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INSEPARABLE GAUSS MAPS AND DORMANT OPERS

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Abstract. The present paper aims to generalize a result by H. Kaji on Gauss maps in positive characteristic and establish an interaction with the study of dormant opers and Frobenius-projective structures. We prove a correspondence between dormant opers on a smooth projective variety and closed immersions into a projective space with purely inseparable Gauss map. By using this, we determine the subfields of the function field of a smooth curve in positive characteristic induced by Gauss maps. Moreover, this correspondence gives us a Frobenius-projective structure on a Fermat hypersurface.

Introduction

0.1. Let X be an algebraic variety of dimension $l > 0$ over an algebraically closed field k embedded in the projective space $\mathbb{P}^L$ for some $L > 0$. Denote by $\mathrm{Grass}(l+1, L+1)$ the Grassmann variety classifying $(l+1)$-dimensional quotient spaces of the k-vector space k^{L+1}; it may be identified with the space of l-planes in $\mathbb{P}^L$. The *Gauss map* is the rational morphism $\gamma : X \dashrightarrow \mathrm{Grass}(l+1, L+1)$ that assigns to a smooth point x the embedded tangent space to X at x in $\mathbb{P}^L$.

The notion of Gauss map is generalized (cf. § 1.1) by using linear spaces tangent to higher order, often called the osculating spaces; see, e.g., [4], [5], and [30]. Also, a different generalization of Gauss map can be found in, e.g., [40]. The study of the Gauss map and such generalizations have been a subject of algebraic geometry for a long time.

It is a well-known fact that the Gauss map of a smooth non-linear subvariety of a projective space in characteristic 0 is finite and birational onto its image (cf. [40, (I, 2.8)]). On the other hand, when the base field has positive characteristic, the birationality is no longer true in general, and the Gauss map can be inseparable. Various properties of Gauss maps in positive characteristic have been investigated by many mathematicians; see, e.g., [6], [7], [17], [18], [19], [21], [27], and [28].

For example, a result by H. Kaji (cf. [18, Corollary 6.2]) asserts that giving a closed immersion $X \hookrightarrow \mathbb{P}^L$ from a given smooth projective curve X with purely inseparable Gauss map of degree p^N $(N > 0)$ is equivalent to giving a rank 2 vector bundle on the N-th Frobenius twist $X^{(N)}$ of X

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satisfying certain conditions. As mentioned in Remark 2.4 of the present paper, this data may be interpreted as a *dormant* GL_2*-oper of level* N, in the sense of [37, Definition 4.2.1]; that is to say, it gives a certain rank 2 vector bundle on X equipped with both an action of the sheaf $\mathcal{D}_X^{(N-1)}$ (:= the ring of differential operators on X of level $N-1$, introduced in [2]) and a Hodge subbundle satisfying a strict form of Griffiths transversality. We refer the reader to, e.g., [26], [35] for the study of dormant GL_2-opers on curves (which are also known as *dormant indigenous bundles*), and the higher-rank cases were investigated in, e.g., [14], [16], and [38].

0.2. The present paper aims to refine and generalize Kaji's result in order to build an interaction between the studies of dormant opers and Gauss maps in positive characteristic. To do this in a unified formulation involving multi-dimensional varieties, we introduce the notion of a *dormant* (n, N)*-oper* (cf. Definition 2.1), which extends the classical notion of a higher-level dormant oper. (However, our discussion deals essentially only with the case where $n = 2$ or the underlying variety has dimension 1.)

Let X be a smooth projective variety over an algebraically closed field k of characteristic $p > 2$ and $\chi := (n, N, d)$ a triple of positive integers with $1 < n \leq p$. We shall write

$$\mathrm{Op}^{\mathrm{Zzz...}}_{\chi,+\mathrm{imm}} \tag{0.1}$$

(cf. (2.14)) for the set of isomorphism classes of dormant (n, N)-opers on X equipped with certain additional data. On the other hand, we write

$$\mathrm{Gau}^{\mathrm{F}}_{\lambda} \tag{0.2}$$

(cf. (1.7)), where $\lambda := (n - 1, N, d)$, for the set of isomorphism classes of closed immersions $\iota : X \hookrightarrow \mathbb{P}^L$ (for some $L > 0$) of degree d whose Gauss maps of order $n - 1$ factor through the N-th relative Frobenius morphism. Then, the main result in the first half of the present paper is the following assertion.

Theorem A (cf. Theorem 2.9 for the full statement)**.** Suppose that the quadruple (X, n, N, d) satisfies one of the conditions (a) and (b) described in § 2.5. Then, there exists a canonical injection of sets

$$\Xi_\chi : \mathrm{Gau}^{\mathrm{F}}_{\lambda} \hookrightarrow \mathrm{Op}^{\mathrm{Zzz...}}_{\chi,+\mathrm{imm}}. \tag{0.3}$$

Moreover, this map is bijective when $n = 2$.

0.3. We here describe two applications of the above theorem proved in the second half of the present paper. As the first application, we use the bijection $\Xi_{(2,N,d)}$ to determine the subfields of the function field of a given curve induced by Gauss maps.

Now, let k be as above and X a smooth projective curve over k of genus $g > 1$. Denote by $K(X)$ the function field of X and by $\mathcal{K}$ the set of subfields K of $K(X)$ satisfying the following condition: There exists a closed immersion $\iota : X \hookrightarrow \mathbb{P}^L$ (for some $L > 1$) such that the extension of function fields defined by the Gauss map associated to (X, ι) coincides with $K(X)/K$.

H. Kaji proved (cf. [18, Corollaries 2.3 and 4.4]) that $K(X)$ itself belongs to $\mathcal{K}$, and that any subfield in $\mathcal{K}$ is of the form $K(X)^{p^N} := \{v^{p^N} \mid v \in K(X)\}$ for some integer $N \geq 0$, i.e., the inclusion relation $\mathcal{K} \subseteq \{K(X)^{p^N} \mid N \geq 0\}$ holds.

To improve this result, we combine the above theorem with the previous study of higher-level dormant opers on curves developed in [36], in which we have shown the existence of a dormant $(2, N)$-opers for every $N > 0$. The resulting assertion is described as follows.

Theorem B (= Theorem 3.3)**.** Let us keep the above notation, and suppose that $2 < p$ and $p \nmid (g - 1)$. Then, the following equality of sets holds:

$$\mathcal{K} = \left\{ K(X)^{p^N} \,\middle|\, N \geq 0 \right\}. \tag{0.4}$$

In particular, for every nonnegative integer N, there exists a closed immersion $X \hookrightarrow \mathbb{P}^L$ (for some positive integer L) such that the extension of function fields defined by the Gauss map coincides with $K(X)/K(X)^{p^N}$.

0.4. The second application concerns higher-dimensional varieties. We shall recall from [36, Definition 1.2.1] (or [9, Definition 2.1]) the notion of an F^N-projective structure; this is a positive characteristic analogue of the classical notion of a projective structure on a complex manifold discussed in, e.g., [8], [22], and [23]. Roughly speaking, an F^N-projective structure on a smooth variety X (for $N > 0$) is a maximal collection of étale coordinate charts on X valued in $\mathbb{P}^{\mathrm{dim}(X)}$ whose transition functions descend to the N-th Frobenius twist $X^{(N)}$ of X.

One ultimate goal of the study of F^N-projective structures is to give a complete answer to (the positive characteristic version of) the classification problem, starting with S. Kobayashi and T. Ochiai (cf. [22], [23]), of varieties admitting projective structures. In [36], we developed the classification for some classes of varieties, including curves, surfaces, and Abelian varieties. The difficulty is that, unlike the 1-dimensional case, there are nontrivial obstructions for the existence of an F^N-projective structure. Indeed, because of our lack of technical knowledge, only a few examples have been previously found for higher dimensions.

However, the bijection $\Xi_{(2,N,d)}$ asserted in Theorem A enables us to construct an F^N-projective structure by using an example of a projective variety whose Gauss map is in a certain special situation. The assertion obtained in the present paper is described as follows.

Theorem C (= Theorem 4.1)**.** Let N be a positive integer and L an integer with $L > 1$, $L \neq 3$. Denote by X the Fermat hypersurface of degree $p^N + 1$ in $\mathbb{P}^L$ (cf. (4.1)). Then, X admits an F^N-projective structure

$$\mathcal{S}^{\blacklozenge}_{\mathrm{Gau}} \tag{0.5}$$

arising from the Gauss map associated to the natural closed immersion $X \hookrightarrow \mathbb{P}^L$. Moreover, if $p \nmid L(L+1)$, then X admits no F^{2N+1}-projective structures.

As mentioned in Remark 4.3, the existence of such an F^N-projective structure may be thought of as an exotic phenomenon of algebraic geometry in positive characteristic. In fact, any unirational projective complex manifold which is not isomorphic to a projective space, such as a Fermat hypersurface, admits no projective structure.

Also, note that the only previous examples of F^N-projective structures on higher-dimensional varieties except for those on projective spaces were derived from F^N-affine structures on Abelian varieties or smooth curves equipped with a Tango structure. By calculating Chern classes on the Fermat hypersurface X, we see that $\mathcal{S}^{\blacklozenge}_{\mathrm{Gau}}$ cannot be constructed in that way (cf. Remark 4.4). This means that the F^N-projective structure asserted in the above theorem is essentially a new example.

Notation and Conventions. Throughout the present paper, we fix a prime number p and an algebraically closed field k of characteristic p.

By a *variety (over k)*, we mean a connected integral scheme of finite type over k. Moreover, by a *curve*, we mean a variety over k of dimension 1. Unless stated otherwise, we will always be working over k; for example, products of varieties will be taken over k, i.e., $X_1 \times X_2 := X_1 \times_k X_2$.

Let X be a variety over k. We shall write Ω_X (resp., $\mathcal{T}_X$) for the sheaf of 1-forms (resp., the sheaf of vector fields) on X over k. If $\mathcal{V}$ is a vector bundle (i.e., a locally free coherent sheaf) on X, then we denote by $\mathbb{P}(\mathcal{V})$ the projective bundle over X associated to $\mathcal{V}$.

Next, let N be a positive integer. We shall denote by $X^{(N)}$ the *N-th Frobenius twist* of X, i.e., the base-change of X by the p^N-th power map $k \to k$. The *N-th relative Frobenius morphism* is denoted by $F^{(N)}_{X/k} : X \to$

$X^{(N)}$. When $N = 1$, we write $F_{X/k}$ instead of $F^{(1)}_{X/k}$. Also, we set $X^{(0)} := X$ and $F^{(0)}_{X/k} = \mathrm{id}_X$ for simplicity.

Recall from [2, § 2.2] the sheaf of differential operators $\mathcal{D}^{(N-1)}_X := \mathcal{D}^{(N-1)}_{X/\mathrm{Spec}(k)}$ on X of level $N-1$, where $\mathrm{Spec}(k)$ is equipped with the trivial $(N-1)$-PD structure. If ∇ is a left $\mathcal{D}^{(0)}_X$-action on an $\mathcal{O}_X$-module $\mathcal{F}$ extending its $\mathcal{O}_X$-module structure, then we will use the same notation to denote the corresponding connection $\mathcal{F} \to \Omega_X \otimes \mathcal{F}$. Also, for a $\mathcal{D}^{(N-1)}_X$-action ∇ on $\mathcal{F}$ extending its $\mathcal{O}_X$-module structure, we shall write $\nabla^{(0)}$ for the $\mathcal{D}^{(0)}_X$-action (or equivalently, the connection) on $\mathcal{F}$ induced by ∇ via the natural morphism $\mathcal{D}^{(0)}_X \to \mathcal{D}^{(N-1)}_X$.

For an $\mathcal{O}_{X^{(N)}}$-module $\mathcal{G}$, there exists a canonical left $\mathcal{D}^{(N-1)}_X$-action

$$\nabla^{(N-1)}_{\mathcal{G},\mathrm{can}} : \mathcal{D}^{(N-1)}_X \to \mathcal{E}nd_k(F^{(N)*}_{X/k}(\mathcal{G})) \tag{0.6}$$

on the pull-back $F^{(N)*}_{X/k}(\mathcal{G})$ with vanishing p-$(N-1)$-curvature (cf. [24, Definition 3.1.1 and Corollary 3.2.4]). Given an $\mathcal{O}_X$-module $\mathcal{F}$ and a left $\mathcal{D}^{(N-1)}_X$-action ∇ on $\mathcal{F}$ extending its $\mathcal{O}_X$-module structure, we shall write $\mathcal{F}^\nabla$ for the subsheaf of $\mathcal{F}$ on which $\mathcal{D}^{(N-1)+}_X$ acts as zero, where $\mathcal{D}^{(N-1)+}_X$ denotes the kernel of the canonical projection $\mathcal{D}^{(N-1)}_X \twoheadrightarrow \mathcal{O}_X$. Note that $\mathcal{F}^\nabla$ may be regarded as an $\mathcal{O}_{X^{(N)}}$-module via the underlying homeomorphism of $F^{(N)}_{X/k}$. The trivial $\mathcal{D}^{(N-1)}_X$-action on $\mathcal{O}_X$ will be denoted by $\nabla^{(N-1)}_{\mathrm{triv}}$.

