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NOTE ON SMOOTHNESS CONDITION ON TROPICAL ELLIPTIC CURVES OF SYMMETRIC TRUNCATED CUBIC FORMS

Rani Sasmita Tarmidi

Abstract. In this work, we provide explicit conditions for the coefficients of a symmetric truncated cubic to give a smooth tropical curve. We also examine non-smooth cases corresponding to some specific subdivision types.

1. Introduction

Let $\mathbb{K}$ be a field with a valuation $\mathrm{val} : \mathbb{K} \to \mathbb{Q} \cup \{\infty\}$. In this paper, we work with a truncated cubic

$$f(x, y) = c_1 xy^2 + c_2 x^2 y + c_3 x^2 + c_4 y^2 + c_5 xy + c_6 x + c_7 y + c_8 \in \mathbb{K}[x, y]$$

with symmetric condition $f(x, y) = f(y, x)$ and its associated tropical polynomial $\mathrm{trop}(f)$. For the sake of convenience, we shall represent f in the form

$$f(x, y) = c_{12}(xy^2 + x^2 y) + c_{34}(x^2 + y^2) + c_5 xy + c_{67}(x + y) + c_8 \tag{1.1}$$

with $c_{12}, c_{34}, c_5, c_{67}, c_8 \in \mathbb{K}$. The purpose of this paper is to determine when its associated tropical polynomial

$$\begin{aligned} \mathrm{trop}(f)(X, Y) = \min(&v_{12} + X + 2Y, v_{12} + 2X + Y, v_{34} + 2X, \\ &v_{34} + 2Y, v_5 + X + Y, v_{67} + X, v_{67} + Y, v_8) \end{aligned} \tag{1.2}$$

where $v_k = \mathrm{val}(c_k)$ for $k \in \{12, 34, 5, 67, 8\}$, has a smooth tropical curve. The tropical curve $C(\mathrm{trop}(f))$ of tropical polynomial $\mathrm{trop}(f) : \mathbb{R}^2 \to \mathbb{R}$ is the collection of its singular loci. The structure of a tropical curve is determined by the subdivision of the Newton polygon of $\mathrm{trop}(f)$. The smooth tropical curves are dual to unimodular subdivisions. Since the Newton polygon of $\mathrm{trop}(f)$ is symmetric around $y = x$ and truncated, there are five unimodular subdivisions.

Theorem 1.1. *Let f be the symmetric truncated cubic in (1.1). Then the possible cycles appearing in the tropical curve of $\mathrm{trop}(f)$ are triangles, squares, pentagons, hexagons and heptagons. Each of these cycles occurs if and only if for $k \in \{12, 34, 5, 67, 8\}$, the coefficients v_k satisfy inequalities listed in Table 1.*

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	Cycle shape	Conditions of v_k
(a)	Triangle	$-v_{34} + 2v_{67} - v_8 < 0$ $v_{12} - v_5 - v_{67} + v_8 < 0$ $-2v_{12} + 3v_5 - v_8 < 0$
(b)	Square	$-v_5 + 2v_{67} - v_8 < 0$ $-v_{12} + 2v_5 - v_{67} < 0$ $v_{12} - v_{34} - v_5 + v_{67} < 0$
(c)	Pentagon	$v_5 - 2v_{67} + v_8 < 0$ $-v_{12} + v_5 + v_{67} - v_8 < 0$ $v_{12} - v_{34} - v_5 + v_{67} < 0$
(d)	Hexagon	$-v_5 + 2v_{67} - v_8 < 0$ $-v_{34} + v_5 < 0$ $-v_{12} + v_{34} + v_5 - v_{67} < 0$
(e)	Heptagon	$v_5 - 2v_{67} + v_8 < 0$ $-v_{34} + 2v_{67} - v_8 < 0$ $-v_{12} + v_{34} + v_5 - v_{67} < 0$

Table 1. Conditions of $v_{12}, v_{34}, v_5, v_{67}, v_8$ for all smooth tropical curves of trop(f).

Meanwhile, the non-smooth tropical curves of trop(f) are the duals of non-unimodular subdivisions. We also investigate possible non-unimodular subdivisions and provide the conditions of $(v_{12}, v_{34}, v_5, v_{67}, v_8)$ for some selected subdivisions. See Theorem 4.2 below. The above results will be applied in another paper [6] that studies the tropicalization of a certain two-parameter family of Edwards elliptic curves closely. See §5.4 for more details.

The contents of this paper are organized as follows. In Section 2, we provide a necessary overview of the general definitions pertaining to tropical curves. Moving on to Section 3, we present the characterization of smooth tropical curves of our symmetric truncated cubic. In Section 4, we delve into a discussion on the non-smooth tropical curves associated with our cubic. Lastly, in Section 5, we showcase the utilization of an integral unimodular transformation on f and demonstrate the practical applications of our main findings.

2. Preliminaries on tropical curves

Let $\mathbb{K}$ be a field with a valuation val : $\mathbb{K} \to \mathbb{Q} \cup \{\infty\}$. For a Laurent polynomial

$$f(x, y) = \sum a_{ij} x^i y^j \in \mathbb{K}[x^{\pm 1}, y^{\pm 1}],$$

we use the following definitions [4].

Definition 2.1. The tropicalization of polynomial $f(x, y)$ is obtained by replacing each coefficient with its valuation and altering the operations $(\cdot, +)$ with operations $(+, \min)$. It is written as

$$\operatorname{trop}(f)(X, Y) = \min(\operatorname{val}(a_{ij}) + i \cdot X + j \cdot Y).$$

The tropical curve of $\operatorname{trop}(f)$ that is denoted by $C(\operatorname{trop}(f))$, is the collection of coordinates $(X, Y) \in \mathbb{R}^2$ where $\operatorname{trop}(f)$ is not differentiable. It forms a collection of vertices, bounded edges, and rays.

