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LOCALLY SERIALLY COALESCENT CLASSES OF LIE ALGEBRAS

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Abstract. We assume that a basic field $\mathfrak{k}$ has zero characteristic. We show that any Fitting class is serially coalescent for locally finite Lie algebras. We also show that any class $\mathfrak{X}$ satisfying $\mathfrak{N} \leq \mathfrak{X} \leq \hat{\mathfrak{Gr}}$ (e.g. $\mathfrak{F}t$, $\mathfrak{B}$, $\mathfrak{Z}$, $\mathfrak{Gr}$, $\mathrm{L}\mathfrak{N}$, $\mathrm{R}\mathfrak{N}$, $\grave{\mathrm{E}}(\lhd)\hat{\mathfrak{A}}$, $\hat{\mathrm{E}}(\lhd)\hat{\mathfrak{A}}$, $\grave{\mathfrak{Gr}}$) is locally serially coalescent for locally finite Lie algebras, and, for any locally finite Lie algebra L, the $\mathfrak{X}$-ser radical of L is locally nilpotent.

1. Introduction

Let $\mathfrak{X}$ be a class of Lie algebras. It is said that $\mathfrak{X}$ is ascendantly coalescent if in any Lie algebra L the join of two ascendant $\mathfrak{X}$-subalgebras of L is always an ascendant $\mathfrak{X}$-subalgebra of L. It is also said that $\mathfrak{X}$ is locally ascendantly coalescent if for any two ascendant $\mathfrak{X}$-subalgebras H, K of a Lie algebra L and for any finitely generated subalgebra Y of $J = \langle H, K \rangle$ there exists an ascendant $\mathfrak{X}$-subalgebra X of L such that $Y \leq X \leq J$. In [4] we introduced the concept of serially coalescent (resp. locally serially coalescent) classes of Lie algebras corresponding to that of ascendantly coalescent (resp. locally ascendantly coalescent) classes and proved that the classes

$$\mathfrak{F} \cap \mathfrak{N}, \quad \mathfrak{F}, \quad \mathfrak{F} \cap \mathrm{E}\mathfrak{A}, \quad \mathrm{Min}, \quad \mathrm{Min}\text{-}\lhd^{\sigma}\ (\sigma \geq 2),$$
$$\mathrm{Min}\text{-si}, \quad \mathrm{Min}\text{-asc}, \quad \mathrm{Min}\text{-ser}, \quad \mathrm{Min}\text{-}\lhd \cap \mathrm{Max}\text{-}\lhd$$

are serially coalescent for locally finite Lie algebras over any field of characteristic zero. These classes are also locally serially coalescent for locally finite Lie algebras.

The purpose of this paper is to find some more serially coalescent classes and locally serially coalescent classes for locally finite Lie algebras, and to consider some properties of the $\mathfrak{X}$-ser radical $\mathrm{R}_{\mathfrak{X}\text{-ser}}(L)$ of a locally finite Lie algebra L.

In Section 3 we shall prove that any Fitting class $\mathfrak{X}$ is serially coalescent for locally finite Lie algebras over any field of characteristic zero (Theorem 1). In Section 4 we shall show that any class $\mathfrak{X}$ satisfying $\mathfrak{N} \leq \mathfrak{X} \leq \hat{\mathfrak{Gr}}$ is locally serially coalescent for locally finite Lie algebras over any field of characteristic zero (Theorem 5). In Section 5 we shall show that if L is locally finite over any field of characteristic zero and $\mathfrak{X}$ is any class such that

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$\mathfrak{N} \le \mathfrak{X} \le \hat{\mathfrak{G}}\mathfrak{r}$, then $\mathrm{R}_{\mathfrak{X}\text{-ser}}(L) = \mathrm{R}_{\mathfrak{F}\cap\mathfrak{N}\text{-ser}}(L)$ and $\mathrm{R}_{\mathfrak{X}\text{-ser}}(L) \in \mathrm{L}\mathfrak{N}$ (Theorem 7 and Corollary 8).

2. Notation and terminology

Throughout the paper Lie algebras are not necessarily finite-dimensional over a field $\mathfrak{k}$ of arbitrary characteristic unless otherwise specified. We mostly follow [2] for the use of notation and terminology.