Finally, for each positive integer L, we denote the L-dimensional projective space over k by

$$\begin{aligned} \mathbb{P}^L &:= \mathrm{Proj}(k[t_0, \cdots, t_L]) \\ &\quad (= \{[t_0 : t_1 : \cdots : t_L] \mid (t_0, \cdots, t_L) \neq (0, \cdots, 0)\}) . \end{aligned} \tag{0.7}$$

1. Gauss maps in positive characteristic

In this section, we recall the higher-order Gauss map associated to a closed subvariety of a projective space. After that, we will observe that, under a certain assumption, the Gauss map induces an action of the ring of differential operators on a jet bundle (cf. (1.11)).

1.1. Let X be a smooth projective variety over k of dimension $l > 0$. Denote by $\mathcal{I}$ the ideal sheaf defining the diagonal in $X \times X$. Also, for each $i = 1, 2$, we denote by pr_i the i-th projection $X \times X \to X$.

Let us fix a line bundle $\mathcal{L}$ on X and a nonnegative integer m. The sheaf

$$J_m(\mathcal{L}) := \mathrm{pr}_{1*}(\mathrm{pr}_2^*(\mathcal{L}) \otimes \mathcal{O}_{X\times X}/\mathcal{I}^{m+1}) \tag{1.1}$$

forms a vector bundle on X of rank $\binom{l+m}{m}$, and it is called the **m-jet bundle of $\mathcal{L}$**. This sheaf is equipped with an $(m+1)$-step decreasing filtration

$$\{J_m(\mathcal{L})^j\}_{j=0}^{m+1} \tag{1.2}$$

given by putting $J_m(\mathcal{L})^0 := J_m(\mathcal{L})$ and $J_m(\mathcal{L})^j := \mathrm{Ker}\,(J_m(\mathcal{L}) \twoheadrightarrow J_{j-1}(\mathcal{L}))$ $(j = 1, \cdots, m+1)$. For each $j = 0, \cdots, m$, we have an isomorphism of $\mathcal{O}_X$-modules

$$S^j(\Omega_X) \otimes \mathcal{L} \xrightarrow{\sim} J_m(\mathcal{L})^j / J_m(\mathcal{L})^{j+1}, \tag{1.3}$$

where $S^j(\Omega_X)$ denotes the j-th symmetric product of Ω_X over $\mathcal{O}_X$.

Note that $\mathrm{pr}_{1*}(\mathrm{pr}_2^*(\mathcal{L}))$ is canonically isomorphic to the vector bundle $H^0(X, \mathcal{L}) \otimes_k \mathcal{O}_X$. By applying the functor $\mathrm{pr}_{1*}(\mathrm{pr}_2^*(\mathcal{L}) \otimes (-))$ to the quotient $\mathcal{O}_{X \times X} \twoheadrightarrow \mathcal{O}_{X \times X}/\mathcal{I}^{m+1}$, we obtain an $\mathcal{O}_X$-linear morphism

$$H^0(X, \mathcal{L}) \otimes_k \mathcal{O}_X \twoheadrightarrow J_m(\mathcal{L}). \tag{1.4}$$

Next, suppose that we are given a closed immersion $\iota : X \hookrightarrow \mathbb{P}^L$ for some positive integer L. This induces the composite

$$\begin{aligned} \alpha_\iota^m : \mathcal{O}_X^{\oplus(L+1)} & \left(= H^0(\mathbb{P}^L, \mathcal{O}_{\mathbb{P}^L}(1)) \otimes_k \mathcal{O}_X\right) \\ & \to H^0(X, \iota^*(\mathcal{O}_{\mathbb{P}^L}(1))) \otimes_k \mathcal{O}_X \\ & \xrightarrow{(1.4)} J_m(\iota^*(\mathcal{O}_{\mathbb{P}^L}(1))), \end{aligned} \tag{1.5}$$

where the first arrow is the morphism induced from the natural morphism $\mathcal{O}_{\mathbb{P}^L}(1) \to \iota_*(\iota^*(\mathcal{O}_{\mathbb{P}^L}(1)))$. Let $\mathrm{Grass}(\binom{l+m}{m}, L+1)$ denote the Grassmann variety classifying $\binom{l+m}{m}$-dimensional quotient spaces of the k-vector space k^{L+1}. If U_ι^m denotes the open locus of X where α_ι^m is surjective, then the restriction of α_ι^m to U_ι^m determines a morphism

$$\gamma_\iota^m : U_\iota^m \to \mathrm{Grass}(\tbinom{l+m}{m}, L+1). \tag{1.6}$$

We call it the **Gauss map of order** m associated to (X, ι). When $m = 1$, the morphism γ_ι^m coincides with the Gauss map in the classical sense (cf. Introduction).

Given a triple of positive integers $\lambda := (m, N, d)$, we shall denote by

$$\mathrm{Gau}_\lambda^{\mathrm{F}} \tag{1.7}$$

the set of isomorphism classes of closed immersions $\iota : X \hookrightarrow \mathbb{P}^L$ (for some $L > 0$) satisfying the following two conditions:

- The closed subvariety $\mathrm{Im}(\iota)$ of $\mathbb{P}^L$ has degree d;
- $U_\iota^m = X$ and γ_ι^m factors through $F_{X/k}^{(N)}$.

Here, two such closed immersions $\iota_i : X \hookrightarrow \mathbb{P}^{L_i}$ $(i = 1, 2)$ are said to be *isomorphic* if there exists an isomorphism $h : \mathbb{P}^{L_1} \xrightarrow{\sim} \mathbb{P}^{L_2}$ satisfying $\iota_2 = h \circ \iota_1$. Thus, it makes sense to speak of the *isomorphism class* of a closed immersion $\iota : X \hookrightarrow \mathbb{P}^L$ as above.

By putting $\lambda_{N'} := (m, N', d)$ for each positive integer N', we obtain the following sequence of inclusions:

$$\mathrm{Gau}^{\mathrm{F}}_{\lambda_1} \supseteq \mathrm{Gau}^{\mathrm{F}}_{\lambda_2} \supseteq \mathrm{Gau}^{\mathrm{F}}_{\lambda_3} \supseteq \cdots \supseteq \mathrm{Gau}^{\mathrm{F}}_{\lambda_N} \supseteq \cdots . \tag{1.8}$$

1.2. Let $\iota : X \hookrightarrow \mathbb{P}^L$ (where $L > 0$) be a closed immersion, m a nonnegative integer, and N a positive integer. Suppose that $U^m_\iota = X$ and γ^m_ι factors through $F^{(N)}_{X/k}$. Then, we can find a unique morphism $\breve{\gamma} : X^{(N)} \to \mathrm{Grass}(\binom{l+m}{m}, L+1)$ with $\breve{\gamma} \circ F^{(N)}_{X/k} = \gamma^m_\iota$. Let us denote the universal quotient on $\mathrm{Grass}(\binom{l+m}{m}, L+1)$ by

$$q_{\mathrm{univ}} : \mathcal{O}^{\oplus(L+1)}_{\mathrm{univ}} \twoheadrightarrow \mathcal{Q}_{\mathrm{univ}}, \tag{1.9}$$

where $\mathcal{O}_{\mathrm{univ}}$ denotes the structure sheaf. The pull-back of q_{univ} by $\breve{\gamma}$ defines an $\mathcal{O}_{X^{(N)}}$-linear surjection $q_0 : \mathcal{O}^{\oplus(L+1)}_{X^{(N)}} \twoheadrightarrow \mathcal{Q}_0$. It follows from the definition of γ^m_ι that there exists a unique isomorphism $\tau : F^{(N)*}_{X/k}(\mathcal{Q}_0) \xrightarrow{\sim} J_m(\iota^*(\mathcal{O}_{\mathbb{P}^L}(1)))$ which makes the following diagram commute:

$$\begin{array}{ccc} & \mathcal{O}^{\oplus(L+1)}_X & \\ {\scriptstyle F^{(N)*}_{X/k}(q_0)} \swarrow & & \searrow {\scriptstyle \alpha^m_\iota} \\ F^{(N)*}_{X/k}(\mathcal{Q}_0) & \xrightarrow[\tau]{\sim} & J_m(\iota^*(\mathcal{O}_{\mathbb{P}^L}(1))). \end{array} \tag{1.10}$$

The $\mathcal{D}^{(N-1)}_X$-action $\nabla^{(N-1)}_{\mathcal{Q}_0,\mathrm{can}}$ (cf. (0.6)) corresponds, via τ, a $\mathcal{D}^{(N-1)}_X$-action

$$\nabla^{(N-1)}_{\iota,\mathrm{Gau}} : \mathcal{D}^{(N-1)}_X \to \mathcal{E}nd_k(J_m(\iota^*(\mathcal{O}_{\mathbb{P}^L}(1)))) \tag{1.11}$$

on the m-th jet bundle $J_m(\iota^*(\mathcal{O}_{\mathbb{P}^L}(1)))$ of $\iota^*(\mathcal{O}_{\mathbb{P}^L}(1))$. By definition, the $\mathcal{D}^{(N-1)}_X$-action $\nabla^{(N-1)}_{\iota,\mathrm{Gau}}$ has vanishing p-$(N-1)$-curvature (cf. § 2.1 discussed below).

2. Dormant (n, N)-opers on a variety

In this section, the classical definition of a (higher-level) dormant oper is generalized to multi-dimensional varieties. The main result of this section describes a relationship between higher-order Gauss maps and generalized dormant opers (cf. Theorem 2.9).

2.1. Let X be a smooth projective variety over k of dimension $l > 0$, and let N, n be two integers with $N > 0$, $p \geq n > 1$. For a rank n vector bundle $\mathcal{V}$ on X and an integer a with $1 \leq a < p$, we shall write $T^a(\mathcal{V})$ (cf. [34, Definition 3.4]) for the subbundle of $\mathcal{V}^{\otimes a}$ generated locally by various sections $\sum_{\sigma \in \mathfrak{S}_a} \breve{e}_{\sigma(1)} \otimes \cdots \otimes \breve{e}_{\sigma(a)}$, where $\mathfrak{S}_a$ denotes the symmetric group of a letters and each $\breve{e}_i$ $(i = 1, \cdots, a)$ is an element in a fixed local basis $\{e_1, \cdots, e_n\}$ of $\mathcal{V}$. Also, we set $T^0(\mathcal{V}) := \mathcal{O}_X$. Note that the subbundle $T^a(\mathcal{V})$ does not depend on the choice of the local basis $\{e_1, \cdots, e_n\}$, and that since $a < p$ it is isomorphic to the a-th symmetric product $S^a(\mathcal{V})$ of $\mathcal{V}$ over $\mathcal{O}_X$ via the composite of natural morphisms $T^a(\mathcal{V}) \hookrightarrow \mathcal{V}^{\otimes a} \twoheadrightarrow S^a(\mathcal{V})$.

Next, we shall recall from [24] the higher-level generalization of p-curvature. Denote by $\mathcal{K}_X^{(N-1)}$ the kernel of the morphism $\mathcal{D}_X^{(N-1)} \to \mathcal{E}nd_k(\mathcal{O}_X)$ defining the trivial $\mathcal{D}_X^{(N-1)}$-action on $\mathcal{O}_X$. If $\mathcal{V}$ is as above and ∇ is a $\mathcal{D}_X^{(N-1)}$-action on $\mathcal{V}$ extending its $\mathcal{O}_X$-module structure, then we shall refer to the composite

$$
{}^p\psi_\nabla : \mathcal{K}_X^{(N-1)} \hookrightarrow \mathcal{D}_X^{(N-1)} \xrightarrow{\nabla} \mathcal{E}nd_k(\mathcal{V}) \tag{2.1}
$$

as the p-$(N-1)$-**curvature** of ∇ (cf. [24, Definition 3.1.1]).

Now, let us consider a collection of data

$$
\mathscr{F}^\heartsuit := (\mathcal{F}, \nabla, \{\mathcal{F}^j\}_{j=0}^n) \tag{2.2}
$$

consisting of a vector bundle $\mathcal{F}$ on X, a $\mathcal{D}_X^{(N-1)}$-action ∇ on $\mathcal{F}$ extending its $\mathcal{O}_X$-module structure, and an n-step decreasing filtration $\{\mathcal{F}^j\}_{j=0}^n$ on $\mathcal{F}$ such that $\mathcal{F}^0 = \mathcal{F}$, $\mathcal{F}^n = 0$, and $\mathcal{F}/\mathcal{F}^1$ is a line bundle.

