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# THE BEST CONSTANT OF THE SOBOLEV INEQUALITY CORRESPONDING TO A BENDING PROBLEM OF A STRING WITH A RECTANGULAR SPRING CONSTANT

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Abstract. The Sobolev inequality shows that the supremum of a function defined on a whole line is estimated from the above by constant multiples of the potential energy. Among such constants, the smallest constant is the best constant. If we replace a constant by the best constant in the Sobolev inequality, then the equality holds for the best function. The aim of this paper is to find the best constant and the best function. In the background, there is a bending problem of a string with a rectangular spring constant. The Green function is an important function because the best constant and the best function consist of the Green function.

## 1. Introduction

Our study of the Sobolev inequality starts with [4, 5], in which we considered the best constant of Sobolev inequality in an $n$-dimensional Euclidean space. Afterwards, we have the best constant of the Sobolev inequality corresponding to some differential equations with boundary conditions. The bending of a string on an interval $(-(d/dx)^2 + a_0^2)u = f(x)$ $(x \in (0, L))$ with Dirichlet [16], Neumann [16] and periodic [7, 16] boundary conditions, and its discrete version [21, 22] are given. Moreover, we consider some spring constants $q(x)$ for the bending of a string on an whole line $(-(d/dx)^2 + q(x))u = f(x)$ $(x \in \mathbb{R})$. The spring constants are the constant function [1], the delta function [8], the step function [15], and the rectangular function in this paper. The bending of a beam on a half line $(-(d/dx)^2 + a_0^2)(-(d/dx)^2 + a_1^2)u = f(x)$ $(x \in (0, \infty))$ with clamped, Dirichlet, Neumann and free boundary conditions [6, 10, 13] are given. The bending of a beam on an interval $(-(d/dx)^2 + a_0^2)(-(d/dx)^2 + a_1^2)u = f(x)$ $(x \in (0, L))$ with clamped [20], Dirichlet [16, 20], Neumann [16, 20], free [20], and periodic [16] boundary conditions are given. Here, we introduce the characteristic polynomial $P(z) = (z + a_0^2) \cdots (z + a_{M-1}^2)$, where $0 < a_0 < a_1 < \cdots < a_{M-1}$. $2M$-th order differential equation $P(-(d/dx)^2)u = f(x)$ on a whole line [1] and on an interval with the preriodic boundary conditions [17] are given. $2M$-th order simple type $(a_0 = \cdots = a_{M-1} = 0)$ differential

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equation $(-1)^M (d/dx)^{2M} u = f(x)$ on an interval with clamped [14], Dirichlet [19], Neumann [19], free [12] and periodic [19] (the discrete version [9]) boundary conditions are given. In addition, we have the best constant of the Sobolev-type inequality corresponding to a 2nd order differential equation $(-\Delta + a^2)u = f(x)$ $(x \in \mathbb{R}^N)$ [18], a $M$-th order differential equation $P(d/dx)u = f(x)$ $(x \in \mathbb{R})$ [3] and $P(d/dx)u = f(x)$ $(x \in (0,1))$ with periodic boundary condition [11] are given. We collect our studies into [2].

A string is supported by springs with spring constant $q(x)$. The spring constant $q(x)$ is a rectangular function (Figure 1, 2) as

$$q(x) = \begin{cases} b^2 & (L < x < \infty), \\ a^2 & (-L < x < L), \\ b^2 & (-\infty < x < -L), \end{cases}$$

where $a$, $b$ and $L$ are positive constants.

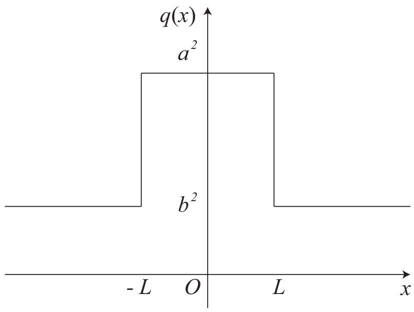


**Figure 1.** $0 < b < a < \infty$.

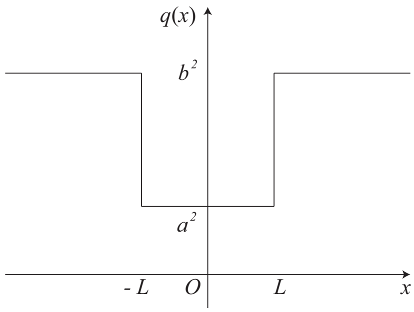


**Figure 2.** $0 < a < b < \infty$.

Under a density of a load $f(x)$, a bending of a string $u(x)$ satisfies the bending problem of a string (Figure 3):

BVP (Boundary Value Problem)

$$\begin{cases} -u'' + q(x)u = f(x) & (x \in \mathbb{R}), \\ u,\ u' \ :\ \text{bounded} & (x \in \mathbb{R}), \\ u^{(i)}(\pm L + 0) = u^{(i)}(\pm L - 0) & (i = 0, 1). \end{cases}$$

Concerning the uniqueness and existence of the solution to BVP, we have the following theorem in this paper. We define the hyperbolic functions $\mathrm{ch}(x) = \cosh(x)$, $\mathrm{sh}(x) = \sinh(x)$ and $\mathrm{th}(x) = \tanh(x)$ for short. We introduce the maximum function $x \vee y = \max\{x, y\}$ and the minimum function $x \wedge y = \min\{x, y\}$, which satisfies

$$x + y = x \vee y + x \wedge y, \qquad |x - y| = x \vee y - x \wedge y.$$

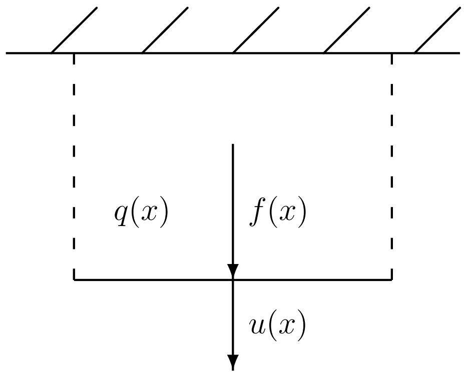


**Figure 3.** *Bending of a string.*

**Theorem 1.1.** *For any bounded continuous function $f(x)$ $(x \in \mathbb{R})$,* BVP *has a unique solution $u(x)$ expressed as*

$$u(x) = \int_{-\infty}^{\infty} G(x,y) f(y) dy \qquad (x \in \mathbb{R}). \tag{1.1}$$

*The Green function $G(x,y)$ is*

$$G(x,y) = g(x \vee y) g(-x \wedge y) \qquad (x, y \in \mathbb{R}). \tag{1.2}$$

*The concrete form of the Green function is given as*

$$G(x,y) = \tag{1.3}$$
$$\begin{cases} g_+(x \vee y)\, g_-(-x \wedge y) & (L < y < \infty) \\ g_+(x)\, g_0(-y) & (-L < y < L) \\ g_+(x)\, g_+(-y) & (-\infty < y < -L) \end{cases} \qquad (L < x < \infty),$$

$$G(x,y) = \tag{1.4}$$
$$\begin{cases} g_+(y)\, g_0(-x) & (L < y < \infty) \\ g_0(x \vee y)\, g_0(-x \wedge y) & (-L < y < L) \\ g_0(x)\, g_+(-y) & (-\infty < y < -L) \end{cases} \qquad (-L < x < L),$$

$$G(x,y) = \tag{1.5}$$
$$\begin{cases} g_+(y)\, g_+(-x) & (L < y < \infty) \\ g_0(y)\, g_+(-x) & (-L < y < L) \\ g_-(x \vee y)\, g_+(-x \wedge y) & (-\infty < y < -L) \end{cases} \qquad (-\infty < x < -L),$$

*where the fundamental solution $g(x)$ is*

$$g(x) = \tag{1.6}$$

$$\begin{cases} g_+(x) = \dfrac{1}{\sqrt{b(\alpha+\beta)}} e^{-b(x-L)} & (L < x < \infty), \\ g_0(x) = \dfrac{1}{\sqrt{b(\alpha+\beta)}} \Big[\mathrm{ch}(a(-x+L)) + a^{-1}b\,\mathrm{sh}(a(-x+L))\Big] & \\ & (-L < x < L), \\ g_-(x) = \dfrac{1}{\sqrt{b(\alpha+\beta)}} \Big[\alpha\,\mathrm{ch}(b(-x-L)) + \beta\,\mathrm{sh}(b(-x-L))\Big] & \\ & (-\infty < x < -L). \end{cases}$$