Remark 2.2. It is common to find other sources in literature that express the tropicalization of a polynomial by using operations $(+, \max)$. The tropical curve of the tropical polynomial in the form

$$\operatorname{trop}(f')(X, Y) = \max(-\operatorname{val}(a_{ij}) + i \cdot X + j \cdot Y)$$

and $C(\operatorname{trop}(f))$ are point-symmetric with respect to the origin O.

Proof. Let (X, Y) be a point on $C(\operatorname{trop}(f))$. Then there exist $i_1 j_1$ and $i_2 j_2$ such that

$$\operatorname{val}(a_{i_1 j_1}) + i_1 X + j_1 Y = \operatorname{val}(a_{i_2 j_2}) + i_2 X + j_2 Y$$

and less than other terms $\operatorname{val}(a_{ij}) + iX + jY$. Thus we have

$$-\operatorname{val}(a_{i_1 j_1}) + i_1(-X) + j_1(-Y) = -\operatorname{val}(a_{i_2 j_2}) + i_2(-X) + j_2(-Y)$$

and greater than other terms of $-\operatorname{val}(a_{ij}) + i(-X) + j(-Y)$. In other words, (X, Y) is a point on $C(\operatorname{trop}(f))$ if and only if $(-X, -Y)$ is a point on $C(\operatorname{trop}(f'))$. Thus, the tropical curves are point-symmetric with respect to the origin O. □

To determine the conditions of $\operatorname{val}(a_{ij})$ for a specific tropical curve, we will use its relationship with the subdivision of the following Newton polygon of $\operatorname{trop}(f)$.

Definition 2.3. The set

$$\operatorname{Supp}_f = \{(i, j) \in \mathbb{Z}^2 : a_{ij} \neq 0\}$$

denotes the support of a tropical polynomial $\operatorname{trop}(f)$. The Newton polygon of $\operatorname{trop}(f)$, that is denoted by Δ_f, is the convex hull of Supp_f.

The subdivision of Δ_f plays a crucial role in understanding the structure of the tropical curves. The definition of a regular subdivision depends on the valuations of non-zero coefficients a_{ij} in the following manner.

Definition 2.4. Let $v = (\text{val}(a_{ij}) | a_{ij} \neq 0) \in \mathbb{R}^{\text{Supp}_f}$. Furthermore, let Δ_f be the Newton polygon of trop(f) and $\overline{\Delta_f}$ be the convex hull of

$$\{(i, j, \text{val}(a_{ij})) \mid (i, j) \in \text{Supp}_f\} \subseteq \mathbb{Z}^2 \times \mathbb{R}.$$

The regular subdivision Subdiv_v is the image of the corner edges of the upper part of $\overline{\Delta_f}$ under the projection to $\mathbb{Z}^2$ that subdivide Δ_f into smaller polygons.

Each smaller polygon is called a cell. A cell is primitive when all of its lattice points are its vertices. It is unimodular if it is a triangle of area half. A subdivision is primitive (resp. unimodular) when all cells are primitive (resp. unimodular).

The collection of vectors v that yield the same regular subdivision forms a polyhedral cone in $\mathbb{R}^{\text{Supp}_f}$. The collection of these cones defines the secondary fan of the Newton polygon Δ_f.

Unimodular subdivisions are also the finest subdivisions. Thus, they correspond to the top-dimensional cones of the secondary fan. We observe that a unimodular cell or subdivision is always primitive, but the converse does not hold in general. Furthermore, the coarsest subdivision is the Newton polygon itself. The tropical curve $C(\text{trop}(f))$ is dual to Subdiv_v (cf. [3, 7]). Therefore, there is a one-to-one correspondence between the edges of a regular subdivision and the edges of a tropical curve. This relation is our main tool to analyze the structure of tropical curves of trop(f).

3. Smooth tropical curves characterization

As mentioned earlier, let f be the truncated cubic polynomial

$$f(x, y) = c_{12}(xy^2 + x^2y) + c_{34}(x^2 + y^2) + c_5xy + c_{67}(x + y) + c_8. \tag{3.1}$$

Its tropicalization is the piece-wise linear function

$$\begin{aligned} \text{trop}(f)(X, Y) = \min(&v_{12} + X + 2Y, v_{12} + 2X + Y, v_{34} + 2X, \\ &v_{34} + 2Y, v_5 + X + Y, v_{67} + X, v_{67} + Y, v_8) \end{aligned} \tag{3.2}$$

where $v_k = \text{val}(c_k)$ for $k \in \{12, 34, 5, 67, 8\}$. We assume $c_k \neq 0$ unless stated otherwise. The Newton polygon Δ_f of trop(f) in Figure 1 is the convex hull of the set

$$\text{Supp}_f = \{(1, 2), (2, 1), (2, 0), (0, 2), (1, 1), (1, 0), (0, 1), (0, 0)\}.$$

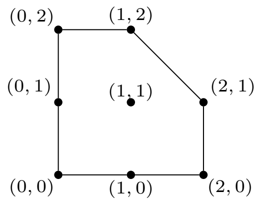


FIGURE 1. Newton polygon Δ_f.

Definition 3.1. A tropical curve is smooth when its dual subdivision is unimodular.

In this section, we will give all combinatorial possibilities of smooth tropical curve of trop(f). The Figure 2 below illustrates the possible unimodular subdivisions of Δ_f.

