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THE QUILLEN MODEL STRUCTURE ON THE CATEGORY OF DIFFEOLOGICAL SPACES

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Abstract. We construct on the category of diffeological spaces a Quillen model structure having smooth weak homotopy equivalences as the class of weak equivalences.

1. Introduction

The theory of model category was introduced by Quillen (cf. [9] and [10]), and is a crucial notion that forms the framework of modern homotopy theory. A model category is just a category with three specified classes of morphism, called fibrations, cofibrations and weak equivalences, which satisfy several axioms that are deliberately reminiscent of typical properties appearing in homotopy theory of topological spaces. It is shown in [2, 8.3] that the category **Top** of topological spaces has, so called, the Quillen model structure, under which a map $f\colon X \to Y$ is defined to be

(1) a weak equivalence if f is a weak homotopy equivalence [13, p.404],
(2) a fibration if f is a Serre fibration [11], and
(3) a cofibration if f has the left lifting property with respect to trivial fibrations.

The objective of this paper is to introduce a model category structure on the category **Diff** of diffeological spaces which closely resembles the original Quillen model structure of **Top** as in the following sense: A smooth map between diffeological spaces is a weak equivalence if it induces isomorphisms between smooth homotopy groups, and is a fibration if it satisfies (a sort of) homotopy lifting properties for pairs of cubical complexes (I^n, J^{n-1}), where I^n is the unit n-cube and $J^{n-1} = \partial I^{n-1} \times I \cup I^{n-1} \times \{1\}$. Our construction is directly connected to the smooth homotopy theory of diffeological spaces, and seems to be suited to combine homotopy theoretical methods with methods of differential topology and geometry. In a subsequent part of this work [4] (see also our original preprint [3]), we show that our model structure is Quillen equivalent to the Quillen model structure of **Top** under the adjunction $\mathbf{Top} \rightleftarrows \mathbf{Diff}$.

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The paper is organized as follows. In Section 2 we study homotopy sets of smooth maps between diffeological spaces. In particular, smooth homotopy groups $\pi_n(X, x_0)$ and relative homotopy groups $\pi_n(X, A, x_0)$ are defined to be the homotopy sets of smooth maps $(I^n, \partial I^n) \to (X, x_0)$ and $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$, respectively. Our definition of homotopy groups is different from, but turns out to be equivalent to, the one given by [5].

Since J^{n-1} is not a smooth retract of I^n, the treatment of smooth homotopy groups is slightly harder compared with the case of continuous homotopy groups. Still, as we shall see in Section 3, basic properties of continuous homotopy groups mostly hold in the smooth case. Especially, there exists a *homotopy long exact sequence* associated with a pair of diffeological spaces (cf. Proposition 3.8).

In section 4 we introduce the notion of fibrations and investigate its basic properties. Briefly, fibration is a smooth version of Serre fibration and is characterized by the right lifting property (in certain restricted sense) with respect to the inclusions $J^{n-1} \to I^n$, and its examples include trivial maps, diffeological fiber bundles, and (tame version of) path-loop fibrations (see Example 4.5). In particular, every diffeological space X is fibrant because the constant map $X \to *$ is a fibration (cf. Corollary 4.3).

Based on the results obtained in preceding sections, we prove in Section 5 that the category **Diff** has a model category structure such that a smooth map is (i) a weak equivalence if it is a weak homotopy equivalence, (ii) a fibration if it is a fibration defined in the preceding section, and (iii) a cofibration if it has the left lifting property with respect to trivial fibrations. The main theorem (Theorem 5.2) is proved by straightforward verification of axioms given by Dwyer-Spalinski [2] except for **MC5**, that is, a factorization of a smooth map f in two ways: (i) $f = p \circ i$, where i is a cofibration and p is a trivial fibration, and (ii) $f = p \circ i$, where i is a trivial cofibration and p is a fibration. The lack of smooth retraction of I^n onto J^{n-1} makes, again, the verification of **MC5** far more complicated than the case of **Top**.

In Section 6 we complete the proof of our main theorem by giving a detailed verification of **MC5**.

Several authors have attempted the construction of a model category related to diffeological spaces. In [1], Dan Christensen and Enxin Wu discuss smooth homotopy theory with respect to behaviors of fibrations, cofibrations and weak equivalences. However, they have yet to introduce a model structure. In [14], Enxin Wu presents a cofibrantly generated model structure on the category of diffeological chain complexes. In [6], Hiroshi Kihara proves that there exists a cofibrantly generated model structure (cf. [6, Theorem 1.3

and Lemma 9.6]) in which fibrations are smooth maps enjoying the right lifting property with respect to inclusions $\Lambda^p_k \to \Delta^p$, where Δ^p and Λ^p_k are the p-simplex and its horn, respectively, equipped with certain (non-standard) diffeology. Thus there are two model structures which give rise to distinct model categories $\mathbf{Diff}_{\mathrm{HS}}$ and $\mathbf{Diff}_{\mathrm{Kihara}}$. We constructed in [3] a Quillen equivalence $T\colon \mathbf{Diff}_{\mathrm{HS}} \rightleftarrows \mathbf{Top}\colon D$. On the other hand, Kihara constructed in [7, Theorem 1.5] a Quillen equivalence $|\ |_D\colon \mathcal{S} \rightleftarrows \mathbf{Diff}_{\mathrm{Kihara}}\colon S^D$, where $\mathcal{S}$ is the model category of simplicial sets. Thus, by passing to homotopy categories, there is a chain of categorical equivalences

$$\mathbf{Ho}(\mathbf{Diff}_{\mathrm{HS}}) \to \mathbf{Ho}(\mathbf{Top}) \leftarrow \mathbf{Ho}(\mathcal{S}) \to \mathbf{Ho}(\mathbf{Diff}_{\mathrm{Kihara}})$$

in which the middle arrow is induced by the Quillen equivalence $|\ |\colon \mathcal{S} \rightleftarrows \mathbf{Top}\colon S$.

The authors wish to thank Dan Christensen and Hiroshi Kihara for helpful discussions while preparing the article. In particular, Kihara pointed out in [6] that there exists a gap in the proof of Lemma 5.6 of our original manuscript [3]. We correct this gap by suitably modifying the infinite gluing construction (see Section 6).

2. Diffeological spaces and Homotopy sets

In this section we introduce homotopy sets of a diffeological space in a slightly different manner than the one given in [5, Chapter 5] and [1, Section 3].

Recall from [12] that a diffeological space consists of a set and its *diffeology*, that is, the family of *plots* (defined on Euclidean open sets) satisfying the conditions similar to, but much more relaxed than, the charts of a smooth manifold. Any subset of a Euclidean space can be regarded as a diffeological space with respect to the *standard diffeology* consisting of all smooth parametrizations. If X and Y are diffeological spaces, then a smooth map from X to Y is a set map $f\colon X \to Y$ such that for any plot $P\colon U \to X$ of X the composition $f \circ P\colon U \to Y$ is a plot of Y. Denote by $C^\infty(X, Y)$ the set of smooth maps from X to Y. Then $C^\infty(X, Y)$ can be regarded as a diffeological space with respect to the coarsest (i.e. weakest) diffeology such that the evaluation map

$$\mathrm{ev}\colon C^\infty(X, Y) \times X \to Y, \quad \mathrm{ev}(f, x) = f(x)$$

is smooth. Let **Diff** be the category with diffeological spaces as objects and smooth maps as morphisms. The theorem below plays an essential role in our ongoing argument. See [5] for the proof.

Theorem 2.1. *The following hold:*

(1) **Diff** *is self-enriched in the sense that the inclusion* $X \to C^\infty(X,X)$ *and the composition* $C^\infty(Y,Z)\times C^\infty(X,Y) \to C^\infty(X,Z)$ *are smooth for all* X, Y, $Z \in \mathbf{Diff}$.
(2) **Diff** *is closed under small limits and colimits.*
(3) **Diff** *is a cartesian closed category with* $C^\infty(X,Y)$ *as exponential objects; in fact, there is a natural isomorphism*

$$\alpha\colon\ C^\infty(X\times Y,Z) \to C^\infty(X, C^\infty(Y,Z))$$

given by the formula: $\alpha(f)(x)(y) = f(x,y)$ *for* $x \in X$ *and* $y \in Y$.

Now, let $\mathbf{R}$ be the real line equipped with the standard diffeology, and let I be the unit interval $[0,1] \subset \mathbf{R}$ equipped with the subspace diffeology. Suppose f_0, $f_1\colon X \to Y$ are smooth maps between diffeological spaces. We say that f_0 and f_1 are homotopic, written $f_0 \simeq f_1$, if there is a smooth map $F\colon X\times I \to Y$ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$ hold for every $x \in X$. Such a smooth map F is called a homotopy between f_0 and f_1. A map $f\colon X \to Y$ is called a homotopy equivalence if there is a smooth map $g\colon Y \to X$ satisfying

$$g\circ f \simeq 1\colon X \to X,\quad f\circ g \simeq 1\colon Y \to Y.$$

We say that X and Y are homotopy equivalent, written $X \simeq Y$, if there exists a homotopy equivalence $f\colon X \to Y$. Let X be a diffeological space and A a subspace of X. Then A is called a *retract* of X if there exists a smooth map $\gamma\colon X \to A$, called a *retraction*, which restricts to the identity on A. If, moreover, there exists a homotopy $H\colon X\times I \to X$ such that

$$H(x,0) = x,\ H(x,1) = \gamma(x),\ H(a,t) = a \quad (x \in X,\ a \in A)$$

then A is called a *deformation retract* of X and γ a *deformation retraction.*

We show that the notion of homotopy introduced above is equivalent to the one given in [5, Chapter 5] and [1, Section 3]. Recall from [8, §4] that there is a non-decreasing smooth function $\lambda\colon \mathbf{R} \to I$ which satisfies $\lambda(t) = 0$ for $t \le 0$, $\lambda(t) = 1$ for $1 \le t$, $\lambda(1-t) = 1-\lambda(t)$ for every t, and is strongly increasing on $[0,1]$.