*Here, we use constants*

$$(1.7)\quad \alpha = \mathrm{ch}(2aL) + a^{-1}b\,\mathrm{sh}(2aL) > 0,$$

$$(1.8)\quad \beta = \mathrm{ch}(2aL) + ab^{-1}\,\mathrm{sh}(2aL) > 0,$$

$$(1.9)\quad \alpha + \beta = 2\mathrm{ch}(2aL) + \big(ab^{-1} + a^{-1}b\big)\,\mathrm{sh}(2aL) > 0,$$

$$(1.10)\quad \alpha - \beta = -\frac{a^2-b^2}{ab}\mathrm{sh}(2aL) \begin{cases} < 0 & (0 < b < a < \infty), \\ = 0 & (a = b), \\ > 0 & (0 < a < b < \infty). \end{cases}$$

In order to present the Sobolev inequality, we introduce the Sobolev space

$$H = \Big\{ u \,\Big|\, u,\ u' \in L^2(\mathbb{R}) \Big\},$$

the Sobolev inner product

$$(u,v)_H = \int_{-\infty}^{\infty} \Big[ u'(x)\,\overline{v}'(x) \ + \ q(x)\,u(x)\,\overline{v}(x) \Big] dx,$$

and the Sobolev energy

$$\|\, u \,\|_H^2 = \int_{-\infty}^{\infty} \Big[\, \big|\, u'(x) \,\big|^2 \ + \ q(x)\,|\, u(x) \,|^2 \,\Big] dx.$$

**Theorem 1.2.** *There exists a positive constant $C$ such that for any $u \in H$ the Sobolev inequality*

$$(1.11)\quad \left( \sup_{y\in\mathbb{R}} |u(y)| \right)^2 \le C\, \|u\|_H^2$$

*holds. Among such $C$ the best constant $C_0$ is*

$$(1.12)\quad C_0 = \sup_{y\in\mathbb{R}} G(y,y) =$$

$$\begin{cases} G(\pm\infty, \pm\infty) = \dfrac{1}{2b} & (0 < b < a < \infty), \\ G(0,0) = \dfrac{(a\,\mathrm{ch}(aL) + b\,\mathrm{sh}(aL))^2}{a\,(2ab\,\mathrm{ch}(2aL) + (a^2+b^2)\mathrm{sh}(2aL))} & (0 < a \le b < \infty). \end{cases}$$

*If we replace $C$ by $C_0$ in the above inequality* (1.11), *the equality holds if and only if the constant multiple of the best function*

$$(1.13)\quad u(x) = \begin{cases} \textit{no existence} & (0 < b < a < \infty) \\ G(|x|, 0) & (0 < a \le b < \infty) \end{cases} \qquad (x \in \mathbb{R}),$$

$$(1.14)\quad G(|x|, 0) = \frac{\mathrm{ch}(aL) + a^{-1}b\,\mathrm{sh}(aL)}{b(\alpha+\beta)} \times \begin{cases} e^{-b(|x|-L)} & (|x| \ge L), \\ \mathrm{ch}(a(-|x|+L)) + a^{-1}b\,\mathrm{sh}(a(-|x|+L)) & (|x| \le L). \end{cases}$$

The engineering meaning of the Sobolev inequality is that the square of the maximum bending of a string is estimated from the above by the constant multiple of the potential energy. Among these constants, the best constant is the maximum of the diagonal value of the Green function.

This paper is organized as follows. In section 2, we prove Theorem 1.1 concerning the uniqueness of the solution of BVP. In section 3, we prove Theorem 1.1 concerning the existence of the solution of BVP. In section 4, we prove Theorem 1.2. In section 5, we show the relations of the Green functions corresponding to some boundary value problems.

## 2. The uniqueness of the solution of BVP

In this section, we prove Theorem 1.1 concerning the uniqueness of the solution of BVP. For the proof of Theorem 1.1, we prepare Lemma 2.1 to 2.3.

**Lemma 2.1.** *The ordinary differential equation*

$$(2.1)\quad -u'' + b^2 u = f(x) \qquad (L < x < \infty)$$

*has a solution and a boundary value as*

$$(2.2)\quad u(x) = \frac{1}{2b}(bu - u')(L+0)e^{-b(x-L)} + \int_L^\infty \frac{1}{2b} e^{-b|x-y|} f(y)dy \qquad (L < x < \infty),$$

$$(2.3)\quad (bu + u')(L+0) = \int_L^\infty e^{-b(y-L)} f(y)dy.$$

**Lemma 2.2.** *The ordinary differential equation*

$$(2.4)\quad -u'' + b^2 u = f(x) \qquad (-\infty < x < -L)$$

*has a solution and a boundary value as*

$$(2.5)\quad u(x) = \frac{1}{2b}(bu + u')(-L-0)e^{b(x+L)} + \int_{-\infty}^{-L} \frac{1}{2b} e^{-b|x-y|} f(y)dy \qquad (-\infty < x < -L),$$

$$(2.6)\quad (bu-u')(-L-0)=\int_{-\infty}^{-L} e^{b(y+L)}f(y)dy.$$

**Lemma 2.3.** *The ordinary differential equation*

$$(2.7)\quad -u''+a^2u=f(x) \qquad (-L<x<L)$$

*has a solution*

$$(2.8)\quad u(x)=A(x)(bu+u')(L-0)+B(x)(bu-u')(-L+0)+ \int_{-L}^{L} G_0(x,y)f(y)dy \qquad (-L<x<L),$$

*where $A(x)$, $B(x)$, and $G_0(x,y)$ are the fundamental solutions as*

$$(2.9)\quad D=b(\alpha+\beta)=b\Big[2\mathrm{ch}(2aL)+(a^{-1}b+ab^{-1})\mathrm{sh}(2aL)\Big]>0,$$

$$(2.10)\quad A(x)=\frac{1}{\sqrt{D}}g_0(-x),$$

$$(2.11)\quad B(x)=\frac{1}{\sqrt{D}}g_0(x),$$

$$(2.12)\quad G_0(x,y)=DA(x\wedge y)B(x\vee y)=g_0(x\vee y)\,g_0(-x\wedge y).$$

**Proof of Lemma 2.1** Applying (2.1) to the new functions $u_0=u$ and $u_1=u'$, we have

$$(2.13)\quad \begin{cases} u_0'=u_1 \\ u_1'=b^2u_0-f(x) \end{cases} \quad\Leftrightarrow\quad \begin{pmatrix} u_0\\ u_1\end{pmatrix}'=\begin{pmatrix}0&1\\ b^2&0\end{pmatrix}\begin{pmatrix}u_0\\ u_1\end{pmatrix}-\begin{pmatrix}0\\ 1\end{pmatrix}f(x).$$

Let

$$\boldsymbol{u}=\begin{pmatrix}u_0\\ u_1\end{pmatrix},\qquad \boldsymbol{B}=\begin{pmatrix}0&1\\ b^2&0\end{pmatrix},\qquad \boldsymbol{e}_1=\begin{pmatrix}0\\ 1\end{pmatrix}.$$

We rewrite (2.13) as

$$(2.14)\quad \boldsymbol{u}'=\boldsymbol{B}\boldsymbol{u}-\boldsymbol{e}_1f(x) \qquad (L<x<\infty).$$

The eigenvalues $b_i$ $(i=0,1)$ of $\boldsymbol{B}$ are $b_0=b,\ b_1=-b$. Moreover, ${}^t(1,b_i)$ $(i=0,1)$ are eigenvectors of $\boldsymbol{B}$ corresponding to $b_i$ $(i=0,1)$. Using

$$\boldsymbol{W}=\begin{pmatrix}1&1\\ b_0&b_1\end{pmatrix}=\begin{pmatrix}1&1\\ b&-b\end{pmatrix},\qquad \boldsymbol{W}^{-1}=\frac{1}{2b}\begin{pmatrix}b&1\\ b&-1\end{pmatrix},$$

$$\widehat{\boldsymbol{B}}=\begin{pmatrix}b_0&0\\ 0&b_1\end{pmatrix}=\begin{pmatrix}b&0\\ 0&-b\end{pmatrix},$$

we have the Jordan canonical form $\boldsymbol{B}=\boldsymbol{W}\widehat{\boldsymbol{B}}\boldsymbol{W}^{-1}$. We define $\boldsymbol{v}$ and $\boldsymbol{d}$ as

$$(2.15)\quad \boldsymbol{v}=\boldsymbol{W}^{-1}\boldsymbol{u} \qquad\Leftrightarrow\qquad \begin{pmatrix}v_0\\ v_1\end{pmatrix}=\frac{1}{2b}\begin{pmatrix}bu+u'\\ bu-u'\end{pmatrix},$$

$$\boldsymbol{d} = \boldsymbol{W}^{-1}\boldsymbol{e}_1 \qquad \Leftrightarrow \qquad \begin{pmatrix} d_0 \\ d_1 \end{pmatrix} = \frac{1}{2b}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Using $\boldsymbol{u} = \boldsymbol{W}\boldsymbol{v}$ and $\boldsymbol{e}_1 = \boldsymbol{W}\boldsymbol{d}$, we rewrite (2.14) as

$$\boldsymbol{v}' = \widehat{\boldsymbol{B}}\boldsymbol{v} - \boldsymbol{d}f(x) \qquad (L < x < \infty),$$

$$\Leftrightarrow \quad v_i' = b_i v_i - d_i f(x) \qquad (i = 0, 1,\ L < x < \infty).$$