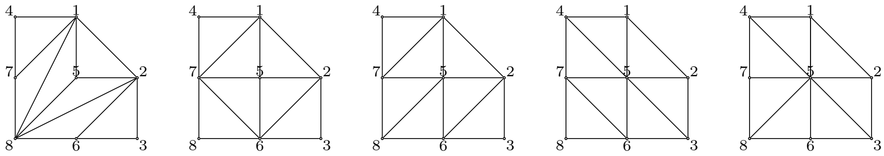


FIGURE 2. The unimodular subdivisions of Δ_f.

Proof of Theorem 1.1. We will prove the case (d). Tropical curve $C(\text{trop}(f))$ has a hexagonal cycle when it is the dual of Figure 2(IV). This subdivision can be written as

$$\mathcal{S} = \{[1,2,5],[1,4,5],[2,3,5],[4,5,7],[3,5,6],[5,6,7],[6,7,8]\}.$$

Each vertex of $C(\text{trop}(f))$ corresponds to a cell of the subdivision as shown in Table 2.

Vertex $(-v_{12}+v_5, -v_{12}+v_5)$ in Figure 3 corresponds to cell $[1,2,5]$ means this vertex is the solution of the system of linear equations

$$v_{12} + X + 2Y = v_{12} + 2X + Y = v_5 + X + Y$$

which are the $1^{\text{st}}, 2^{\text{nd}}, 5^{\text{th}}$ terms of trop$(f)(X,Y)$. Furthermore,

$$\begin{aligned} &\text{trop}(f)(-v_{12}+v_5, -v_{12}+v_5) \\ =&\min(-2v_{12}+3v_5, -2v_{12}+3v_5, v_{34}-2v_{12}+2v_5, v_{34}-2v_{12}+2v_5, \\ &-2v_{12}+3v_5, v_{67}-v_{12}+v_5, v_{67}-v_{12}+v_5, v_8) \\ =&-2v_{12}+3v_5, \end{aligned}$$

Cell	X coordinate	Y coordinate
$[1,2,5]$	$-v_{12}+v_5$	$-v_{12}+v_5$
$[1,4,5]$	$-v_{12}+v_{34}$	$-v_{12}+v_5$
$[2,3,5]$	$-v_{12}+v_5$	$-v_{12}+v_{34}$
$[4,5,7]$	$-v_5+v_{67}$	$-v_{34}+v_{67}$
$[3,5,6]$	$-v_{34}+v_{67}$	$-v_5+v_{67}$
$[5,6,7]$	$-v_5+v_{67}$	$-v_5+v_{67}$
$[6,7,8]$	$-v_{67}+v_8$	$-v_{67}+v_8$

TABLE 2. (X,Y) coordinates of smooth $C(\text{trop}(f))$ with a hexagonal cycle.

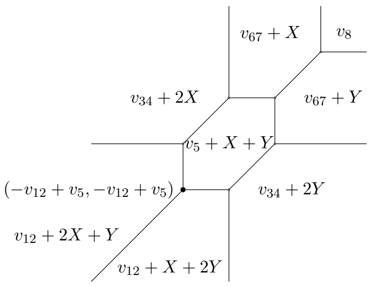


FIGURE 3. Tropical curve $C(\text{trop}(f))$ with a hexagonal cycle.

which is the $1^{\text{st}}, 2^{\text{nd}}, 5^{\text{th}}$ terms of $\text{trop}(f)(-v_{12}+v_5,-v_{12}+v_5)$. Cell $[1,2,5]$ implies that $-2v_{12}+3v_5$ is less than the other terms. It gives inequalities

$$(3.3)\qquad -2v_{12}+3v_5-v_8<0 \qquad v_5-v_{34}<0 \qquad -v_{12}+2v_5-v_{67}<0.$$

After applying the same procedure to the other six cells of $\mathcal{S}$, we have the regular subdivision $\mathcal{S}$ occurs if and only if inequalities

$$\begin{array}{ccc} -2v_{12}+v_{34}+2v_5-v_8<0 & -v_{12}+2v_5-v_{67}<0 & -2v_{12}+3v_5-v_8<0 \\ -v_{12}+v_{34}+v_5-v_{67}<0 & -v_{34}+2v_{67}-v_8<0 & -v_5+2v_{67}-v_8<0 \\ -v_{12}+3v_{67}-2v_8<0 & v_5-v_{34}<0 & \end{array}$$

hold. The above inequalities are equivalent to polyhedron

$$(3.4)\qquad -v_5+2v_{67}-v_8<0 \qquad -v_{34}+v_5<0 \qquad -v_{12}+v_{34}+v_5-v_{67}<0.$$

Thus, the tropical curve $C(\text{trop}(f))$ is smooth with a hexagonal cycle if and only if (3.4) holds. The same arguments hold for the cases of other cycles. □

4. Non-smooth tropical curves

A tropical curve $C(\text{trop}(f))$ is non-smooth when its dual is a non-unimodular subdivision. These subdivisions do not correspond to the top-dimensional cones of the secondary fan of Δ_f. Thus, some or all of the inequalities in the conditions of $(v_{12}, v_{34}, v_5, v_{67}, v_8)$ will be non-strict inequalities. Table 3