Definition 2.1. (i) We say that $\mathscr{F}^\heartsuit$ is an (n, N)**-oper** on X if it satisfies the following conditions:

- For each $j = 1, \cdots, n-1$, the inclusion relation $\nabla^{(0)}(\mathcal{F}^j) \subseteq \Omega_X \otimes \mathcal{F}^{j-1}$ holds and the $\mathcal{O}_X$*-linear* morphism

$$
\mathrm{KS}^j_{\mathscr{F}^\heartsuit} : \mathcal{F}^j/\mathcal{F}^{j+1} \to \Omega_X \otimes (\mathcal{F}^{j-1}/\mathcal{F}^j) \tag{2.3}
$$

  induced naturally by $\nabla^{(0)}$ is injective. We call $\mathrm{KS}^j_{\mathscr{F}^\heartsuit}$ the j*-th Kodaira-Spencer map* associated to $\mathscr{F}^\heartsuit$.
- For each $j = 0, \cdots, n-1$, the image of the $\mathcal{O}_X$-linear morphism

$$
\mathcal{F}^j/\mathcal{F}^{j+1} \to \Omega_X^{\otimes j} \otimes (\mathcal{F}/\mathcal{F}^1) \tag{2.4}
$$

  obtained by composing various $\mathrm{KS}^{j'}_{\mathscr{F}^\heartsuit}$'s coincides with $T^j(\Omega_X) \otimes (\mathcal{F}/\mathcal{F}^1) \subseteq \Omega_X^{\otimes j} \otimes (\mathcal{F}/\mathcal{F}^1)$. (If either $l = 1$ or $n = 2$ is satisfied, then this condition is equivalent to the condition that (2.4) is an isomorphism for every j.)

Moreover, the notion of an isomorphism between two (n, N)-opers can be defined in a natural manner, so we will omit the details of that definition.

(ii) An (n, N)-oper $\mathscr{F}^\heartsuit := (\mathcal{F}, \nabla, \{\mathcal{F}^j\}_{j=0}^n)$ is called **dormant** if ∇ has vanishing p-$(N-1)$-curvature, i.e., the equality ${}^p\psi_\nabla = 0$ holds.

Remark 2.2. If the underlying variety X has dimension 1, then the definition of an oper using the ring of higher-level differential operators can be found in [37, Definition 4.2.1]. In the cace of $n = 2$ (but X is arbitrary), an equivalent definition was discussed in [36, Definition 2.3.1] under the name of *(dormant) indigenous $\mathcal{D}_X^{(N-1)}$-module.* Also, it follows from [36, Theorem A] that, when $p \nmid (\dim(X)+1)$, equivalence classes (with respect to a certain equivalence relation) of dormant $(2, N)$-opers are in bijection with what we call *F^N-projective structures.* (In § 4.1, we will mention briefly the definition of an F^N-projective structure.)

Remark 2.3. Here, let us recall a typical example of a dormant $(p, 1)$-oper on a multi-dimensional variety provided by X. Sun (cf. [34, Theorem 3.7]); this is a generalization of the dormant $(p, 1)$-oper on a curve discussed in [16, § 5].

Given a line bundle $\mathcal{L}$ on X, we shall set $\mathcal{V} := F_{X/k}^*(F_{X/k*}(\mathcal{L}))$. The $\mathcal{O}_X$-module $\mathcal{V}$ forms a vector bundle on X and admits a connection $\nabla := \nabla^{(0)}_{F_{X/k*}(\mathcal{L}),\mathrm{can}}$ with vanishing p-curvature. This sheaf is equipped with a p-step decreasing filtration $\{\mathcal{V}^j\}_{j=0}^p$ given by the following construction:

- $\mathcal{V}^0 := \mathcal{V}$ and $\mathcal{V}^1$ is the kernel of the morphism $\mathcal{V} \twoheadrightarrow \mathcal{L}$ corresponding to the identity morphism $\mathrm{id}_{F_{X/k*}(\mathcal{L})}$ via the adjunction relation "$F_{X/k}^*(-) \dashv F_{X/k*}(-)$".
- For each $j = 2, \cdots, p$, we define $\mathcal{V}^j$ inductively as follows:

$$\mathcal{V}^j := \mathrm{Ker}\left(\mathcal{V}^{j-1} \xrightarrow{\nabla} \Omega_X \otimes \mathcal{V}^{j-2} \xrightarrow{\text{quotient}} \Omega_X \otimes (\mathcal{V}^{j-2}/\mathcal{V}^{j-1})\right). \tag{2.5}$$

Then, the resulting collection $(\mathcal{V}, \nabla, \{\mathcal{V}^j\}_{j=0}^p)$ forms a dormant $(p, 1)$-oper on X.

Remark 2.4. Suppose that X is a smooth projective curve of genus $g > 1$ and N is a positive integer. According to [18, Corollary 6.2], the existence of a closed immersion $X \hookrightarrow \mathbb{P}^L$ with purely inseparable Gauss map of degree p^N is equivalent to the existence of a rank 2 vector bundle $\mathcal{G}$ on $X^{(N)}$ satisfying the following condition:

$(*)_\mathcal{G}$: $F_{X^{(1)}/k}^{(N-1)*}(\mathcal{G})$ is stable and $F_{X/k}^{(N)*}(\mathcal{G}) \cong J_1(\mathcal{L})$ for some line bundle $\mathcal{L}$ on X.

In this Remark, we shall examine the relationship between such vector bundles $\mathcal{G}$ and dormant $(2, N)$-opers. Let us take a dormant $(2, N)$-oper $\mathscr{F}^\heartsuit := (\mathcal{F}, \nabla, \{\mathcal{F}^j\}_{j=0}^2)$ on X.

First, we shall prove the claim that, *for every positive integer N' with $N' \leq N$, the rank 2 vector bundle $F^{(N-N')*}_{X^{(N')}/k}(\mathcal{F}^\nabla)$ on $X^{(N')}$ is stable.* (The following discussion is available for a general n, but we focus on the rank 2 case for simplicity.) Suppose, on the contrary, that $F^{(N-N')*}_{X^{(N')}/k}(\mathcal{F}^\nabla)$ is unstable, i.e., there exists a line subbundle $\mathcal{L}$ of $F^{(N-N')*}_{X^{(N')}/k}(\mathcal{F}^\nabla)$ of degree $\geq \frac{1}{2} \cdot a$, where $a := \deg(F^{(N-N')*}_{X^{(N')}/k}(\mathcal{F}^\nabla))$. Then, $\deg(F^{(N')*}_{X/k}(\mathcal{L})) = p^{N'} \cdot \deg(\mathcal{L}) \geq \frac{p^{N'}}{2} \cdot a$. Since $\mathrm{KS}^1_{\mathscr{F}^\heartsuit}$ is an isomorphism, we have

$$\begin{aligned} \deg(\mathcal{F}^1) - \deg(\mathcal{F}/\mathcal{F}^1) &= \deg(\Omega_X \otimes (\mathcal{F}/\mathcal{F}^1)) - \deg(\mathcal{F}/\mathcal{F}^1) \\ &= 2g-2. \end{aligned} \tag{2.6}$$

On the other hand, the following equalities hold:

$$\begin{aligned} \deg(\mathcal{F}^1) + \deg(\mathcal{F}/\mathcal{F}^1) &= \deg(\mathcal{F}) \\ &= \deg(F^{(N')*}_{X/k}(F^{(N-N')*}_{X^{(N')}/k}(\mathcal{F}^\nabla))) \\ &= p^{N'} \cdot a, \end{aligned} \tag{2.7}$$

where the second equality follows from the isomorphism

$$\left(F^{(N')*}_{X/k}(F^{(N-N')*}_{X^{(N')}/k}(\mathcal{F}^\nabla)) =\right) F^{(N)*}_{X/k}(\mathcal{F}^\nabla) \xrightarrow{\sim} \mathcal{F} \tag{2.8}$$

resulting from [24, Corollary 3.2.4]. It follows from (2.6) and (2.7) that $\deg(\mathcal{F}^1) = \frac{p^{N'}}{2} \cdot a + g - 1$ and $\deg(\mathcal{F}/\mathcal{F}^1) = \frac{p^{N'}}{2} \cdot a - g + 1$. By comparing the respective degrees of $F^{(N')*}_{X/k}(\mathcal{L})$ and $\mathcal{F}/\mathcal{F}^1$, we see that the composite of natural morphisms

$$F^{(N')*}_{X/k}(\mathcal{L}) \hookrightarrow F^{(N')*}_{X/k}(F^{(N-N')*}_{X^{(N')}/k}(\mathcal{F}^\nabla)) \xrightarrow{(2.8)} \mathcal{F} \twoheadrightarrow \mathcal{F}/\mathcal{F}^1 \tag{2.9}$$

coincides with the zero map. Hence, the composite of the first two morphisms in (2.9), which we shall denote by h, factors through the inclusion $\mathcal{F}^1 \hookrightarrow \mathcal{F}$. The resulting morphism $F^{(N')*}_{X/k}(\mathcal{L}) \to \mathcal{F}^1$ is injective and an isomorphism at the generic point η of X. Let us identify each local section of $F^{(N')*}_{X/k}(\mathcal{L})$ with its image via this injection. Then, the stalk of $\mathcal{F}^1$ at η is closed under the connection $\nabla^{(0)}_{\mathcal{L},\mathrm{can}}$. Since $\nabla^{(0)}_{\mathcal{L},\mathrm{can}}$ is compatible with ∇ via h, the stalk of $\mathcal{F}^1$ at η is also closed under ∇. But, it contradicts the assumption that $\mathrm{KS}^1_{\mathscr{F}^\heartsuit}$ is an isomorphism. Consequently, $F^{(N-N')*}_{X^{(N')}/k}(\mathcal{F}^\nabla)$

turns out to be stable, and the proof of the claim is completed. In particular, $F^{(N-1)*}_{X^{(1)}/k}(\mathcal{F}^\nabla)$ is stable.

Next, observe that, since $\mathrm{KS}^1_{\mathscr{F}^\heartsuit}$ is an isomorphism, the following composite is an isomorphism:

$$\mathcal{D}^{(N-1)}_{X,\leq 1} \otimes (\mathcal{F}/\mathcal{F}^1)^\vee \xrightarrow{\text{inclusion}} \mathcal{D}^{(N-1)}_X \otimes \mathcal{F}^\vee \to \mathcal{F}^\vee, \tag{2.10}$$

where $\mathcal{D}^{(N-1)}_{X,\leq 1}$ denotes the subsheaf of $\mathcal{D}^{(N-1)}_X$ consisting of differential operators of order ≤ 1 and the second arrow denotes the morphism induced naturally by the dual of ∇. If we write $\mathcal{L} := \mathcal{F}/\mathcal{F}^1$, then the dual of this composite determines an isomorphism $\mathcal{F} \xrightarrow{\sim} J_1(\mathcal{L})$. (This isomorphism preserves the filtration, i.e., restricts to an isomorphism $\mathcal{F}^1 \xrightarrow{\sim} J_1(\mathcal{L})^1$.) Thus, we conclude that the rank 2 vector bundle $\mathcal{F}^\nabla$ on $X^{(N)}$ satisfies the condition $(*)_{\mathcal{F}^\nabla}$ described above.

One may verify that *the resulting assignment $\mathscr{F}^\heartsuit \mapsto \mathcal{F}^\nabla$ gives a bijective correspondence between dormant $(2, N)$-opers on X and rank 2 vector bundles $\mathcal{G}$ on $X^{(N)}$ satisfying $(*)_\mathcal{G}$.*

2.2. We shall prove the following assertion concerning dormant (n, N)-opers.

Proposition 2.5. Let $\mathscr{F}^\heartsuit := (\mathcal{F}, \nabla, \{\mathcal{F}^j\}_j)$ be a dormant (n, N)-oper on X. Denote by σ the morphism $X \to \mathbb{P}(\mathcal{F})$ induced, via projectivization, from the natural quotient $\mathcal{F} \to \mathcal{F}/\mathcal{F}^1$. Also, denote by $\overline{\sigma}$ the composite

$$\overline{\sigma} : X \xrightarrow{\sigma} \mathbb{P}(\mathcal{F}) \left(= X \times_{X^{(N)}} \mathbb{P}(\mathcal{F}^\nabla)\right) \xrightarrow{\text{projection}} \mathbb{P}(\mathcal{F}^\nabla). \tag{2.11}$$

Then, the following assertions hold:

(i) Suppose that $\dim(X) = 1$. Then, $\overline{\sigma}$ is birational onto its image.
(ii) Suppose that $n = 2$ (but $\dim(X)$ is arbitrary). Then, $\overline{\sigma}$ is a closed immersion.