Solving this differential equation, we have

$$v_i(x) = v_i(L)e^{b_i(x-L)} - \int_L^x d_i e^{b_i(x-y)} f(y)dy \qquad (i = 0, 1,\ L < x < \infty). \tag{2.16}$$

Here, because $u_i$ $(i = 0, 1)$ are bounded, $v_i$ $(i = 0, 1)$ are also bounded from (2.15). We consider the case $i = 0$ in (2.16). Taking the limit $x \to +\infty$ on both sides of

$$e^{-b_0 x} v_0(x) = e^{-b_0 L} v_0(L) - \int_L^x d_0 e^{-b_0 y} f(y)dy,$$

we have

$$0 = e^{-b_0 L} v_0(L) - \int_L^\infty d_0 e^{-b_0 y} f(y)dy.$$

Subtracting the above two relations, we have

$$v_0(x) = \int_x^\infty d_0 e^{b_0(x-y)} f(y)dy \qquad (L < x < \infty).$$

In the case of $i = 1$ in (2.16), we have

$$v_1(x) = v_1(L)e^{b_1(x-L)} - \int_L^x d_1 e^{b_1(x-y)} f(y)dy \qquad (L < x < \infty).$$

Taking a sum of the above relations, we have

$$\begin{aligned} &u(x) = u_0(x) = v_0(x) + v_1(x) = \\ &v_1(L)e^{b_1(x-L)} - \int_L^x d_1 e^{b_1(x-y)} f(y)dy + \int_x^\infty d_0 e^{b_0(x-y)} f(y)dy = \\ &v_1(L)e^{-b(x-L)} + \int_L^\infty \frac{1}{2b} e^{-b|x-y|} f(y)dy \qquad (L < x < \infty). \end{aligned}$$

Applying $v_1(L) = (bu - u')(L)/(2b)$ from (2.15) to the above $u(x)$, we have (2.2). Applying $x = L$ to (2.2), we have (2.3). This completes the proof of Lemma 2.1. ■

**Proof of Lemma 2.2** Applying

$$u(x)\Big|_{x=-\xi} = v(\xi)\Big|_{\xi=-x}, \qquad f(x)\Big|_{x=-\xi} = g(\xi)\Big|_{\xi=-x}$$

to (2.1) to (2.3) in Lemma 2.1, we have (2.4) to (2.6). We have proved Lemma 2.2. ■

**Proof of Lemma 2.3** Applying (2.7) to the new functions $u_0 = u$ and $u_1 = u'$, we have

$$\left\{ \begin{array}{l} u_0' = u_1 \\ u_1' = a^2 u_0 - f(x) \end{array} \right. \quad \Leftrightarrow \quad \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ a^2 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x).$$

If we put

$$\boldsymbol{u} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \qquad \boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ a^2 & 0 \end{pmatrix}, \qquad \boldsymbol{e}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

then we have

$$\text{(2.17)} \quad \boldsymbol{u}' = \boldsymbol{A}\boldsymbol{u} - \boldsymbol{e}_1 f(x) \qquad (-L < x < L).$$

The eigenvalues $a_i$ $(i = 0, 1)$ of $\boldsymbol{A}$ are $a_0 = a$, $a_1 = -a$. Moreover, ${}^t(1, a_i)$ $(i = 0, 1)$ are eigenvectors of $\boldsymbol{A}$ corresponding to $a_i$ $(i = 0, 1)$. Using

$$\boldsymbol{W} = \begin{pmatrix} 1 & 1 \\ a_0 & a_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ a & -a \end{pmatrix}, \qquad \boldsymbol{W}^{-1} = \frac{1}{2a} \begin{pmatrix} a & 1 \\ a & -1 \end{pmatrix},$$

$$\widehat{\boldsymbol{A}} = \begin{pmatrix} a_0 & 0 \\ 0 & a_1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix},$$

we have the Jordan canonical form $\boldsymbol{A} = \boldsymbol{W}\widehat{\boldsymbol{A}}\boldsymbol{W}^{-1}$.

We introduce $K_0(x)$ and its successive derivatives $K_j(x) = K_0^{(j)}(x)$ $(x \in \mathbb{R})$ as

$$K_0(x) = a^{-1}\mathrm{sh}(ax), \qquad K_1(x) = \mathrm{ch}(ax), \qquad K_2(x) = a\,\mathrm{sh}(ax).$$

The initial values $K_j(0)$ are as

$$K_0(0) = 0, \qquad K_1(0) = 1, \qquad K_2(0) = 0.$$

$K_j(x)$ satisfy $-K_j''(x) + a^2 K_j(x) = 0$ and $K_j(-x) = (-1)^{j+1} K_j(x)$. We introduce the fundamental solution matrix

$$\boldsymbol{E}(x) = \exp(\boldsymbol{A}x) = \exp(\boldsymbol{W}\widehat{\boldsymbol{A}}\boldsymbol{W}^{-1}x) = \boldsymbol{W}\exp(\widehat{\boldsymbol{A}}x)\boldsymbol{W}^{-1} =$$
$$\begin{pmatrix} \mathrm{ch}(ax) & a^{-1}\mathrm{sh}(ax) \\ a\,\mathrm{sh}(ax) & \mathrm{ch}(ax) \end{pmatrix} = \begin{pmatrix} K_1 & K_0 \\ K_2 & K_1 \end{pmatrix}(x) =$$
$$\begin{pmatrix} K_0 & K_1 \\ K_1 & K_2 \end{pmatrix}(x) \begin{pmatrix} K_0 & K_1 \\ K_1 & K_2 \end{pmatrix}^{-1}(0) \qquad (x \in \mathbb{R})$$

which satisfies the initial value problem and additional formula

$$\left\{ \begin{array}{l} \boldsymbol{E}'(x) = \boldsymbol{A}\boldsymbol{E}(x) \\ \boldsymbol{E}(0) = \boldsymbol{I} \end{array} \right., \qquad \boldsymbol{E}(x+y) = \boldsymbol{E}(x)\boldsymbol{E}(y) \qquad (x, y \in \mathbb{R}),$$

where $\boldsymbol{I}$ is a unit matrix.

Solving the differential equation (2.17), we have

$$\text{(2.18)}\quad \boldsymbol{u}(x) = \boldsymbol{E}(x+L)\boldsymbol{u}(-L) - \int_{-L}^{x} \boldsymbol{E}(x-y)\boldsymbol{e}_1 f(y)dy \qquad \Leftrightarrow$$

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}(x) = \begin{pmatrix} K_1 & K_0 \\ K_2 & K_1 \end{pmatrix}(x+L)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}(-L) - \int_{-L}^{x} \begin{pmatrix} K_0 \\ K_1 \end{pmatrix}(x-y)f(y)dy.$$

Substituting $x = L$ into (2.18), we have

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}(L) = \begin{pmatrix} K_1 & K_0 \\ K_2 & K_1 \end{pmatrix}(2L)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}(-L) - \int_{-L}^{L} \begin{pmatrix} K_0 \\ K_1 \end{pmatrix}(L-y)f(y)dy.$$

For (2.3) and (2.6), we have

$$\begin{pmatrix} (b\,u_0 + u_1)(L) \\ (b\,u_0 - u_1)(-L) \end{pmatrix} = \begin{pmatrix} bK_1 + K_2 & bK_0 + K_1 \\ b & -1 \end{pmatrix}(2L)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}(-L) -$$
$$\int_{-L}^{L} (bK_0 + K_1)(L-y)\begin{pmatrix} 1 \\ 0 \end{pmatrix} f(y)dy.$$

From

$$\det\begin{pmatrix} bK_1 + K_2 & bK_0 + K_1 \\ b & -1 \end{pmatrix}(2L) = -(b^2K_0 + 2bK_1 + K_2)(2L) =$$
$$-\,b\Big[2\mathrm{ch}(2aL) + (a^{-1}b + ab^{-1})\mathrm{sh}(2aL)\Big] = -b(\alpha+\beta) = -D < 0,$$

we have

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}(-L) = \begin{pmatrix} bK_1 + K_2 & bK_0 + K_1 \\ b & -1 \end{pmatrix}^{-1}(2L)\Big\{$$
$$\begin{pmatrix} (b\,u_0 + u_1)(L) \\ (b\,u_0 - u_1)(-L) \end{pmatrix} + \int_{-L}^{L} (bK_0 + K_1)(L-y)\begin{pmatrix} 1 \\ 0 \end{pmatrix} f(y)dy\Big\}.$$