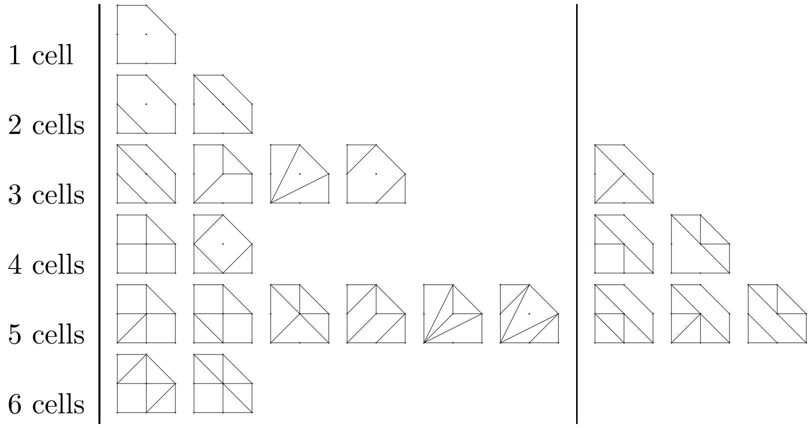


Table 3. The non-unimodular subdivisions of Δ_f.

presents the subdivisions of Δ_f into polygons, some of its cells have areas greater than half. However, it is important to note that the corresponding tropical curves $C(\text{trop}(f))$ associated with the subdivisions listed in the right column do not occur in practice.

Proposition 4.1. *Let* $\text{trop}(f)$ *be as defined in (3.2) and* Δ_f *be its Newton polygon. Then, the subdivisions on the right column of Table 3 never occur as the regular subdivisions of* Δ_f *for any* $v = (v_{12}, v_{34}, v_5, v_{67}, v_8)$.

Proof. The proof can be accomplished by examining the shape of the subdivision. Let us assume the subdivisions are viable. In doing so, we observe that the interior point $(1,1)$ forms a vertex of the Newton polygon Δ_f. However, it is evident that its dual cannot form a closed cycle in a tropical curve. □

Theorem 4.2. *The Table 4 below shows the necessary and sufficient conditions for valuations* $(v_{12}, v_{34}, v_5, v_{67}, v_8)$ *to give specific non-smooth tropical curve* $C(\text{trop}(f))$ *listed in the left column.*

Proof of Theorem 4.2. We will prove case (h) in detail, the non-smooth tropical curve with a trivalent pentagonal cycle. The collection of vectors v corresponding to it and the collection of vectors v yielding the subdivision in

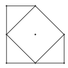
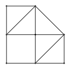
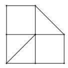
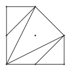
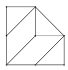
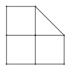
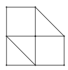
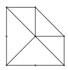
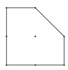
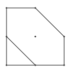
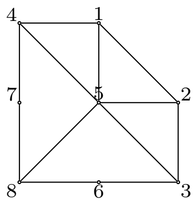
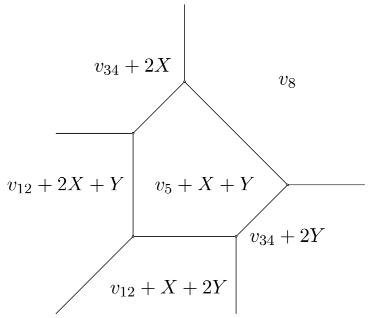
	Non-smooth tropical curves	*Subdivisions*	*Conditions of v_k*
(a)	*Four vertices with no cycle*		$v_{12} - 2v_{34} + v_{67} < 0$ $-v_{12} + 3v_{67} - 2v_8 < 0$ $v_{12} - 2v_5 + v_{67} \leq 0$
(b)	*A square cycle with two bounded edges*		$-v_{12} + 3v_{67} - 2v_8 < 0$ $v_{12} - v_{34} - v_{67} + v_8 < 0$ $v_5 - 2v_{67} + v_8 = 0$
(c)	*A pentagon cycle with seven rays*		$-v_{34} + 2v_{67} - v_8 < 0$ $v_{12} - v_{34} - v_{67} + v_8 < 0$ $-v_{12} + v_{34} + v_5 - v_{67} = 0$
(d)	*Five vertices with no cycle*		$-v_{34} + 2v_{67} - v_8 < 0$ $v_{12} - 3v_{67} + 2v_8 < 0$ $2v_{12} - 3v_5 + v_8 \leq 0$
(e)	*A triangle cycle with seven rays*		$v_{12} - 3v_{67} + 2v_8 < 0$ $-v_{34} + 2v_{67} - v_8 < 0$ $-v_{12} + v_5 + v_{67} - v_8 = 0$
(f)	*A square cycle with no bounded edge*		$v_{12} - 2v_{34} + v_{67} < 0$ $v_{12} - v_{34} - v_{67} + v_8 = 0$ $v_{12} - v_{34} - v_5 + v_{67} = 0$
(g)	*A square cycle with one bounded edge*		$v_{12} - 2v_{34} + v_{67} < 0$ $-v_{12} + v_{34} + v_{67} - v_8 < 0$ $-v_{12} + v_{34} + v_5 - v_{67} = 0$
(h)	*A trivalent pentagon cycle*		$-v_{34} + v_5 < 0$ $-2v_{12} + v_{34} + 2v_5 - v_8 < 0$ $v_{34} - 2v_{67} + v_8 \leq 0$
(i)	*No bounded edge*		$u_8 + 2u_{12} - 3u_{34} = 0$ $-u_{12} + 2u_{34} - u_{67} \leq 0$ $-u_5 + u_{34} \leq 0$
(j)	*One bounded edge with seven rays*		$-u_8 - u_{34} + 2u_{67} < 0$ $-u_5 + u_{34} \leq 0$ $u_{12} - 2u_{34} + u_{67} = 0$

TABLE 4. Some non-smooth tropical curves trop(f).

Figure 4(I) coincide. In this case, we have triangle cells that are not primitive. However, notice that a triangle cell is dual to a vertex that separates three areas in the tropical curve. Specifically, the non-smooth tropical curve of case (h) is shown in Figure 4(II). Thus, we always name the cells of a subdivision according to the points of Δ_f that form vertices of the cell and



(I) Subdivision $\mathcal{S}$.