Proof. First, we shall consider assertion (i). Denote by Y the nomalization of the image $\mathrm{Im}(\overline{\sigma})$ of $\overline{\sigma}$. Suppose that the field extension $K(X)/K(Y)$ is nontrivial. Since $F^{(N)}_{X/k} : X \to X^{(N)}$ factors through a morphism $h : X \to Y$, there exists an integer M with $1 \leq M \leq N$ such that $Y = X^{(M)}$ and $h = F^{(M)}_{X/k}$. Denote by $\mathcal{G}$ the pull-back of $\mathcal{F}^\nabla$ to $X^{(M)}$. In particular, its pull-back $F^{(M)*}_{X/k}(\mathcal{G})$ may be canonically identified with $\mathcal{F}$. The section $(Y =) X^{(M)} \to \mathbb{P}(\mathcal{G})$ induced by the composite of the normalization $Y \to \mathrm{Im}(\overline{\sigma})$ and the inclusion $\mathrm{Im}(\overline{\sigma}) \hookrightarrow \mathbb{P}(\mathcal{F}^\nabla)$ determines a surjection $\mathcal{G} \twoheadrightarrow \mathcal{Q}$ for some line bundle $\mathcal{Q}$ on $X^{(M)}$. Under the identification $F^{(M)*}_{X/k}(\mathcal{G}) = \mathcal{F}$, the pull-back of this surjection $\mathcal{G} \twoheadrightarrow \mathcal{Q}$ by $F^{(M)}_{X/k}$ coincides with the natural projection $\mathcal{F} \twoheadrightarrow \mathcal{F}/\mathcal{F}^1$, and the equality $\nabla^{(0)} = (\nabla^{(M-1)}_{\mathcal{G},\mathrm{can}})^{(0)}$ holds. Hence, the subbundle $\mathcal{F}^1$ of $\mathcal{F}$ (which may be identified with the pull-back of $\mathrm{Ker}\,(\mathcal{G} \twoheadrightarrow \mathcal{Q})$) is closed under $\nabla^{(0)}$. But, this contradicts the fact that $\mathrm{KS}^1_{\mathscr{F}^\heartsuit}$ is nonzero. Thus, we have $K(X) = K(Y)$, meaning that $\overline{\sigma}$ is birational onto its image. This completes the proof of assertion (i).

Next, we shall prove assertion (ii). For each point x of X, we can find an open neighborhood U of x in X such that there exists an $\mathcal{O}_{U^{(N)}}$-linear isomorphism $\mathcal{F}^\nabla|_{U^{(N)}} \xrightarrow{\sim} \mathcal{O}^{\oplus l+1}_{U^{(N)}}$. Let us consider the isomorphism of $\mathbb{P}^l$-bundles

$$\mathbb{P}(\mathcal{F}^\nabla|_{U^{(N)}}) \xrightarrow{\sim} U^{(N)} \times \mathbb{P}^l \tag{2.12}$$

induced by this isomorphism. Since $\mathrm{KS}^1_{\mathscr{F}^\heartsuit}$ is an isomorphism, the composite

$$U \xrightarrow{\overline{\sigma}|_U} \mathbb{P}(\mathcal{F}^\nabla|_{U^{(N)}}) \xrightarrow{(2.12)} U^{(N)} \times \mathbb{P}^l \xrightarrow{\text{projection}} \mathbb{P}^l \tag{2.13}$$

is étale (cf. [36, Corollary 1.6.2]). This implies that the restriction $\overline{\sigma}|_U$ of $\overline{\sigma}$ is unramified. By applying this argument to various points x of X, we see that $\overline{\sigma}$ is unramified. Moreover, since $F^{(N)}_{X/k}$ factors through $\overline{\sigma}$, the morphism $\overline{\sigma}$ is universally injective. Thus, $\overline{\sigma}$ turns out to be a closed immersion. This completes the proof of assertion (ii). □

Given a triple of positive integers $\chi := (n, N, d)$ with $1 < n \leq p$, we shall write

$$\mathrm{Op}^{\text{Zzz...}}_{\chi,+\mathrm{bir}} \ \left(\text{resp., } \mathrm{Op}^{\text{Zzz...}}_{\chi,+\mathrm{imm}}\right) \tag{2.14}$$

for the set of isomorphism classes of pairs $f := (\mathscr{F}^\heartsuit, q)$ consisting of a dormant (n, N)-oper $\mathscr{F}^\heartsuit := (\mathcal{F}, \nabla, \{\mathcal{F}^j\}_j)$ on X and a surjective morphism of $\mathcal{D}^{(N-1)}_X$-modules $q : (\mathcal{O}_X, \nabla^{(N-1)}_{\mathrm{triv}})^{\oplus(L_f+1)} \twoheadrightarrow (\mathcal{F}, \nabla)$ for some $L_f > 0$ that satisfies the following two conditions:

- The morphism
$$\iota_f : X \to \mathbb{P}^{L_f} \tag{2.15}$$
determined, via projectivization, by the composite of q and the surjection $\mathcal{F} \twoheadrightarrow \mathcal{F}/\mathcal{F}^1$ is birational onto its image (resp., a closed immersion);
- The degree of the closed subvariety $\mathrm{Im}(\iota_f)$ of $\mathbb{P}^{L_f}$ is equal to d. (This is equivalent to the condition that $\mathcal{F}/\mathcal{F}^1$ has degree d with respect to the ample line bundle $\iota_f^*(\mathcal{O}_{\mathbb{P}^{L_f}}(1))$, in the sense of [11, Definition 1.2.11].)

Here, two such pairs $f_i := (\mathscr{F}_i^{\heartsuit}, q_i)$ $(i = 1, 2)$ are said to be *isomorphic* if there exists a pair $(h_{\mathscr{F}}, h_{\mathcal{O}})$ consisting of an isomorphism of (n, N)-opers $h_{\mathscr{F}} : \mathscr{F}_1^{\heartsuit} \xrightarrow{\sim} \mathscr{F}_2^{\heartsuit}$ and an isomorphism of $\mathcal{D}_X^{(N-1)}$-modules

$$h_{\mathcal{O}} : (\mathcal{O}_X, \nabla_{\mathrm{triv}}^{(N-1)})^{\oplus(L_{f_1}+1)} \xrightarrow{\sim} (\mathcal{O}_X, \nabla_{\mathrm{triv}}^{(N-1)})^{\oplus(L_{f_2}+1)} \tag{2.16}$$

satisfying $q_2 \circ h_{\mathcal{O}} = h_{\mathscr{F}} \circ q_1$. Thus, we can define the *isomorphism class* of a pair $f := (\mathscr{F}^{\heartsuit}, q)$ as above. It is clear that $\overset{\mathrm{Zzz...}}{\mathrm{Op}}_{\chi,+\mathrm{imm}} \subseteq \overset{\mathrm{Zzz...}}{\mathrm{Op}}_{\chi,+\mathrm{bir}}$.

2.3. Hereinafter, we shall use the notation $\Box$ to denote either "bir" or "imm". Let $f := (\mathscr{F}^{\heartsuit}, q)$ (where $\mathscr{F}^{\heartsuit} := (\mathcal{F}, \nabla, \{\mathcal{F}^j\}_j)$) be a pair classified by $\overset{\mathrm{Zzz...}}{\mathrm{Op}}_{\chi,+\Box}$. For a positive integer N' with $N' < N$, denote by $\nabla^{(N'-1)}$ the $\mathcal{D}_X^{(N'-1)}$-action on $\mathcal{F}$ induced from ∇ via the natural morphism $\mathcal{D}_X^{(N'-1)} \to \mathcal{D}_X^{(N-1)}$. Then, the collection $\mathscr{F}^{\heartsuit(N')} := (\mathcal{F}, \nabla^{(N'-1)}, \{\mathcal{F}^j\}_j)$ forms a dormant (n, N')-oper and the morphism q determines a surjective morphism of $\mathcal{D}_X^{(N'-1)}$-modules $(\mathcal{O}_X, \nabla_{\mathrm{triv}}^{(N'-1)})^{\oplus(L_f+1)} \twoheadrightarrow (\mathcal{F}, \nabla^{(N'-1)})$. In particular, the pair $f^{(N')} := (\mathscr{F}^{\heartsuit(N')}, q)$ is an element of $\overset{\mathrm{Zzz...}}{\mathrm{Op}}_{\chi,+\Box}$, where $\chi' := (n, N', d)$. The map of sets $\overset{\mathrm{Zzz...}}{\mathrm{Op}}_{\chi,+\Box} \to \overset{\mathrm{Zzz...}}{\mathrm{Op}}_{\chi',+\Box}$ given by $f \mapsto f^{(N')}$ is verified to be injective. This injection allows us to regard $\overset{\mathrm{Zzz...}}{\mathrm{Op}}_{\chi,+\Box}$ as a subset of $\overset{\mathrm{Zzz...}}{\mathrm{Op}}_{\chi',+\Box}$.

Thus, by putting $\chi_{N'} := (n, N', d)$ for each positive integer N', we obtain the following diagram of inclusions:

$$\begin{array}{ccccccccc} \overset{\mathrm{Zzz...}}{\mathrm{Op}}_{\chi_1,+\mathrm{bir}} & \supseteq & \overset{\mathrm{Zzz...}}{\mathrm{Op}}_{\chi_2,+\mathrm{bir}} & \supseteq & \overset{\mathrm{Zzz...}}{\mathrm{Op}}_{\chi_3,+\mathrm{bir}} & \supseteq \cdots \supseteq & \overset{\mathrm{Zzz...}}{\mathrm{Op}}_{\chi_N,+\mathrm{bir}} & \supseteq \cdots \\ \cup\!| & & \cup\!| & & \cup\!| & \cdots & \cup\!| & \cdots \\ \overset{\mathrm{Zzz...}}{\mathrm{Op}}_{\chi_1,+\mathrm{imm}} & \supseteq & \overset{\mathrm{Zzz...}}{\mathrm{Op}}_{\chi_2,+\mathrm{imm}} & \supseteq & \overset{\mathrm{Zzz...}}{\mathrm{Op}}_{\chi_3,+\mathrm{imm}} & \supseteq \cdots \supseteq & \overset{\mathrm{Zzz...}}{\mathrm{Op}}_{\chi_N,+\mathrm{imm}} & \supseteq \cdots . \end{array} \tag{2.17}$$

The following assertion gives a necessary condition for the set $\overset{\mathrm{Zzz...}}{\mathrm{Op}}_{\chi,+\mathrm{bir}}$ being nonempty.

Proposition 2.6. Suppose that there exists a pair $f := (\mathscr{F}^{\heartsuit}, q)$ classified by $\overset{\mathrm{Zzz...}}{\mathrm{Op}}_{\chi,+\mathrm{bir}}$. Denote by $\deg(\Omega_X)$ the degree of Ω_X with respect to the ample divisor H determined by $\iota_f^*(\mathcal{O}_{\mathbb{P}^{L_f}}(1))$, i.e., $\deg(\Omega_X) := c_1(\Omega_X) \cdot H^{l-1}$. Then, we have

$$\frac{1}{p^N} \cdot \left(\deg(\Omega_X) + \frac{d(l+1)}{n-1}\right) \cdot \binom{l+n-1}{n-2} \in \mathbb{Z}. \tag{2.18}$$

In particular, if X is a smooth projective curve of genus g, then $\overset{\mathrm{Zzz...}}{\mathrm{Op}}_{\chi,+\mathrm{bir}} = \overset{\mathrm{Zzz...}}{\mathrm{Op}}_{\chi,+\mathrm{imm}} = \emptyset$ unless $p^N \mid n((n-1)(g-1)+d)$.