Proposition 2.2. *Let* f_0, $f_1\colon X \to Y$ *be smooth maps. Then* f_0 *and* f_1 *are homotopic if and only if there exists a smooth map* $G\colon X\times\mathbf{R} \to Y$ *such that* $G(x,0) = f_0(x)$ *and* $G(x,1) = f_1(x)$ *hold for every* $x \in X$.

Proof. Suppose there is a smooth map $G\colon X\times\mathbf{R} \to Y$ such that $G(x,0) = f_0(x)$ and $G(x,1) = f_1(x)$ hold for every $x \in X$. Then the restriction of G to $X\times I$ gives a homotopy $f_0 \simeq f_1$. On the other hand, if there is a homotopy $F\colon X\times I \to Y$ between f_0 and f_1 then the composition $G = F\circ(1\times\lambda)$ is a smooth map $X\times\mathbf{R} \to Y$ satisfying $G(x,0) = f_0(x)$ and $G(x,1) = f_1(x)$. □

Suppose F is a homotopy between f_0, $f_1 \colon X \to Y$ and G a homotopy between f_1, $f_2 \colon X \to Y$. Let us define $F * G \colon X \times I \to Y$ by the formula

$$F * G(x,t) = \begin{cases} F(x, \lambda(3t)), & 0 \le t \le 1/2, \\ G(x, \lambda(3t-2)), & 1/2 \le t \le 1. \end{cases}$$

Then $F * G$ is smooth all over $X \times I$, hence gives a homotopy between f_0 and f_2. It follows that the relation "$\simeq$" is an equivalence relation. The resulting equivalence classes are called homotopy classes.

In particular, if P consists of a single point then smooth maps from P to X are just the points of X and their homotopies are smooth paths $I \to X$.

Definition 2.3. Given a diffeological space X, we denote by $\pi_0 X$ the set of path components of X, that is, equivalence classes of points of X, where x and y are equivalent if there is a smooth path $\alpha \colon I \to X$ such that $\alpha(0) = x$ and $\alpha(1) = y$ hold.

For given pairs of diffeological spaces (X, X_1) and (Y, Y_1), put

$$[X, X_1 \,; Y, Y_1] = \pi_0 \, C^\infty((X, X_1), (Y, Y_1)),$$

where $C^\infty((X, X_1), (Y, Y_1))$ is the subspace of $C^\infty(X, Y)$ consisting of maps of pairs $(X, X_1) \to (Y, Y_1)$. Similarly, we put

$$[X, X_1, X_2 \,; Y, Y_1, Y_2] = \pi_0 \, C^\infty((X, X_1, X_2), (Y, Y_1, Y_2)),$$

where $C^\infty((X, X_1, X_2), (Y, Y_1, Y_2))$ is the subspace consisting of maps of triples. Clearly, we have $[X, X_1 \,; Y, Y_1] = [X, X_1, \emptyset \,; Y, Y_1, \emptyset]$. Then, by using the cartesian closedness of **Diff** we can prove the following.

Proposition 2.4. *The elements of* $[X, X_1, X_2 \,; Y, Y_1, Y_2]$ *are in one-to-one correspondence with the homotopy classes of maps* $(X, X_1, X_2) \to (Y, Y_1, Y_2)$.

We also have the following proposition.

Proposition 2.5. *Suppose* $f \colon (X, X_1, X_2) \to (Y, Y_1, Y_2)$ *is a homotopy equivalence. Then for every* (Z, Z_1, Z_2) *the precomposition and postcomposition by* f *induce homotopy equivalences*

$$f^* \colon \; C^\infty((Y, Y_1, Y_2), (Z, Z_1, Z_2)) \to C^\infty((X, X_1, X_2), (Z, Z_1, Z_2))$$
$$f_* \colon \; C^\infty((Z, Z_1, Z_2), (X, X_1, X_2)) \to C^\infty((Z, Z_1, Z_2), (Y, Y_1, Y_2))$$

Proof. For any $\boldsymbol{X} = (X, X_1, X_2)$ and fixed $\boldsymbol{Z} = (Z, Z_1, Z_2)$, put

$$F\boldsymbol{X} = C^\infty(\boldsymbol{X}, \boldsymbol{Z}) = C^\infty((X, X_1, X_2), (Z, Z_1, Z_2)).$$

We shall show that the contravariant functor F from the category of triples of diffeological spaces to **Diff** preserves homotopies. This of course implies that $f^* \colon F\boldsymbol{Y} \to F\boldsymbol{X}$ is a homotopy equivalence if so is $f \colon \boldsymbol{X} \to \boldsymbol{Y}$.

The contravariant functor F is enriched in the sense that the map

$$C^\infty(\boldsymbol{X}, \boldsymbol{Y}) \to C^\infty(F\boldsymbol{Y}, F\boldsymbol{X}),$$

which takes $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ to the induced map $f^*\colon F\boldsymbol{Y} \to F\boldsymbol{X}$, is smooth. This follows from Theorem 2.1 because the map above is adjoint to the composition $C^\infty(\boldsymbol{Y}, \boldsymbol{Z}) \times C^\infty(\boldsymbol{X}, \boldsymbol{Y}) \to C^\infty(\boldsymbol{X}, \boldsymbol{Z})$.

Suppose $h\colon \boldsymbol{X} \times I \to \boldsymbol{Y}$ is a homotopy between f and g. Then by Theorem 2.1 together with the enrichedness of F the composite map

$$I \to C^\infty(\boldsymbol{X}, \boldsymbol{Y}) \to C^\infty(F\boldsymbol{Y}, F\boldsymbol{X}),$$

which takes $t \in I$ to $h_t^*\colon F\boldsymbol{Y} \to F\boldsymbol{X}$, is smooth. Thus, by passing to the adjoint again, we get a smooth map $F\boldsymbol{Y} \times I \to F\boldsymbol{X}$ giving a homotopy between f^* and g^*.

Quite similarly, we can prove that the covariant functor $\boldsymbol{X} \to C^\infty(\boldsymbol{Z}, \boldsymbol{X})$ preserves homotopies. □

Corollary 2.6. *The homotopy set* $[X, X_1, X_2\,; Y, Y_1, Y_2]$ *is homotopy invariant with respect to both* (X, X_1, X_2) *and* (Y, Y_1, Y_2).

We are now ready to define the n-th homotopy set of a diffeological space. Let ∂I^n be the boundary of the unit n-cube $I^n \subset \mathbf{R}^n$, and let

$$J^{n-1} = \partial I^{n-1} \times I \cup I^{n-1} \times \{1\} \quad (n \geq 1).$$

Definition 2.7. Given a pointed diffeological space (X, x_0), we put

$$\pi_n(X, x_0) = [I^n, \partial I^n\,; X, x_0], \quad n \geq 0.$$

Similarly, given a pointed pair of diffeological spaces (X, A, x_0), we put

$$\pi_n(X, A, x_0) = [I^n, \partial I^n, J^{n-1}\,; X, A, x_0], \quad n \geq 1.$$

For $n \geq 1$, $\pi_n(X, x_0)$ is isomorphic to $\pi_n(X, x_0, x_0)$, and $\pi_0(X, x_0)$ is isomorphic to the set of path components $\pi_0 X$, regardless of the choice of basepoint x_0. Note, however, that we consider $\pi_0(X, x_0)$ as a pointed set with basepoint $[x_0] \in \pi_0 X$. We now introduce a group structure on $\pi_n(X, A, x_0)$. Suppose ϕ and ψ are smooth maps from $(I^n, \partial I^n, J^{n-1})$ to (X, A, x_0). If $n \geq 2$, or if $n \geq 1$ and $A = x_0$, then there is a smooth map $\phi * \psi\colon I^n \to X$ which takes $(t_1, t_2, \ldots, t_n) \in I^n$ to

$$\begin{cases} \phi(\lambda(3t_1), t_2, \ldots, t_n), & 0 \leq t_1 \leq 1/2 \\ \psi(\lambda(3t_1 - 2), t_2, \ldots, t_n), & 1/2 \leq t_1 \leq 1. \end{cases}$$

It is clear that $\phi * \psi$ defines a map of triples $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$, and there is a multiplication on $\pi_n(X, A, x_0)$ given by the formula

$$[\phi] \cdot [\psi] = [\phi * \psi] \in \pi_n(X, A, x_0).$$

Proposition 2.8. *With respect to the multiplication* $([\phi],[\psi]) \mapsto [\phi]\cdot[\psi]$, *the homotopy set* $\pi_n(X,A,x_0)$ *is a group if* $n \geq 2$ *or if* $n \geq 1$ *and* $A = x_0$, *and is an abelian group if* $n \geq 3$ *or if* $n \geq 2$ *and* $A = x_0$. *Moreover, for every smooth map* $f\colon (X,A,x_0) \to (Y,B,y_0)$ *the induced map*

$$f_*\colon \pi_n(X,A,x_0) \to \pi_n(Y,B,y_0)$$

is a group homomorphism whenever its source and target are groups.