Let L be a Lie algebra over $\mathfrak{k}$ and let H be a subalgebra of L. For a totally ordered set Σ, a series from H to L of type Σ is a collection $\{\Lambda_\sigma, V_\sigma \,:\, \sigma \in \Sigma\}$ of subalgebras of L such that

(1) $H \subseteq V_\sigma \subseteq \Lambda_\sigma$ for all $\sigma \in \Sigma$,
(2) $\Lambda_\tau \subseteq V_\sigma$ if $\tau < \sigma$,
(3) $L\backslash H = \cup_{\sigma\in\Sigma}(\Lambda_\sigma\backslash V_\sigma)$,
(4) $V_\sigma \lhd \Lambda_\sigma$ for all $\sigma \in \Sigma$.

H is a serial subalgebra of L, which we denote by H ser L, if there exists a series from H to L.

For an ordinal σ, H is a σ-step ascendant subalgebra of L, denoted by $H \lhd^\sigma L$, if there exists an ascending chain $(H_\alpha)_{\alpha\le\sigma}$ of subalgebras of L such that

(1) $H_0 = H$ and $H_\sigma = L$,
(2) $H_\alpha \lhd H_{\alpha+1}$ for any ordinal $\alpha < \sigma$,
(3) $H_\lambda = \cup_{\alpha<\lambda} H_\alpha$ for any limit ordinal $\lambda \le \sigma$.

H is an ascendant subalgebra of L, denoted by H asc L if $H \lhd^\sigma L$ for some ordinal σ. When σ is finite, H is a subideal of L and denoted by H si L. For an ordinal α, we denote by L^α the α-th term of the transfinite lower central series of L and by $L^{(\alpha)}$ the α-th term of the transfinite derived series of L.

Let $\mathfrak{X}$ be a class of Lie algebras and let Δ be any of the relations $\le, \lhd$, si, asc, ser. A Lie algebra L is said to lie in $\mathrm{L}(\Delta)\mathfrak{X}$ if for any finite subset X of L there exists an $\mathfrak{X}$-subalgebra H of L such that $X \subseteq H \;\Delta\; L$. In particular we write $\mathrm{L}\mathfrak{X}$ for $\mathrm{L}(\le)\mathfrak{X}$. When $L \in \mathrm{L}\mathfrak{X}$, L is called a locally $\mathfrak{X}$-algebra. We write Max-Δ (resp. Min-Δ) for the classes of Lie algebras satisfying the maximal (resp. minimal) condition for Δ-subalgebras. $\mathfrak{F}, \mathfrak{G}, \mathfrak{A}, \mathfrak{N}, \mathfrak{Z}, \mathrm{E}\mathfrak{A}$ and $\mathfrak{D}$ are the classes of Lie algebras which are finite-dimensional, finitely generated, abelian, nilpotent, hypercentral, soluble and all Lie algebras respectively. The $\mathfrak{X}$-residual $\lambda_{\mathfrak{X}}(L)$ of L is the intersection of the ideals I of L such that $L/I \in \mathfrak{X}$. Let A be a closure operation. We say that $\mathfrak{X}$ is A-closed if $\mathfrak{X} = \mathrm{A}\mathfrak{X}$. The class $\mathrm{S}\mathfrak{X}$ (resp. $\mathrm{Q}\mathfrak{X}$) consists of all subalgebras (resp. quotients) of $\mathfrak{X}$-algebras.

We say that $\mathfrak{X}$ is *serially coalescent* if in any Lie algebra L,

(SC) for any two serial $\mathfrak{X}$-subalgebras H, K of L, $J = \langle H, K\rangle$ is always a serial $\mathfrak{X}$-subalgebra of L.

We also say that $\mathfrak{X}$ is *locally serially coalescent* if in any Lie algebra L,

(LSC) for any two serial $\mathfrak{X}$-subalgebras H, K of L and for any finitely generated subalgebra Y of $J = \langle H, K\rangle$, there exists a serial $\mathfrak{X}$-subalgebra X of L such that $Y \leq X \leq J$.

3. Serial coalescence and local serial coalescence

Let $\mathfrak{X}$ be a class of Lie algebras. In order to study serially (resp. locally serially) coalescent classes of Lie algebras we need to restrict ourselves to locally finite Lie algebras. We say that $\mathfrak{X}$ is serially coalescent *for locally finite Lie algebras* (or *for* $\mathrm{L}\mathfrak{F}$*-algebras*) if in any locally finite Lie algebra L, the condition (SC) holds. We also say that $\mathfrak{X}$ is locally serially coalescent *for locally finite Lie algebras* (or *for* $\mathrm{L}\mathfrak{F}$*-algebras*) if in any locally finite Lie algebra L, the condition (LSC) holds.