Proof. Let $(\mathcal{F}, \nabla, \{\mathcal{F}^j\}_j)$ be the collection of data defining $\mathscr{F}^\heartsuit$. It follows from [34, Lemma 4.3] that for each integer j with $0 \leq j \leq n-1 \,(< p)$, the following equalities hold:

$$\begin{aligned}(2.19) \qquad & \deg(T^j(\Omega_X) \otimes (\mathcal{F}/\mathcal{F}^1)) \\ &= \deg(T^j(\Omega_X)) + \deg(\mathcal{F}/\mathcal{F}^1) \cdot \mathrm{rk}(T^j(\Omega_X)) \\ &= \deg(\Omega_X) \cdot \binom{l+j-1}{j-1} + d \cdot \binom{l+j-1}{j},\end{aligned}$$

where $\deg(-) = c_1(-) \cdot H^{l-1}$. Hence, we have

$$\begin{aligned}(2.20) \qquad \deg(\mathcal{F}) &= \sum_{j=0}^{n-1} \deg(\mathcal{F}^j/\mathcal{F}^{j+1}) \\ &= \sum_{j=0}^{n-1} \deg(T^j(\Omega_X) \otimes (\mathcal{F}/\mathcal{F}^1)) \\ &\overset{(2.19)}{=} \sum_{j=0}^{n-1} \left(\deg(\Omega_X) \cdot \binom{l+j-1}{j-1} + d \cdot \binom{l+j-1}{j}\right) \\ &= \deg(\Omega_X) \cdot \binom{l+n-1}{n-2} + d \cdot \binom{l+n-1}{n-1} \\ &= \left(\deg(\Omega_X) + \frac{d(l+1)}{n-1}\right) \binom{l+n-1}{n-2}.\end{aligned}$$

On the other hand, according to [24, Corollary 3.2.4], the natural morphism $F^{(N)*}_{X/k}(\mathcal{F}^\nabla) \to \mathcal{F}$ extending the inclusion $\mathcal{F}^\nabla \hookrightarrow \mathcal{F}$ is an isomorphism. This implies that $\deg(\mathcal{F})$ is equal to $p^N \cdot \deg(\mathcal{F}^\nabla)$ and hence divisible by p^N. By this fact together with (2.20), the proof of the assertion is completed. $\square$

2.4. Denote by $\mathrm{Pic}(X^{(N)})$ the group of line bundles on $X^{(N)}$. If $\mathrm{Op}^{\mathrm{Zzz...}}_{n,N}$ denotes the set of isomorphism classes of dormant (n,N)-opers on X, then we can define an action of $\mathrm{Pic}(X^{(N)})$ on $\mathrm{Op}^{\mathrm{Zzz...}}_{n,N}$ as follows: Let $\mathcal{N}$ be a line bundle on $X^{(N)}$ and $\mathscr{F}^\heartsuit := (\mathcal{F}, \nabla, \{\mathcal{F}^j\}_j)$ a dormant (n,N)-oper on X. Denote by $\nabla^{(N-1)}_{\mathcal{N},\mathrm{can}} \otimes \nabla$ the $\mathcal{D}^{(N-1)}_X$-action on the tensor product $F^{(N)*}_{X/k}(\mathcal{N}) \otimes \mathcal{F}$ induced by $\nabla^{(N-1)}_{\mathcal{N},\mathrm{can}}$ and ∇ in a natural manner. Then, one may verify that the collection of data

$$(2.21) \qquad \mathscr{F}^\heartsuit_{\otimes \mathcal{N}} := (F^{(N)*}_{X/k}(\mathcal{N}) \otimes \mathcal{F}, \nabla^{(N-1)}_{\mathcal{N},\mathrm{can}} \otimes \nabla, \{F^{(N)*}_{X/k}(\mathcal{N}) \otimes \mathcal{F}^j\}^n_{j=0})$$

forms a dormant (n,N)-oper on X. The resulting assignment $(\mathcal{N}, \mathscr{F}^\heartsuit) \mapsto \mathscr{F}^\heartsuit_{\otimes \mathcal{N}}$ defines a desired action of $\mathrm{Pic}(X^{(N)})$ on $\mathrm{Op}^{\mathrm{Zzz...}}_{n,N}$. In particular, we obtain the quotient set

$$\overline{\mathrm{Op}}^{\text{Zzz...}}_{n,N} := \mathrm{Op}^{\text{Zzz...}}_{n,N} / \mathrm{Pic}(X^{(N)}). \tag{2.22}$$

We write $[\mathscr{F}^\heartsuit]$ for the element of $\overline{\mathrm{Op}}^{\text{Zzz...}}_{n,N}$ represented by $\mathscr{F}^\heartsuit$. In the case of $\dim(X) = 1$, this set may be identified with the set of dormant PGL_n-opers of level N on X, in the sense of [37, Definition 4.2.5].

Proposition 2.7. Suppose that $\dim(X) = 1$ (resp., $n = 2$). Then, the map of sets

$$\coprod_{d \in \mathbb{Z}_{>0}} \mathrm{Op}^{\text{Zzz...}}_{(n,N,d),+\mathrm{bir}} \to \overline{\mathrm{Op}}^{\text{Zzz...}}_{n,N} \tag{2.23}$$

$$\left(\text{resp., } \coprod_{d \in \mathbb{Z}_{>0}} \mathrm{Op}^{\text{Zzz...}}_{(2,N,d),+\mathrm{imm}} \to \overline{\mathrm{Op}}^{\text{Zzz...}}_{2,N}\right)$$

given by assigning $(\mathscr{F}^\heartsuit, q) \mapsto [\mathscr{F}^\heartsuit]$ is surjective.

Proof. We only consider the resp'd assertion since the non-resp'd assertion can be proved by an entirely similar argument (by applying assertion (i) of Proposition 2.5 instead of (ii)).

Let $\mathscr{F}^\heartsuit := (\mathcal{F}, \nabla, \{\mathcal{F}^j\}_j)$ be a dormant $(2, N)$-oper on X. Denote by $\pi : \mathbb{P}(\mathcal{F}^\nabla) \to X^{(N)}$ the natural projection. Then, we can find a very ample line bundle $\mathcal{N}$ on $X^{(N)}$ such that the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{F}^\nabla)}(1) \otimes \pi^*(\mathcal{N})$ on $\mathbb{P}(\mathcal{F}^\nabla)$ is very ample. Write $\sigma_\mathbb{P}$ for the closed immersion $\mathbb{P}(\mathcal{F}^\nabla) \hookrightarrow \mathbb{P}^L$ (where $L > 0$) defined by the complete linear system associated to $\pi^*(\mathcal{N}) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{F}^\nabla)}(1)$. Since $\pi_*(\pi^*(\mathcal{N}) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{F}^\nabla)}(1))$ may be identified with $\mathcal{N} \otimes \mathcal{F}^\nabla$ by the projection formula, $\sigma_\mathbb{P}$ coincides with the morphism induced, via projectivization, from the natural morphism

$$H^0(X^{(N)}, \mathcal{N} \otimes \mathcal{F}^\nabla) \otimes_k \mathcal{O}_{X^{(N)}} \to \mathcal{N} \otimes \mathcal{F}^\nabla. \tag{2.24}$$

Hence, the pull-back of (2.24) by $F^{(N)}_{X/k}$ gives, after choosing an identification $H^0(X^{(N)}, \mathcal{N} \otimes \mathcal{F}^\nabla) = k^{L+1}$, a surjective morphism of $\mathcal{D}^{(N-1)}_X$-modules

$$q : (\mathcal{O}_X, \nabla^{(N-1)}_{\mathrm{triv}})^{\oplus(L+1)} \twoheadrightarrow (F^{(N)*}_{X/k}(\mathcal{N}) \otimes \mathcal{F}, \nabla^{(N-1)}_{\mathcal{N},\mathrm{can}} \otimes \nabla). \tag{2.25}$$

Since the composite $\overline{\sigma} : X \xrightarrow{\sigma} \mathbb{P}(\mathcal{F}^\nabla) \xrightarrow{\sigma_\mathbb{P}} \mathbb{P}^L$ is a closed immersion by Proposition 2.5, (ii), the pair $(\mathscr{F}^\heartsuit_{\otimes\mathcal{N}}, q)$ specifies an element of $\coprod_{d \in \mathbb{Z}_{>0}} \mathrm{Op}^{\text{Zzz...}}_{(2,N,d),+\mathrm{imm}}$ mapped to $[\mathscr{F}^\heartsuit] \in \overline{\mathrm{Op}}^{\text{Zzz...}}_{2,N}$ via (2.23). This implies the surjectivity of (2.23), so the proof of this proposition is completed. ∎

2.5. Let us construct, under a certain condition, a dormant (n, N)-oper by using the Gauss map of order $n-1$ on X associated to a closed immersion into a projective space. To this end, we shall consider the following two conditions on a quadruple (X, n, N, d):

- (a) X is a smooth projective curve over k of genus $g > 1$ and n, N, d are positive integers with $1 < n < p$ and $p \nmid (g-1)$;
- (b) X is a smooth projective variety over k whose tangent bundle $\mathcal{T}_X$ is stable with respect to some ample line bundle and n, N, d are positive integers with $n = 2 < p$, $p \nmid d$.

Then, the following assertion holds.

Proposition 2.8. Let (X, n, N, d) be a quadruple satisfying one of the conditions (a), (b) described above. Let $\mathcal{L}$ be a line bundle on X of degree d with respect to some closed immersion from X to a projective space. Also, let ∇ be a left $\mathcal{D}_X^{(N-1)}$-action on $J_{n-1}(\mathcal{L})$ extending its $\mathcal{O}_X$-module structure whose p-$(N-1)$-curvature vanishes identically. Then, the collection

$$(J_{n-1}(\mathcal{L}), \nabla, \{J_{n-1}(\mathcal{L})^j\}_{j=0}^n) \tag{2.26}$$

(cf. (1.2) for the definition of the filtration $\{J_{n-1}(\mathcal{L})^j\}_j$) forms a dormant (n, N)-oper on X.

Proof. First, we shall consider the case where the condition (a) is satisfied. Let B be the subset of $\{1, \cdots, n-1\}$ consisting of integers j satisfying $\nabla^{(0)}(J_{n-1}(\mathcal{L})^j) \subseteq \Omega_X \otimes J_{n-1}(\mathcal{L})^{j-1}$. Suppose that $B \neq \{1, \cdots, n-1\}$. Then, there exists the minimum number j_0 in $\{1, \cdots, n-1\} \setminus B$. Since $1 \in B$, we have $j_0 \geq 2$. The integer $j_0 - 1$ belongs to B, so the following $\mathcal{O}_X$-linear composite can be defined:

$$\begin{aligned} & J_{n-1}(\mathcal{L})^{j_0-1} \\ & \xrightarrow{\nabla^{(0)}} \Omega_X \otimes J_{n-1}(\mathcal{L})^{j_0-2} \\ & \twoheadrightarrow \Omega_X \otimes (J_{n-1}(\mathcal{L})^{j_0-2}/J_{n-1}(\mathcal{L})^{j_0-1}) \left(\overset{(1.3)}{\cong} \Omega_X^{\otimes(j_0-1)} \otimes \mathcal{L} \right). \end{aligned} \tag{2.27}$$

It follows from $\deg(\Omega_X) > 0$ and (1.3) that this composite becomes the zero map when restricted to $J_{n-1}(\mathcal{L})^{j_0}$ $\left(\subseteq J_{n-1}(\mathcal{L})^{j_0-1}\right)$. This implies $\nabla^{(0)}(J_{n-1}(\mathcal{L})^{j_0}) \subseteq \Omega_X \otimes J_{n-1}(\mathcal{L})^{j_0-1}$, which contradicts the fact that $j_0 \notin B$. Hence, the equality $B = \{1, \cdots, n-1\}$ holds.

Now, let us fix $j \in \{1, \cdots, n-1\}$, and denote by KS^j the j-th Kodaira-Spencer map (cf. (2.3)) associated to the collection (2.26). We shall prove the claim that KS^j is nonzero. Suppose, on the contrary, that $\mathrm{KS}^j = 0$. Then, $J_{n-1}(\mathcal{L})^j$ is closed under $\nabla^{(0)}$, and we can define a connection

$\nabla_j^{(0)}$ on $J_{n-1}(\mathcal{L})/J_{n-1}(\mathcal{L})^j$ induced from $\nabla^{(0)}$ via the quotient $J_{n-1}(\mathcal{L}) \twoheadrightarrow J_{n-1}(\mathcal{L})/J_{n-1}(\mathcal{L})^j$. Since $\nabla_j^{(0)}$ has vanishing p-curvature, the $\mathcal{O}_X$-linear morphism

$$F_{X/k}^*(\mathrm{Ker}(\nabla_j^{(0)})) \to J_{n-1}(\mathcal{L})/J_{n-1}(\mathcal{L})^j \tag{2.28}$$

extending the inclusion $\mathrm{Ker}(\nabla_j^{(0)}) \hookrightarrow J_{n-1}(\mathcal{L})/J_{n-1}(\mathcal{L})^j$ (regarded as an $\mathcal{O}_{X^{(1)}}$-linear morphism via the underlying homeomorphism of $F_{X/k}$) is an isomorphism (cf. [20, Theorem (5.1)]). By putting $\nabla_n^{(0)} := \nabla^{(0)}$, we obtain the following sequence of equalities for each $j' \in \{j, n\}$:

$$\begin{aligned} p \cdot \deg(\mathrm{Ker}(\nabla_{j'}^{(0)})) &= \deg(F_{X/k}^*(\mathrm{Ker}(\nabla_{j'}^{(0)}))) \\ &= \det(J_{n-1}(\mathcal{L})/J_{n-1}(\mathcal{L})^{j'}) \\ &= \sum_{i=0}^{j'-1} \deg(J_{n-1}(\mathcal{L})^i/J_{n-1}(\mathcal{L})^{i+1}) \\ &= \sum_{i=0}^{j'-1} \deg(\Omega_X^{\otimes i} \otimes \mathcal{L}) \\ &= \sum_{i=0}^{j'-1} (i \cdot (2g-2) + d) \\ &= j' \cdot ((j'-1) \cdot (g-1) + d). \end{aligned} \tag{2.29}$$

This implies (from the assumption $n < p$) that both $(j-1) \cdot (g-1) + d$ and $(n-1) \cdot (g-1) + d$ are divisible by p. In particular, the integer

$$(g-1)(n-j) \, (= ((n-1) \cdot (g-1) + d) - ((j-1) \cdot (g-1) + d)) \tag{2.30}$$

is divisible by p. This contradicts the assumption that $p \nmid (g-1)$. Hence, KS^j turns out to be nonzero, and this completes the proof of the claim.