Remark. Our definition of $\pi_n(X,x_0)$ is equivalent to the one given in [5], in which it is defined as the set of path components of $\mathrm{Loops}^n(X,x_0)$, where

$$\mathrm{Loops}(Y,y) = C^\infty((\mathbf{R},0,1),(Y,y,y))$$

for any pointed diffeological space (Y,y). On the other hand, our $\pi_n(X,x_0)$ is isomorphic to the set of path components of $\Omega^n(X,x_0)$, where

$$\Omega(Y,y) = C^\infty((I,0,1),(Y,y,y)).$$

But the inclusion $(I,0,1) \to (\mathbf{R},0,1)$ is a homotopy equivalence because it has a homotopy inverse $\lambda\colon (\mathbf{R},0,1) \to (I,0,1)$. Thus, by Proposition 2.5 we have $\mathrm{Loops}(Y,y) \simeq \Omega(Y,y)$, hence $\mathrm{Loops}^n(X,x_0) \simeq \Omega^n(X,x_0)$ for all $n \geq 0$. The situation is similar for the homotopy sets of pairs $\pi_n(X,A,x_0)$.

3. Tame maps and approximate retractions

In the case of topological spaces, the fact that J^{n-1} is a retract of I^n is crucial for developing homotopy theory (e.g. homotopy exact sequence and homotopy extension property). Unfortunately, this scenario does not work in the smooth case because there is no smooth retraction $I^n \to J^{n-1}$. Still, we can retrieve most of the ingredients of homotopy theory by replacing retractions with a more relaxed notion of *approximate retractions*.

For every $1 \leq j \leq n$ and $\alpha \in \{0,1\}$, let π_j^α denote the orthogonal projection of I^n onto its $(n-1)$-dimensional face $\{(t_1,\cdots,t_n) \in I^n \mid t_j = \alpha\}$.

Definition 3.1. (1) Let K be a subset of I^n and $0 < \epsilon \leq 1/2$. A smooth map $f\colon K \to X$ is said to be ϵ-*tame* if for every $P = (t_1,\ldots,t_n) \in K$ we have $f(P) = f(\pi_j^\alpha(P))$ whenever $|t_j - \alpha| \leq \epsilon$ and $\pi_j^\alpha(P) \in K$ hold.

(2) A smooth homotopy $H\colon X \times I \to Y$ is said to be ϵ-*tame* if so is its adjoint $I \to C^\infty(X,Y)$, that is, there hold $H(x,t) = H(x,0)$ for $0 \leq t \leq \epsilon$ and $H(x,t) = H(x,1)$ for $1-\epsilon \leq t \leq 1$.

Note that ϵ-tameness implies σ-tameness for all $\sigma < \epsilon$. We use the abbreviation "tame" to mean ϵ-tame for some $\epsilon > 0$.

For $0 < \epsilon \leq 1/2$, let $I^n(\epsilon) = [\epsilon, 1-\epsilon]^n$ and call it the ϵ-*chamber* of I^n. More generally, if K is a cubical subcomplex (i.e. a union of faces) of I^n then its ϵ-chamber $K(\epsilon)$ is defined to be the union of ϵ-chambers of its maximal faces. Thus we have $(\partial I^n)(\epsilon) = \bigcup F(\epsilon)$, where F runs through the $(n-1)$-dimensional faces of I^n, and $J^{n-1}(\epsilon) = (\partial I^n)(\epsilon) \cap J^{n-1}$.

It is evident that the following holds.

Lemma 3.2. *Let f and g be smooth maps from a cubical subcomplex K of I^n to a diffeological space X. Suppose both f and g are ϵ-tame. Then f and g coincide on K if and only if they coincide on $K(\epsilon)$.*

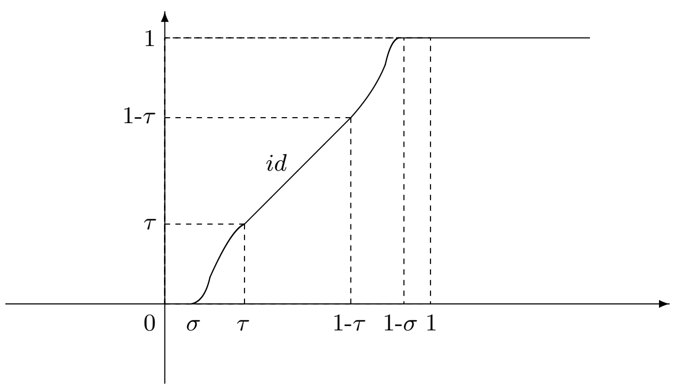
We show that any tame map defined on J^{n-1} is extendable over I^n. For this purpose, and to proceed further, we need a refinement of λ.

Lemma 3.3. *Suppose $0 \le \sigma < \tau \le 1/2$. Then there exists a non-decreasing smooth function $T_{\sigma,\tau} \colon \mathbf{R} \to I$ satisfying the following conditions:*

1. *$T_{\sigma,\tau}(t) = 0$ for $t \le \sigma$,*
2. *$T_{\sigma,\tau}(t) = t$ for $\tau \le t \le 1-\tau$,*
3. *$T_{\sigma,\tau}(t) = 1$ for $1-\sigma \le t$, and*
4. *$T_{\sigma,\tau}(1-t) = 1 - T_{\sigma,\tau}(t)$ for all t.*

Proof. For every $t \in \mathbf{R}$, put
$$F(t) = \int_0^t \lambda\left(\frac{\tau x - \sigma}{\tau - \sigma}\right) dx + \frac{\tau + \sigma}{2\tau}\lambda\left(\frac{\tau t - \sigma}{\tau - \sigma}\right).$$
Then $F \colon \mathbf{R} \to \mathbf{R}$ is a non-decreasing smooth function such that $F(t)$ has value 0 for $t \le \sigma/\tau$ and has value t for $t \ge 1$. Now, let us define a function $T_{\sigma,\tau} \colon \mathbf{R} \to I$ by putting $T_{\sigma,\tau}(t) = F(t/\tau)$ for $t \le 1/2$, and $T_{\sigma,\tau}(t) = 1 - F((1-t)/\tau)$ for $1/2 \le t$. As we have $T_{\sigma,\tau}(t) = t$ for $\tau \le t \le 1-\tau$, the function $T_{\sigma,\tau}$ is smooth all over $\mathbf{R}$ and satisfies the desired conditions. $\square$

Note that $T_{0,1/2}$ differs from but can be used as an replacement of λ thanks to its properties. The graph of $T_{\sigma,\tau}$ looks as follows:



Lemma 3.4. *Let K be a cubical subcomplex of I^n. Then for any smooth map $f \colon K \to X$ and $0 < \sigma < \epsilon \le 1/2$, there exists a homotopy $f \simeq g$ relative*

to $K(\epsilon)$ such that g is σ-tame. If f is ϵ-tame on a subcomplex L of K then the homotopy can be taken to be relative to $L \cup K(\epsilon)$.

Proof. Let $g = f \circ T^n_{\sigma,\epsilon}|K \colon K \to X$. Then g is σ-tame and there is a homotopy $f \simeq g$ relative to $K(\epsilon)$ given by the map $K \times I \to X$ which takes $(v,t) \in K \times I$ to $f((1-t)v + tT^n_{\sigma,\epsilon}(v))$. If f is ϵ-tame on L then the homotopy $f \simeq g$ is relative to $L \cup K(\epsilon)$ because g coincides with f on L. □

It follows, in particular, that any element of $\pi_n(X, A, x_0)$ is represented by a tame map $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$.

A map $I^n \to J^{n-1}$ is called an *ϵ-approximate retraction* if it restricts to the identity on the ϵ-chamber $J^{n-1}(\epsilon)$.

Lemma 3.5. *For any real numbers ϵ, σ such that $0 < \sigma < \epsilon < 1/2$, there exists an ϵ-approximate retraction $R_{\sigma,\epsilon} \colon I^n \to J^{n-1}$ which is σ-tame.*

Proof. For any $t = (t_1, \cdots, t_{n-1}) \in I^{n-1}$ and $u \in I$, put

$$L(t) = \prod\nolimits_{1 \le k \le n-1} T_{0,1/2}\left(\frac{t_k}{\sigma}\right) T_{0,1/2}\left(\frac{1-t_k}{\sigma}\right)$$
$$v(t,u) = L(t) + (1 - L(t)) T_{\sigma,\epsilon}(u)$$

Then $L(t)$ has value 1 for $t \in I^{n-1}(\sigma)$ and 0 for $t \in \partial I^{n-1}$, and consequently, $v(t,u)$ has value 1 for $(t,u) \in I^{n-1}(\sigma) \times I$, and $T_{\sigma,\epsilon}(u)$ for $(t,u) \in \partial I^{n-1} \times I$. Clearly, the smooth map $R_{\sigma,\epsilon} \colon I^n \to J^{n-1}$ given by the formula

$$R_{\sigma,\epsilon}(t,u) = (T_{\sigma,\epsilon}(t_1), \cdots, T_{\sigma,\epsilon}(t_{n-1}), v(t,u)).$$

is σ-tame and restricts to the identity on $J^{n-1}(\epsilon)$. □

By combining this with Lemma 3.2, we obtain the following.

Proposition 3.6. *Any ϵ-tame map $f \colon J^{n-1} \to X$ can be extended to a σ-tame map $g \colon I^n \to X$ for any $\sigma < \epsilon$.*

Proof. Define $g \colon I^n \to X$ as the composition $f \circ R_{\sigma,\epsilon}$. Then g is σ-tame and coincides with f on J^{n-1} by Lemma 3.2. □

The following is an immediate consequence of the proposition above.