(II) The dual of $\mathcal{S}$.

FIGURE 4. The subdivision and the tropical curve of case (h).

we have

$$\mathcal{S} = \{[1,2,5],[1,4,5],[2,3,5],[4,5,8],[3,5,8]\}.$$

Cell $[1,2,5]$ is defined by inequalities (3.3). Cells $[1,4,5]$ and $[2,3,5]$ are defined by inequalities

$$\begin{aligned} -v_{12} + v_{34} + v_5 - v_{67} &< 0 & -v_{34} + v_5 &< 0 \\ -2v_{12} + v_{34} + 2v_5 - v_8 &< 0 & -v_{12} + 2v_5 - v_{67} &< 0. \end{aligned} \tag{4.1}$$

Lastly, cells [4,5,8] and [3,5,8] are defined by inequalities

$$\begin{aligned} -v_{34} + 2v_5 - 2v_{67} + v_8 &< 0 & -v_{34} + v_5 &< 0 \\ -2v_{12} - v_{34} + 4v_5 - v_8 &< 0 & v_{34} - 2v_{67} + v_8 &< 0 \\ -2v_{12} + v_{34} + 2v_5 - v_8 &< 0. \end{aligned} \tag{4.2}$$

Inequalities (3.3), (4.1), and (4.2) form a polyhedral cone that can be represented by

$$-v_{34} + v_5 < 0 \qquad -2v_{12} + v_{34} + 2v_5 - v_8 < 0 \qquad v_{34} - 2v_{67} + v_8 < 0. \tag{4.3}$$

Next, we have to determine the extreme rays of cone (4.3). We will do this by evaluating the following three polyhedral cones with lower dimensions.

$$-v_{34} + v_5 = 0 \qquad -2v_{12} + v_{34} + 2v_5 - v_8 < 0 \qquad v_{34} - 2v_{67} + v_8 < 0, \tag{4.4}$$

$$-v_{34} + v_5 < 0 \qquad -2v_{12} + v_{34} + 2v_5 - v_8 = 0 \qquad v_{34} - 2v_{67} + v_8 < 0, \tag{4.5}$$

$$-v_{34} + v_5 < 0 \qquad -2v_{12} + v_{34} + 2v_5 - v_8 < 0 \qquad v_{34} - 2v_{67} + v_8 = 0. \tag{4.6}$$

Cones (4.4) and (4.5) have coordinates $v = (1,-1,-1,1,-1)$ and $v = (-2,0,-3,2,-2)$, respectively, that correspond to tropical curves differ from Figure 4(II). Meanwhile, let

$$v_8 = -v_{34} + 2v_{67}$$

as mentioned in cone (4.6) and substitute to (3.2) to have

$$\begin{aligned}\operatorname{trop}(f)(X,Y) = \min(&v_{12}+X+2Y, v_{12}+2X+Y, v_{34}+2X,\\ &v_{34}+2Y, v_5+X+Y, v_{67}+X, v_{67}+Y, -v_{34}+2v_{67}).\end{aligned}$$

The relations between each cell, its dual coordinate (X,Y), and the value of $\operatorname{trop}(f)(X,Y)$ are as shown in Table 5. In each cell, we see that the terms of

Cell	Corresponding (X,Y)	The minimum value of $\operatorname{trop}(f)(X,Y)$
$[1,2,5]$	$(-v_{12}+v_5, -v_{12}+v_5)$	$1^{\text{st}}, 2^{\text{nd}}$, and 5^{th} terms
$[1,4,5]$	$(-v_{12}+v_{34}, -v_{12}+v_5)$	$1^{\text{st}}, 4^{\text{th}}$, and 5^{th} terms
$[2,3,5]$	$(-v_{12}+v_5, -v_{12}+v_{34})$	$2^{\text{nd}}, 3^{\text{rd}}$, and 5^{th} terms
$[4,5,8]$	$(-v_5+v_{67}, -v_{34}+v_{67})$	$4^{\text{th}}, 5^{\text{th}}, 7^{\text{th}}$, and 8^{th} terms
$[3,5,8]$	$(-v_{34}+v_{67}, -v_5+v_{67})$	$3^{\text{rd}}, 5^{\text{th}}, 6^{\text{th}}$, and 8^{th} terms

Table 5. Coordinates of Figure 4(II).

$\operatorname{trop}(f)$ that determine the value of $\operatorname{trop}(f)(X,Y)$ do not exceed the points of Δ_f that are covered by the cell. Thus, we have the subdivision in case (h) occurs if and only if

$$-v_{34}+v_5<0 \quad -2v_{12}+v_{34}+2v_5-v_8<0 \quad v_{34}-2v_{67}+v_8\le 0$$

hold. ☐

5. Applications

5.1. **Symmetric Honeycomb.** Chan-Sturmfels [1] considered a cubic in the form of

$$g(x,y)=a(x^3+y^3+1)+b(x^2y+x^2+xy^2+x+y^2+y)+xy,$$

and showed that $C(\operatorname{trop}(g))$ is a symmetric honeycomb if and only if $\operatorname{val}(a) > 2\operatorname{val}(b) > 0$. Here $C(\operatorname{trop}(g))$ is called in honeycomb form if it contains a trivalent hexagonal cycle. Moreover, a tropical curve in honeycomb form is called *symmetric* when the lattice lengths of the six edges of the hexagon are equal, and the lattice lengths of the three bounded edges emerging from the hexagon are also equal. In this subsection, we examine our truncated cubic

$$f(x,y)=c_{12}(xy^2+x^2y)+c_{34}(x^2+y^2)+c_5xy+c_{67}(x+y)+c_8 \tag{5.1}$$

and investigate analogous conditions for $C(\operatorname{trop}(f))$.