Moreover, by comparing the degrees of the line bundles $J_{n-1}(\mathcal{L})^j/J_{n-1}(\mathcal{L})^{j+1}$ and $\Omega_X \otimes (J_{n-1}(\mathcal{L})^{j-1})/J_{n-1}(\mathcal{L})^j)$, we see that KS^j is an isomorphism. Consequently, the collection (2.26) forms a dormant (n, N)-oper on X.

Next, let us consider the case where the condition (b) is satisfied. Suppose that the 1-st Kodaira-Spencer map $\mathrm{KS}^1 : J_1(\mathcal{L})^1 \to \Omega_X \otimes (J_1(\mathcal{L})/J_1(\mathcal{L})^1)$ associated to (2.26) coincides with the zero map. This implies that $J_1(\mathcal{L})^1$ is closed under $\nabla^{(0)}$, so we can define a connection $\nabla_1^{(0)}$ on $\mathcal{L} \left(= J_1(\mathcal{L})/J_1(\mathcal{L})^1\right)$ induced naturally from $\nabla^{(0)}$. By an argument similar to the above argument, we have $d \,(= \deg(\mathcal{L})) = p \cdot \deg(\mathrm{Ker}(\nabla_1^{(0)}))$, which contradicts the assumption that $p \nmid d$. Hence, KS^1 specifies a *nonzero* endomorphism of $\Omega_X \otimes \mathcal{L} \left(= J_1(\mathcal{L})^1 = \Omega_X \otimes (J_1(\mathcal{L})/J_1(\mathcal{L})^1)\right)$. Since $\mathcal{T}_X$ (hence also $\Omega_X \otimes \mathcal{L}$)

is stable, KS^1 must be an isomorphism. That is to say, the collection (2.26) defines a dormant $(2,N)$-oper. This completes the proof of the proposition. $\square$

By applying the above proposition, we obtain the following assertion, which is the main result of this section.

Theorem 2.9. Let (X,n,N,d) be a quadruple satisfying one of the conditions (a), (b). We shall set $\chi := (n,N,d)$ and $\lambda := (n-1,N,d)$. Then, the following assertions hold:

(i) Let $\iota : X \hookrightarrow \mathbb{P}^L$ (where $L > 0$) be a closed immersion classified by $\mathrm{Gau}^{\mathrm{F}}_\lambda$. Write $\mathcal{L} := \iota^*(\mathcal{O}_{\mathbb{P}^L}(1))$, and we shall set

$$\mathscr{F}^{\heartsuit}_\iota := (J_{n-1}(\mathcal{L}), \nabla^{(N-1)}_{\iota,\mathrm{Gau}}, \{J_{n-1}(\mathcal{L})^j\}_{j=0}^n). \tag{2.31}$$

Then, the pair $f_\iota := (\mathscr{F}^{\heartsuit}_\iota, \alpha^{n-1}_\iota)$ (cf. (1.5) for the definition of α^{n-1}_ι) specifies an element of $\mathrm{Op}^{\mathrm{Zzz...}}_{\chi,+\mathrm{imm}}$. Moreover, the map of sets

$$\Xi_\chi : \mathrm{Gau}^{\mathrm{F}}_\lambda \rightarrow \mathrm{Op}^{\mathrm{Zzz...}}_{\chi,+\mathrm{imm}}. \tag{2.32}$$

given by $\iota \mapsto f_\iota$ is injective.

(ii) Let N' be a positive integer with $N' < N$. We shall set $\chi' := (n,N',d)$ and $\lambda' := (n-1,N',d)$. Then, the following square diagram is commutative:

$$\begin{array}{ccc} \mathrm{Gau}^{\mathrm{F}}_\lambda & \xrightarrow{\Xi_\chi} & \mathrm{Op}^{\mathrm{Zzz...}}_{\chi,+\mathrm{imm}} \\ {\scriptstyle \mathrm{inclusion}}\downarrow & & \downarrow{\scriptstyle \mathrm{inclusion}} \\ \mathrm{Gau}^{\mathrm{F}}_{\lambda'} & \xrightarrow[\Xi_{\chi'}]{} & \mathrm{Op}^{\mathrm{Zzz...}}_{\chi',+\mathrm{imm}}. \end{array} \tag{2.33}$$

(iii) Suppose further that $n = 2$. Then, the map Ξ_χ is bijective.

Proof. First, we shall consider assertion (i). It follows from Proposition 2.8 that $\mathscr{F}^{\heartsuit}_\iota$ forms a dormant (n,N)-oper on X. Since γ^{n-1}_ι factors through $F^{(N)}_{X/k}$, there exists a morphism $h : X^{(N)} \rightarrow \mathrm{Grass}(\binom{l+n-1}{n-1}, L+1)$ with $h \circ F^{(N)}_{X/k} = \gamma^{n-1}_\iota$. If $q_0 : \mathcal{O}^{\oplus(L+1)}_{X^{(N)}} \twoheadrightarrow \mathcal{Q}$ denotes the pull-back of q_{univ} (cf. (1.9)) by h, then $F^{(N)*}_{X/k}(q_0)$ may be identified with α^{n-1}_ι. This implies that α^{n-1}_ι defines a surjection of $\mathcal{D}^{(N-1)}_X$-modules $(\mathcal{O}_X, \nabla^{(N-1)}_{\mathrm{triv}})^{\oplus(L+1)} \twoheadrightarrow (J_{n-1}(\mathcal{L}), \nabla^{(N-1)}_{\iota,\mathrm{Gau}})$. Thus, the pair $(\mathscr{F}^{\heartsuit}_\iota, \alpha^{n-1}_\iota)$ turns out to be an element of $\mathrm{Op}^{\mathrm{Zzz...}}_{\chi,+\mathrm{imm}}$. Moreover, the injectivity of Ξ_χ follows immediately from the observation that each closed immersion $\iota : X \hookrightarrow \mathbb{P}^L$ in $\mathrm{Gau}^{\mathrm{F}}_{\chi}$ may be reconstructed as the projectvization of the composite of α_ι^{n-1} and the natural quotient $J_{n-1}(\iota^*(\mathcal{O}_{\mathbb{P}^L}(1))) \twoheadrightarrow \iota^*(\mathcal{O}_{\mathbb{P}^L}(1))$. This completes the proof of assertion (i).

Also, assertion (ii) follows from the definition of Ξ_χ.

Finally, we shall prove assertion (iii), i.e., the surjectivity of Ξ_χ under the assumption that $n = 2$. Let $f := (\mathscr{F}^\heartsuit, q)$ (where $\mathscr{F}^\heartsuit := (\mathcal{F}, \nabla, \{\mathcal{F}^j\}_{j=0}^2)$) be a pair classified by $\mathrm{Op}^{\mathrm{Zzz...}}_{\chi,+\mathrm{imm}}$. It follows from Proposition 2.5, (ii), that the morphism $\iota := \iota_f : X \to \mathbb{P}^{L_f}$ (cf. (2.15)) is a closed immersion. Let us write $\mathcal{L} := \iota^*(\mathcal{O}_{\mathbb{P}^{L_f}}(1))$. Also, write $\sigma : X \to \mathbb{P}(\mathcal{F})$ (resp., $\overline{\sigma} : X \to \mathbb{P}(\mathcal{F}^\nabla)$) for the morphism induced from the surjection $\mathcal{F} \twoheadrightarrow \mathcal{F}/\mathcal{F}^1$ (resp., the composite $\mathcal{F}^\nabla \hookrightarrow \mathcal{F} \twoheadrightarrow \mathcal{F}/\mathcal{F}^1$) as defined in Proposition 2.5. Then, the surjection $\iota^*(\Omega_{\mathbb{P}^L}) \twoheadrightarrow \Omega_X$ induced by ι can be decomposed as $\iota^*(\Omega_{\mathbb{P}^L}) \twoheadrightarrow \overline{\sigma}^*(\Omega_{\mathbb{P}(\mathcal{F}^\nabla)}) \twoheadrightarrow \Omega_X$. Since the differential of the composite $X \xrightarrow{\overline{\sigma}} \mathbb{P}(\mathcal{F}^\nabla) \xrightarrow{\text{projection}} X^{(N)}$ (which coincides with $F^{(N)}_{X/k}$) is the zero map, the surjection $\overline{\sigma}^*(\Omega_{\mathbb{P}(\mathcal{F}^\nabla)}) \twoheadrightarrow \Omega_X$ factors through the quotient $\overline{\sigma}^*(\Omega_{\mathbb{P}(\mathcal{F}^\nabla)}) \twoheadrightarrow \overline{\sigma}^*(\Omega_{\mathbb{P}(\mathcal{F}^\nabla)/X^{(N)}})$. The resulting morphism between line bundles $\big(\sigma^*(\Omega_{\mathbb{P}(\mathcal{F})/X}) =\big)\, \overline{\sigma}^*(\Omega_{\mathbb{P}(\mathcal{F}^\nabla)/X^{(N)}}) \to \Omega_X$ is surjective, hence it is also bijective. This implies that the families of linear subvarieties in $\mathbb{P}^{L_f}$ (parametrized by X) given by q and α_ι^{n-1}, respectively, are identical, i.e., $\mathbb{P}(J_1(\mathcal{L})) = \mathbb{P}(\mathcal{F})$. It follows from the various definitions involved that the associated isomorphism $J_{n-1}(\mathcal{L}) \xrightarrow{\sim} \mathcal{F}$ between quotient bundles of $\mathcal{O}_X^{\oplus(L_f+1)}$ defines an isomorphism $f \cong (\mathscr{F}_\iota^\heartsuit, \alpha_\iota^{n-1})$. This shows the surjectivity of Ξ_χ, so the proof of assertion (iii) is completed. $\square$

3. Purely inseparable Gauss maps on a curve

In this section, we consider a sufficient condition for the nonemptiness of the set $\mathrm{Op}^{\mathrm{Zzz...}}_{\chi,+\Box}$ (where $\Box \in \{\mathrm{bir}, \mathrm{imm}\}$) in the case of $\dim(X) = 1$. As an application of this result, we show (cf. Theorem 3.3) that, for any $N > 0$, there always exists a closed immersion $X \hookrightarrow \mathbb{P}^L$ with purely inseparable Gauss map of degree p^N.

3.1. Let $\chi := (n, N, d)$ be a triple of positive integers with $n > 1$, and let X be a smooth projective curve over k of genus $g > 1$.

Proposition 3.1. Let L be a positive integer. Suppose that there exists an integer a satisfying

$$\frac{L+1}{n} + g - 1 \geq \frac{d + (g-1)(n-1)}{p^N} = a \geq \frac{(g-1)(n-1)}{p^N} + 2g + 1. \tag{3.1}$$

Also, suppose that $n < p$ (resp., $n = 2 < p$). Then, there exists an element $f := (\mathscr{F}^\heartsuit, q)$ of $\mathrm{Op}^{\mathrm{Zzz...}}_{\chi,+\mathrm{bir}}$ (resp., $\mathrm{Op}^{\mathrm{Zzz...}}_{\chi,+\mathrm{imm}}$) such that $L_f = L$ and the underlying vector bundle of $\mathscr{F}^\heartsuit$ has degree $p^N \cdot n \cdot a$. In particular, if d is sufficiently large relative to g, n, N and satisfies $d \equiv -(g-1)(n-1)$ mod p^N, then the set $\mathrm{Op}^{\mathrm{Zzz...}}_{\chi,+\mathrm{bir}}$ (resp., $\mathrm{Op}^{\mathrm{Zzz...}}_{\chi,+\mathrm{imm}}$) is nonempty.