Theorem 3.7. *Let (X, A) be a pair of cubical subcomplexes of I^m, and $f \colon X \to Y$ be a tame map. Suppose there is a tame homotopy $h \colon A \times I \to Y$ satisfying $h_0 = f|A$. Then there exists a tame homotopy $H \colon X \times I \to Y$ satisfying $H_0 = f$ and $H|A \times I = h$.*

Proof. $X \times I$ is obtained from $A \times I \cup X \times \{0\}$ by successively attaching cubes of the form I^n along its subcomplex $L^{n-1} = \partial I^{n-1} \times I \cup I^{n-1} \times \{0\}$. Thus, Proposition 3.6 (but with J^{n-1} replaced by its copy L^{n-1}) enables us to extend $h \cup f \colon A \times I \cup X \times \{0\} \to Y$ to a tame homotopy $H \colon X \times I \to Y$. □

For any pointed pair of diffeological spaces (X, A, x_0), let

$$i_*\colon \pi_n(A, x_0) \to \pi_n(X, x_0), \quad j_*\colon \pi_n(X, x_0) \to \pi_n(X, A, x_0)$$

be the maps induced, respectively, by the inclusions $i\colon (A, x_0) \to (X, x_0)$ and $j\colon (X, x_0, x_0) \to (X, A, x_0)$, and let

$$\Delta\colon \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0) \quad (n \geq 1)$$

be the map which takes the class of $\phi\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ to the class of its restriction $\phi|I^{n-1}\colon (I^{n-1}, \partial I^{n-1}) \to (A, x_0)$. Here, we identify I^{n-1} with $I^{n-1} \times \{0\} \subset I^n$. Clearly, Δ is a group homomorphism for $n \geq 2$.

Since any element of the homotopy group has a tame representative, we can obtain the homotopy exact sequence by arguing as in the case of topological spaces. (Compare [5, 5.19].)

Proposition 3.8. *Given a pointed pair of diffeological spaces (X, A, x_0), there is an exact sequence of pointed sets*

$$\cdots \to \pi_{n+1}(X, A, x_0) \xrightarrow{\Delta} \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \to \cdots$$
$$\cdots \to \pi_1(X, A, x_0) \xrightarrow{\Delta} \pi_0(A, x_0) \xrightarrow{i_*} \pi_0(X, x_0).$$

There is an alternative interpretation of $\pi_n(X, x_0)$ in terms of basepointed maps $(\partial I^{n+1}, e) \to (X, x_0)$, where $e = (1, \cdots, 1) \in \partial I^{n+1}$.

Lemma 3.9. *For every $n \geq 0$, there is a natural isomorphism*

$$\pi_n(X, x_0) \cong [\partial I^{n+1}, e\,; X, x_0].$$

Proof. Consider the commutative diagram

$$\begin{tikzcd}
{[I^n, \partial I^n\,; X, x_0]} & {[\partial I^{n+1}, J^n\,; X, x_0]} \arrow[l, "i^*"'] \arrow[r, "j^*"] & {[\partial I^{n+1}, e\,; X, x_0]} \\
{[I^n/\partial I^n, *\,; X, x_0]} \arrow[u, "\cong"'] & {[\partial I^{n+1}/J^n, *\,; X, x_0]} \arrow[l, "\cong"'] \arrow[u, "\cong"'] &
\end{tikzcd}$$

induced by the evident inclusions and projections. One easily observes that the vertical arrows are isomorphisms by the property of subductions, and the lower horizontal arrow is an isomorphism induced by a diffeomorphism. It follows by the commutativity of the left hand square that i^* is an isomorphism. Hence, to prove the lemma, it suffice to show that j^* is also an isomorphism.

To see that j_* is surjective, take a tame map $f\colon (\partial I^{n+1}, e) \to (X, x_0)$. Since $I^{n-1} \times \{1\}$ is a deformation retract of J^n and is contractible to e, there exists a tame contracting homotopy $r\colon J^n \times I \to J^n$ of J^n onto e. By applying Theorem 3.7 to the map f and the homotopy $h = (f|J^n) \circ r\colon J^n \times$

$I \to X$, we obtain a homotopy $f \simeq g$ relative to e such that $g(J^n) = x_0$. Hence we have $[f] = j^*([g]) \in [\partial I^{n+1}, e\,; X, x_0]$, meaning that j^* is surjective.

Injectivity of j^* is proved as follows. Let f and g be two tame maps $(\partial I^{n+1}, J^n) \to (X, x_0)$ such that $j^*([f]) = j^*([g])$. Then there are a tame homotopy $H\colon (\partial I^{n+1}, e) \times I \to (X, x_0)$ between $f \circ j$ and $g \circ j$, and a tame homotopy (of homotopies) from $H|(J^n, e) \times I$ to the trivial homotopy $c_{x_0}\colon (J^n, e) \times I \to (x_0, x_0) \subset (X, x_0)$ induced by r. Thus, by Theorem 3.7 again, H is homotopic to H' such that $H'|(J^n, e) \times I = c_{x_0}$, implying that $H'_0 \simeq H'_1\colon (\partial I^{n+1}, J^n) \to (X, x_0)$ and hence $[f] = [H'_0] = [H'_1] = [g]$. □

Suppose $f\colon (\partial I^{n+1}, e) \to (X, x_0)$ is a tame representative of an element of $\pi_n(X, x_0)$, and $\ell\colon I \to X$ a tame path from x_0 to x_1. Then, by applying Proposition 3.6 to the tame homotopy $e \times I \to X$ given by l, we obtain a homotopy $f \simeq g$ such that $g(e) = x_1$ holds. Thus we can construct

$$\ell_\sharp\colon \pi_n(X, x_0) \to \pi_n(X, x_1)$$

to be the map which takes $[f] \in \pi_n(X, x_0)$ to the class $[g] \in \pi_n(X, x_1)$.

We leave it to the reader to verify the following.

Proposition 3.10. *To every tame path $\ell\colon I \to X$ joining x_0 to x_1, there attached a group isomorphism $\ell_\sharp\colon \pi_n(X, A, x_0) \to \pi_n(X, x_1)$. If $\ell'\colon I \to X$ is another tame path joining x_1 to x_2 then we have $(\ell * \ell')_\sharp = \ell'_\sharp \circ \ell_\sharp$.*

4. Fibrations

In this section, we introduce the notion of smooth fibrations by mimicking Serre fibrations in **Top**.

Definition 4.1. Let K be a subset of I^n and let $0 < \epsilon \leq 1/2$. Then a smooth map $f\colon K \to X$ is said to be *ϵ-admissible* if its restriction $f|K \cap F$ is $\epsilon^{\dim F}$-tame for every positive dimensional face F of I^n.

Clearly, ϵ-tameness implies ϵ-admissibility and, conversely, ϵ-admissibility implies ϵ^d-tameness for some $d \leq n$. But admissibility has an advantage over tameness as exhibited by the next proposition and its corollary.

Proposition 4.2. *Let K be a cubical subcomplex of I^n, and $f\colon K \to X$ be a smooth map. Suppose f is ϵ-admissible on a cubical subcomplex L of K. Then there is a homotopy $f \simeq g$ relative to L such that g is ϵ-admissible.*

Proof. It is easy to construct a homotopy $f \simeq f'$ rel L such that f' is σ-tame for $\sigma < \epsilon^{\dim L}$ (cf. Lemma 3.4). Hence we may assume from the beginning that f is a tame map. For $0 \leq j \leq \dim K$, let $\bar{K}^j = L \cup K^j$ be the union of L and the j-skeleton of K. Starting from the constant homotopy $\tilde{h}^0\colon \bar{K}^0 \times I \to X$, we inductively construct a tame homotopy $\tilde{h}^j\colon \bar{K}^j \times I \to X$ from $f|\bar{K}^j$ to an ϵ-admissible map g^j relative to L.

Suppose $\tilde{h}^{j-1}$ exists. Let F be a j-dimensional face not contained in L. As $\partial F \subset \bar{K}^{j-1}$, there is a tame map $h^F\colon (\partial F \times I) \cup (F \times \{0\}) \to X$, which takes (t,u) to $\tilde{h}^{j-1}(t,u)$ if $t \in \partial F$ and to $f(t)$ if $u = 0$. But as $g^{j-1}|\partial F$ is ϵ^{j-1}-tame and $(\partial F \times I) \cup (F \times \{0\})$ is linearly diffeomorphic to J^{j-1}, we can apply Proposition 3.6 with sufficiently small σ and $\sigma' = \epsilon^j$ to obtain a tame extension $\tilde{h}^F\colon F \times I \to X$ such that $g^F = \tilde{h}^F|F \times \{1\}\colon F \to X$ is ϵ-admissible. Thus, if we define $\tilde{h}^j\colon \bar{K}^j \times I \to X$ to be the union $\bigcup_F \tilde{h}^F$, where F runs through j-dimensional faces of K not contained in L, then $\tilde{h}^j$ gives a tame homotopy $f|\bar{K}^j \simeq g^j$ rel L such that $g^j = \bigcup_F g^F$ is ϵ-admissible, proving the induction step. □

This, together with Proposition 3.6, implies the following.

Corollary 4.3. *Any ϵ-admissible map $f\colon J^{n-1} \to X$ can be extended to an ϵ-admissible map $I^n \to X$.*

Consider the following classes of inclusions:

$$\mathcal{I} = \{i_n\colon \partial I^n \to I^n \mid n \geq 0\}, \quad \mathcal{J} = \{j_n\colon J^{n-1} \to I^n \mid n \geq 1\}.$$

Definition 4.4. Let $\mathcal{K}$ be either $\mathcal{I}$ or $\mathcal{J}$. A smooth map $p\colon X \to Y$ is called a *$\mathcal{K}$-fibration* if for every member $K^{n-1} \to I^n$ of $\mathcal{K}$ (i.e. $K^{n-1} = \partial I^n$ or J^{n-1}) and every pair of ϵ-admissible maps $f\colon K^{n-1} \to X$ and $g\colon I^n \to Y$ satisfying $p \circ f = g|K^{n-1}$, there exists an ϵ-admissible map $h\colon I^n \to X$ which makes the two triangles in the diagram below commutative:

$$\begin{tikzcd} K^{n-1} \arrow[r, "f"] \arrow[d] & X \arrow[d, "p"] \\ I^n \arrow[r, "g"] \arrow[ur, dotted, "h"] & Y. \end{tikzcd} \tag{4.1}$$

In particular, $\mathcal{J}$-fibrations are analogy of Serre fibrations in **Top**.

