N}_\alpha\left(\mathrm{G}^l\,\psi\right) = -\mathcal{N}_\alpha\left(\mathrm{G}^k\,\psi\right)$. In other word, the sequence $\left\{\mathcal{N}_\alpha\left(\mathrm{G}^k\,\psi\right)\right\}_{k\geq 0}$ alternates between 1 and -1 except for 0. Moreover, when we denote $\#\mathfrak{fpl}(n,\tau_-;\mathcal{O}(\psi))$ as m, 1 and -1 appear the same number of times between the 0th term and the mth term in $\left\{\mathcal{N}_\alpha\left(\mathrm{G}^k\,\psi\right)\right\}_{k\geq 0}$ because the color of e is reversed an even number times while the gyration is repaeted m times. Since the left hand side of (5.6) is equal to the sum of the 0th through mth terms of $\left\{\mathcal{N}_\alpha\left(\mathrm{G}^k\,\psi\right)\right\}_{k\geq 0}$, then (5.6) follows. When α is even, we can show the same by replacing G_1 with G_0 and G with G^{-1}.

Next, we consider two distinct interior plaquette they have a common edge e. Let α (resp. β) be an odd (resp. even) interior plaquette which has the edge e. We set two sequences $\{\mu_n\}_{n\geq 0}$ and $\{\nu_n\}_{n\geq 0}$ as following:

$$\mu_n := \begin{cases} \mathcal{N}_\alpha\left(\mathrm{G}^k\,\psi\right) & (n = 2k) \\ \mathcal{N}_\beta\left(\mathrm{G}_1\,\mathrm{G}^k\,\psi\right) & (n = 2k+1) \end{cases}, \tag{5.7}$$

$$\nu_n := \begin{cases} \mathrm{G}^k\,\psi(e) & (n = 2k) \\ \mathrm{G}_1\,\mathrm{G}^k\,\psi(e) & (n = 2k+1) \end{cases}. \tag{5.8}$$

Note that $\nu_n \neq \nu_{n+1}$ if and only if $\mu_n \neq 0$. Similar to previous discussion, $\{\mu_n\}_{n\geq 0}$ alternates between 1 and -1 except for 0, and the sum of the 0th through $(2m-1)$th terms of $\{\mu_n\}_{n\geq 0}$ is equal to 0. Since $\mathrm{G}_1\,\mathrm{G}^k\,\psi$ is equal to $\mathrm{G}^{-k}\,\mathrm{G}_1\,\psi$, we hold the following equation:

$$\sum_{0\leq k\leq m-1} \mathcal{N}_\alpha\left(\mathrm{G}^k\,\psi\right) + \sum_{0\leq k\leq m-1} \mathcal{N}_\beta\left(\mathrm{G}^{-k}\,\mathrm{G}_1\,\psi\right) = 0. \tag{5.9}$$

Since the cardinarity of $\mathfrak{fpl}(n,\tau_-;\mathcal{O}(\mathrm{G}_1\,\psi))$ is equal to m, the equation (5.9) is equivalent to the following equation:

$$\sum_{\varphi\in\mathfrak{fpl}(n,\tau_-;\mathcal{O}(\psi))} \mathcal{N}_\alpha(\varphi) + \sum_{\varphi'\in\mathfrak{fpl}(n,\tau_-;\mathcal{O}(\mathrm{G}_1\,\psi))} \mathcal{N}_\beta\left(\varphi'\right) = 0. \tag{5.10}$$

Combined with the previous discussion, proposition 5.3 follows. □

6. The vector space which has link patterns as basis

We consider the $\mathbb{C}$-vector space which has link pattern as basis to refine a enumeration of FPLs. Let n be a positive integer. We denote $\mathbb{C}^{\mathcal{F}(2n)}$ as the $\mathbb{C}$-vector space which is spaned by $\mathcal{F}(2n)$, and we will write each element $x \in \mathbb{C}^{\mathcal{F}(2n)}$ as $|x\rangle$ for emphasis. We also denote $\mathbb{C}^{\mathfrak{fpl}(n,\tau)}$ as the $\mathbb{C}$-vector space which is spaned by $\mathfrak{fpl}(n,\tau)$ when a boundary condition $\tau \in \{b,w\}^{4n}$ is given, and we will write each element $y \in \mathbb{C}^{\mathfrak{fpl}(n,\tau)}$ as $\|y\rangle\!\rangle$ for emphasis. In particular, we denote $\sum_{\psi\in\mathfrak{fpl}(n,\tau_-)} \left|\pi_{b,-}(\psi)\right\rangle \in \mathbb{C}^{\mathcal{F}(2n)}$ as $|s_n\rangle$, and $\sum_{\psi\in\mathfrak{fpl}(n,\tau_-)} \|\psi\rangle\!\rangle \in \mathbb{C}^{\mathfrak{fpl}(n,\tau_-)}$ as $\|s_{n,\tau_-}\rangle\!\rangle$.