Corollary 5.1. *The tropical curve of* $\operatorname{trop}(f)$ *is in honeycomb form if and only if*

$$-v_5+2v_{67}-v_8<0 \qquad -v_{34}+v_5<0 \qquad -v_{12}+v_{34}+v_5-v_{67}<0.$$

Proof. Tropical curve $C(\text{trop}(f))$ contains a trivalent hexagonal cycle if and only if its dual is a regular subdivision containing cells

$$\{[1,2,5],[1,4,5],[2,3,5],[4,5,7],[3,5,6],[5,6,7]\}.$$

Thus, this is the case (d) of Table 1. □

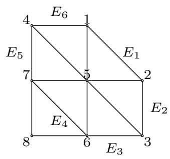
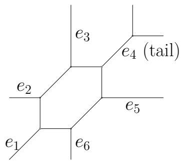
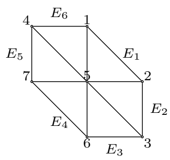
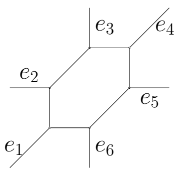
	Edges emenating from the hexagon	Subdivision	Tropical curve
(a)	Five rays and one bounded edge		
(b)	Six rays		

TABLE 6. Two types of truncated honeycomb.

Proposition 5.2 (Two types of truncated honeycomb)**.** *Let f be as defined in (5.1), and suppose the conditions outlined in Corollary 5.1 are satisfied by* $\text{trop}(f)$*. In this case, the six edges emanating from the hexagonal cycle can be classified as either:*

(a) five rays and one bounded edge (called the tail), or

(b) six rays,

as illustrated in Table 6. The cases (a),(b) occur according to whether $c_8 \neq 0$, $c_8 = 0$, respectively.

Proof. The six edges emenating from the hexagonal cycle are the duals of edges $E_1, \ldots, E_6$ of the subdivisions on Table 6. For $i = 1, 2, 3, 5, 6$, the dual of edges E_i are the rays e_i since E_i are parts of the border of Δ_f. When $c_8 \neq 0$, the Newton polygon Δ_f takes the form shown in case (a). In this scenario, edge E_4 does not lie on the border of Δ_f, resulting in its dual edge, e_4, being a bounded edge. If $c_8 = 0$, Δ_f exhibits the shape depicted in case (b). In this case, edge E_4 is part of the border of Δ_f, causing e_4 to form a ray. □

We shall say a truncated honeycomb $C(\text{trop}(f))$ to be *quasi-symmetric* if the six sides of the hexagon have the same lattice length. A quasi-symmetric truncated honeycomb is *symmetric* (following the definition in [1]) if and only if the hexagon has six emanating rays and does not possess a tail, that is of type (b) of Proposition 5.2.

Proposition 5.3 (Quasi-symmetric truncated honeycombs)**.** *Let f be as in (5.1) and suppose $C(\text{trop}(f))$ is a truncated honeycomb. Then $C(\text{trop}(f))$ is quasi-symmetric if and only if*

$$2v_{34} = v_{12} + v_{67} \text{ and } -v_5 + 2v_{67} < v_8.$$

The lattice length of the hexagon's side is $|v_{34} - v_5|$ and the tail is equal to $|v_5 - 2v_{67} + v_8|$. In particular, $C(\text{trop}(f))$ is symmetric if and only if

$$2v_{34} = v_{12} + v_{67} \text{ and } v_8 = \infty.$$

Proof. From case (a) of Table 6, a truncated honeycomb tropical curve $C(\text{trop}(f))$ is quasi-symmetric if and only if the lattice lengths of the edges dual to $[5, 1]$, $[5, 4]$, and $[5, 7]$ are equal. The lattice length can be determined by the differences of coordinates X or Y. From Table 2 in the proof of Theorem 1.1, we have Table 7 that implies the six edges of the hexagon are equal if and only if $|v_{34} - v_5| = |v_{12} - v_{34} - v_5 + v_{67}|$. From the last two inequalities of Corollary 5.1, we have $v_{34} - v_5 = v_{12} - v_{34} - v_5 + v_{67}$, thus $2v_{34} = v_{12} + v_{67}$. Together with the first inequality of Corollary 5.1, the result follows. Thus, the lattice length of the hexagon's side is $|v_{34} - v_5|$, while the tail is the dual of edge $[6, 7]$ whose lattice length is $|v_5 - 2v_{67} + v_8|$.

Edges of Δ_f	The lattice length of its duals
$[5, 1]$	$\|v_{34} - v_5\|$
$[5, 4]$	$\|v_{12} - v_{34} - v_5 + v_{67}\|$
$[5, 7]$	$\|v_{34} - v_5\|$
$[6, 7]$	$\|v_5 - 2v_{67} + v_8\|$

TABLE 7. The lattice length of some edges on $C(\text{trop}(f))$.

Meanwhile, truncated honeycomb $C(\text{trop}(f))$ is symmetric if and only if the dual of edge $[6, 7]$ has infinite lattice length. That is $|v_5 - 2v_{67} + v_8| = v_5 - 2v_{67} + v_8 = \infty$. Hence, $v_5 = \infty$ or $v_8 = \infty$. If $v_5 = \infty$, the edges $[5, i]$, where $i = 1, 2, 3, 4, 6, 7$, of the regular subdivisions on Table 6 do not exist. Thus, $v_8 = \infty$. □

Example 5.4. Let $(v_{12}, v_{34}, v_5, v_{67}) = (3, 2, 0, 1)$ such that $v_{12} \neq v_{34} \neq v_{67}$. If $v_8 = 3$, $C(\text{trop}(f))$ is a quasi-symmetric truncated honeycomb where the hexagon's sides have length 2 and the tail has length 1 as shown on

Figure 5(I). If $v_8 = \infty$, tropical curve $C(\mathrm{trop}(f))$ is a symmetric truncated honeycomb as illustrated in Figure 5(II).