Example 4.5. (1) It follows by Corollary 4.3 that for any diffeological space X the constant map $X \to *$ is a $\mathcal{J}$-fibration.

(2) If $p\colon E \to B$ is a diffeological fiber bundle with fiber F then its pullback by a smooth map from I^n to B is trivial (cf. [5, 8.19, Lemma 2]). But (1) implies that a trivial fiber bundle is a $\mathcal{J}$-fibration, hence so is p.

(3) Given a diffeological space X with basepoint x_0, let $P(X, x_0)$ be the subset of $C^\infty((I,\{1\}),(X,x_0))$ consisting of tame paths $l\colon I \to X$ satisfying $l(1) = x_0$. Then the map $p\colon P(X,x_0) \to X$, which takes a path l to its initial point $l(0)$, is a $\mathcal{J}$-fibration. To see this, let $u\colon J^{n-1} \to P(X,x_0)$ and $v\colon I^n \to X$ be ϵ-admissible maps satisfying $p \circ u = v|J^{n-1}$. Let $K = J^{n-1} \times I \cup I^n \times \{0,1\}$ and $u'\colon K \to X$ be a tame map which takes $(t,s) \in K$ to $u(t)(s)$ if $t \in J^{n-1}$, to $v(t)$ if $s = 0$, and to x_0 if $s = 1$. To obtain

an ϵ-admissible lift $I^n \to P(X, x_0)$ of (u, v), it suffices to extend u' to a tame map $\tilde{u}\colon I^n \times I \to X$ which is ϵ-admissible with respect to the first n coordinates. We accomplish this by extending u' in several steps. Let $A = I^n(\epsilon^n) \times I$, $B = I^{n-1}(\epsilon^n) \times [0, 1-\epsilon^n] \times I$, and $C = I^n \times I - \mathrm{Int}B$. As we have $C = K \cup (\bar{J}^{n-1} \times I)$, where $\bar{J}^{n-1}$ is the closure of the ϵ^n-neighborhood of J^{n-1}, and v is ϵ^n-tame, u' can be extended to a tame map $\tilde{u}'\colon C \to X$ in an evident manner. But as $(A, A \cap C) \cong (I^{n+1}, J^n)$ and $\tilde{u}'$ is tame on $A \cap C$, there exists an extension $\tilde{u}''\colon A \to X$ of $\tilde{u}'|A \cap C$ having enough tameness on $A \cap (I^{n-1} \times \{\epsilon^n\} \times I)$ (cf. Proposition 3.6). It is now clear that $\tilde{u}''$ can be extended trivially to $\tilde{u}'''\colon B \to X$, and the resulting map $\tilde{u} = \tilde{u}' \cup \tilde{u}'''\colon I^{n+1} = C \cup B \to X$ extends u' and is ϵ-admissible with respect to the first n coordinates as desired.

We say that a smooth map $p\colon X \to Y$ is a weak homotopy equivalence if the induced map $p_*\colon \pi_n(X, x) \to \pi_n(Y, p(x))$ is a bijection for every $x \in X$ and $n \geq 0$.

Proposition 4.6. *A smooth map $p\colon X \to Y$ is an $\mathcal{I}$-fibration if and only if it is a $\mathcal{J}$-fibration and a weak homotopy equivalence.*

Proof. Suppose $p\colon X \to Y$ is an $\mathcal{I}$-fibration. We show that p is a weak equivalence, that is, the induced map $p_*\colon \pi_n(X, x) \to \pi_n(Y, p(x))$ is bijective for every $n \geq 0$ and $x \in X$. Let $\gamma\colon (I^n, \partial I^n) \to (Y, p(x))$ be a tame map, and let $c_x\colon \partial I^n \to X$ be the constant map with value $x \in X$. Then we have a commutative square

$$\begin{array}{ccc} \partial I^n & \xrightarrow{c_x} & X \\ \downarrow{\scriptstyle i_n} & & \downarrow{\scriptstyle p} \\ I^n & \xrightarrow{\gamma} & Y. \end{array}$$

Since c_x and γ are ϵ-admissible for some $\epsilon > 0$, there is a lift $\tilde{\gamma}\colon I^n \to X$ satisfying $\tilde{\gamma} \circ i_n = c_x$ and $p \circ \tilde{\gamma} = \gamma$. Thus we have $p_*([\tilde{\gamma}]) = [\gamma]$, implying that p_* is a surjection. To see that p_* is injective, let γ_0 and γ_1 be tame maps $(I^n, \partial I^n) \to (X, x)$ such that $p_*([\gamma_0]) = p_*([\gamma_1])$ holds in $\pi_n(Y, p(x))$. Then there exists a tame homotopy $H\colon I^n \times I \to Y$ between $p \circ \gamma_0$ and $p \circ \gamma_1$ relative to ∂I^n. Let $\gamma\colon \partial I^{n+1} \to X$ be a tame map which takes (t, s) to $\gamma_s(t)$ if $(t, s) \in I^n \times \{0, 1\}$, and to x if $(t, s) \in \partial I^n \times I$. Then we have a commutative square

$$\begin{array}{ccc} \partial I^{n+1} & \xrightarrow{\gamma} & X \\ \downarrow{\scriptstyle i_{n+1}} & & \downarrow{\scriptstyle p} \\ I^n \times I & \xrightarrow{H} & Y. \end{array}$$

Hence there exists a lift $\tilde{H}\colon I^n \times I \to X$ which gives a homotopy $\gamma_0 \simeq \gamma_1$. Thus we have $[\gamma_0] = [\gamma_1]$, showing that p_* is injective.

We now show that p is a $\mathcal{J}$-fibration. Let $0 < \epsilon \leq 1/2$ and take ϵ-admissible maps $f\colon J^{n-1} \to X$ and $g\colon I^n \to Y$ satisfying $p \circ f = g \circ i_n$. Then we have a commutative square

$$\begin{tikzcd}
\partial I^{n-1} \times \{0\} \arrow[r, "f|\partial I^{n-1}\times\{0\}"] \arrow[d, "i_{n-1}"] & X \arrow[d, "p"] \\
I^{n-1} \times \{0\} \arrow[r, "g|I^{n-1}\times\{0\}"] & Y.
\end{tikzcd}$$

Since $f|\partial I^{n-1}\times\{0\}$ and $g|I^{n-1}\times\{0\}$ are ϵ-admissible, there is an ϵ-admissible lift $\tilde{g}\colon I^{n-1} \times \{0\} \to X$, and consequently we can define $\tilde{f}\colon \partial I^n \to X$ to be the union $f \cup \tilde{g}\colon J^{n-1} \cup I^{n-1} \times \{0\} \to X$. Clearly, $\tilde{f}$ is ϵ-admissible, and hence there exists an ϵ-admissible lift $G\colon I^n \to X$ satisfying $p \circ G = g$ and $G \circ i_n = \tilde{f}$. But this means $G|J^{n-1} = f$, implying that p is a $\mathcal{J}$-fibration.

Conversely, suppose $p\colon X \to Y$ is a $\mathcal{J}$-fibration and a weak homotopy equivalence. Let $f\colon \partial I^n \to X$ and $g\colon I^n \to Y$ be ϵ-admissible maps satisfying $p \circ f = g \circ i_n$. We need to show that there is an ϵ-admissible lift $G\colon I^n \to X$ satisfying $p \circ G = g$ and $G \circ i_n = f$. Let $e = (1, \cdots, 1) \in \partial I^n$ and $x = f(e) \in X$. Since $p \circ f = g|\partial I^n$ is null homotopic and p is a weak equivalence, there exists, by Lemma 3.9, a tame homotopy $F\colon \partial I^n \times I \to X$ from f to the constant map. Let us define $H\colon \partial I^n \times I \cup I^n \times \{0\} \to Y$ by

$$H(t,s) = \begin{cases} g(t), & (t,s) \in I^n \times \{0\} \\ p(F(t,s)), & (t,s) \in \partial I^n \times I. \end{cases}$$

Since H is ϵ-admissible, it can be extended by Corollary 4.3 to an ϵ-admissible homotopy $H'\colon I^n \times I \to Y$ from g to $\gamma'\colon (I^n, \partial I^n) \to (Y, p(x))$. But as p is a weak equivalence, there exist a tame map $\gamma\colon (I^n, \partial I^n) \to (X, x)$ and an ϵ-admissible homotopy $H''\colon I^n \times I \to Y$ from γ' to $p \circ \gamma$ relative to ∂I^n. By tameness, we can define smooth maps $F'\colon J^n \to X$ and $G'\colon I^n \times I \to Y$ by the formula,

$$F'(t,s) = \begin{cases} F(t,2s), & (t,s) \in \partial I^n \times [0,1/2] \\ x, & (t,s) \in \partial I^n \times [1/2,1] \\ \gamma(t), & (t,s) \in I^n \times \{1\}, \end{cases}$$

$$G'(t,s) = \begin{cases} H'(t,2s)), & 0 \leq s \leq 1/2 \\ H''(t,2s-1), & 1/2 \leq s \leq 1. \end{cases}$$

Let $F''\colon J^n \to X$ and $G''\colon I^n \times I \to Y$ be ϵ-admissible maps defined by $F''(t,s) = F'(t, \lambda_\epsilon(s))$ and $G''(t,s) = G'(t, \lambda_\epsilon(s))$. Then there exists an ϵ-admissible lift $\tilde{G}\colon I^n \times I \to X$ satisfying $\tilde{G}|J^n = F''$ and $p \circ \tilde{G} = G''$, since $p \circ F'' = G''|J^n$ and p is a $\mathcal{J}$-fibration. Hence we have an ϵ-admissible lift $G = \tilde{G}|I^n \times \{0\}\colon I^n \to X$ satisfying $G \circ i_n = f$ and $p \circ G = g$. $\square$

5. Model category of diffeological spaces

In this section we shall show that the category **Diff** has a model structure by arguing as in the proof of [2, Proposition 8.3].