6.1. **Operators on the vector space.** We define some operators on the vector space which are introduced above. First, we define $\Pi_-\colon \mathbb{C}^{\mathfrak{fpl}(n,\tau_-)} \to \mathbb{C}^{\mathcal{F}(2n)}$ (Resp. $\Pi_+\colon \mathbb{C}^{\mathfrak{fpl}(n,\tau_+)} \to \mathbb{C}^{\mathcal{F}(2n)}$) as follows:

$$\Pi_-\left(\sum_{\psi\in\mathfrak{fpl}(n,\tau_-)} c_\psi \|\psi\rangle\!\rangle\right) := \sum_{\psi\in\mathfrak{fpl}(n,\tau_-)} c_\psi \left|\pi_{b,-}(\psi)\right\rangle, \tag{6.1a}$$

$$\Pi_+\left(\sum_{\psi\in\mathfrak{fpl}(n,\tau_+)} c_\psi \|\psi\rangle\!\rangle\right) := \sum_{\psi\in\mathfrak{fpl}(n,\tau_+)} c_\psi \left|\pi_{b,+}(\psi)\right\rangle. \tag{6.1b}$$

Especially, we hold $\Pi_-(\|s_{n,\tau_-}\rangle\!\rangle) = |s_n\rangle$. Second, we define $\tilde{\mathcal{N}_\alpha}\colon \mathbb{C}^{\mathfrak{fpl}(n,\tau)} \to \mathbb{C}^{\mathfrak{fpl}(n,\tau)}$ for each interior plaquette α as following:

$$\tilde{\mathcal{N}_\alpha}\left(\sum_{\psi\in\mathfrak{fpl}(n,\tau)} b_\psi \|\psi\rangle\!\rangle\right) := \sum_{\psi\in\mathfrak{fpl}(n,\tau)} \mathcal{N}_\alpha b_\psi \|\psi\rangle\!\rangle. \tag{6.2}$$

Next, we make operators on $\mathcal{F}(2n)$ extend linearly. When the operator $X\colon \mathcal{F}(2n) \to \mathcal{F}(2n)$ is given, we define $\hat{X}\colon \mathbb{C}^{\mathcal{F}(2n)} \to \mathbb{C}^{\mathcal{F}(2n)}$ as follows:

$$\hat{X}\left(\sum_{\mu\in\mathcal{F}(2n)} a_\mu |\mu\rangle\right) := \sum_{\mu\in\mathcal{F}(2n)} a_\mu |X(\mu)\rangle. \tag{6.3}$$

Similarly, we make operators on $\mathfrak{fpl}(n,\tau)$ extend linearly. When a boundary condition $\tau \in \{b,w\}^{4n}$ and the operator $Y\colon \mathfrak{fpl}(n,\tau) \to \mathfrak{fpl}(n,\tau)$ is given, we define $\hat{Y}\colon \mathbb{C}^{\mathfrak{fpl}(n,\tau)} \to \mathbb{C}^{\mathfrak{fpl}(n,\tau)}$ as follows:

$$\hat{Y}\left(\sum_{\psi\in\mathfrak{fpl}(n,\tau)} b_\psi \|\psi\rangle\!\rangle\right) := \sum_{\psi\in\mathfrak{fpl}(n,\tau)} b_\psi \|Y(\psi)\rangle\!\rangle. \tag{6.4}$$

We define operators $\mathrm{Sym}\colon \mathbb{C}^{\mathcal{F}(2n)} \to \mathbb{C}^{\mathcal{F}(2n)}$ and $\mathrm{H}_n\colon \mathbb{C}^{\mathcal{F}(2n)} \to \mathbb{C}^{\mathcal{F}(2n)}$ respectively as $\mathrm{Sym} := \sum_{k=0}^{2n-1} \left(\hat{\mathrm{R}}\right)^k$, $\mathrm{H}_n := \sum_{k=1}^{2n} \hat{e_k}$. Then we call Sym *symmetrise* *operator*, and we call H_n *Hamiltonian*. Now, the following claim, stated in [3], hold as a corollary of proposition 5.3.

Proposition 6.1. *Let α be an interior plaquette. We have the following equation:*

$$\operatorname{Sym} \Pi_{-} \hat{\mathcal{N}_\alpha} \left\| s_{n,\tau_-} \right\rangle\!\rangle = 0. \tag{6.5}$$

Proof. First, we define a equivalence on $\mathcal{F}(2n)$. Let μ and ν be link patterns of size n. We define μ and ν are equivalent if there exists a non-negative integer k such that $\nu = \mathrm{R}^k \mu$, and we write $\mu \sim \nu$. Then we denote $\mathcal{F}(2n)/\sim$ as $\mathcal{F}^*(2n)$, and we denote the complete set of $\mathcal{F}^*(2n)$ as $\{\mu_1, \mu_2, \ldots, \mu_m\}$.

Next, we shall transforme the left hand side of (6.5).

$$\begin{aligned} &\operatorname{Sym} \Pi_{-} \hat{\mathcal{N}_\alpha} \left\| s_{n,\tau_-} \right\rangle\!\rangle \\ &= \sum_{k=0}^{2n-1} \sum_{\psi \in \mathfrak{fpl}(n,\tau_-)} \mathcal{N}_\alpha(\psi) \left| \mathrm{R}^k \pi(\psi) \right\rangle \\ &= \sum_{i=1}^{m} \sum_{\substack{\psi \in \mathfrak{fpl}(n,\tau_-) \\ \pi(\psi) \sim \mu_i}} \mathcal{N}_\alpha(\psi) \sum_{k=0}^{2n-1} \left| \mathrm{R}^k \mu_i \right\rangle \end{aligned} \tag{6.6}$$

We remark that the set of FPLs the link pattern of which equivalent to μ_i is represented as disjonit union of some orbits of FPL for each $1 \le i \le m$. Therefore $\sum_{\substack{\psi \in \mathfrak{fpl}(n,\tau_-) \\ \pi(\psi) \sim \mu_i}} \mathcal{N}_\alpha(\psi) = 0$ follows to proposition 5.3, thus (6.5) follows. $\square$

7. Futurework

Althought we deal with the ordinary type of ASM in this paper, we are interested in half-turn ASM. We will define a poset similar to $\mathbb{P}_n$ for half-turn ASM. Further more, we aim to make half-turn ASM correspond to a root systems.

Acknowledgment

I would like to express my deep gratitude to Prof. Masao Ishikawa and Yoshiki Takayama for helpful suggestions and advices.

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(*Received May 17, 2024*)
(*Accepted October 5, 2024*)