Substituting this ${}^t(u_0, u_1)(-L)$ into $u(x) = u_0(x)$ in (2.18) and introducing the following functions

$$\begin{pmatrix} A & B \end{pmatrix}(x) = \begin{pmatrix} K_1 & K_0 \end{pmatrix}(L+x)\begin{pmatrix} bK_1 + K_2 & bK_0 + K_1 \\ b & -1 \end{pmatrix}^{-1}(2L),$$
$$G_0(x,y) = A(x)(bK_0 + K_1)(L-y) - Y(x-y)K_0(x-y),$$

we have (2.8). Here, $Y(x) = 1$ $(0 \le x < \infty)$, $0$ $(-\infty < x < 0)$ is the Heaviside step function. Applying (2.7) and (2.8) in Lemma 2.3 to $f(x) = 0$, we have the fact that the homogeneous equation

$$\text{(2.19)}\quad -u'' + a^2u = 0 \qquad (-L < x < L)$$

has a unique solution as

$$\text{(2.20)}\quad u(x) = A(x)(bu + u')(L) + B(x)(bu - u')(-L) \qquad (-L < x < L).$$

The function $u(x) = (b\,K_0 + K_1)(L + x)$ satisfies (2.19). From (2.20), $u(x) = (b\,K_0 + K_1)(L + x)$ is expressed as

$$(b\,K_0 + K_1)(L + x) =$$
$$A(x)\Big(b(b\,K_0 + K_1) + (b\,K_1 + K_2)\Big)(2L) +$$
$$B(x)\Big(b(b\,K_0 + K_1) - (b\,K_1 + K_2)\Big)(0) = D\,A(x),$$

where $D$ is

$$D = (b^2 K_0 + 2bK_1 + K_2)(2L) =$$
$$b\Big[2\mathrm{ch}(2aL) + (a^{-1}b + ab^{-1})\mathrm{sh}(2aL)\Big] = b(\alpha + \beta).$$

So we have (2.9). The fundamental solution $A(x)$ is

$$A(x) = D^{-1}(b\,K_0 + K_1)(L + x) =$$
$$\frac{1}{D}\Big[\mathrm{ch}(a(x + L)) + a^{-1}b\,\mathrm{sh}(a(x + L))\Big] = \frac{1}{\sqrt{D}} g_0(-x).$$

So we have (2.10). The function $u(x) = (b\,K_0 + K_1)(L - x)$ satisfies (2.19). From (2.20), $u(x) = (b\,K_0 + K_1)(L - x)$ is expressed as

$$(b\,K_0 + K_1)(L - x) =$$
$$A(x)\Big(b(b\,K_0 + K_1) - (b\,K_1 + K_2)\Big)(0) +$$
$$B(x)\Big(b(b\,K_0 + K_1) + (b\,K_1 + K_2)\Big)(2L) = D\,B(x).$$

The fundamental solution $B(x)$ is

$$B(x) = D^{-1}(b\,K_0 + K_1)(L - x) = A(-x).$$

So we have (2.11). The function $u(x) = K_0(x - y)$ satisfies (2.19). From (2.20), $u(x) = K_0(x - y)$ is expressed as

$$K_0(x - y) =$$
$$A(x)(bK_0 + K_1)(L - y) + B(x)(bK_0 - K_1)(-L - y) =$$
$$A(x)(bK_0 + K_1)(L - y) - B(x)(bK_0 + K_1)(L + y) =$$
$$D\Big[A(x)B(y) - B(x)A(y)\Big].$$

Using the $K_0(x - y)$, we have (2.12) as

$$G_0(x, y) = A(x)(bK_0 + K_1)(L - y) - Y(x - y)K_0(x - y) =$$
$$A(x)DB(y) - Y(x - y)D\Big[A(x)B(y) - B(x)A(y)\Big] =$$
$$D\Big[(1 - Y(x - y))A(x)B(y) + Y(x - y)B(x)A(y)\Big] =$$

$$D\Big[Y(y-x)A(x)B(y)+Y(x-y)B(x)A(y)\Big]=DA(x\wedge y)B(x\vee y).$$

This completes of the proof of Lemma 2.3. ■

**Proof of Theorem 1.1 (The uniqueness of the solution of** BVP**)** Using Lemma 2.1 to 2.3, we can show the uniqueness of the solution of BVP.

Using the connecting conditions $u^{(i)}(\pm L+0)=u^{(i)}(\pm L-0)$ $(i=0,1)$ of BVP, (2.3), and (2.6), we have the following relations

$$\begin{aligned}
&(bu+u')(L-0)=(bu+u')(L+0)=\\
&\int_L^\infty e^{-b(y-L)}f(y)dy=\int_L^\infty \sqrt{D}\,g_+(y)f(y)dy,\\
&(bu-u')(-L+0)=(bu-u')(-L-0)=\\
&\int_{-\infty}^{-L} e^{b(y+L)}f(y)dy=\int_{-\infty}^{-L} \sqrt{D}\,g_+(-y)f(y)dy.
\end{aligned}$$

Applying these relations to (2.8), we have the solution (1.1) $(-L<x<L)$ and the Green function (1.4) as

$$G(x,y)=$$
$$\begin{cases}
\sqrt{D}\,A(x)\,g_+(y)=g_+(y)\,g_0(-x) & (-L<x<L,\ L<y<\infty),\\
G_0(x,y)=g_0(x\vee y)\,g_0(-x\wedge y) & (-L<x<L,\ -L<y<L),\\
\sqrt{D}\,B(x)\,g_+(-y)=g_0(x)\,g_+(-y) & (-L<x<L,\ -\infty<y<-L).
\end{cases}$$

We consider the solution (1.1) $(-\infty<x<-L,\ L<x<\infty)$. If we prepare

$$G(x,y)=\begin{cases}
g_+(y)\,g_0(-x) & (-L<x<L<y<\infty),\\
g_0(y)\,g_0(-x) & (-L<x<y<L),\\
g_0(x)\,g_0(-y) & (-L<y<x<L),\\
g_0(x)\,g_+(-y) & (-\infty<y<-L<x<L),
\end{cases}$$

then we obtain

$$\partial_x G(x,y)=\begin{cases}
-g_+(y)\,g_0'(-x) & (-L<x<L<y<\infty),\\
-g_0(y)\,g_0'(-x) & (-L<x<y<L),\\
g_0'(x)\,g_0(-y) & (-L<y<x<L),\\
g_0'(x)\,g_+(-y) & (-\infty<y<-L<x<L).
\end{cases}$$

Hence we have

$$\Big(b\,G(x,y)-\partial_x G(x,y)\Big)\Big|_{x=L-0}=$$
$$\begin{cases} g_+(y)\Big(b\,g_0(-L)+g_0'(-L)\Big) & (L<y<\infty),\\ g_0(-y)\Big(b\,g_0(L)-g_0'(L)\Big) & (-L<y<L),\\ g_+(-y)\Big(b\,g_0(L)-g_0'(L)\Big) & (-\infty<y<-L),\end{cases}$$

and

$$\Big(b\,G(x,y)+\partial_x G(x,y)\Big)\Big|_{x=-L+0}=$$
$$\begin{cases} g_+(y)\Big(b\,g_0(L)-g_0'(L)\Big) & (L<y<\infty),\\ g_0(y)\Big(b\,g_0(L)-g_0'(L)\Big) & (-L<y<L),\\ g_+(-y)\Big(b\,g_0(-L)+g_0'(-L)\Big) & (-\infty<y<-L).\end{cases}$$

We prepare

$$g_0(x)=\frac{1}{\sqrt{D}}\Big[\mathrm{ch}(a(-x+L))+a^{-1}b\,\mathrm{sh}(a(-x+L))\Big]\qquad(-L<x<L),$$
$$g_0'(x)=-\frac{1}{\sqrt{D}}\Big[b\,\mathrm{ch}(a(-x+L))+a\,\mathrm{sh}(a(-x+L))\Big]\qquad(-L<x<L),$$

and the constants

$$b\,g_0(-L)+g_0'(-L)=\frac{b(\alpha-\beta)}{\sqrt{D}},\qquad b\,g_0(L)-g_0'(L)=\frac{2b}{\sqrt{D}}.$$

Using these constants, we have

$$\Big(b\,G(x,y)-\partial_x G(x,y)\Big)\Big|_{x=L-0}=$$
$$\frac{1}{\sqrt{D}}\begin{cases} b(\alpha-\beta)\,g_+(y) & (L<y<\infty),\\ 2b\,g_0(-y) & (-L<y<L),\\ 2b\,g_+(-y) & (-\infty<y<-L),\end{cases}$$

and

$$\Big(b\,G(x,y)+\partial_x G(x,y)\Big)\Big|_{x=-L+0}=$$
$$\frac{1}{\sqrt{D}}\begin{cases} 2b\,g_+(y) & (L<y<\infty),\\ 2b\,g_0(y) & (-L<y<L),\\ b(\alpha-\beta)\,g_+(-y) & (-\infty<y<-L).\end{cases}$$