It is said that $\mathfrak{X}$ is a Fitting class (cf. [2, p. 259]) if $\mathfrak{X} \leq \mathfrak{F}$ and $\{\mathrm{N}_0, \mathrm{I}\}\mathfrak{X} = \mathfrak{X}$. For any Fitting class $\mathfrak{X}$ and any $\mathfrak{F}$-algebra L, the $\mathfrak{X}$-radical of L, denoted by $\rho_{\mathfrak{X}}(L)$, is the unique maximal $\mathfrak{X}$-ideal of L. Obviously the classes $\mathfrak{F}\cap\mathfrak{N}$, $\mathfrak{F}$ and $\mathfrak{F} \cap \mathrm{E}\mathfrak{A}$ are Fitting classes. Therefore the following first main theorem generalises [4, Theorems 4, 5 and 6].

Theorem 1. *Over any field of characteristic zero, any Fitting class $\mathfrak{X}$ is serially coalescent for* $\mathrm{L}\mathfrak{F}$*-algebras.*

Proof. Let $L \in \mathrm{L}\mathfrak{F}$ and suppose that H, K ser L and $H, K \in \mathfrak{X}$. Since $\mathfrak{X} \leq \mathfrak{F}$, it follows from [4, Theorem 5] that $\langle H, K\rangle$ ser L and $\langle H, K\rangle \in \mathfrak{F}$. As H, K ser $\langle H, K\rangle$, we have H, K si $\langle H, K\rangle$. Hence using [2, Lemma 13.3.3], we get $H, K \leq \rho_{\mathfrak{X}}(\langle H, K\rangle)$, that is, $\langle H, K\rangle = \rho_{\mathfrak{X}}(\langle H, K\rangle)$. Therefore we obtain $\langle H, K\rangle \in \mathfrak{X}$. Thus $\mathfrak{X}$ is serially coalescent for $\mathrm{L}\mathfrak{F}$-algebras. □

The following proposition is analogous to [7, Proposition 2.1] which shows some relations between serial coalescence and local serial coalescence over any field.

Proposition 2. (1) *If $\mathfrak{X}$ is locally serially coalescent, then $\mathfrak{X} \cap \mathfrak{G}$ is serially coalescent.*

(2) *If $\mathfrak{X}$ and $\mathfrak{Y}$ are* S*-closed and locally serially coalescent, then $\mathfrak{X} \cap \mathfrak{Y}$ is also locally serially coalescent.*

(3) *If $\mathfrak{X}$ is* $\mathrm{L}(\mathrm{ser})$*-closed and locally serially coalescent and $\mathfrak{Y}$ is serially coalescent, then $\mathfrak{X} \cap \mathfrak{Y}$ is serially coalescent.*

Proof. We set $J = \langle H, K\rangle$ for subalgebras H and K of a Lie algebra L.

(1) Let $\mathfrak{X}$ be locally serially coalescent. Suppose that H, K ser L and $H, K \in \mathfrak{X} \cap \mathfrak{G}$. As $J \in \mathfrak{G}$, there is a serial subalgebra X of L such that $X \in \mathfrak{X}$ and $J \leq X \leq J$. Hence we get $J = X \in \mathfrak{X} \cap \mathfrak{G}$ and J ser L.

(2) Suppose that H, K ser L and $H, K \in \mathfrak{X} \cap \mathfrak{Y}$. Let F be a finitely generated subalgebra of J. Since $\mathfrak{X}$ and $\mathfrak{Y}$ are locally serially coalescent, there are serial subalgebras X and Y of L such that

$$F \leq X \leq J, \quad F \leq Y \leq J, \quad X \in \mathfrak{X} \text{ and } Y \in \mathfrak{Y}.$$

Therefore, we have $F \leq X \cap Y \in \mathfrak{X} \cap \mathfrak{Y}$ since $\mathfrak{X}$ and $\mathfrak{Y}$ are S-closed. Moreover we obtain $X \cap Y$ ser Y ser L, so $X \cap Y$ ser L. Thus $\mathfrak{X} \cap \mathfrak{Y}$ is locally serially coalescent.