Proof. We only consider the non-resp'd assertion since the resp'd assertion can be proved by an entirely similar argument (by applying assertion (ii) of Proposition 2.5 instead of (i).)

Let us take a theta characteristic of X, i.e., a line bundle Θ on X together with an isomorphism $\Theta^{\otimes 2} \xrightarrow{\sim} \Omega_X$. According to [36, Theorem 7.5.2], there exists a dormant $(2, N)$-oper $\mathscr{F}^\heartsuit := (\mathcal{F}_0, \nabla_0, \{\mathcal{F}_0^j\}_{j=0}^2)$ on X with $\mathcal{F}_0^1 = \Theta$ and $\mathcal{F}_0^0/\mathcal{F}_0^1 = \Theta^\vee$. Denote by $S^{n-1}(\mathcal{F}_0)$ the $(n-1)$-st symmetric product of $\mathcal{F}_0$ over $\mathcal{O}_X$. Note that $S^{n-1}(\mathcal{F}_0)$ forms a rank n vector bundle on X of degree 0 and admits a $\mathcal{D}_X^{(N-1)}$-action $S^{n-1}(\nabla_0)$ induced naturally by ∇_0. Moreover, $S^{n-1}(\mathcal{F}_0)$ is equipped with an n-step decreasing filtration $\{S^{n-1}(\mathcal{F}_0)^j\}_{j=0}^n$ induced from $\{\mathcal{F}_0^j\}_j$; to be precise, we set $S^{n-1}(\mathcal{F}_0)^0 := S^{n-1}(\mathcal{F}_0)$, $S^{n-1}(\mathcal{F}_0)^n := 0$, and $S^{n-1}(\mathcal{F}_0)^j$ (for each $j = 1, \cdots, n-1$) is defined as the image of $(\mathcal{F}_0^1)^{\otimes j} \otimes \mathcal{F}_0^{\otimes(n-j-1)}$ via the natural quotient $\mathcal{F}_0^{\otimes(n-1)} \twoheadrightarrow S^{n-1}(\mathcal{F}_0)$. This filtration satisfies that

$$S^{n-1}(\mathcal{F}_0)^j / S^{n-1}(\mathcal{F}_0)^{j+1} \cong \Theta^{\otimes(1-n)} \otimes \Omega_X^{\otimes j} \tag{3.2}$$

for every $j = 0, \cdots, n-1$. Since $\mathrm{KS}^1_{\mathscr{F}_0^\heartsuit}$ is an isomorphism, the assumption $n < p$ implies that the Kodaira-Spencer maps associated to the collection

$$(S^{n-1}(\mathcal{F}_0), S^{n-1}(\nabla_0), \{S^{n-1}(\mathcal{F}_0)^j\}_{j=0}^n) \tag{3.3}$$

are verified to be isomorphisms. That is to say, this collection forms a dormant (n, N)-oper on X.

Since ∇_0 (hence also $S^{n-1}(\nabla_0)$) has vanishing p-$(N-1)$-curvature, the inclusion $S^{n-1}(\mathcal{F}_0)^\nabla \hookrightarrow S^{n-1}(\mathcal{F}_0)$ extends to an isomorphism of $\mathcal{D}_X^{(N-1)}$-modules

$$(F^{(N)*}_{X/k}(S^{n-1}(\mathcal{F}_0)^\nabla), \nabla^{(N-1)}_{S^{n-1}(\mathcal{F}_0)^\nabla, \mathrm{can}}) \xrightarrow{\sim} (S^{n-1}(\mathcal{F}_0), S^{n-1}(\nabla_0)) \tag{3.4}$$

(cf. [24, Corollary 3.2.4]). Hence, the faithful flatness of $F^{(N)}_{X/k}$ implies that $S^{n-1}(\mathcal{F}_0)^\nabla$ forms a rank n vector bundle of degree 0 $\left(= \frac{1}{p^N} \cdot \deg(S^{n-1}(\mathcal{F}_0))\right)$ on $X^{(N)}$.

Now, let us choose a line bundle $\mathcal{N}$ on $X^{(N)}$ of degree a and a quotient line bundle $\mathcal{M}$ of $\mathcal{N} \otimes S^{n-1}(\mathcal{F}_0)^{\nabla}$ having minimal degree. Write

$$\mathcal{F} := F_{X/k}^{(N)*}(\mathcal{N}) \otimes S^{n-1}(\mathcal{F}_0) \quad \text{and} \quad \mathcal{F}^j := F_{X/k}^{(N)*}(\mathcal{N}) \otimes S^{n-1}(\mathcal{F}_0)^j \tag{3.5}$$

$(j = 0, \cdots, n)$. The degree of $\mathcal{F}$ is given by

$$\begin{aligned} \deg(\mathcal{F}) &= n \cdot \deg(F_{X/k}^{(N)*}(\mathcal{N})) + \deg(S^{n-1}(\mathcal{F}_0)) \\ &= n \cdot p^N \cdot \deg(\mathcal{N}) + 0 \\ &= p^N \cdot n \cdot a. \end{aligned} \tag{3.6}$$

Also, we have

$$\begin{aligned} \deg(\mathcal{F}/\mathcal{F}^1) &= \deg(F_{X/k}^{(N)*}(\mathcal{N}) \otimes \Theta^{\otimes(1-n)}) \\ &= p^N \cdot a + (g-1)(1-n) \\ &= d. \end{aligned} \tag{3.7}$$

By (3.2) and (3.4), $F_{X/k}^{(N)*}((\mathcal{N} \otimes S^{n-1}(\mathcal{F}_0)^{\nabla})^{\vee})$ may be identified with $\mathcal{F}^{\vee}$ and has a filtration whose graded pieces are isomorphic to the line bundles

$$F_{X/k}^{(N)*}(\mathcal{N}^{\vee}) \otimes \Theta^{\otimes(n-1)} \otimes \Omega_X^{\otimes(-j)} \tag{3.8}$$

$(j = 0, \cdots, n-1)$. Hence, since $F_{X/k}^{(N)*}(\mathcal{M}^{\vee})$ specifies a line subbundle of $F_{X/k}^{(N)*}((\mathcal{N} \otimes S^{n-1}(\mathcal{F}_0)^{\nabla})^{\vee})$, we have

(3.9)

$$\begin{aligned} &\deg(\mathcal{M}) \\ &= -\frac{1}{p^N} \cdot \deg(F_{X/k}^{(N)*}(\mathcal{M}^{\vee})) \\ &\geq -\frac{1}{p^N} \cdot \max\left\{\deg(F_{X/k}^{(N)*}(\mathcal{N}^{\vee}) \otimes \Theta^{\otimes(n-1)} \otimes \Omega_X^{\otimes(-j)}) \,\middle|\, 0 \leq j \leq n-1\right\} \\ &= -\frac{1}{p^N} \cdot \deg(F_{X/k}^{(N)*}(\mathcal{N}^{\vee}) \otimes \Theta^{\otimes(n-1)}) \\ &= a - \frac{1}{p^N} \cdot (n-1)(g-1) \\ &\geq 2g+1, \end{aligned}$$

where the last inequality follows from the assumption (3.1). This implies that $\mathcal{N} \otimes S^{n-1}(\mathcal{F}_0)^{\nabla}$ is globally generated and very ample (cf. [13, Proposition 2, (iii) and (iv)]), and moreover, the following equalities hold:

$$h^1(\mathcal{N} \otimes S^{n-1}(\mathcal{F}_0)^{\nabla}) = h^0((\mathcal{N} \otimes S^{n-1}(\mathcal{F}_0)^{\nabla})^{\vee} \otimes \Omega_X) = 0. \tag{3.10}$$

By the Riemann-Roch theorem, we have

$$\begin{aligned} &h^0(\mathcal{N}\otimes S^{n-1}(\mathcal{F}_0)^\nabla) \\ &= h^0(\mathcal{N}\otimes S^{n-1}(\mathcal{F}_0)^\nabla) - h^1(\mathcal{N}\otimes S^{n-1}(\mathcal{F}_0)^\nabla) \\ &= na + n(1-g) \\ &\leq L+1, \end{aligned} \tag{3.11}$$

where the last inequality follows from the assumption (3.1). Hence, there exists an $\mathcal{O}_{X^{(N)}}$-linear surjection

$$q_0 : \mathcal{O}_{X^{(N)}}^{\oplus(L+1)} \twoheadrightarrow \mathcal{N}\otimes S^{n-1}(\mathcal{F}_0)^\nabla \tag{3.12}$$

such that the associated morphism

$$\mathbb{P}(\mathcal{N}\otimes S^{n-1}(\mathcal{F}_0)^\nabla) \to \mathbb{P}^L \tag{3.13}$$

is a closed immersion. It follows from Proposition 2.5, (i), that the following composite is birational onto its image:

$$\begin{aligned} \iota : X \to \mathbb{P}(\mathcal{F}) \bigl(= X \times_{X^{(N)}} \mathbb{P}(\mathcal{N}\otimes S^{n-1}(\mathcal{F}_0)^\nabla)\bigr) \\ \xrightarrow{\text{projection}} \mathbb{P}(\mathcal{N}\otimes S^{n-1}(\mathcal{F}_0)^\nabla) \\ \xrightarrow{(3.13)} \mathbb{P}^L, \end{aligned} \tag{3.14}$$

where the first arrow denotes the morphism arising from the quotient $\mathcal{F} \twoheadrightarrow \mathcal{F}/\mathcal{F}^1$. If ∇ denotes the $\mathcal{D}_X^{(N-1)}$-action on $\mathcal{F}$ induced by $\nabla_{\mathcal{N},\mathrm{can}}^{(N-1)}$ and $S^{n-1}(\nabla_0)$, then the collection of data

$$\mathscr{F}^\heartsuit := (\mathcal{F}, \nabla, \{\mathcal{F}^j\}_{j=0}^n), \tag{3.15}$$

forms a dormant (n, N)-oper on X (cf. [37, § 4.2]). Moreover, the pull-back of q_0 by $F_{X/k}^{(N)}$ defines, via (3.4), a surjective morphism of $\mathcal{D}_X^{(N-1)}$-modules

$$q : (\mathcal{O}_X, \nabla_{\mathrm{triv}}^{(N-1)})^{\oplus(L+1)} \twoheadrightarrow (\mathcal{F}, \nabla). \tag{3.16}$$

It follows that the pair $(\mathscr{F}^\heartsuit, q)$ specifies an element of $\mathrm{Op}_{\mathcal{X},+\mathrm{bir}}^{\mathrm{Zzz...}}$. This completes the proof of the assertion. $\square$

Corollary 3.2. Suppose that $2 < p$ and $p \nmid (g-1)$. Also, let d be an integer satisfying $p^N(2g+1) \leq d$ and $p^N \mid (d+g-1)$. Then, $\mathrm{Gau}_{(1,N,d)}^{\mathrm{F}}$ is nonempty. If, moreover, the integer d satisfies $p^{N'} \nmid (d+g-1)$ for a positive integer $N' \, (> N)$, then $\mathrm{Gau}_{(1,N',d)}^{\mathrm{F}}$ is empty.

Proof. The assertion follows immediately from the resp'd portion of Proposition 3.1 and Theorem 2.9, (iii), in the case where (X, n, N, d) satisfies the condition (a). $\square$

3.2. Let us describe an assertion improving a result by H. Kaji (cf. Introduction). We shall denote by $K(X)$ the function field of X and by $\mathcal{K}$ the set of subfields K of $K(X)$ satisfying the following condition: There exists a closed immersion ι from X to some projective space such that the extension of function fields $K(X)/K(\mathrm{Im}(\gamma_\iota^1))$ defined by the 1-st order (i.e., classical) Gauss map γ_ι^1 associated to (X, ι) coincides with $K(X)/K$.

Theorem 3.3 (= Theorem B)**.** Let X be a smooth projective curve over k of genus $g > 1$. Suppose that $2 < p$, $p \nmid (g-1)$. Then, the following equality of sets holds:

$$\mathcal{K} = \left\{ K(X)^{p^N} \,\middle|\, N \geq 0 \right\}, \tag{3.17}$$

where $K(X)^{p^N} := \{v^{p^N} \mid v \in K(X)\}$.