(I) $(v_{12}, v_{34}, v_5, v_{67}, v_8) = (3, 2, 0, 1, 3)$ (II) $(v_{12}, v_{34}, v_5, v_{67}, v_8) = (3, 2, 0, 1, \infty)$

FIGURE 5. Quasi-symmetric and symmetric truncated honeycombs in Example 5.4.

5.2. **Nobe's one-parameter family f_k.** In [5], Nobe studied a certain piecewise linear dynamical system called the ultradiscrete QRT map and found its invariant curve to be identified with the cycle of a tropical elliptic curve. Fix $(v_{12}, v_{34}, c_{67}, v_8) \in \mathbb{R}^4$ and consider a one-parameter family of tropical curves $\{C(\mathrm{trop}(f_k))\}_{k\in\mathbb{R}}$ with

$$\mathrm{trop}(f_k)(X,Y) = \min(k + X + Y, v_{12} + X + 2Y, v_{12} + 2X + Y, \\ v_{34} + 2X, v_{34} + 2Y, v_{67} + X, v_{67} + Y, v_8).$$

According to [5] Lemma 1, there is a one-parameter family of ultradiscrete QRT maps whose invariant curve I_k coincides with the cycle part of $C(\mathrm{trop}(f_k))$ for each $k \in \mathbb{R}$.

Example 5.5 (Nobe, [5, Example 1])**.** Since we are dealing with operations $(+, \min)$ while [5] works with $(+, \max)$, substitute negative values of Nobe's parameters as follows, (See Remark 2.2),

$$v_{12} = -10 \quad v_{34} = 0 \quad v_{67} = -5 \quad v_8 = 0.$$

The invariant curves $I_k (k \in \mathbb{R})$ are classified into heptagons, pentagons, squares or (degeneration to) a point respectively for

$$k \in (-\infty, -15), [-15, -10), [-10, -7.5), [-7.5, \infty).$$

If k is in $(-\infty, -15), (-15, -10), (-10, -7.5)$, then $C(\mathrm{trop}(f_k))$ is smooth according to Theorem 1.1 (e), (c), (b), respectively. If $k = -15, -10$, and $k \geq -7.5$, then $C(\mathrm{trop}(f_k))$ is non-smooth and the case corresponds to Theorem 4.2 (c), (b), (a), respectively.

Example 5.6. Let us present the case $(v_{12}, v_{34}, v_{67}, v_8) = (0, 14, 4, 0)$. According to Theorem 1.1 (e), the cycle part of $C(\mathrm{trop}(f_k))$ forms a heptagon for $k < -10$. Similarly, Theorem 1.1 (c) and Theorem 4.2 (c) reveal that the cycle part becomes a pentagon for $-10 \leq k < -4$. Moving further, for the range $-4 \leq k < 0$, Theorem 1.1 (a) and Theorem 4.2 (e) indicate that the cycle part takes the shape of a triangle. Notably, Theorem 4.2 (d) states that when $k \geq 0$, the tropical curve $C(\text{trop}(f_k))$ does not contain any cycle. See Figure 6.

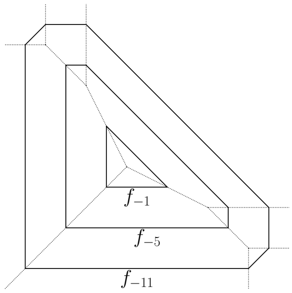


FIGURE 6. $C(\text{trop}(f_k))$ for $k = -1, -5, -11$ in Example 5.6.

5.3. **Unimodular transformation.** Let
$$g(x,y) = x^2y^2 \cdot f\left(\frac{1}{x}, \frac{1}{y}\right)$$
be the result of an integral unimodular transformation on f and we have
$$g(x,y) = c_{12}(x+y) + c_{34}(x^2+y^2) + c_5xy + c_{67}(x^2y + y^2x) + c_8x^2y^2.$$
The tropicalization of g is
$$\begin{aligned}\text{trop}(g)(X,Y) = \min(&v_{12}+X, v_{12}+Y, v_{34}+2X, v_{34}+2Y, v_5+X+Y,\\ &v_{67}+2X+Y, v_{67}+X+2Y, v_8+2X+2Y).\end{aligned}$$
In this section we will show that the conditions of Theorem 1.1 and Theorem 4.2 are invariant after such transformation.

Lemma 5.7. *Let f and g be the Laurent polynomials that are defined above. Then*
$$C(\text{trop}(f)) = -1 \cdot C(\text{trop}(g))$$
holds for the same collection of coefficients (v_{ij}).