(3) Suppose that H, K ser L and $H, K \in \mathfrak{X} \cap \mathfrak{Y}$. Because $\mathfrak{Y}$ is serially coalescent, we get J ser L and $J \in \mathfrak{Y}$. For any finite subset F of J, there is a serial subalgebra X of L such that $F \subseteq X \leq J$ and $X \in \mathfrak{X}$, since $\mathfrak{X}$ is locally serially coalescent. From X ser J, we also obtain $J \in \mathrm{L}(\mathrm{ser})\mathfrak{X} = \mathfrak{X}$. Thus it follows that J ser L and $J \in \mathfrak{X} \cap \mathfrak{Y}$. □

Let $\mathfrak{X}$ be a class of Lie algebras. In [6] the following two classes $\mathfrak{X}_{(\omega)}$ and $\mathfrak{X}_\omega$ are defined as follows: $\mathfrak{X}_{(\omega)}$(resp. $\mathfrak{X}_\omega$) is the class of Lie algebras L such that $L/L^{(\omega)} \in \mathfrak{X}$ (resp. $L/L^\omega \in \mathfrak{X}$).

The following proposition is analogous to [7, Theorem 2.2].

Proposition 3. *Let $\mathfrak{k}$ be any field of characteristic zero. If $\mathfrak{X}$ is Q-closed and locally serially coalescent for $\mathrm{L}\mathfrak{F}$-algebras, then $\mathfrak{X}_{(\omega)} \cap \mathfrak{F}$ and $\mathfrak{X}_\omega \cap \mathfrak{F}$ are serially coalescent for $\mathrm{L}\mathfrak{F}$-algebras.*

Proof. Let $L \in \mathrm{L}\mathfrak{F}$ and suppose that H, K ser L and $H, K \in \mathfrak{X}_{(\omega)} \cap \mathfrak{F}$ (resp. $\mathfrak{X}_\omega \cap \mathfrak{F}$). Owing to [4, Lemma 1 and Lemma 2(1)] we get

$$H^{(\omega)} = \lambda_{\mathrm{LE}\mathfrak{A}}(H) \lhd L \text{ and } K^{(\omega)} = \lambda_{\mathrm{LE}\mathfrak{A}}(K) \lhd L$$
$$(\text{resp. } H^\omega = \lambda_{\mathrm{L}\mathfrak{N}}(H) \lhd L \text{ and } K^\omega = \lambda_{\mathrm{L}\mathfrak{N}}(K) \lhd L).$$

Put $I = H^{(\omega)} + K^{(\omega)}$ (resp. $I = H^\omega + K^\omega$). Then $I \lhd L$. Set $J = \langle H, K\rangle$. Then it is clear that $J \in \mathfrak{F}$. We also have $(H+I)/I$ ser L/I and $(K+I)/I$ ser L/I by virtue of [4, Lemma 3]. Since $H^{(\omega)} \leq H \cap I$ and $K^{(\omega)} \leq K \cap I$ (resp. $H^\omega \leq H \cap I$ and $K^\omega \leq K \cap I$), we obtain

$$(H+I)/I \cong H/H \cap I \cong (H/H^{(\omega)})/(H \cap I/H^{(\omega)}) \in \mathrm{Q}\mathfrak{X} = \mathfrak{X},$$
$$(K+I)/I \cong K/K \cap I \cong (K/K^{(\omega)})/(K \cap I/K^{(\omega)}) \in \mathrm{Q}\mathfrak{X} = \mathfrak{X}$$

(resp.

$$(H+I)/I \cong H/H \cap I \cong (H/H^\omega)/(H \cap I/H^\omega) \in \mathrm{Q}\mathfrak{X} = \mathfrak{X},$$
$$(K+I)/I \cong K/K \cap I \cong (K/K^\omega)/(K \cap I/K^\omega) \in \mathrm{Q}\mathfrak{X} = \mathfrak{X}).$$

Because $H, K \in \mathfrak{F}$, we conclude that

$$(H+I)/I, (K+I)/I \in \mathfrak{X} \cap \mathfrak{F} \text{ and } (H+I)/I, (K+I)/I \text{ ser } L/I \in \mathrm{L}\mathfrak{F}.$$

Here, since $\mathfrak{F}$ is serially coalescent for $\mathrm{L}\mathfrak{F}$-algebras ([4, Theorem 5]), it follows that

$$J/I = \langle (H+I)/I, (K+I)/I \rangle \in \mathfrak{F} \ \text{ and}$$
$$J/I = \langle (H+I)/I, (K+I)/I \rangle \text{ ser } L/I.$$