Definition 5.1. Let $\mathcal{K}$ be either $\mathcal{I}$ or $\mathcal{J}$. A smooth map $i\colon X \to Y$ is called a $\mathcal{K}$-*cofibration* if it has the left lifting property with respect to $\mathcal{K}$-fibrations, that is, for every commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "i"] & E \arrow[d, "p"] \\ Y \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

such that $p\colon E \to B$ is a $\mathcal{K}$-fibration, there exists a lift $Y \to E$ making the two triangles commutative.

Theorem 5.2. *The category* **Diff** *has a structure of a model category by defining a smooth map $h\colon X \to Y$ to be*

(1) *a weak equivalence if h is a weak homotopy equivalence,*
(2) *a fibration if h is a $\mathcal{J}$-fibration, and*
(3) *a cofibration if h is an $\mathcal{I}$-cofibration.*

Observe that every diffeological space X is *fibrant* in sense that the constant map $X \to *$ is a $\mathcal{J}$-fibration (cf. Example 4.5).

We prove Theorem 5.2 by verifying the axioms below (cf. [2]). A fibration or a cofibration is called to be *trivial* if it is a weak homotopy equivalence.

MC1 Finite limits and colimits exist.
MC2 If f and g are maps such that $g \circ f$ is defined and if two of the three maps f, g, $g \circ f$ are weak equivalences, then so is the third.
MC3 If f is a retract of g and g is a fibration, cofibration, or a weak equivalence, then so is f.
MC4 Given a commutative square of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

the dotted arrow exists so as to make the two triangles commutative if either (i) i is a cofibration and p is a trivial fibration, or (ii) i is a trivial cofibration and p is a fibration.

MC5 Any map f can be factored in two ways: (i) $f = p \circ i$, where i is a cofibration and p is a trivial fibration, and (ii) $f = p \circ i$, where i is a trivial cofibration and p is a fibration.

Axiom **MC1** follows from the fact that **Diff** has small limits and colimits, and **MC2** follows from the functoriality of induced maps combined with the change of basepoint homomorphism (Proposition 3.10). Axiom **MC3** is straightforward from the definitions (cf. [2, 8.10]). In order to verify **MC4** and **MC5**, we need several lemmas and propositions.

By Proposition 4.6, all cofibrations have the left lifting property with respect to trivial $\mathcal{J}$-fibrations. Hence we have the following.

Corollary 5.3. *Axiom* **MC4** *holds under the condition* (i).

The rest of the axioms follow from the theorem below, whose proof is deferred until the next section.

Theorem 5.4. *Let $\mathcal{K}$ be either $\mathcal{I}$ or $\mathcal{J}$. Then any smooth map $f\colon X \to Y$ can be factorized as a composition*

$$X \xrightarrow{i_\infty} G^\infty(\mathcal{K}, f) \xrightarrow{p_\infty} Y$$

such that i_∞ is a $\mathcal{K}$-cofibration and p_∞ is a $\mathcal{K}$-fibration. Moreover, i_∞ can be taken as a trivial cofibration when $\mathcal{K} = \mathcal{J}$.

Since $\mathcal{I}$-fibration is a trivial fibration by Proposition 4.6, the factorization

$$X \xrightarrow{i_\infty} G^\infty(\mathcal{I}, f) \xrightarrow{p_\infty} Y$$

implies **MC5** (1), while on the other hand,

$$X \xrightarrow{i_\infty} G^\infty(\mathcal{J}, f) \xrightarrow{p_\infty} Y$$

implies **MC5** (2). Finally, we prove **MC4** (ii), that is,

Proposition 5.5. *Every trivial cofibration has the left lifting property with respect to fibrations.*

Proof. Suppose $i\colon X \to Y$ is a trivial cofibration and $p\colon A \to B$ a fibration. Let $f\colon X \to A$ and $g\colon Y \to B$ be smooth maps such that $p \circ f = g \circ i$ holds. Let us take the factorization $i = p_\infty \circ i_\infty\colon X \to G^\infty(\mathcal{J}, f) \to Y$, where i_∞ is a trivial cofibration and p_∞ is a fibration. Because i and i_∞ are weak equivalences, p_∞ is a weak equivalence, and hence a trivial fibration.

Now, consider the commutative square

$$\begin{tikzcd} X \arrow[r, "i_\infty"] \arrow[d, "i"] & G^\infty(\mathcal{J}, i) \arrow[d, "p_\infty"] \\ Y \arrow[r, equal] \arrow[ur, dotted] & Y. \end{tikzcd}$$

As i is a cofibration, there exists by MC4 (i) a lift $h\colon Y \to G^\infty(\mathcal{J}, i)$ such that $p_\infty \circ h = 1$ and $h \circ i = i_\infty$. Hence we obtain a commutative diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "i"] & X \arrow[r, equal] \arrow[d, "i_\infty"] & X \arrow[r, "f"] \arrow[d, "i"] & A \arrow[d, "p"] \\ Y \arrow[r, "h"] & G^\infty(\mathcal{J}, i) \arrow[r, "p_\infty"] \arrow[urr, dotted] & Y \arrow[r, "g"] & B. \end{tikzcd}$$

As i_∞ is a $\mathcal{J}$-cofibration, there exists a lift $g'\colon G^\infty(\mathcal{J}, i) \to A$ making the diagram commutative. Thus we obtain a desired lift $g' \circ h\colon Y \to A$. $\square$

This completes the proof of Theorem 5.2.

6. Infinite gluing construction

We prove Theorem 5.4 by applying infinite gluing construction to define a factorization $X \to G^\infty(\mathcal{K}, f) \to Y$ for $\mathcal{K} = \mathcal{I}$ and $\mathcal{J}$.

For $0 < \epsilon < \tau \leq 1/2$, let $\tilde{I}^n_{\epsilon,\tau}$ be the n-cube equipped with the diffeology generated by the smooth map $T^n_{\epsilon,\tau}\colon \mathbf{R}^n \to I^n$ (cf. Lemma 3.3). By the definition, $T^n_{\epsilon,\tau}$ restricts to a subduction $I^n \to \tilde{I}^n_{\epsilon,\tau}$.

Lemma 6.1. *For any ϵ-tame map $f\colon I^n \to X$, there exists a smooth map $\tilde{f}\colon \tilde{I}^n_{\epsilon,\tau} \to X$ satisfying $f = \tilde{f} \circ T^n_{\epsilon,\tau}$.*

Proof. Since $T^n_{\epsilon,\tau}$ restricts to bijection $[\epsilon, 1-\epsilon]^n \to \tilde{I}^n_{\epsilon,\tau}$, and since f is ϵ-tame, there is a well defined map $\tilde{f} = f \circ (T^n_{\epsilon,\tau})^{-1}\colon \tilde{I}^n_{\epsilon,\tau} \to X$ which satisfies $f = \tilde{f} \circ T^n_{\epsilon,\tau}$. But as $T^n_{\epsilon,\tau}$ is a subduction, $\tilde{f}$ is smooth by [5, 1.51]. $\square$

Proposition 6.2. *The map $T^n_{\epsilon,\tau}\colon I^n \to \tilde{I}^n_{\epsilon,\tau}$ gives a homotopy inverse to the inclusion $1_{\epsilon,\tau}\colon \tilde{I}^n_{\epsilon,\tau} \to I^n$.*

Proof. Define $F\colon I^n \times I \to I^n$ by $F(t,u) = (1-u)t + uT^n_{\epsilon,\tau}(t)$. Then F gives a homotopy $1 \simeq 1_{\epsilon,\tau} \circ T^n_{\epsilon,\tau}$. On the other hand, if we define $G\colon \tilde{I}_{\epsilon,\tau} \times I \to \tilde{I}_{\epsilon,\tau}$ by $G(t,u) = T^n_{\epsilon,\tau}(F((T^n_{\epsilon,\tau})^{-1}(t), u))$, then G is smooth because its composition with the subduction $T^n_{\epsilon,\tau} \times 1$ is a smooth map $T^n_{\epsilon,\tau} \circ F$, and gives a homotopy $1 \simeq T^n_{\epsilon,\tau} \circ 1_{\epsilon,\tau}$. $\square$

For $0 < \delta < 1/2$, let $J^{n-1}_\delta = \partial I^n \setminus (\delta, 1-\delta)^{n-1} \times \{0\}$.