Hence we have

$$(2.21)\quad \frac{1}{2b}(bu-u')(L+0)e^{-b(x-L)} = (bu-u')(L-0)\,\frac{\sqrt{D}}{2b}g_+(x) =$$
$$\int_{-\infty}^{\infty}\Big(b\,G(x,y)-\partial_x G(x,y)\Big)\Big|_{x=L-0} f(y)dy\,\frac{\sqrt{D}}{2b}g_+(x) =$$
$$\int_L^{\infty}\frac{\alpha-\beta}{2}g_+(x)\,g_+(y)f(y)dy+\int_{-L}^{L}g_+(x)\,g_0(-y)f(y)dy+$$
$$\int_{-\infty}^{-L}g_+(x)\,g_+(-y)f(y)dy$$

and

$$(2.22)\quad \frac{1}{2b}(bu+u')(-L-0)e^{b(x+L)} = (bu+u')(-L+0)\,\frac{\sqrt{D}}{2b}g_+(-x) =$$
$$\int_{-\infty}^{\infty}\Big(b\,G(x,y)+\partial_x G(x,y)\Big)\Big|_{x=-L+0} f(y)dy\,\frac{\sqrt{D}}{2b}g_+(-x) =$$
$$\int_L^{\infty}g_+(-x)\,g_+(y)f(y)dy+\int_{-L}^{L}g_+(-x)\,g_0(y)f(y)dy+$$
$$\int_{-\infty}^{-L}\frac{\alpha-\beta}{2}g_+(-x)\,g_+(-y)f(y)dy.$$

We note that

$$(2.23)\quad \frac{1}{2b}e^{-b|x-y|}+\frac{\alpha-\beta}{2}g_+(x)g_+(y) = \frac{1}{2b}\left[e^{-b|x-y|}+\frac{\alpha-\beta}{\alpha+\beta}e^{b(2L-x-y)}\right] =$$
$$\frac{1}{2D}\Big[(\alpha+\beta)e^{-b(x\vee y-x\wedge y)}+(\alpha-\beta)e^{b(2L-x\vee y-x\wedge y)}\Big] =$$
$$\frac{1}{2D}e^{-b(x\vee y-L)}\Big[(\alpha+\beta)e^{b(x\wedge y-L)}+(\alpha-\beta)e^{-b(x\wedge y-L)}\Big] =$$
$$\frac{1}{D}e^{-b(x\vee y-L)}\Big[\alpha\,\mathrm{ch}(b(x\wedge y-L))+\beta\,\mathrm{sh}(b(x\wedge y-L))\Big] =$$
$$g_+(x\vee y)\,g_-(-x\wedge y)$$

and

$$(2.24)\quad \frac{1}{2b}e^{-b|x-y|}+\frac{\alpha-\beta}{2}g_+(-x)g_+(-y) =$$
$$g_+((-x)\vee(-y))\,g_-(-(-x)\wedge(-y)) = g_+(-x\wedge y)\,g_-(x\vee y).$$

Substituting (2.2) into (2.21) and (2.23), we have the solution (1.1) ($L < x < \infty$) and the Green function (1.3). Substituting (2.5) into (2.22) and (2.24), we have the solution (1.1) ($-\infty < x < -L$) and the Green function (1.5).

This completes the proof of Theorem 1.1 (The uniqueness of the solution of BVP). ■

## 3. The existence of the solution of BVP

In this section, we prove Theorem 1.1 concerning the existence of the solution of BVP. For the proof of this theorem, we prepare Lemma 3.1 and 3.2.

**Lemma 3.1.** *The fundamental solution* $g(x)$ *in* (1.6) *satisfies*

$$-g''(x)+q(x)g(x)=0 \qquad (x\in\mathbb{R}), \tag{3.1}$$

$$g^{(i)}(x) \quad \to \quad 0 \qquad (i=0,1, \quad x\to+\infty), \tag{3.2}$$

$$g^{(i)}(\pm L+0)=g^{(i)}(\pm L-0) \qquad (i=0,1), \tag{3.3}$$

$$g'(x)g(-x)+g(x)g'(-x)=-1 \qquad (x\in\mathbb{R}). \tag{3.4}$$

*The fundamental solution satisfies the differential equation* (3.1)*, the bounded condition* (3.2)*, the connecting condition* (3.3)*, and the jumping condition* (3.4)*.*

**Proof of Lemma 3.1** The fundamental solution (1.6) is derived by (3.1) to (3.4).

The ordinary differential equation (3.1) in $x\in(L,\infty)$ has a general solution given

$$g(x)=Ae^{b(x-L)}+Be^{-b(x-L)} \qquad (L<x<\infty),$$

where $A$ and $B$ are arbitrary constants. Using the bounded condition (3.2), we have $A=0$. By replacing $C$ with $B$, we have $g(x)$, $g'(x)$ and $g^{(i)}(L+0)$ as

$$\begin{cases} g(x)=Ce^{-b(x-L)} & (L<x<\infty),\\ g'(x)=-bC\,e^{-b(x-L)} & (L<x<\infty),\\ g(L+0)=C, \\ g'(L+0)=-b\,C. \end{cases} \tag{3.5}$$

The ordinary differential equation (3.1) in $x\in(-L,L)$ has a general solution given as

$$\begin{cases} g(x)=Ae^{a(x-L)}+Be^{-a(x-L)} & (-L<x<L),\\ g'(x)=aAe^{a(x-L)}-aBe^{-a(x-L)} & (-L<x<L),\\ g(L-0)=A+B, \\ g'(L-0)=a(A-B), \end{cases}$$

where $A$ and $B$ are arbitrary constants. By applying the connecting condition (3.3), we have

$$\begin{cases} C = g(L+0) = g(L-0) = A+B, \\ -b\,C = g'(L+0) = g'(L-0) = a(A-B), \end{cases} \Leftrightarrow \begin{cases} A = \dfrac{a-b}{2a}C, \\ B = \dfrac{a+b}{2a}C. \end{cases}$$

Hence $g(x)$ is given as

$$g(x) = \frac{a-b}{2a}Ce^{a(x-L)} + \frac{a+b}{2a}Ce^{-a(x-L)} = \frac{C}{2a}\Big[2a\,\mathrm{ch}(a(x-L)) - 2b\,\mathrm{sh}(a(x-L))\Big].$$

Thus, we have

$$\begin{cases} g(x) = C\Big[\mathrm{ch}(a(x-L)) - a^{-1}b\,\mathrm{sh}(a(x-L))\Big] & (-L<x<L), \\ g'(x) = C\Big[a\,\mathrm{sh}(a(x-L)) - b\,\mathrm{ch}(a(x-L))\Big] & (-L<x<L), \\ g(L-0) = C, \qquad g'(L-0) = -b\,C, \\ g(-L+0) = C\,\alpha, \qquad g'(-L+0) = -C\,b\,\beta, \end{cases} \tag{3.6}$$

where $\alpha$ and $\beta$ are (1.7) and (1.8).

The ordinary differential equation (3.1) in $x \in (-\infty, -L)$ has a general solution given as

$$g(x) = Ae^{b(x+L)} + Be^{-b(x+L)} \qquad (-\infty < x < -L),$$

where $A$ and $B$ are arbitrary constants. If we put $C_0 = A+B$ and $C_1 = A-B$, then we have

$$\begin{cases} g(x) = C_0\,\mathrm{ch}(b(x+L)) + C_1\,\mathrm{sh}(b(x+L)) & (-\infty<x<-L), \\ g'(x) = C_0\,b\,\mathrm{sh}(b(x+L)) + C_1\,b\,\mathrm{ch}(b(x+L)) & (-\infty<x<-L), \\ g(-L-0) = C_0, \\ g'(-L-0) = C_1\,b. \end{cases}$$

By applying the connecting condition (3.3), we have

$$\begin{cases} C_0 = g(-L-0) = g(-L+0) = \alpha\,C, \\ C_1\,b = g'(-L-0) = g'(-L+0) = -b\,\beta\,C, \end{cases} \Leftrightarrow \begin{cases} C_0 = \alpha\,C, \\ C_1 = -\beta\,C. \end{cases}$$

Thus we have

$$(3.7)\quad \begin{cases} g(x) = C\Big[\alpha\,\mathrm{ch}(b(x+L)) - \beta\,\mathrm{sh}(b(x+L))\Big] & (-\infty < x < -L), \\ g'(x) = C\,b\Big[\alpha\,\mathrm{sh}(b(x+L)) - \beta\,\mathrm{ch}(b(x+L))\Big] & (-\infty < x < -L), \\ g(-L-0) = C\,\alpha, \\ g'(-L-0) = -C\,b\,\beta. \end{cases}$$

From (3.5), (3.6) and (3.7), $g(x)$ is given as

$$(3.8)\quad g(x) = C \begin{cases} e^{-b(x-L)} & (L < x < \infty), \\ \mathrm{ch}(a(x-L)) - a^{-1}b\,\mathrm{sh}(a(x-L)) & (-L < x < L), \\ \alpha\,\mathrm{ch}(b(x+L)) - \beta\,\mathrm{sh}(b(x+L)) & (-\infty < x < -L). \end{cases}$$