Moreover the assumption that $\mathfrak{X}$ is locally serially coalescent for $\mathrm{L}\mathfrak{F}$-algebras leads that there exists a subalgebra X of L containing I such that

$$X/I \text{ ser } L/I,\ X/I \in \mathfrak{X} \text{ and } \langle (H+I)/I, (K+I)/I \rangle \le X/I \le J/I.$$

Hence $J/I = X/I \in \mathfrak{X}$ and so $J/I \in \mathfrak{X} \cap \mathfrak{F}$. Since $I = H^{(\omega)} + K^{(\omega)} \le J^{(\omega)}$ (resp. $I = H^{\omega} + K^{\omega} \le J^{\omega}$), we have

$$J/J^{(\omega)} \cong (J/I)/(J^{(\omega)}/I) \in \mathrm{Q}\mathfrak{X} = \mathfrak{X}$$
$$(\text{resp. } J/J^{\omega} \cong (J/I)/(J^{\omega}/I) \in \mathrm{Q}\mathfrak{X} = \mathfrak{X}),$$

that is to say, $J \in \mathfrak{X}_{(\omega)} \cap \mathfrak{F}$ (resp. $J \in \mathfrak{X}_{\omega} \cap \mathfrak{F}$). Because J ser L, we conclude that $\mathfrak{X}_{(\omega)} \cap \mathfrak{F}$ (resp. $\mathfrak{X}_{\omega} \cap \mathfrak{F}$) is serially coalescent for $\mathrm{L}\mathfrak{F}$-algebras. □

4. Several locally serially coalescent classes

In this section, we find several locally serially coalescent classes for $\mathrm{L}\mathfrak{F}$-algebras. First we begin with the class of nilpotent Lie algebras.

Lemma 4. *Over any field of characteristic zero, the class $\mathfrak{N}$ is locally serially coalescent for $\mathrm{L}\mathfrak{F}$-algebras.*

Proof. Let $L \in \mathrm{L}\mathfrak{F}$ and suppose that H, K ser L and $H, K \in \mathfrak{N}$. We set $J = \langle H, K \rangle$. Let Y be any finitely generated subalgebra of J. Then there are finitely generated subalgebras A of H and B of K such that $Y \le \langle A, B \rangle \le J$. Since $H \in \mathfrak{N}$, we get A si H and $A \in \mathfrak{G} \cap \mathfrak{N} = \mathfrak{F} \cap \mathfrak{N}$. Therefore we obtain A ser L. Similarly we have B ser L and $B \in \mathfrak{F} \cap \mathfrak{N}$. It follows from [4, Theorem 4] that $\langle A, B \rangle$ ser L and $\langle A, B \rangle \in \mathfrak{F} \cap \mathfrak{N}$. That is to say, we get

$$Y \le \langle A, B \rangle \le J, \quad \langle A, B \rangle \text{ ser } L \ \text{ and } \ \langle A, B \rangle \in \mathfrak{N}.$$

Thus $\mathfrak{N}$ is locally serially coalescent for $\mathrm{L}\mathfrak{F}$-algebras. □

We recall that $\mathfrak{Gr}$ is the class of Gruenberg algebras L, that is to say, in which every 1-dimensional subalgebra of L is ascendant in L (cf. [1]). In [3] we analogously defined the classes $\grave{\mathfrak{Gr}}$ and $\hat{\mathfrak{Gr}}$ of Lie algebras as follows:

$L \in \grave{\mathfrak{Gr}}$ iff every 1-dimensional subalgebra of L is descendant in L,

$L \in \hat{\mathfrak{Gr}}$ iff every 1-dimensional subalgebra of L is serial in L.