Lemma 6.3. (1) *Any ϵ-admissible map $f\colon J^{n-1} \to X$ can be extended to an ϵ-admissible map $f_\epsilon\colon J^{n-1}_{\epsilon^{n-1}} \to X$.*

(2) *For any smooth map $p\colon X \to Y$ the following conditions are equivalent with each other.*

(a) *p is a $\mathcal{J}$-fibration.*

(b) *For every pair of ϵ-admissible maps $f\colon J^{n-1}_{\epsilon^{n-1}} \to X$ and $g\colon I^n \to Y$ satisfying $p \circ f = g|J^{n-1}_{\epsilon^{n-1}}$, there exists an ϵ-admissible map $h\colon I^n \to X$ such that $h|J^{n-1}_{\epsilon^{n-1}} = f$ and $p \circ h = g$ hold.*

Proof. (1) Since f is ϵ^{n-2}-tame on

$$J^{n-1}_{\epsilon^{n-1}} \cap I^{n-1} \times \{0\} = I^{n-1} \setminus (\epsilon^{n-1}, 1 - \epsilon^{n-1})^{n-1},$$

we can extend f to an ϵ^{n-1}-admissible map $f_\epsilon\colon J^{n-1}_{\epsilon^{n-1}} \to X$ by assigning

$$f_\epsilon(t, 0) = f(T^{n-1}_{\epsilon^{n-1},\epsilon^{n-2}}(t), 0) \quad \text{for} \quad (t, 0) \in J^{n-1}_{\epsilon^{n-1}} \cap I^{n-1} \times \{0\}.$$

(2) Suppose $p\colon X \to Y$ is a $\mathcal{J}$-fibration. Let $f\colon J^{n-1}_{\epsilon^{n-1}} \to X$ and $g\colon I^n \to Y$ be ϵ-admissible maps satisfying $p \circ f = g|J^{n-1}_{\epsilon^{n-1}}$. Then we have $p \circ f|J^{n-1} = g|J^{n-1}$, and hence there exists an ϵ-admissible map $h\colon I^n \to X$ satisfying $h|J^{n-1} = f|J^{n-1}$ and $p \circ h = g$. But as h is ϵ^{n-1}-tame on $I^{n-1} \times \{0\}$, it must coincides with f on $J^{n-1}_{\epsilon^{n-1}} \cap I^{n-1} \times \{0\}$. Thus we have $h|J^{n-1}_{\epsilon^{n-1}} = f$, implying that p satisfies the condition (b).

Conversely, suppose p satisfies (b). Let $f\colon J^{n-1} \to X$ and $g\colon I^n \to Y$ be ϵ-admissible maps satisfying $p \circ f = g|J^{n-1}$, and let $f_\epsilon\colon J^{n-1}_{\epsilon^{n-1}} \to X$ be an ϵ^{n-1}-admissible extension of f given by (1). Then we have $p \circ f_\epsilon = g|J^{n-1}_{\epsilon^{n-1}}$ because f_ϵ and g are ϵ^{n-1}-tame on $J^{n-1}_{\epsilon^{n-1}} \cap I^{n-1} \times \{0\}$. Thus, there exists by (b) an ϵ-admissible map $h\colon I^n \to X$ satisfying $h|J^{n-1} = f$ and $p \circ h = g$, meaning that p is a $\mathcal{J}$-fibration. ∎

In the sequel, we denote $\tilde{I}^n_{\epsilon^n} = \tilde{I}^n_{\epsilon^n,\epsilon^{n-1}}$, $T^n_{\epsilon^n} = T^n_{\epsilon^n,\epsilon^{n-1}}\colon I^n \to \tilde{I}^n_{\epsilon^n}$, and by K^{n-1}_ϵ either ∂I^n or $J^{n-1}_{\epsilon^{n-1}}$ according as $\mathcal{K}$ is $\mathcal{I}$ or $\mathcal{J}$. For any smooth map $\phi_\epsilon\colon K^{n-1}_\epsilon \to X$, let $X \cup_{\phi_\epsilon} \tilde{I}^n_{\epsilon^n}$ denote the adjunction space given by a pushout square:

$$\begin{tikzcd}
K^{n-1}_\epsilon \arrow[r, "\phi_\epsilon"] \arrow[d, "T^n_{\epsilon^n}"] & X \arrow[d, "i"] \\
\tilde{I}^n_{\epsilon^n} \arrow[r, "\Phi"] & X \cup_{\phi_\epsilon} \tilde{I}^n_{\epsilon^n}.
\end{tikzcd}$$

Proposition 6.4. *Let $\mathcal{K}$ be either $\mathcal{I}$ or $\mathcal{J}$, and K^{n-1}_ϵ be ∂I^n or $J^{n-1}_{\epsilon^{n-1}}$ according as $\mathcal{K}$ is $\mathcal{I}$ or $\mathcal{J}$. Then for any ϵ-admissible map $\phi_\epsilon\colon K^{n-1}_\epsilon \to X$ the inclusion $i\colon X \to X \cup_{\phi_\epsilon} \tilde{I}^n_{\epsilon^n}$ is a $\mathcal{K}$-cofibration.*

Proof. Let $p\colon E \to B$ be a $\mathcal{K}$-fibration and let $f\colon X \to E$ and $g\colon Y \to B$ be smooth maps satisfying $p \circ f = g \circ i$. Then we have a commutative diagram

$$\begin{tikzcd} K_\epsilon^{n-1} \arrow[r, "="] \arrow[d] & K_\epsilon^{n-1} \arrow[r, "\phi_\epsilon"] \arrow[d, "T^n_{\epsilon^n}"] & X \arrow[r, "f"] \arrow[d, "i"] & E \arrow[d, "p"] \\ I^n \arrow[r, "T^n_{\epsilon^n}"] & \tilde{I}^n_{\epsilon^n} \arrow[r, "\Phi"] & X \cup_{\phi_\epsilon} \tilde{I}^n_{\epsilon^n} \arrow[r, "g"] & B. \end{tikzcd} \tag{6.1}$$

Since p is a $\mathcal{K}$-fibration, and since $f \circ \phi_\epsilon$ and $g \circ \Phi \circ T^n_{\epsilon^n}$ are ϵ-admissible, there exists by Lemma 6.3 (2) an ϵ-admissible (hence ϵ^n-tame) lift $h'\colon I^n \to E$ making the diagram commutative, which in turn induces by Lemma 6.1, a smooth map $\tilde{h}'\colon \tilde{I}^n_{\epsilon^n} \to E$ satisfying $h' = \tilde{h}' \circ T^n_{\epsilon^n}$. Now, we have

(1) $h' \circ T^n_{\epsilon^n} | K^{n-1}_\epsilon = h' | K^{n-1}_\epsilon = f \circ \phi_\epsilon$, and
(2) $p \circ \tilde{h}' = p \circ h' \circ (T^n_{\epsilon^n})^{-1} = g \circ \Phi \circ T^n_{\epsilon^n} \circ (T^n_{\epsilon^n})^{-1} = g \circ \Phi$.

Thus, by the property of pushouts, there is a lift $h\colon X \cup_{\phi_\epsilon} \tilde{I}^n_{\epsilon^n} \to E$ such that $h \circ i = f$ and $p \circ h = g$ hold. □

Proposition 6.5. *Suppose $n \geq 1$ and $0 < \epsilon \leq 1/2$. If $\phi_\epsilon\colon J^{n-1}_{\epsilon^{n-1}} \to X$ is an ϵ-admissible map then X is a deformation retract of $X \cup_{\phi_\epsilon} \tilde{I}^n_{\epsilon^n}$.*

Proof. Let $R\colon I^n \to J^{n-1}$ be an ϵ^{n-1}-approximate retraction, say $R = R_{\epsilon^n, \epsilon^{n-1}}$ (cf. Lemma 3.5), and define $h\colon I^n \times I \to I^n$ by the formula

$$h(t,u) = (1-u)t + uR(t), \quad (t,u) \in I^n \times I.$$

Then the following hold.

(1) For each $u \in I$, $h_u = h | I^n \times \{u\}\colon I^n \to I^n$ maps $J^{n-1}_{\epsilon^{n-1}}$ into $J^{n-1}_{\epsilon^{n-1}}$, and restricts to the identity on its ϵ^{n-1}-chamber $J^{n-1}(\epsilon^{n-1})$.
(2) $h_0 = 1$ and h_1 is an ϵ^{n-1}-approximate retraction of I^n onto J^{n-1}.

Let $\tilde{h} = T^n_{\epsilon^n} \circ h \circ (T^n_{\epsilon^n} \times 1)\colon I^n \times I \to \tilde{I}^n_{\epsilon^n}$. Then $\tilde{h}_u$ is ϵ^n-tame for all $u \in I$, and hence there exists by Lemma 6.1 a homotopy $G\colon \tilde{I}^n_{\epsilon^n} \times I \to X \cup_{\phi_\epsilon} \tilde{I}^n_{\epsilon^n}$ such that the diagram below is commutative:

$$\begin{tikzcd} I^n \times I \arrow[r, "T^n_{\epsilon^n} \times 1"] \arrow[d, "\tilde{h}"] & \tilde{I}^n_{\epsilon^n} \times I \arrow[d, "G"] \\ \tilde{I}^n_{\epsilon^n} \arrow[r, "\Phi"] & X \cup_{\phi_\epsilon} \tilde{I}^n_{\epsilon^n}. \end{tikzcd}$$

But then we have

$$G_u \circ T^n_{\epsilon^n} = \Phi \circ \tilde{h}_u = \Phi \circ T^n_{\epsilon^n} \circ h_u \circ T^n_{\epsilon^n} = i \circ \phi_\epsilon \circ h_u \circ T^n_{\epsilon^n} = i \circ \phi_\epsilon \text{ on } J^{n-1}_{\epsilon^{n-1}}$$

by Lemma 3.2, because $i \circ \phi_\epsilon \colon J^{n-1}_{\epsilon^{n-1}} \to X \cup_{\phi_\epsilon} \tilde{I}^n_{\epsilon^n}$ is ϵ^{n-1}-tame and $h_u \circ T^n_{\epsilon^n}$ restricts to the identity on $J^{n-1}(\epsilon^{n-1})$. Hence there exists a map

$$H \colon (X \cup_{\phi_\epsilon} \tilde{I}^n_{\epsilon^n}) \times I \to X \cup_{\phi_\epsilon} \tilde{I}^n_{\epsilon^n}$$

such that the diagram below is commutative.