From (3.4), we determine $C$. Substituting $x = 0$ into the jumping condition (3.4), we have

$$2g(0)g'(0) = -1.$$

Putting $x = 0$ into (3.6), we have

$$g(0) = C\Big[\mathrm{ch}(aL) + a^{-1}b\,\mathrm{sh}(aL)\Big],$$
$$g'(0) = -\,C\,b\Big[\mathrm{ch}(aL) + ab^{-1}\mathrm{sh}(aL)\Big].$$

Hence, we have

$$-\,1 = 2g(0)g'(0) = -b\,C^2(\alpha+\beta) \quad \Leftrightarrow \quad C = \frac{1}{\sqrt{b\,(\alpha+\beta)}} = \frac{1}{\sqrt{D}}.$$

Substituting this $C$ into (3.8), we have (1.6). This is the complete proof of Lemma 3.1. ■

**Lemma 3.2.** *The Green function $G(x, y)$ satisfies the following properties:*

$$(3.9)\quad G(x,y) = G(y,x) \qquad (x, y \in \mathbb{R}),$$

$$(3.10)\quad (-\partial_x^2 + q(x))G(x,y) = 0 \qquad (x, y \in \mathbb{R},\ x \neq y),$$

$$(3.11)\quad \partial_x^i G(x,y) : \text{bounded} \qquad (i = 0, 1,\ x \in \mathbb{R},\ \textit{For any } y \textit{ fixed}),$$

$$(3.12)\quad \partial_x^i G(x,y)\Big|_{x=\pm L+0} = \partial_x^i G(x,y)\Big|_{x=\pm L-0} \qquad (i = 0, 1,\ y \in \mathbb{R}),$$

$$(3.13)\quad \partial_x^i G(x,y)\Big|_{y=x-0} - \partial_x^i G(x,y)\Big|_{y=x+0} = \begin{cases} 0 & (i=0) \\ -1 & (i=1) \end{cases} \quad (x \in \mathbb{R}).$$

*The Green function satisfies the symmetry of variables* (3.9)*, the differential equation* (3.10)*, the bounded condition* (3.11)*, the connecting condition* (3.12)*, and the jumping condition* (3.13)*.*

**Proof of Lemma 3.2** The symmetry of variables (3.9) is obvious from (1.2). We rewrite (1.2) as

$$G(x,y)=\begin{cases} g(x)g(-y) & (y<x), \\ g(y)g(-x) & (x<y), \end{cases}$$

$$\partial_x G(x,y)=\begin{cases} g'(x)g(-y) & (y<x), \\ -g(y)g'(-x) & (x<y), \end{cases}$$

$$\partial_x^2 G(x,y)=\begin{cases} g''(x)g(-y) & (y<x), \\ g(y)g''(-x) & (x<y). \end{cases}$$

From (3.1), the differential equation (3.10) is given as

$$(-\partial_x^2+q(x))G(x,y)=$$
$$\left\{\begin{array}{ll} (-g''(x)+q(x)g(x))g(-y) & (y<x) \\ g(y)(-g''(-x)+q(x)g(-x)) & (x<y) \end{array}\right\}=0.$$

From (3.2), the bounded condition (3.11) follows from

$$\partial_x^i G(x,y)=\left\{\begin{array}{ll} g^{(i)}(x)g(-y) & (y<x) \\ (-1)^i g(y)g^{(i)}(-x) & (x<y) \end{array}\right\}\to$$
$$\left\{\begin{array}{ll} g^{(i)}(\infty)g(-y) & (x\to\infty) \\ (-1)^i g(y)g^{(i)}(\infty) & (x\to-\infty) \end{array}\right\}=0 \qquad (i=0,1).$$

From (3.3), the connecting condition (3.12) follows from

$$\partial_x^i G(x,y)\Big|_{x=\pm L+0}=$$
$$\left\{\begin{array}{ll} g^{(i)}(\pm L+0)g(-y)=g^{(i)}(\pm L-0)g(-y) & (y<x) \\ (-1)^i g(y)g^{(i)}(\mp L-0)=(-1)^i g(y)g^{(i)}(\mp L+0) & (x<y) \end{array}\right\}=$$
$$\partial_x^i G(x,y)\Big|_{x=\pm L-0} \qquad (i=0,1).$$

From (3.4), the jumping condition (3.13) follows from

$$\partial_x^i G(x,y)\Big|_{y=x-0}-\partial_x^i G(x,y)\Big|_{y=x+0}=$$
$$\begin{cases} g(x)g(-x)-g(x)g(-x)=0 & (i=0), \\ g'(x)g(-x)+g(x)g'(-x)=-1 & (i=1). \end{cases}$$

This completes the proof of Lemma 3.2. ■

**Proof of Theorem 1.1 (The existence of the solution of** BVP**)** We show the existence of the solution of BVP.

We rewrite (1.1) as

$$u(x) = \int_{-\infty}^{x} G(x,y)f(y)dy + \int_{x}^{\infty} G(x,y)f(y)dy.$$

Differentiating $u(x)$, we have

$$\begin{aligned}
&u'(x) = \\
&\int_{-\infty}^{x} \partial_x G(x,y)f(y)dy + \int_{x}^{\infty} \partial_x G(x,y)f(y)dy + \\
&\left\{ G(x,y)\Big|_{y=x-0} - G(x,y)\Big|_{y=x+0} \right\} f(x) = \\
&\int_{-\infty}^{x} \partial_x G(x,y)f(y)dy + \int_{x}^{\infty} \partial_x G(x,y)f(y)dy,
\end{aligned}$$

where we use $i = 0$ of (3.13). Differentiating $u'(x)$, we have

$$\begin{aligned}
&u''(x) = \\
&\int_{-\infty}^{x} \partial_x^2 G(x,y)f(y)dy + \int_{x}^{\infty} \partial_x^2 G(x,y)f(y)dy + \\
&\left\{ \partial_x G(x,y)\Big|_{y=x-0} - \partial_x G(x,y)\Big|_{y=x+0} \right\} f(x) = \\
&\int_{-\infty}^{x} \partial_x^2 G(x,y)f(y)dy + \int_{x}^{\infty} \partial_x^2 G(x,y)f(y)dy - f(x),
\end{aligned}$$

where we use $i = 1$ of (3.13). Therefore, we have

$$\begin{aligned}
&- u'' + q(x)u = \\
&\int_{-\infty}^{x} \left(-\partial_x^2 + q(x)\right) G(x,y)f(y)dy + \\
&\int_{x}^{\infty} \left(-\partial_x^2 + q(x)\right) G(x,y)f(y)dy + f(x) = f(x),
\end{aligned}$$

where we use (3.10). Using

$$u^{(i)}(x) = \int_{-\infty}^{\infty} \partial_x^i G(x,y)f(y)dy \qquad (i = 0, 1,\ x \in \mathbb{R}),$$

we have (3.11) and (3.12). This completes the proof of Theorem 1.1. ■

## 4. Sobolev inequality

In this section, we prove Theorem 1.2.

**Lemma 4.1.** *For any $u \in H$ and fixed $y \in \mathbb{R}$, the reproducing relation*

$$u(y) = (\, u(\cdot), G(\cdot, y)\,)_H, \tag{4.1}$$

$$G(y, y) = (\, G(\cdot, y), G(\cdot, y)\,)_H = \|\, G(\cdot, y)\,\|_H^2 \tag{4.2}$$

*holds.*

This means that the Green function $G(x, y)$ is a reproducing kernel for $H$ and $(\cdot, \cdot)_H$.

**Proof of Lemma 4.1** We put $v = v(x) = G(x, y) \in H$ and remark $\overline{v} = v$. Thus, we have

$$(u, v)_H = \int_{-\infty}^{\infty} \Big[ u'(x)v'(x) + q(x)u(x)v(x) \Big] dx =$$

$$\int_{-\infty}^{\infty} \Big[ \Big( u(x)v'(x) \Big)' - u(x)v''(x) + q(x)u(x)v(x) \Big] dx =$$

$$u'(x)\, v'(x) \left\{ \Big|_{x=-\infty}^{x=y-0} + \Big|_{x=y+0}^{x=\infty} \right\} +$$

$$\left\{ \int_{-\infty}^{y} + \int_{y}^{\infty} \right\} u(x) \Big( - v''(x) + q(x)v(x) \Big) dx.$$

Setting $u = u(x) \in H$ and using Lemma 3.2, we have (4.1). Applying $u(x) = G(x, y) \in H$ to (4.1), we have (4.2). This completes the proof of Lemma 4.1. ■

**Proof of Theorem 1.2** Applying the Schwarz inequality to (4.1) and using (4.2), we have

$$|\, u(y)\,|^2 \leq \|\, G(\cdot, y)\,\|_H^2 \;\|\, u\,\|_H^2 = G(y, y)\,\|\, u\,\|_H^2\,.$$