Proof. By [18, Corollaries 2.3 and 4.4] (cf. the discussion preceding Theorem B) together with the fact mentioned in Remark 2.4, the problem is reduced to proving that, for every positive integer N, there exists $d > 0$ with $\mathrm{Gau}^{\mathrm{F}}_{1,N,d} \setminus \mathrm{Gau}^{\mathrm{F}}_{1,N+1,d} \neq \emptyset$. (In fact, the extension of function fields associated to a closed immersion in $\mathrm{Gau}^{\mathrm{F}}_{1,N,d} \setminus \mathrm{Gau}^{\mathrm{F}}_{1,N+1,d}$ must be equal to $K(X)/K(X)^{p^N}$.) However, we can always find an integer d with $p^N(2g+1) \leq d$, $p^N \mid (d+g-1)$, and $p^{N+1} \nmid (d+g-1)$, and Corollary 3.2 implies that such an integer d satisfies the required condition. □

4. A Frobenius-projective structure on a Fermat hypersurface

In this final section, we construct a Frobenius-projective structure on a Fermat hypersurface by applying Theorem 2.9 and the previous study of Gauss maps in positive characteristic. We also show that this Frobenius-projective structure cannot lift to sufficiently high levels.

4.1. Let N be a positive integer and X a smooth projective variety over k of dimension $l > 0$. Denote by PGL_{l+1} the projective linear group over k of rank $l+1$, which can be identified with the automorphism group of $\mathbb{P}^l$. Let us denote by $(\mathrm{PGL}_{l+1})_X^{(N)}$ the Zariski sheaf of groups on X given by $U \mapsto \mathrm{PGL}_{l+1}(U^{(N)})$ for each open subscheme U of X. Also, denote by $\mathcal{P}_X^{\text{ét}}$ the Zariski sheaf of sets on X that assigns, to each open subscheme U of X, the set of étale morphisms $U \to \mathbb{P}^l$. Note that the sheaf $\mathcal{P}_X^{\text{ét}}$ has a natural $(\mathrm{PGL}_{l+1})_X^{(N)}$-action (cf. [36, § 1.2]).

Recall that a subsheaf $\mathcal{S}^\blacklozenge$ of $\mathcal{P}_X^{\text{ét}}$ is said to be a **Frobenius-projective structure of level** N (or, F^N**-projective structure**, for short) on X if it is closed under the $(\mathrm{PGL}_{l+1})_X^{(N)}$-action on $\mathcal{P}_X^{\text{ét}}$ and forms a $(\mathrm{PGL}_{l+1})_X^{(N)}$-torsor with respect to the resulting $(\mathrm{PGL}_{l+1})^{(N)}_X$-action on $\mathcal{S}^\blacklozenge$ (cf. [9, Definition 2.1], [36, Definition 1.2.1]).

According to [36, Theorem A], there exists an assignment from a dormant $(2,N)$-oper on X (i.e., a dormant indigenous $\mathcal{D}^{(N-1)}_X$-modules, in the sense of [36, Definitions 2.3.1 and 3.2.1]) to an F^N-projective structure (cf. Remark 2.2); moreover, if $p \nmid (l+1)$, then this assignment defines a bijective correspondence between the set of F^N-projective structures on X and the set of equivalence classes of dormant $(2,N)$-opers. (We here omit the details of the *equivalence relation* on dormant $(2,N)$-opers. When $\dim(X) = 1$, each such equivalence class was referred, in [37, Definition 4.2.5], to as a *dormant* PGL_2*-opers of level* N.)

4.2. As an application of Theorem 2.9, we can construct an F^N-projective structure by using the Gauss map of a certain Fermat hypersurface (cf. [39, § 7]).

Hereinafter, let us fix an integer $L > 1$, and suppose that X is the Fermat hypersurface of degree p^N+1 in the projective space $\mathbb{P}^L$, i.e., the smooth hypersurface defined by the homogenous polynomial

$$f_N := t_0^{p^N+1} + \cdots + t_L^{p^N+1}. \tag{4.1}$$

Write $\iota : X \hookrightarrow \mathbb{P}^L$ for the natural closed immersion. Let us identify $\mathrm{Grass}(L, L+1)$ with $\mathbb{P}^L$ in such a way that if an L-plane in $\mathbb{P}^L$ (i.e., a point of $\mathrm{Grass}(L, L+1)$) is given by an equation $\sum_{i=0}^L v_i \cdot t_i = 0$ ($v_0, \cdots, v_L \in k$, $(v_0, \cdots, v_L) \neq (0, \cdots, 0)$), then it corresponds to the point $[v_0 : \cdots : v_L]$ of $\mathbb{P}^L$. Under this identification, the 1-st order Gauss map $\gamma^1_\iota : X \to \mathbb{P}^L$ associated to (X, ι) can be described as the assignment

$$a := [a_0 : \cdots : a_L] \mapsto \left(\left[\frac{\partial f_N}{\partial t_0}(a) : \cdots : \frac{\partial f_N}{\partial t_L}(a) \right] = \right) [a_0^{p^N} : \cdots : a_L^{p^N}]. \tag{4.2}$$

That is to say, the morphism $X \to \mathrm{Im}(\gamma^1_\iota)$ induced by γ^1_ι coincides with the N-th relative Frobenius morphism $F^{(N)}_{X/k}$ of X. It follows that the closed immersion ι defines an element of $\mathrm{Gau}^{\mathrm{F}}_{(1,N,p^N+1)}$.

Theorem 4.1. Suppose that $L \neq 3$ and $p > 2$. Then, the Fermat hypersurface X of degree p^N+1 in $\mathbb{P}^L$ admits an F^N-projective structure

$$\mathcal{S}^\blacklozenge_{\mathrm{Gau}}, \tag{4.3}$$

which corresponds to the dormant $(2,N)$-oper $\mathscr{F}^\heartsuit_\iota$ (cf. (2.31)). Moreover, if $p \nmid L(L+1)$, then X admits no F^{2N+1}-projective structures.

Proof. By the assumption $L \neq 3$, the tangent bundle $\mathcal{T}_X$ is stable (cf. [25, Remark 3.2] or [29, Corollary 0.3]). In particular, the quadruple $(X, 2, N, p^N +$

1) satisfies the condition (b) described in § 2.5. It follows that we can define the map $\Xi_{(2,N,p^N+1)}$ asserted in Theorem 2.9, (i), and the image via this map of the element $\iota \in \mathrm{Gau}^{\mathrm{F}}_{(1,N,p^N+1)}$ determines a dormant $(2,N)$-oper, or equivalently, an F^N-projective structure, on X. This completes the proof of the former assertion.

Next, let us consider the latter assertion. For each vector bundle $\mathcal{V}$ and an integer $m > 0$, we shall use the notation "$c_m^{\mathrm{crys}}(\mathcal{V})$" to denote the m-th crystalline Chern class of $\mathcal{V}$, which is an element of the $2m$-th crystalline cohomology group $H^{2m}_{\mathrm{crys}}(X/W)$ (W denotes the ring of Witt vectors over k). Denote by H the restriction to X of $c_1^{\mathrm{crys}}(\mathcal{O}_{\mathbb{P}^L}(1))$. Then, the Chern polynomial $c_t^{\mathrm{crys}}(\mathcal{T}_X)$ of $\mathcal{T}_X$ is given by

(4.4)
$$\begin{aligned} & c_t^{\mathrm{crys}}(\mathcal{T}_X) \\ & = (1+Ht)^{L+1}(1+(p^N+1)Ht)^{-1} \\ & = (1+Ht)^{L+1}(1-(p^N+1)Ht + ((p^N+1)Ht)^2 - ((p^N+1)Ht)^3 + \cdots), \end{aligned}$$

where the first equality follows from the natural short exact sequence

$$0 \longrightarrow \mathcal{T}_X \longrightarrow \iota^*(\mathcal{T}_{\mathbb{P}^L}) \longrightarrow \mathcal{O}_X(p^N+1) \longrightarrow 0 \tag{4.5}$$

and the Euler sequence on $\mathbb{P}^L$. Since $H^4_{\mathrm{crys}}(X/W) = \{aH^2 \mid a \in W\}$ (cf. [3, Exp. XI, Theorem 1.5], [1, Chap. VII, Remark 1.1.11], [12, Chap. II, Corollary 3.5]), the equality (4.4) implies

$$\begin{aligned} & c_2^{\mathrm{crys}}(\mathcal{T}_X) - \frac{1}{L^2}\cdot \binom{L}{2} \cdot c_1^{\mathrm{crys}}(\mathcal{T}_X)^2 \\ & = \left(p^{2N} - p^N L + p^N + \frac{L^2-L}{2}\right) H^2 - \left((L-p)^2 \cdot \frac{L-1}{2L}\right) H^2 \\ & = \frac{p^{2N}\cdot (L+1)}{2L} H^2 \\ & \not\equiv 0 \ (\mathrm{mod}\ p^{2N+1}), \end{aligned} \tag{4.6}$$

where the last "$\not\equiv$" follows from the assumption $p \nmid L(L+1)$. Thus, the assertion follows from [36, Theorem 3.7.1]. □

Remark 4.2. In the case of $L = 2$, the variety X defined by (4.1) is known as a *Hermitian curve*; this is a smooth projective curve of genus $\frac{p^N(p^N-1)}{2}$ having large automorphism groups so that it violates the classical Hurwitz bound (i.e., $\sharp(\mathrm{Aut}(X)) \leq 84(g-1)$). In fact, let us define $\mathrm{U}_3(p^{2N})$ as the subgroup of $\mathrm{GL}_3(\mathbb{F}_{p^{2N}})$ ($\subseteq \mathrm{GL}_3(k)$) leaving (4.1) invariant, and $\mathrm{PGU}_3(p^{2N})$ as the factor of $U_3(p^{2N})$ modulo its center. Then, $\mathrm{PGU}_3(p^{2N})$ coincides with the full automorphism group of X, and its order is given by $p^{3N}(p^{3N}+1)(p^{2N}-1)\left(> 84(\frac{p^N(p^N-1)}{2}-1)\right)$.

Since the closed immersion $\iota : X \hookrightarrow \mathbb{P}^2$ is compatible with the respective $\mathrm{PGU}_3(p^{2N})$-actions on X and $\mathbb{P}^2$, the Gauss map $\gamma^1_\iota : X \to \mathrm{Grass}(2,3)$ is compatible with the respective $\mathrm{PGU}_3(p^{2N})$-actions. By the definition of $\Xi_{(2,N,p^N+1)}$, the dormant $(2,N)$-oper obtained from ι, hence also the F^N-projective structure $\mathcal{S}^\blacklozenge_{\mathrm{Gau}}$, turns out to be invariant under the $\mathrm{PGU}_3(p^{2N})$-action on X. So $\mathcal{S}^\blacklozenge_{\mathrm{Gau}}$ has large symmetry in this sense and descends to any étale quotient of X.

Remark 4.3. According to [31, Corollary] (or, [32, Proposition 1], [33, Theorem III]), the Fermat hypersurface $X \left(\subseteq \mathbb{P}^L\right)$ is unirational when $L > 3$. Recall that any unirational projective variety over the field of complex numbers $\mathbb{C}$ admits no projective structure unless it is isomorphic to a projective space; this is because such a variety contains a rational curve (cf. [15, Theorem 4.1]). In this sense, the example of an F^N-projective structure resulting from the above theorem embodies an exotic phenomenon of algebraic geometry in positive characteristic.

Remark 4.4. Recall that there is an "affine" version of an F^N-projective structure, which is called an *F^N-affine structure* (cf. [10, Definition 2.1], [36, Definition 1.2.1]). By a change of structure group from the group of affine transformations to that of projective transformations, each F^N-affine structure yields an F^N-projective structure. The only previous examples of F^N-projective structures on higher-dimensional varieties except for those on projective spaces were obtained, in that manner, from F^N-affine projective structures on Abelian varieties or smooth curves equipped with a Tango structure via, e.g., taking products, étale coverings, or quotients by a finite group action (cf. [36, §6.1, §6.5, §8.1]). On the other hand, the degree p^N+1 Fermat hypersurface X embedded in $\mathbb{P}^L$ with $p \nmid L$ satisfies $c_1^{\mathrm{crys}}(\mathcal{T}_X) \left(= (L-p^N)H\right) \not\equiv 0 \pmod{p^N}$ in $H^2_{\mathrm{crys}}(X/W) \,(= \{aH \mid a \in W\})$. Hence, it follows from [36, Theorem 3.7.1] that X admits no F^N-affine structures. In particular, the F^N-projective structure $\mathcal{S}^\blacklozenge_{\mathrm{Gau}}$ resulting from the above theorem does not come from any F^N-affine structure via changing the structure group. This means that $\mathcal{S}^\blacklozenge_{\mathrm{Gau}}$ is essentially a new example constructed in a way that has never been done before.

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