$$\begin{tikzcd}
X \times I \coprod \tilde{I}^n_{\epsilon^n} \times I \arrow[r, "i \circ pr \bigcup G"] \arrow[d, "(i\times 1)\bigcup(\Phi\times 1)"'] & X \cup_{\phi_\epsilon} \tilde{I}^n_{\epsilon^n} \\
(X \cup_{\phi_\epsilon} \tilde{I}^n_{\epsilon^n}) \times I \arrow[ur, "H"'] &
\end{tikzcd}$$

Since the vertical map $(i \times 1) \bigcup (\Phi \times 1)$ is a subduction, H gives a smooth homotopy relative to X such that $H_0 | \tilde{I}^n_{\epsilon^n} = \Phi \circ T^n_{\epsilon^n}$ and $H_1(\tilde{I}^n_{\epsilon^n}) \subset X$ hold. Now, we can define a retracting homotopy of $X \cup_{\phi_\epsilon} \tilde{I}^n_{\epsilon^n}$ onto X to be the composition $1 \simeq H_0 \simeq H_1$, where $1 \simeq H_0$ is induced by the homotopy $h \colon \tilde{I}^n_{\epsilon^n} \times I \to \tilde{I}^n_{\epsilon^n}$ given by the formula: $h(x, u) = \Phi((1-u)x + uT^n_{\epsilon^n}(x))$ for $(x, u) \in \tilde{I}^n_{\epsilon^n} \times I$. (Cf. Proposition 6.2.) $\square$

We are ready to construct a factorization $X \to G^\infty(\mathcal{K}, f) \to Y$ of a smooth map $f \colon X \to Y$. Let K^{n-1} be either ∂I^n or J^{n-1} according as $\mathcal{K}$ is $\mathcal{I}$ or $\mathcal{J}$. Let $S_n(\mathcal{K}, f)$ be the set of pairs of admissible maps $\phi \colon K^{n-1} \to X$ and $\psi \colon I^n \to Y$ satisfying $f \circ \phi = \psi | K^{n-1}$. Suppose ϕ and ψ are ϵ-admissible. Then by Lemmas 6.1 and 6.3 (1), there are a smooth map $\tilde{\psi} \colon \tilde{I}^n_{\epsilon^n} \to Y$ satisfying $\tilde{\psi} \circ T^n_{\epsilon^n} = \psi$ and an ϵ-admissible map $\phi_\epsilon \colon K^{n-1}_\epsilon \to X$ satisfying $\phi_\epsilon | K^{n-1} = \phi$. Now, let

$$G^1(\mathcal{K}, f) = \bigcup_{n \geq 0} \bigcup_{(\phi,\psi) \in S_n(\mathcal{K}, f)} X \cup_{\phi_\epsilon} \tilde{I}^n_{\epsilon^n}.$$

Then there are natural maps $i_1 \colon X \to G^1(\mathcal{K}, f)$ and $p_1 \colon G^1(\mathcal{K}, f) \to Y$ induced by the inclusions $X \to X \cup_{\phi_\epsilon} \tilde{I}^n_{\epsilon^n}$ and $f \cup \tilde{\psi} \colon X \coprod \tilde{I}^n_{\epsilon^n} \to Y$, respectively, such that we have

$$f = p_1 \circ i_1 \colon X \xrightarrow{i_1} G^1(\mathcal{K}, f) \xrightarrow{p_1} Y.$$

This process can be repeated to construct $G^l(\mathcal{K}, f)$ for all $l > 1$. Suppose $p_{l-1} \colon G^{l-1}(\mathcal{K}, f) \to Y$ exists. Then there are a space $G^l(\mathcal{K}, f) = G^1(\mathcal{K}, p_{l-1})$ together with a factorization

$$p_{l-1} = p_l \circ i_l \colon G^{l-1}(\mathcal{K}, f) \xrightarrow{i_l} G^l(\mathcal{K}, f) \xrightarrow{p_l} Y.$$

Consequently, we obtain a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "i_1"] \arrow[d, "f"] & G^1(\mathcal{K}, f) \arrow[r, "i_2"] \arrow[d, "p_1"] & G^2(\mathcal{K}, f) \arrow[r, "i_3"] \arrow[d, "p_2"] & \cdots \arrow[r, "i_l"] & G^l(\mathcal{K}, f) \arrow[r] \arrow[d, "p_l"] & \cdots \\
Y \arrow[r, "="] & Y \arrow[r, "="] & Y \arrow[r, "="] & \cdots \arrow[r, "="] & Y \arrow[r] & \cdots .
\end{tikzcd}$$

which in turn induces a factorization

$$f = p_\infty \circ i_\infty \colon X \xrightarrow{i_\infty} G^\infty(\mathcal{K}, f) \xrightarrow{p_\infty} Y$$

where $G^\infty(\mathcal{K}, f) = \operatorname{colim} G^l(\mathcal{K}, f)$. To prove Theorem 5.4, we need to verify the following.

(1) $i_\infty \colon X \to G^\infty(\mathcal{K}, f)$ is a $\mathcal{K}$-cofibration.
(2) $i_\infty \colon X \to G^\infty(\mathcal{J}, f)$ is a weak homotopy equivalence.
(3) $p_\infty \colon G^\infty(\mathcal{K}, f) \to Y$ is a $\mathcal{K}$-fibration.

It follows by Proposition 6.4 that the inclusion $i_l \colon G^{l-1}(\mathcal{K}, f) \to G^l(\mathcal{K}, f)$ is a $\mathcal{K}$-cofibration for every $l > 1$, and hence so is the composition

$$i_\infty \colon X = G^0(\mathcal{K}, f) \to \operatorname{colim} G^l(\mathcal{K}, f) = G^\infty(\mathcal{K}, f).$$

Thus (1) holds. Moreover, when $\mathcal{K} = \mathcal{J}$, each $i_l \colon G^{l-1}(\mathcal{J}, f) \to G^l(\mathcal{J}, f)$ is a deformation retract, hence a weak homotopy equivalence, by Proposition 6.5. Clearly, this implies (2). Finally, to prove (3) we need a further lemma.

Lemma 6.6. *Let K be a cubical subcomplex of I^m and G^∞ be the colimit of a sequence of inclusions of diffeological spaces*

$$G^0 \xrightarrow{i_1} G^1 \xrightarrow{i_2} G^2 \xrightarrow{i_3} \cdots \xrightarrow{i_l} G^l \to \cdots$$

Then for any smooth map $f \colon K \to G^\infty$, there exists an $N > 0$ such that the image of f is contained in G^N.

Proof. By the definition, f is smooth if and only if so are its restrictions to the faces of K. Hence it suffices to prove the case $K = I^n$ $(0 < n \leq m)$. Let $\sigma = \lambda^n \colon \mathbf{R}^n \to I^n$. Then the composite $f \circ \sigma$ is a plot of G^∞. One easily observes, by the definition of colimits in **Diff**, that there exist for any $v \in I^n$ an open neighborhood V_v of v and a plot $P_v \colon V_v \to G^{n(v)}$ $(n(v) > 0)$ such that $f \circ \sigma | V_v$ coincides with the composition of P_v with the inclusion $G^{n(v)} \to G^\infty$. Since $I^n \subset \cup_{v \in I^n} V_v$ and I^n is compact, there exist $v_1, \cdots, v_k \in I^n$ such that

$$f(I^n) = f \circ \sigma(I^n) \subset \bigcup\nolimits_{1 \leq j \leq k} G^{n(v_j)}$$

holds. Thus we have $f(I^n) \subset G^N$ for $N = \max\{n(v_j) \mid 1 \leq j \leq k\}$. $\square$

To see that p_∞ is a $\mathcal{K}$-fibration, suppose $\phi\colon K^{n-1} \to G^\infty(\mathcal{K}, f)$ and $\psi\colon I^n \to Y$ are admissible maps satisfying $p_\infty \circ \phi = \psi | K^{n-1}$, where K^{n-1} is ∂I^n or J^{n-1} according as $\mathcal{K}$ is $\mathcal{I}$ or $\mathcal{J}$. Then the image of ϕ is contained in some $G^l(\mathcal{K}, f)$ by Lemma 6.6, and we have a commutative diagram

$$\begin{tikzcd}
K^{n-1} \arrow[r] \arrow[d] & K^{n-1} \arrow[r, "\phi"] \arrow[d] & G^l(\mathcal{K}, f) \arrow[r, "i_{l+1}"] \arrow[d, "p_l"] & G^{l+1}(\mathcal{K}, f) \arrow[r] \arrow[d, "p_{l+1}"] & G^\infty(\mathcal{K}, f) \arrow[d, "p_\infty"] \\
I^n \arrow[r, "T^n_{\epsilon^n}"] & \tilde{I}^n_{\epsilon^n} \arrow[r, "\tilde{\psi}"] \arrow[urr, dotted] & Y \arrow[r, "="] & Y \arrow[r, "="] & Y,
\end{tikzcd}$$

where ϵ is the largest constant such that both ϕ and ψ are ϵ-admissible, and $\tilde{\psi}$ satisfies $\psi = \tilde{\psi} \circ T^n_{\epsilon^n}$ (cf. Proposition 6.1). As (ϕ, ψ) belongs to $S(n, p_l)$, There exist by Lemma 6.3 (1) an extension $\phi_\epsilon\colon K^{n-1}_{\epsilon^{n-1}} \to G^l(\mathcal{K}, f)$ of ϕ and a smooth map $\Phi\colon \tilde{I}^n_{\epsilon^n} \to G^l(\mathcal{K}, f) \cup_{\phi_\epsilon} \tilde{I}^n_{\epsilon^n} \subset G^{l+1}(\mathcal{K}, f)$ making the diagram commutative. It is now clear that the resulting composition $I^n \to G^\infty(\mathcal{K}, f)$ gives a desired lift for the pair (ϕ, ψ), showing that p_∞ is a $\mathcal{K}$-fibration.

This completes the proof of Theorem 5.4.

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