By [3, Lemma 2.4] we also have

$$\hat{\mathrm{E}}(\lhd)\hat{\mathfrak{A}} = \{ L \in \mathfrak{O} : L \text{ has a central series } \},$$

$$\acute{\mathrm{E}}(\lhd)\hat{\mathfrak{A}} = \{L \in \mathfrak{D} : L \text{ has an ascending central series } \} = \mathfrak{Z},$$
$$\grave{\mathrm{E}}(\lhd)\hat{\mathfrak{A}} = \{L \in \mathfrak{D} : L \text{ has a descending central series } \}.$$

We now assume that the basic field $\mathfrak{k}$ has zero characteristic. $\mathfrak{F}t$ and $\mathfrak{B}$ are the classes of Fitting algebras and Baer algebras respectively (cf. [2, p.114]). By using [2, Theorem 6.2.1] we obtain that if $\mathfrak{X}$ is any of the classes

$$\mathfrak{F}t,\ \mathfrak{B},\ \acute{\mathrm{E}}(\lhd)\hat{\mathfrak{A}} = \mathfrak{Z},\ \mathfrak{Gr},$$

then $\mathfrak{N} \le \mathfrak{X} \le \mathrm{L}\mathfrak{N}$. Furthermore it follows from [3, Corollary 2.7 and Theorem 2.9] that if $\mathfrak{X}$ is any of the classes

$$\mathrm{L}\mathfrak{N},\ \mathrm{R}\mathfrak{N},\ \mathrm{LR}\mathfrak{N},\ \grave{\mathrm{E}}(\lhd)\hat{\mathfrak{A}},\ \mathrm{L}\grave{\mathrm{E}}(\lhd)\hat{\mathfrak{A}},\ \hat{\mathrm{E}}(\lhd)\hat{\mathfrak{A}},\ \grave{\mathfrak{Gr}},$$

then $\mathfrak{N} \le \mathfrak{X} \le \hat{\mathfrak{Gr}}$. Thus we can summarize the results above as follows: If $\mathfrak{X}$ is any of the classes

$$\mathfrak{F}t,\ \mathfrak{B},\ \acute{\mathrm{E}}(\lhd)\hat{\mathfrak{A}} = \mathfrak{Z},\ \mathfrak{Gr},\ \mathrm{L}\mathfrak{N},\ \mathrm{R}\mathfrak{N},\ \mathrm{LR}\mathfrak{N},\ \grave{\mathrm{E}}(\lhd)\hat{\mathfrak{A}},\ \mathrm{L}\grave{\mathrm{E}}(\lhd)\hat{\mathfrak{A}},\ \hat{\mathrm{E}}(\lhd)\hat{\mathfrak{A}},\ \grave{\mathfrak{Gr}},$$

then $\mathfrak{N} \le \mathfrak{X} \le \hat{\mathfrak{Gr}}$.

The following is the second main theorem in this paper.

Theorem 5. *Over any field of characteristic zero, any class $\mathfrak{X}$ satisfying $\mathfrak{N} \le \mathfrak{X} \le \hat{\mathfrak{Gr}}$ is locally serially coalescent for $\mathrm{L}\mathfrak{F}$-algebras.*

Proof. In Lemma 4 we have proved that the class $\mathfrak{N}$ is locally serially coalescent for $\mathrm{L}\mathfrak{F}$-algebras. By virtue of [4, Theorem 13], if a class $\mathfrak{Y}$ satisfies

$$\mathrm{L}\mathfrak{F} \cap \mathfrak{N} \le \mathrm{L}\mathfrak{F} \cap \mathfrak{Y} \le \mathrm{L}\mathfrak{F} \cap \mathrm{L}(\mathrm{ser})\mathfrak{N},$$

then $\mathfrak{Y}$ is locally serially coalescent for $\mathrm{L}\mathfrak{F}$-algebras. Now, let $\mathfrak{X}$ be a class such that $\mathfrak{N} \le \mathfrak{X} \le \hat{\mathfrak{Gr}}$. Then we get

$$\mathrm{L}\mathfrak{F} \cap \mathfrak{N} \le \mathrm{L}\mathfrak{F} \cap \mathfrak{X} \le \mathrm{L}\mathfrak{F} \cap \hat{\mathfrak{Gr}}.$$

Here, [3, Proposition 2.10] and [5, Theorem 4] lead to

$$\mathrm{L}\mathfrak{F} \cap \hat{\mathfrak{Gr}} = \mathrm{L}\mathfrak{N} = \mathrm{L}(\mathrm{ser})(\mathfrak{F} \cap \mathfrak{N}) \le \mathrm{L}\mathfrak{F} \cap \mathrm{L}(\mathrm{ser})\mathfrak{N}.$$

Therefore we conclude

$$\mathrm{L}\mathfrak{F} \cap \mathfrak{N} \le \mathrm{L}\mathfrak{F} \cap \mathfrak{X} \le \mathrm{L}\mathfrak{F} \cap \mathrm{L}(\mathrm{ser})\mathfrak{N}.$$

This proves the theorem. $\square$

By using Theorem 5, we present several specific classes.