Noting

$$C_0 = \sup_{y \in \mathbb{R}} G(y, y) = G(y_0, y_0) \tag{4.3}$$

and taking the supremum with respect to $y \in \mathbb{R}$, we have the Sobolev inequality

$$\left( \sup_{y \in \mathbb{R}} |\, u(y)\,| \right)^2 \leq C_0 \;\|\, u\,\|_H^2\,. \tag{4.4}$$

This inequality shows that $(\cdot, \cdot)_H$ is a positive definite. In fact, $\|u\|_H = 0$ yields $u \equiv 0$ $(x \in \mathbb{R})$. Applying (4.4) to $u(x) = G(x, y_0) \in H$, we have

$$\left( \sup_{y \in \mathbb{R}} |\, G(y, y_0)\,| \right)^2 \leq C_0\,\|\, G(\cdot, y_0)\,\|_H^2 = C_0^2.$$

Combining this with the trivial inequality

$$C_0^2 = (G(y_0, y_0))^2 \le \left( \sup_{y \in \mathbb{R}} |\, G(y, y_0) \,| \right)^2 ,$$

we have

$$\left( \sup_{y \in \mathbb{R}} |\, G(y, y_0) \,| \right)^2 = C_0 \,\|\, G(x, y_0) \,\|_H^2 . \tag{4.5}$$

So we see that the supremum of the diagonal value of the Green function $G(y_0, y_0)$ is the best constant and the Green function $G(x, y_0)$ is the best function.

From (4.3), we can show the best constant (1.12). Using (1.2) to (1.5), we have the diagonal value of the Green function

$$G(y,y) = g(y)g(-y) = \begin{cases} g_+(y)\, g_-(-y) & (L < y < \infty), \\ g_0(y)\, g_0(-y) & (-L < y < L), \\ g_-(y)\, g_+(-y) & (-\infty < y < -L). \end{cases}$$

From $G(-y,-y) = G(y,y)$, $G(y,y)$ is an even function. So we consider $y \in (0, \infty)$ in the following. In the case of $L < y < \infty$, we have

$$G(y,y) = g_+(y)\, g_-(-y) =$$
$$\frac{1}{b(\alpha+\beta)} e^{-b(y-L)} \Big[ \alpha \,\mathrm{ch}(b(y-L)) + \beta \,\mathrm{sh}(b(y-L)) \Big] =$$
$$\frac{1}{2b} \left[ 1 + \frac{\alpha-\beta}{\alpha+\beta} e^{-2b(y-L)} \right] \qquad (L < y < \infty).$$

In the case of $0 < y < L$, we have

$$G(y,y) = g_0(y)\, g_0(-y) =$$
$$\frac{1}{b(\alpha+\beta)} \Big[ \mathrm{ch}(a(-y+L)) + a^{-1} b \,\mathrm{sh}(a(-y+L)) \Big] \times$$
$$\Big[ \mathrm{ch}(a(y+L)) + a^{-1} b \,\mathrm{sh}(a(y+L)) \Big] =$$
$$\frac{1}{b(\alpha+\beta)} \Big[ \mathrm{ch}(a(-y+L))\mathrm{ch}(a(y+L)) +$$
$$a^{-1} b \Big( \mathrm{sh}(a(-y+L))\mathrm{ch}(a(y+L)) + \mathrm{ch}(a(-y+L))\mathrm{sh}(a(y+L)) \Big) +$$
$$a^{-2} b^2 \,\mathrm{sh}(a(-y+L))\mathrm{sh}(a(y+L)) \Big] =$$
$$\frac{1}{2a^2 b(\alpha+\beta)} \Big[ \big(a^2+b^2\big) \,\mathrm{ch}(2aL) + 2ab \,\mathrm{sh}(2aL) + \big(a^2-b^2\big) \,\mathrm{ch}(2ay) \Big]$$
$$(0 < y < L).$$

Differentiating $G(y,y)$, we have

$$\frac{d}{dy}G(y,y)=\frac{1}{\alpha+\beta}\begin{cases}-(\alpha-\beta)e^{-2b(y-L)} & (L<y<\infty),\\ \dfrac{a^2-b^2}{ab}\mathrm{sh}(2ay) & (0<y<L).\end{cases}$$

Using (1.9) and (1.10), we have

$$\frac{d}{dy}G(y,y)\begin{cases}>0 & (0<b<a<\infty)\\ =0 & (b=a)\\ <0 & (0<a<b<\infty)\end{cases}\qquad (0<y<\infty).$$

Taking the supremum of $G(y,y)$ with respect to $y\in(0,\infty)$, we have

$$\sup_{0\le y<\infty} G(y,y)= \tag{4.6}$$
$$\begin{cases}G(\infty,\infty)=\dfrac{1}{2b} & (0<b\le a<\infty),\\ G(0,0)=(g_0(0))^2=\dfrac{(a\,\mathrm{ch}(aL)+b\,\mathrm{sh}(aL))^2}{a^2b(\alpha+\beta)} & (0<a<b<\infty).\end{cases}$$

Because $G(y,y)$ is an even function, (4.6) is equivalently the best constant (1.12).

From (1.2), we can show the best function (1.13). In the case of $0<b<a<\infty$, the best function $G(x,y_0)$ does not exist because

$$\lim_{y_0\to\infty}G(x,y_0)=\left(\lim_{y_0\to\infty}g(y_0)\right)g(-x)=0\qquad(x\in\mathbb{R}).$$

In the case of $0<a\le b<\infty$, because the function

$$G(x,0)=g(x\vee 0)g(-(x\wedge 0))=$$
$$\begin{cases}g_+(x)\,g_0(0) & (L<x<\infty),\\ g_0(x\vee 0)\,g_0(-x\wedge 0) & (-L<x<L),\\ g_0(0)\,g_+(-x) & (-\infty<x<-L)\end{cases}$$

is an even function, we have

$$G(x,0)=\begin{cases}g_+(x)\,g_0(0) & (L<x<\infty),\\ g_0(x)\,g_0(0) & (0<x<L),\end{cases}$$

that is,

$$G(x,0)=\frac{\mathrm{ch}(aL)+a^{-1}b\,\mathrm{sh}(aL)}{b(\alpha+\beta)}\times \tag{4.7}$$
$$\begin{cases}e^{-b(x-L)} & (L<x<\infty),\\ \mathrm{ch}(a(-x+L))+a^{-1}b\,\mathrm{sh}(a(-x+L)) & (0<x<L).\end{cases}$$

Because $G(x,0)$ is an even function, (4.7) is equivalently the best function (1.13).

This completes the proof of Theorem 1.2. ■

## 5. Examination

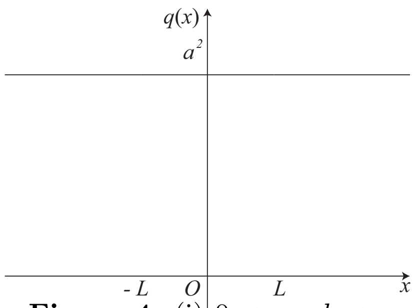


**Figure 4.** (i) $0 < a = b < \infty$.

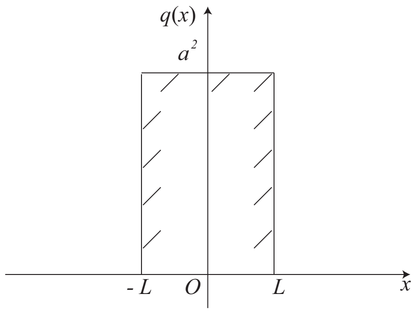


**Figure 5.** (ii) $0 = b < a < \infty$.

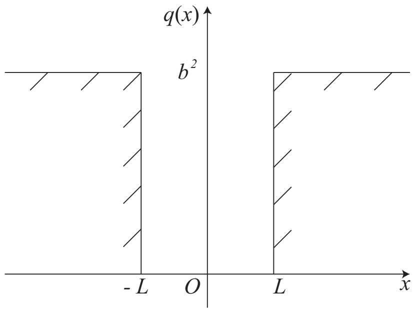


**Figure 6.** (iii) $0 = a < b < \infty$.

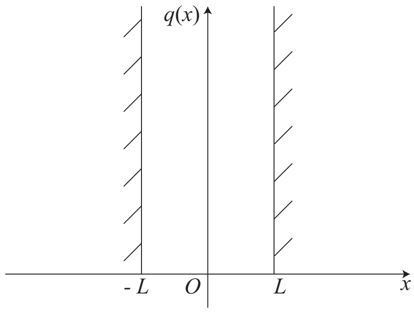


**Figure 7.** (iii) $0 = a < b \to \infty$.

(i) The case of $0 < a = b < \infty$ (Figure 4).