Proof. From the tropicalization of
$$g(x,y) = f\left(\frac{1}{x}, \frac{1}{y}\right) \cdot x^2y^2,$$
we have
$$\text{trop}(g)(X,Y) = \text{trop}(f)(-X,-Y) + 2X + 2Y.$$

Since $2X + 2Y$ does not exhibit any singularities for all (X, Y), we have (X, Y) is a point on $C(\text{trop}(g))$ if and only if $(-X, -Y)$ is a point on $C(\text{trop}(f))$. In other words,

$$C(\text{trop}(g))(X, Y) = -1 \cdot C(\text{trop}(f))(X, Y)$$

holds. □

5.4. **Two-parameter family of Edwards curves $f_{r,s}$.** Let $\mathbb{K}$ be a valuated field, $q \in \mathbb{K}$ with positive valuation, and

$$\epsilon = \prod_{n=1}^{\infty}(1 + q^n) \quad \text{and} \quad \bar{\epsilon} = \prod_{n=1}^{\infty}(1 + (-q)^n).$$

In [6], we demonstrated that for certain values of $d_{12}, d_{34}, d_5, d_{67}, d_8 \in \mathbb{K}$, the polynomial

$$f_{r,s}(x, y) = d_{12}(x + y) + d_{34}(x^2 + y^2) + d_5xy + d_{67}(x^2y + y^2x) + d_8x^2y^2$$

is birationally equivalent to the Edwards curve. This equivalence allows us to gain insight into the tropicalization of $f_{r,s}$ through a theta function parametrization, akin to the approach employed by Kajiwara-Kaneko-Nobe-Tsuda [2]. Let $u_k = \text{val}(d_k)$ for $k \in \{12, 34, 5, 67, 8\}$. The tropicalization of $f_{r,s}$ is the tropical polynomial

$$\begin{aligned}\text{trop}(f_{r,s})(X, Y) = \min(&u_{12} + X, u_{12} + Y, u_{34} + 2X, u_{34} + 2Y, u_5 + X + Y,\\ &u_{67} + 2X + Y, u_{67} + 2Y + X, u_8 + 2X + 2Y).\end{aligned}$$

The coefficients d_k are parametrized by two parameters $r, s \in \mathbb{K}$ in the following way.

$$\begin{aligned} d_{12} &= 2\epsilon\bar{\epsilon}(\epsilon^4 - \bar{\epsilon}^4)(\bar{\epsilon}s - \epsilon r),\\ d_{34} &= (\epsilon^4 - \bar{\epsilon}^4)(\bar{\epsilon}^2s^2 - \epsilon^2r^2),\\ d_5 &= 8\epsilon\bar{\epsilon}(\epsilon r - \bar{\epsilon}s)(\bar{\epsilon}^3r - \epsilon^3s),\\ d_{67} &= 2(\epsilon r - \bar{\epsilon}s)\{(\bar{\epsilon}^4 - \epsilon^4)rs + 2\epsilon\bar{\epsilon}(\bar{\epsilon}^2r^2 - \epsilon^2s^2)\},\\ d_8 &= 2(\epsilon^2s^2 - \bar{\epsilon}^2r^2)(\bar{\epsilon}^2s^2 - \epsilon^2r^2). \end{aligned} \tag{5.2}$$

The values of $u_k = \text{val}(d_k)$ $(k \in \{12, 34, 5, 67, 8\})$ are contingent not only upon the valuations of r and s but also on their individual coefficients in q. The unimodular transformation discussed in §5.3 allows us to investigate $C(\text{trop}(f_{r,s}))$ in our framework using the truncated symmetric

$$\begin{aligned} h_{r,s}(x, y) :=& x^2y^2f_{r,s}(x^{-1}, y^{-1})\\ =& d_{12}(xy^2 + x^2y) + d_{34}(x^2 + y^2) + d_5xy + d_{67}(x + y) + d_8 \end{aligned}$$

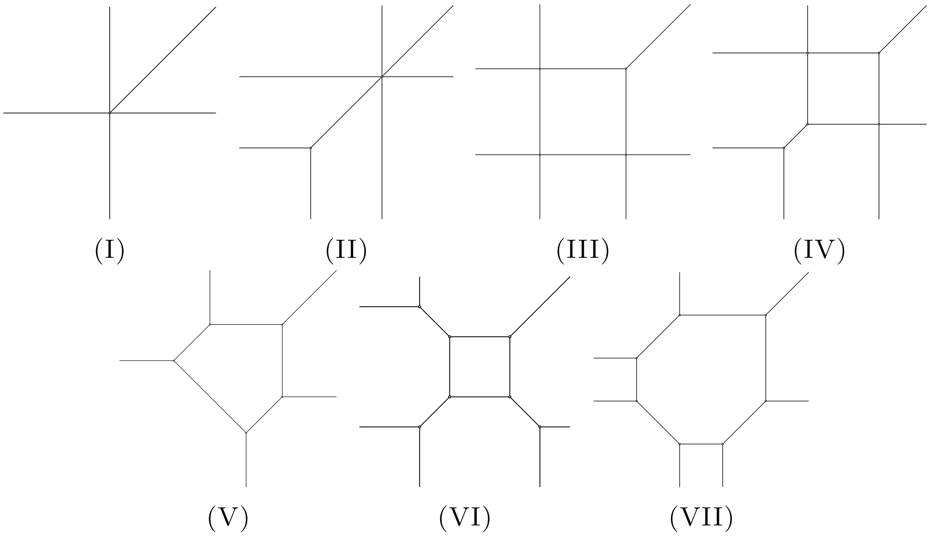


FIGURE 7. All possible tropical curves of $\mathrm{trop}(f_{r,s})$.

and hence to apply Theorem 1.1 and Theorem 4.2. It turns out that the shapes of

$$C(\mathrm{trop}(f_{r,s})) = -1 \cdot C(\mathrm{trop}(h_{r,s})) \tag{5.3}$$

are classified into forms listed in Figure 7. In particular, the cases (III), (IV), (V), (VI), (VII) having nontrivial cycles correspond to Thoerem 4.2 (f), (g), (h) and Theorem 1.1 (b), (e), respectively. We refer the readers to [6, §5] for our subsequent discussions.

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Rani Sasmita Tarmidi
Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka, 560-0043 Japan
e-mail address: u644627d@ecs.osaka-u.ac.jp

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