Corollary 6. *Over any field of characteristic zero, if $\mathfrak{X}$ is any of the classes*

$$\mathfrak{N},\ \mathfrak{F}t,\ \mathfrak{B},\ \acute{\mathrm{E}}(\lhd)\hat{\mathfrak{A}} = \mathfrak{Z},\ \mathfrak{Gr},\ \mathrm{L}\mathfrak{N},\ \mathrm{R}\mathfrak{N},$$
$$\mathrm{LR}\mathfrak{N},\ \grave{\mathrm{E}}(\lhd)\hat{\mathfrak{A}},\ \mathrm{L}\grave{\mathrm{E}}(\lhd)\hat{\mathfrak{A}},\ \hat{\mathrm{E}}(\lhd)\hat{\mathfrak{A}},\ \grave{\mathfrak{Gr}},\ \hat{\mathfrak{Gr}},$$

then $\mathfrak{X}$ is locally serially coalescent for $\mathrm{L}\mathfrak{F}$*-algebras.*

5. Radicals

In [4], for a class $\mathfrak{X}$ of Lie algebras and a Lie algebra L we defined the $\mathfrak{X}$-ser radical of L, denoted by $\mathrm{R}_{\mathfrak{X}\text{-ser}}(L)$, by the subalgebra generated by all the serial $\mathfrak{X}$-subalgebras of L. Then we have the third main theorem in this paper.

Theorem 7. *Let $\mathfrak{X}$ be any class of Lie algebras over any field of characteristic zero such that $\mathfrak{N} \leq \mathfrak{X} \leq \hat{\mathfrak{Gr}}$. If $L \in \mathrm{L}\mathfrak{F}$, then* $\mathrm{R}_{\mathfrak{X}\text{-ser}}(L) = \mathrm{R}_{\mathfrak{F}\cap\mathfrak{N}\text{-ser}}(L)$.

Proof. Let $L \in \mathrm{L}\mathfrak{F}$. Since $\mathfrak{F} \cap \mathfrak{N} \leq \mathfrak{N} \leq \mathfrak{X}$, it is trivial that $\mathrm{R}_{\mathfrak{F}\cap\mathfrak{N}\text{-ser}}(L) \leq \mathrm{R}_{\mathfrak{X}\text{-ser}}(L)$.

Conversely, let H be any serial $\mathfrak{X}$-subalgebra of L. Since $L \in \mathrm{L}\mathfrak{F}$, we have

$$H \in \mathrm{L}\mathfrak{F} \cap \mathfrak{X} \leq \mathrm{L}\mathfrak{F} \cap \hat{\mathfrak{Gr}}.$$

Because $\mathrm{L}\mathfrak{F} \cap \hat{\mathfrak{Gr}} = \mathrm{L}\mathfrak{N}$ by [3, Proposition 2.10], we get $H \in \mathrm{L}\mathfrak{N}$. Thus $H \leq \mathrm{R}_{\mathfrak{F}\cap\mathfrak{N}\text{-ser}}(L)$ by [4, Corollary 15]. This shows that $\mathrm{R}_{\mathfrak{X}\text{-ser}}(L) \leq \mathrm{R}_{\mathfrak{F}\cap\mathfrak{N}\text{-ser}}(L)$. □

As a corollary of Theorem 7, we obtain the following result which corresponds to [2, Theorem 6.2.1].

Corollary 8. *Let $\mathfrak{X}$ be any class of Lie algebras over any field of characteristic zero such that $\mathfrak{N} \leq \mathfrak{X} \leq \hat{\mathfrak{Gr}}$. If $L \in \mathrm{L}\mathfrak{F}$, then* $\mathrm{R}_{\mathfrak{X}\text{-ser}}(L) \in \mathrm{L}\mathfrak{N}$.

Proof. Let $L \in \mathrm{L}\mathfrak{F}$. Owing to Theorem 7 we have $\mathrm{R}_{\mathfrak{X}\text{-ser}}(L) = \mathrm{R}_{\mathfrak{F}\cap\mathfrak{N}\text{-ser}}(L)$. On the other hand, it follows from [4, Proposition 14 (1)] that $\mathrm{R}_{\mathfrak{F}\cap\mathfrak{N}\text{-ser}}(L) \in \mathrm{L}\mathfrak{N}$. This proves the corollary. □

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