Taking limit as $b \to a$, we have

$$\alpha = \beta \quad \to \quad \mathrm{ch}(2aL) + \mathrm{sh}(2aL) = e^{2aL}$$

and

$$g(x) = \frac{1}{\sqrt{2a}} e^{-ax} \qquad (x \in \mathbb{R}).$$

Thus we have the Green function

$$G(x,y) = g(x \vee y) g(-x \wedge y) = \frac{1}{\sqrt{2a}} e^{-a|x-y|} \qquad (x, y \in \mathbb{R})$$

of the following boundary value problem

$$\begin{cases} -u'' + a^2 u = f(x) & (x \in \mathbb{R}), \\ u,\ u' \ : \ \text{bounded} & (x \in \mathbb{R}). \end{cases}$$

(ii) The case of $0 = b < a < \infty$ (Figure 5).
Taking limit as $b \to 0$, we have

$$\alpha \;\to\; \mathrm{ch}(2aL), \qquad \beta\,\mathrm{sh}(b(x+L)) \;\to\; a\,\mathrm{sh}(2aL)\,(x+L),$$
$$b(\alpha+\beta) \;\to\; a\,\mathrm{sh}(2aL)$$

and

$$g(x) =$$
$$\begin{cases} g_+(x) = \dfrac{1}{\sqrt{a\,\mathrm{sh}(2aL)}} & (L < x < \infty), \\ g_0(x) = \dfrac{1}{\sqrt{a\,\mathrm{sh}(2aL)}}\,\mathrm{ch}(a(-x+L)) & (-L < x < L), \\ g_-(x) = \dfrac{1}{\sqrt{a\,\mathrm{sh}(2aL)}}\Big[\mathrm{ch}(2aL) - a\,\mathrm{sh}(2aL)\,(x+L)\Big] & \\ & (-\infty < x < -L). \end{cases}$$

The Green function

$$G(x,y) =$$
$$\frac{1}{a\,\mathrm{sh}(2aL)}\mathrm{ch}(a(-x\vee y+L))\,\mathrm{ch}(a(x\wedge y+L)) \qquad (-L < x,y < L)$$

of the following boundary value problem

$$\begin{cases} -u'' + a^2 u = f(x) \qquad (-L < x < L), \\ u'(\pm L) = 0. \end{cases}$$

(iii) The case of $0 = a < b \to \infty$ (Figure 7).
Taking limit as $a \to 0$ (Figure 6), we have

$$\alpha \;\to\; 1+2bL, \qquad \beta \;\to\; 1, \qquad b(\alpha+\beta) \;\to\; 2b(1+bL)$$

and

$$g(x) =$$
$$\begin{cases} g_+(x) = \dfrac{1}{\sqrt{2b(1+bL)}}e^{-b(x-L)} & (L < x < \infty), \\ g_0(x) = \dfrac{1}{\sqrt{2b(1+bL)}}\Big[1 + b(L-x)\Big] & (-L < x < L), \\ g_-(x) = \dfrac{1}{\sqrt{2b(1+bL)}}\Big[(1+2bL)\,\mathrm{ch}(b(-x-L)) + \mathrm{sh}(b(-x-L))\Big] & \\ & (-\infty < x < -L). \end{cases}$$

Taking limit as $b \to \infty$, we have the Green function

$$G(x,y) = g_0(x \vee y) g_0(-x \wedge y) =$$

$$\frac{1}{2L}\left(-x \vee y+L\right)\left(x \wedge y+L\right) \qquad (-L<x,y<L)$$

of the following boundary value problem

$$\begin{cases} -u''=f(x) & (-L<x<L), \\ u(\pm L)=0. \end{cases}$$

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## References

[1] Y. Kametaka, A. Nagai, K. Watanabe, K. Takemura and H. Yamagishi, *Giambelli's formula and the best constant of Sobolev inequality in one dimensional Euclidean space*, Sci. Math. Jpn., **e-2009** (2009), 621–635.

[2] Y. Kametaka, A. Nagai and H. Yamagishi, *Green function*, Shokabo (2022) [in Japanese].

[3] Y. Kametaka, K. Takemura, H. Yamagishi, A. Nagai and K. Watanabe, *Heaviside cable, Thomson cable and the best constant of a Sobolev-type inequality*, Sci. Math. Jpn., **e-2007** (2007), 739–755.

[4] Y. Kametaka, K. Watanabe and A. Nagai, *The best constant of Sobolev inequality in an n dimensional Euclidean space*, Proc. Japan Acad., **81**, Ser.A, No.3 (2005), 57–60.

[5] Y. Kametaka, K. Watanabe, A. Nagai and S. Pyatkov, *The best constant of Sobolev inequality in an n dimensional Euclidean space*, Sci. Math. Jpn., **e-2004** (2004), 295–303.

[6] Y. Kametaka, K. Watanabe, A. Nagai, K. Takemura and H. Yamagishi, *Positivity and Hierarchical Structure of four Green Functions Corresponding to a Bending Problem of a Beam on a half line*, Math. J. Okayama Univ., **65** (2023) 145–173.

[7] Y. Kametaka, K. Watanabe, A. Nagai, H. Yamagishi and K. Takemura, *The best constant of Sobolev inequality which correspond to a bending problem of a string with periodic boundary condition*, Sci. Math. Jpn., **e-2007** (2007), 283–300.

[8] Y. Kametaka, H. Yamagishi, K. Watanabe, A. Nagai, K. Takemura and M. Arai, *The best constant of Sobolev inequality which corresponds to a Schrödinger operator with Dirac delta potential*, Sci. Math. Jpn., **e-2008** (2008), 541–555.

[9] A. Nagai, Y. Kametaka, H. Yamagishi, K. Takemura and K. Watanabe, *Discrete Bernoulli polynomials and the best constant of discrete Sobolev inequality*, Funkcial. Ekvac., **51** (2008), 307–327.

[10] K. Takemura, Y. Kametaka, K. Watanabe, A. Nagai and H. Yamagishi, *The best constant of Sobolev inequality corresponding to a bending problem of a beam on a half line*, Far East J. Appl. Math., **51** No.1 (2011), 45–71.

[11] K. Takemura, Y. Kametaka, K. Watanabe, A. Nagai and H. Yamagishi, *Sobolev type inequalities of time-periodic boundary value problems for Heaviside and Thomson Cables*, Bound. Value Probl., (2012), 2012:95.

[12] K. Takemura, A. Nagai, Y. Kametaka, K. Watanabe and H. Yamagishi, *Sobolev inequality of free boundary value problem for* $(-1)^M(d/dx)^{2M}$, J. Inequal. Appl., (2015) 2015:54, DOI 10.1186/s13660-015-0568-9.

[13] K. Watanabe, Y. Kametaka, A. Nagai, K. Takemura and H. Yamagishi, *The best constants of Sobolev and Kolmogorov type inequalities on a half line*, Far East J. Appl. Math., **52** No.2 (2011), 101–129.

[14] K. Watanabe, Y. Kametaka, H. Yamagishi, A. Nagai and K. Takemura, *The best constant of Sobolev inequality corresponding to clamped boundary value problem*, Bound. Value Probl., (2011), Article ID 875057, 17 pages.

[15] H. Yamagishi, *The Best Constant of Sobolev Inequality on Bending Problem of a String with Step Type Spring Constant*, Rep. RIAM Symp., **28AO-S6** (2017), 53–60 [in Japanese].

[16] H. Yamagishi, *The Best Constant of Sobolev Inequality Corresponding to a Bending Problem of a Beam under Tension on an Elastic Foundation II*, Trans. Jpn. Soc. Ind. Appl. Math., **29** (2019), 294–324 [in Japanese].

[17] H. Yamagishi and Y. Kametaka, *The best constant of Sobolev inequality corresponding to $2M$-th order differential equation with periodic boundary condition*, Kumamoto J. Math., **36** (2023), 1–16.

[18] H. Yamagishi and Y. Kametaka, *The Best Constant of the $L^p$ Sobolev-type Inequality corresponding to Elliptic Operator in $\mathbf{R}^N$*, Hiroshima Math. J., **54** (2024), 87–102.

[19] H. Yamagishi, Y. Kametaka, A. Nagai, K. Watanabe and K. Takemura, *Riemann zeta function and the best constants of five series of Sobolev inequalities*, RIMS Kokyuroku Bessatsu, **B13** (2009), 125–139.

[20] H. Yamagishi, Y. Kametaka, K. Takemura, K. Watanabe and A. Nagai, *The best constant of Sobolev inequality corresponding to a bending problem of a beam under tension on an elastic foundation*, Trans. Jpn. Soc. Ind. Appl. Math., **19** (2009), 489–518 [in Japanese].

[21] H. Yamagishi and A. Nagai, *The best constant of discrete Sobolev inequality corresponding to a discrete bending problem of a string*, Saitama Math. J., **34** (2022), 19–46.

[22] H. Yamagishi, A. Nagai, K. Watanabe, K. Takemura and Y. Kametaka, *The best constant of discrete Sobolev inequality corresponding to a bending problem of a string*, Kumamoto J. Math., **25** (2012), 1–15.

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